FREE HILBERT TRANSFORMS

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Abstract
We study Fourier multipliers of Hilbert transform type on free groups. We prove that they are completely bounded on noncommutative L^p-spaces associated with the free group von Neumann algebras for all $1 < p < \infty$. This implies that the decomposition of the free group $\mathbf{F}_\infty$ into reduced words starting with distinct free generators is completely unconditional in L^p. We study the case of Voiculescu's amalgamated free products of von Neumann algebras as well. As by-products, we obtain a positive answer to a compactness problem posed by Ozawa, a length-independent estimate for Junge–Parcet–Xu's free Rosenthal's inequality, a Littlewood–Paley–Stein-type inequality for geodesic paths of free groups, and a length reduction formula for L^p-norms of free group von Neumann algebras.

1. Introduction

The Hilbert transform is a fundamental and influential object in mathematical analysis and signal processing. It was originally defined for periodic functions. Given a trigonometric polynomial $f(z) = \sum_{k=-N}^{N} a_k z^k$, let $P_+ f = \sum_{k=0}^{N} a_k z^k$ be its analytic part, and let $P_- f = \sum_{k=-N}^{-1} a_k z^k$ be its antianalytic part. The Hilbert transform is formally defined as

$$H = -iP_+ + iP_-$$

and clearly extends to a unitary on $L^2(\mathbf{T})$. The case of L^p, $1 < p < \infty$, is more subtle. Riesz first proved that H extends to a bounded operator on $L^p(\mathbf{T})$ for all $1 < p < \infty$. It is also well known that H is unbounded on $L^p(\mathbf{T})$ at the endpoint $p = 1, \infty$ but is of weak type $(1, 1)$. In modern harmonic analysis, the Hilbert transform is considered a basic example of *Calderón–Zygmund singular integrals*. Its analogues have been studied in much more general situations with connections to L^p-approximation, Hardy/BMO spaces, and more applied subjects.

The Hilbert transform appears also as the key tool to define conjugate functions in abstract settings such as for Dirichlet algebras. In operator algebras, it shows

DUKE MATHEMATICAL JOURNAL
Vol. 166, No. 11, © 2017 DOI 10.1215/00127094-2017-0007
Received 5 July 2016. Revision received 10 February 2017.
First published online 28 April 2017.
2010 *Mathematics Subject Classification*. Primary 46L07; Secondary 46L54, 46L52.

up through Arveson's concept of maximal subdiagonal algebra of a von Neumann algebra $\mathcal{M}$. Its L^p-boundedness is well known (see [12]) and the weak-type $(1,1)$-estimate was obtained by Randrianantoanina in [13].

This article describes a natural analogue of the Hilbert transform in the context of amalgamated free products of von Neumann algebras. The study is from a different viewpoint to Arveson's and is motivated by questions in the theory of L^p-Herz–Schur multipliers on free groups.

Our model case is the von Neumann algebra $(\mathcal{L}(\mathbf{F}_\infty), \tau)$ of free group with a countable set of generators $g_1, g_2, \dots$. The associated L^p-space $L^p(\hat{\mathbf{F}}_\infty)$ is a noncommutative analogue of $L^p(\hat{\mathbf{Z}}) = L^p(\mathbf{T})$. Let $\mathcal{L}_{g_i}, \mathcal{L}_{g_i^{-1}}$ be the subsets of $\mathbf{F}_\infty$ of reduced words starting, respectively, with g_i, g_i^{-1}. One can naturally associate to them projections; given a finitely supported function $\hat{x}$ on $\mathbf{F}_\infty$, $\hat{x} = \sum_{g\in\mathbf{F}_\infty} a_g\delta_g, a_g \in \mathbb{C}$, define

$$L_{g_i}\hat{x} = \sum_{g\in\mathcal{L}_{g_i}} a_g\delta_g$$

and $L_{g_i^{-1}}\hat{x}$ similarly. In fact, all of them will obviously extend to norm 1 projections on $\ell_2(\mathbf{F}_\infty) = L^2(\hat{\mathbf{F}}_\infty)$. Natural questions are whether these projections are bounded on $L^p(\hat{\mathbf{F}}_\infty)$ and whether the decomposition $\mathbf{F}_\infty = \{e\}\bigcup_{i\in\mathbf{N},\varepsilon=\pm1}\mathcal{L}_{g_i^\varepsilon}$ is unconditional in $L^p(\hat{\mathbf{F}}_\infty)$. In this sense, we define a free analogue of the classical Hilbert transform as the following map

$$H_\varepsilon = \varepsilon_1 L_{g_1} + \varepsilon_{-1}L_{g_1^{-1}} + \varepsilon_2 L_{g_2} + \varepsilon_{-2}L_{g_2^{-1}} + \cdots \tag{1}$$

for $\varepsilon_i = \pm1$. We are interested in the (complete) boundedness of H_ε on $L^p(\hat{\mathbf{F}}_\infty)$ as well as possible connections to semigroup Hardy/BMO spaces and the L^p-approximation property in the noncommutative setting.

The question of the $L^p(\hat{\mathbf{F}}_\infty)$-boundedness of H_ε has been around for some time. The authors learned from G. Pisier that P. Biane had raised and discussed this question with him during their participation in a research semester at the Institut Henri Poincaré in 2000. Ozawa indicated that the $L^4(\hat{\mathbf{F}}_\infty)$-boundedness of H_ε would provide a positive answer to the problem he posed at the end of [8]. Junge, Parcet, and Xu [5] obtained a length-dependent estimate for a related question in their work on Rosenthal's inequality for amalgamated free products.

The first result of the present article (Theorem 3.5) is a positive answer to the L^p-boundedness question of H_ε in the general case of Voiculescu's amalgamated free products, which includes the free group as a particular case (see Theorem 4.1).

One can also consider two similar Hilbert transforms. One is

$$H_\varepsilon^{Ld} = \varepsilon_e P_{d-1} + \sum_{h,|h|=d} \varepsilon_h L_h$$

with P_d the projection onto the reduced words with length at most d, and with L_h's the projections onto the reduced words starting with h. Another is

$$H_\varepsilon^{(d)} = \varepsilon_e P_{d-1} + \sum_{g,|g|=1} \varepsilon_g L_g^{(d)}$$

with $L_g^{(d)}$'s the projections onto reduced words having g as their dth letter. Their (complete) boundedness on $L^p(\hat{\mathbf{F}}_n)$ can be easily deduced from that of H_ε with constants depending on d. The main result of the present article (Theorem 4.7) says that $H_\varepsilon^{(d)}$'s are completely bounded on $L^p(\hat{\mathbf{F}}_\infty)$ for any $d \geq 1$. While H_ε^{Ld}'s are bounded for all $1 < p < \infty$ but not completely bounded on $L^p(\hat{\mathbf{F}}_\infty)$, for any $p \neq 2, d \geq 2$. The authors also prove a length-reduction formula to compute L^p-norms and a Rosenthal-type inequality with length-independent constants.

A classical argument used in proving the L^p-boundedness of the Hilbert transform H is Cotlar's identity

$$\left|H(f)\right|^2 = |f|^2 + H(\bar{f} Hf + \overline{Hf} f), \tag{2}$$

which allows one to get the result for L^{2p} from that of L^p and implies optimal estimates. This identity holds in a general setting, if one can identify a suitable "analytic" algebra and define the corresponding Hilbert transform as the difference of the two projections on this algebra and its adjoint. This is the case of noncommutative Hilbert transforms associated with Arveson's maximal subdiagonal algebras (see [12, Lemma 8.5]).[1] After obtaining an initial proof of Theorem 4.1, we observed that a free version of Cotlar's identity (see (5)) holds in the context of amalgamated free products for H_ε with $|\varepsilon_k| \leq 1$.[2] We were slightly surprised when this observation came out, given that H_ε, defined in (1), is associated to subsets instead of subalgebras. On the other hand, finding a proof of (5) was not hard once we started to feel it. It is odd that this identity was not noticed earlier.

We introduce notation and necessary preliminaries in Section 2. Theorem 3.5 and a Cotlar-type formula for amalgamated free products are proved in Section 3.1. Section 3.2 includes a few immediate consequences. Section 3.3 obtains a length-independent Rosenthal-type inequality, which was initially proved by Junge, Parcet, and Xu [5, Theorem A] restricted to a fixed length. Section 4.1 proves our main result Theorem 4.7. Corollary 4.6 of that section gives a length reduction formula and generalizes the main result of [9]. Corollary 4.10(iii) gives a positive answer to the problem that Ozawa posed at the end of [8]. Section 4.3 studies Littlewood–Paley–Stein-type

[1] Arveson's "analytic" subalgebras do not seem available for amalgamated free products of von Neumann algebras in general. They are available for free group von Neumann algebras, but the corresponding Hilbert transforms are different from ours and their formulations as Herz–Schur multipliers are difficult to determine.
[2] The classical Cotlar formula fails for $H = -iP_+ + \varepsilon i P_-$ if $\varepsilon \neq \pm 1$.

inequalities. Corollary 4.15 shows that the projection onto a geodesic path of the free group is completely bounded on L^p for $1 < p < \infty$. Theorem 4.19 is a dyadic Littlewood–Paley–Stein inequality for geodesic paths of free groups.

2. Notation and preliminaries

We refer the reader to [15] and [5] for the definition of amalgamated free products and to [12] and the references therein for formal definitions and basic properties of noncommutative L^p-spaces. For simplicity, we will restrict to the case of finite von Neumann algebras, but it should be possible to adapt all the arguments to type III algebras with normal faithful states.

About noncommutative L^p-spaces associated to a finite von Neumann algebra $(\mathcal{A}, \tau)$, we will mainly need duality, interpolation, and the noncommutative Khintchine inequality (see [6], [7]) in $L^p(\mathcal{A})$ as well as p-row and p-column spaces. For simplicity, we denote by $e_{k1} = e_{k,1}$ and $e_{1k} = e_{1,k}$, $e_{kk} = e_{k,k}$ the canonical basis of the column and the row and diagonal subspaces of the Schatten p-class $S_p(\ell_2(\mathbf{N}))$.

We will use the duality $\langle x, y\rangle_{L^p,L^q} = \tau(xy)$ to identify $L^q(\mathcal{A})$ with $L^p(\mathcal{A})^*$ isometrically for $1 \leq p < \infty$. At the operator space level, this gives a complete isometry $L^p(\mathcal{A})^* = L^q(\mathcal{A})^{\mathrm{op}}$ (see [10]).

As $\mathcal{A} = L^\infty(\mathcal{A})$ is finite, the obvious embedding $L^\infty(\mathcal{A}) \subset L^1(\mathcal{A})$ makes $(L^\infty(\mathcal{A}), L^1(\mathcal{A}))$ a compatible couple of Banach spaces. For $1 < p < \infty$, the complex interpolation space between $\mathcal{A}$ and $L^1(\mathcal{A})$ with index $\frac{1}{p}$ is isometric to $L^p(\mathcal{A})$:

$$\big(L^\infty(\mathcal{A}), L^1(\mathcal{A})\big)_{\frac{1}{p}} = L^p(\mathcal{A}). \tag{3}$$

For a sequence (x_k) in $L^p(\mathcal{A})$, we use the classical notation

$$\|(x_k)\|_{L^p(\mathcal{A},\ell_2^c)} = \Big\|\Big(\sum_k |x_k|^2\Big)^{\frac{1}{2}}\Big\|_p, \qquad \|(x_k)\|_{L^p(\mathcal{A},\ell_2^r)} = \Big\|\Big(\sum_k |x_k^*|^2\Big)^{\frac{1}{2}}\Big\|_p,$$

and

$$\|(x_k)\|_{L^p(\mathcal{A},\ell_2^{cr})} = \begin{cases} \max\{\|(x_k)\|_{L^p(\mathcal{A},\ell_2^c)}, \|(x_k^*)\|_{L^p(\mathcal{A},\ell_2^c)}\} & \text{if } 2 \leq p \leq \infty, \\ \inf_{y_k+z_k=x_k} \|(y_k)\|_{L^p(\mathcal{A},\ell_2^c)} + \|(z_k^*)\|_{L^p(\mathcal{A},\ell_2^c)} & \text{if } 0 < p < 2. \end{cases}$$

We may often drop the reference to $\mathcal{A}$ when there is no possibility of confusion. (We refer readers to [10] for noncommutative vector-valued L^p-spaces.) The above definition is justified by the noncommutative Khintchine inequalities, as follows.

LEMMA 2.1 ([6, Théorèmes 1, 3, 4], [7, Theorem 0.1])
Let (ε_k) *be a sequence of independent Rademacher random variables. Then for* $1 \leq p < \infty$,

$$\alpha_p^{-1} E_\varepsilon \Big\| \sum_k \varepsilon_k \otimes x_k \Big\|_p \leq \|(x_k)\|_{L^p(\mathcal{A}, \ell_2^{cr})} \leq \beta_p E_\varepsilon \Big\| \sum_k \varepsilon_k \otimes x_k \Big\|_p. \tag{4}$$

Here ε_k can also be replaced by other orthonormal sequences of some $L^2(\Omega, \mu)$, for example, z^{2^k} on the unit circle or standard Gaussian. For z^{2^k} on the unit circle or standard Gaussian, the best constant β_p is $\sqrt{2}$ for $p = 1$ and is 1 for $p \geq 2$ (see [1]). We have that α_p is 1 for $1 \leq p \leq 2$ and is of order $\sqrt{p}$ as $p \to \infty$. We note that (4) was recently pushed further to the case of $0 < p < 1$ by Pisier and the second author in [11].

If $(\mathcal{A}_k, \tau_k), k \geq 1$ are finite von Neumann algebras with a common sub–von Neumann algebra $(\mathcal{B}, \tau_0)$ with conditional expectation E so that $\tau_k E = \tau_0$, then we denote by $(\mathcal{A}, \tau) = *_{\mathcal{B}}(\mathcal{A}_k, \tau_k)$ the amalgamated free product of $(\mathcal{A}_k, \tau_k)$'s over $\mathcal{B}$. We will briefly recall the construction to fix notation.

For any $x \in \mathcal{A}_k$, we denote $\mathring{x} = x - Ex$ and $\mathring{\mathcal{A}}_k = \{\mathring{x}; x \in \mathcal{A}_k\}$; there is a natural decomposition $\mathcal{A}_k = \mathcal{B} \oplus \mathring{\mathcal{A}}_k$.

The space

$$\mathcal{W} = \mathcal{B} \bigoplus_{n \geq 1} \bigoplus_{\substack{(i_1, \ldots, i_n) \in \mathbf{N}^n \\ i_1 \neq i_2 \cdots \neq i_n}} \mathring{\mathcal{A}}_{i_1} \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} \mathring{\mathcal{A}}_{i_n} = \bigoplus_{n \geq 0} \bigoplus_{\substack{(i_1, \ldots, i_n) \in \mathbf{N}^n \\ i_1 \neq i_2 \cdots \neq i_n}} \mathcal{W}_{\underline{i}}$$

is a $*$-algebra using concatenation and centering with respect to $\mathcal{B}$. The natural projection E onto $\mathcal{B}$ is a conditional expectation, and τE is a trace on $\mathcal{W}$ still denoted by τ. Then $(\mathcal{A}, \tau)$ is the finite von Neumann algebra obtained by the Gelfand–Naimark–Segal construction from $(\mathcal{W}, \tau)$. Thus $\mathcal{W}$ is weak-$*$ dense in $\mathcal{A}$ and dense in $L^p(\mathcal{A})$ for $p < \infty$.

For multi-indices, we write $(i_1, \ldots, i_n) = \underline{i} \preceq \underline{j} = (j_1, \ldots, j_m)$ if $m \geq n$ and $i_k = j_k$ for $k \leq n$. We also put $\underline{i} \prec \underline{j}$ if $\underline{i} \preceq \underline{j}$ and $n < m$, and we put $\underline{i} \not\prec \underline{j}$ otherwise. We extend those relations for nonzero elementary tensors $g \in \mathcal{W}_{\underline{i}}$ and $h \in \mathcal{W}_{\underline{j}}$, and we write $g \prec h$ if $\underline{i} \prec \underline{j}$ and $g \not\prec h$ if $\underline{i} \not\prec \underline{j}$.

For $k \in \mathbf{N}$, put

$$\mathcal{L}_k = \bigoplus_{k \preceq \underline{i}} \mathcal{W}_{\underline{i}} \qquad \text{and} \qquad \mathcal{R}_k = \mathcal{L}_k^*.$$

We denote the associated orthogonal projections on $\mathcal{W}$ by L_k and R_k. We use the convention $L_0 = E$.

Given a sequence of $\varepsilon_k \in \mathcal{B}, k \in \mathbf{N}$, and $x \in \mathcal{W}$, we let

$$H_\varepsilon(x) = \varepsilon_0 E(x) + \sum_{k\in\mathbf{N}} \varepsilon_k L_k(x); \qquad H_\varepsilon^{\mathrm{op}} = E(x)\varepsilon_0^* + \sum_{k\in\mathbf{N}} R_k(x)\varepsilon_k^*.$$

The main theorem says that, for $1 < p < \infty$, H_ε extends to L^p and for any $x \in L^p$,

$$\|H_\varepsilon x\|_p \simeq^{c_p} \|x\|_p,$$

for any choice of unitaries $\varepsilon_k \in \mathcal{Z}(\mathcal{B})$ in the center of $\mathcal{B}$ and $1 < p < \infty$.

3. Amalgamated free products

3.1. A Cotlar-type formula for free products

We start with very basic observations. Recall that $\overset{\circ}{x} = x - Ex$ for $x \in \mathcal{A}$.

PROPOSITION 3.1

For $g \in \mathcal{W}$, and $\varepsilon, \varepsilon'$ sequences in $\mathcal{B}$,

(i) $H_\varepsilon(g^*) = (H_\varepsilon^{\mathrm{op}}(g))^*$,

(ii) $H_\varepsilon(\overset{\circ}{g}) = \overset{\circ}{\overbrace{H_\varepsilon(g)}}$,

(iii) $H_\varepsilon H_{\varepsilon'}^{\mathrm{op}}(g) = H_{\varepsilon'}^{\mathrm{op}} H_\varepsilon(g)$.

Proof

This is clear on elementary tensors. □

We now give the free version of Cotlar's identity.

PROPOSITION 3.2

For elementary tensors $g, h \in \mathcal{W}$,

(iv) $\overset{\circ}{\overbrace{H_\varepsilon(g^* h)}} = \overset{\circ}{\overbrace{H_\varepsilon(g^*) h}}$ *if $g \not\succ h$,*

(v) $\overset{\circ}{\overbrace{H_\varepsilon^{\mathrm{op}}(g^* h)}} = \overset{\circ}{\overbrace{g^* H_\varepsilon^{\mathrm{op}}(h)}}$ *if $h \not\succ g$.*

And for any $g, h \in \mathcal{W}$,

(vi) $\overset{\circ}{\overbrace{H_\varepsilon(g^*) H_{\varepsilon'}^{\mathrm{op}}(h)}} = \overset{\circ}{\overbrace{H_\varepsilon(g^* H_{\varepsilon'}^{\mathrm{op}}(h))}} + \overset{\circ}{\overbrace{H_{\varepsilon'}^{\mathrm{op}}(H_\varepsilon(g^*) h)}} - \overset{\circ}{\overbrace{H_{\varepsilon'}^{\mathrm{op}} H_\varepsilon(g^* h)}}$.

Proof

Let $g = g_1 \otimes \cdots \otimes g_n \in \mathcal{W}_{\underline{i}}$ and $h = h_1 \otimes \cdots \otimes h_m \in \mathcal{W}_{\underline{j}}$ with $\underline{i} = (i_1, \ldots, i_n)$ and $\underline{j} = (i_1, \ldots, j_m)$, $n, m \geq 0$. We start by proving (iv) by induction on $n + m$.

If $n+m=0$, then this is clear as $\overbrace{H_\varepsilon(g^*h)}^{\circ}=\overbrace{H_\varepsilon(g^*)h}^{\circ}=0$. Assume that $n+m\geq 1$ and $g\not\prec h$. Note that necessarily $n\geq 1$.

First case: If $i_1\neq j_1$ or $m=0$, then

$$g^*h=g_n^*\otimes\cdots\otimes g_2^*\otimes g_1^*\otimes h_1\otimes h_2\otimes\cdots\otimes h_m$$

and $H_\varepsilon(g^*h)=H_\varepsilon(g^*)h=\varepsilon_{i_n}g^*h$.

Second case: If $i_1=j_1$, then

$$\begin{aligned}g^*h&=g_n^*\otimes\cdots\otimes g_2^*\otimes(\overbrace{g_1^*h_1}^{\circ})\otimes h_2\otimes\cdots\otimes h_m\\&\quad+(g_n^*\otimes\cdots\otimes g_2^*).\big((Eg_1^*h_1)h_2\otimes\cdots\otimes h_m\big).\end{aligned}$$

Put $\tilde g=\overbrace{h_1^*g_1}^{\circ}\otimes\cdots\otimes g_n$, $\tilde h=h_2\otimes\cdots\otimes h_m$ and $\hat g=g_2\otimes\cdots\otimes g_n$, $\hat h=(Eg_1^*h_1)h_2\otimes\cdots\otimes h_m$ (if $n=1$, $\hat g=1$). Note that $\tilde g\not\prec\tilde h$ (or $\tilde g=0$) and $\hat g\not\prec\hat h$ and that the sum of their length is strictly smaller than $n+m$. We can apply the formula of (iv) to them to get $\overbrace{H_\varepsilon(\tilde g^*\tilde h)}^{\circ}=\overbrace{H_\varepsilon(\tilde g^*)\tilde h}^{\circ}=\varepsilon_{i_n}\tilde g^*\tilde h$ and $\overbrace{H_\varepsilon(\hat g^*\hat h)}^{\circ}=\overbrace{H_\varepsilon(\hat g^*)\hat h}^{\circ}=\varepsilon_{i_n}\overbrace{\hat g^*\hat h}^{\circ}$ (this holds if $n=1$ because then $m=1$ and $\overbrace{\hat g^*\hat h}^{\circ}=0$). Finally,

$$\overbrace{H_\varepsilon(g^*h)}^{\circ}=\varepsilon_{i_n}(\tilde g^*\tilde h+\overbrace{\hat g^*\hat h}^{\circ})=\varepsilon_{i_n}\overbrace{g^*h}^{\circ}=\overbrace{H_\varepsilon(g^*)h}^{\circ}.$$

This completes the proof of (iv). Assertion (v) follows from (iv) by taking adjoints because of (i).

To get (vi) it suffices to do it for elementary tensors by linearity. Assume first that $g\not\prec h$. Then obviously $g\not\prec H_{\varepsilon'}^{\mathrm{op}}(h)$, so by (iv)

$$\overbrace{H_\varepsilon\big(g^*H_{\varepsilon'}^{\mathrm{op}}(h)\big)}^{\circ}=\overbrace{H_\varepsilon(g^*)H_{\varepsilon'}^{\mathrm{op}}(h)}^{\circ},\qquad\overbrace{H_\varepsilon(g^*)h}^{\circ}=\overbrace{H_\varepsilon(g^*h)}^{\circ}.$$

Since the centering operation commutes with $H_{\varepsilon'}^{\mathrm{op}}$ by Proposition 3.1, we get (vi).

If $g\prec h$, then $h\not\prec g$, and we can use (v) and Proposition 3.1(ii) as above and (iii) to get (vi) as

$$\begin{aligned}\overbrace{H_{\varepsilon'}^{\mathrm{op}}\big(H_\varepsilon(g^*)h\big)}^{\circ}&=\overbrace{H_\varepsilon(g^*)H_{\varepsilon'}^{\mathrm{op}}(h)}^{\circ},\ \overbrace{H_{\varepsilon'}^{\mathrm{op}}H_\varepsilon(g^*h)}^{\circ}\\&=\overbrace{H_\varepsilon H_{\varepsilon'}^{\mathrm{op}}(g^*h)}^{\circ}=\overbrace{H_\varepsilon\big(g^*H_{\varepsilon'}^{\mathrm{op}}(h)\big)}^{\circ}.\end{aligned}$$

□

Remark 3.3
By removing the centering, we obtain a Cotlar-type formula for $x = \sum_i g_i, y = \sum_j h_j, g_i, h_j \in \mathcal{W}$ as follows:

$$\begin{aligned} &H_\varepsilon x (H_{\varepsilon'} y)^* - E\big[(H_\varepsilon x - \varepsilon_0 x)(H_{\varepsilon'} y - \varepsilon'_0 y)^*\big] \\ &\quad = H_\varepsilon\big(x H^{\mathrm{op}}_{\varepsilon'}(y^*)\big) + H^{\mathrm{op}}_{\varepsilon'}\big(H_\varepsilon(x) y^*\big) - H^{\mathrm{op}}_{\varepsilon'} H_\varepsilon(x y^*). \end{aligned} \tag{5}$$

Note that the justified Cotlar identity (5) holds for all $\|\varepsilon_k\| \leq 1$, while in the commutative setting the Cotlar formula (2) holds for $\varepsilon_k = \pm 1$ only.

PROPOSITION 3.4
For any $x \in \mathcal{W}$, and any $p \geq 1$, and $\varepsilon_k \in \mathcal{Z}(\mathcal{B})$, $\|\varepsilon_k\| \leq 1$

$$\begin{aligned} &\max\Big\{\big\|E\big(H_\varepsilon x (H_\varepsilon x)^*\big)\big\|_p, \big\|E\big(H_\varepsilon\big(x H^{\mathrm{op}}_\varepsilon(x^*)\big)\big)\big\|_p, \\ &\big\|E\big(H^{\mathrm{op}}_\varepsilon\big(H_\varepsilon(x) x^*\big)\big)\big\|_p, \big\|E\big(H^{\mathrm{op}}_\varepsilon H_\varepsilon(x x^*)\big)\big\|_p\Big\} \leq \big\|E(x x^*)\big\|_p. \end{aligned}$$

Proof
Write $x = \sum_{\underline{i}} g_{\underline{i}}$ with $g_{\underline{i}} \in \mathcal{W}_{\underline{i}}$. Then, by orthogonality of the $\mathcal{W}_{\underline{i}}$ over $\mathcal{B}$, all four elements on the left-hand side are of the form $\sum_{\underline{i}} a_{\underline{i}} E(g_{\underline{i}} g_{\underline{i}}^*) b_{\underline{i}}^*$ with $a_{\underline{i}}, b_{\underline{i}} \in \{1, \varepsilon_{i_n}\} \subset \mathcal{Z}(\mathcal{B})$. But $\sum_{\underline{i}} a_{\underline{i}} E(g_{\underline{i}} g_{\underline{i}}^*) b_i^* = \sum_{\underline{i}} y_{\underline{i}} a_{\underline{i}} b_{\underline{i}}^* y_{\underline{i}}$ with $y_{\underline{i}} = E(g_{\underline{i}} g_{\underline{i}}^*)^{1/2}$ so that the inequality follows by the Hölder inequality as $\sum_{\underline{i}} y_{\underline{i}}^2 = E(x x^*)$. □

We can now prove the main result.

THEOREM 3.5
For $1 < p < \infty$, there is a constant c_p so that for $\varepsilon_k \in \mathcal{Z}(\mathcal{B})$, $\|\varepsilon_k\| \leq 1$, and $x \in \mathcal{W}$

$$\|H_\varepsilon x\|_p \leq c_p \|x\|_p, \qquad \|H^{\mathrm{op}}_\varepsilon x\|_p \leq c_p \|x\|_p. \tag{6}$$

Moreover, the equivalence holds with constant c_p in both directions if the ε_k's are further assumed to be unitaries.

Proof
Assume here that $\|H_\varepsilon\|_{L^p(\mathcal{A}) \to L^p(\mathcal{A})} \leq c_p$. We will now show that $\|H_\varepsilon x\|_{2p} \leq (c_p + \sqrt{2c_p^2 + 4})\|x\|_{2p}$ for all $x \in \mathcal{W}$, and similarly for $H^{\mathrm{op}}_\varepsilon$ by using the $*$-operation. Once this is proved, we get the upper desired estimate for all $p = 2^n, n \in \mathbf{N}$, by induction and the fact that $\|H_\varepsilon x\|_2 = \|H^{\mathrm{op}}_\varepsilon x\|_2 \leq \|x\|_2$. Applying interpolation and duality, we then get the result for all $1 < p < \infty$ (note that the adjoint of H_ε is $H^{\mathrm{op}}_{\varepsilon^*}$). The equivalence holds for unitary ε since $H_\varepsilon H_{\varepsilon^*} = \mathrm{id}_{\mathcal{A}}$ in this case. In fact, the Cotlar-

type formula (5) implies that, for $x, y \in \mathcal{W}$,

$$\overbrace{H_\varepsilon x (H_{\varepsilon'} y)^*}^{\circ} = \overbrace{H_\varepsilon\big(x H_{\varepsilon'}^{\text{op}}(y^*)\big)}^{\circ} + \overbrace{H_{\varepsilon'}^{\text{op}}\big(H_\varepsilon(x) y^*\big)}^{\circ} - \overbrace{H_{\varepsilon'}^{\text{op}} H_\varepsilon(x y^*)}^{\circ}. \tag{7}$$

Applying Hölder's inequality and Proposition 3.4 to this identity for $x = y, \varepsilon = \varepsilon'$, we get

$$\|H_\varepsilon x\|_{2p}^2 \leq 2c_p \|x\|_{2p} \|H_\varepsilon x\|_{2p} + (4 + c_p^2)\|x\|_{2p}^2.$$

That is, $\|H_\varepsilon x\|_{2p} \leq (c_p + \sqrt{2c_p^2 + 4})\|x\|_{2p}$. □

Remark 3.6
As $\prod_{n=0}^\infty \frac{1+\sqrt{2+4/c_{2^n}^2}}{1+\sqrt{2}} < \infty$, one gets that for $p \geq 2$, $c_p \leq Cp^\gamma$ with $\gamma = \frac{\ln(1+\sqrt{2})}{\ln 2}$.

Remark 3.7
By the usual trick to replace $\mathcal{B}, \mathcal{A}_k$ by $\mathcal{B} \otimes M_n$ and $\mathcal{A}_k \otimes M_n$, one gets that the maps H_ε are completely bounded on L^p for $1 < p < \infty$.

Remark 3.8
We can use a slightly more general definition for H_ε by taking $\varepsilon_k \in \mathcal{B} \otimes \mathcal{M}$, where $\mathcal{M}$ is a finite von Neumann algebra. Then $E(x)$ and $L_k(x)$ have to be understood as $E(x) \otimes 1$ and $L_k(x) \otimes 1$. Theorem 3.5 remains valid with the assumption that $\varepsilon \in \mathcal{Z}(\mathcal{B}) \otimes \mathcal{M}$.

3.2. *Corollaries*

In this section, we derive a few direct consequences of Theorem 3.5.

For any $k_0 \in \mathbf{N}$, let $\varepsilon_{k_0} = -1$ and $\varepsilon_k = 1$ for $k \neq k_0$. Then $L_{k_0} = \frac{\mathrm{id}_{\mathcal{A}} - H_\varepsilon}{2}$.

COROLLARY 3.9
For any $1 < p < \infty$,

$$\|L_k x\|_p \leq \frac{1 + c_p}{2} \|x\|_p.$$

COROLLARY 3.10
We have

$$c_p^{-1}\|x\|_p \leq \big\|(L_k x)_{k=0}^\infty\big\|_{L^p(\ell_2^{cr})} \leq \sqrt{2} c_p \|x\|_p, \tag{8}$$

for $1 < p < 2$ *and*

$$(\sqrt{2}c_p)^{-1}\|x\|_p \le \big\|(L_k x)_{k=0}^{\infty}\big\|_{L^p(\ell_2^{cr})} \le c_p\|x\|_p, \tag{9}$$

for $2 \le p < \infty$. *Similar inequalities hold for* $(R_k x)$.

Proof
Let $\varepsilon_k = z^{2^k}$, and let E_ε be the expectation. By Theorem 3.5, for any $x \in L^p$,

$$\frac{1}{c_p} E_\varepsilon\big\|H_\varepsilon(x)\big\|_p \le \|x\|_p \le c_p E_\varepsilon\big\|H_\varepsilon(x)\big\|_p = c_p E_\varepsilon\Big\|\sum_k \varepsilon_k L_k x\Big\|_p.$$

We then have, by the noncommutative Khintchine inequality (4), that

$$(\alpha_p c_p)^{-1}\|x\|_p \le \big\|(L_k x)_{k=0}^{\infty}\big\|_{L^p(\ell_2^{cr})} \le \beta_p c_p\|x\|_p. \tag{10}$$

This implies (8) since $\beta_p \le \sqrt{2}, \alpha_p = 1$ for $1 < p < 2$. Using the identity $\tau x^* y = \tau\sum_{k=0}^{\infty}(L_k x)^* L_k y$, an additional duality argument implies that

$$(\beta_q c_q)^{-1}\|x\|_p \le \big\|(L_k x)_{k=0}^{\infty}\big\|_{L^p(\ell_2^{cr})}, \tag{11}$$

for $1 < p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1$. We then get (9) since $c_p = c_q$ and $\beta_p = 1$ for $2 \le p < \infty$ (see the comment after Lemma 2.1). A similar argument works for $(R_k x)$. □

Remark 3.11
We will prove a variant of Corollary 3.10 in Section 3.3 as Theorem 3.17.

COROLLARY 3.12
For any $1 < p < \infty$, *any sequences* $(i_k) \in \mathbf{N}^{\mathbf{N}}$ *and* $(x_k) \in L^p(\mathcal{A}, \ell_2^c)$, *we have*

$$\Big\|\Big(\sum_{k=1}^{\infty}|L_{i_k} x_k|^2\Big)^{\frac{1}{2}}\Big\|_p \le c_p\Big\|\Big(\sum_{k=1}^{\infty}|x_k|^2\Big)^{\frac{1}{2}}\Big\|_p, \tag{12}$$

$$\Big\|\Big(\sum_{k=1}^{\infty}|R_{i_k} x_k|^2\Big)^{\frac{1}{2}}\Big\|_p \le c_p\Big\|\Big(\sum_{k=1}^{\infty}|x_k|^2\Big)^{\frac{1}{2}}\Big\|_p. \tag{13}$$

Proof
Fix a sequence $\varepsilon_k = \pm 1$, and apply Theorem 3.5 to $x = \sum_j \varepsilon_{i_j} x_j \otimes e_{j1} \in L^p(\mathcal{A} \otimes B(\ell_2))$. We have

$$\Big\|\sum_{k,j}\varepsilon_k\varepsilon_{i_j} L_k(x_j) \otimes e_{j1}\Big\|_p \le c_p\Big\|\sum_j \varepsilon_{i_j} x_j \otimes e_{j1}\Big\|_p = c_p\Big\|\Big(\sum_{j=1}^{\infty}|x_j|^2\Big)^{\frac{1}{2}}\Big\|_p.$$

Now let (ε_k) be a sequence of Rademacher variables. We have

$$\Big\|\Big(\sum_{j=1}^{\infty}|L_{i_j}x_j|^2\Big)^{\frac{1}{2}}\Big\|_p=\Big\|E_\varepsilon\sum_{k,j}\varepsilon_k\varepsilon_{i_j}L_k(x_j)\otimes e_{j1}\Big\|_p\le c_p\Big\|\Big(\sum_{j=1}^{\infty}|x_j|^2\Big)^{\frac{1}{2}}\Big\|_p.$$

The proof of the second inequality is similar. □

Remark 3.13

Corollary 3.12 was proved in [5, Lemma 2.5, Corollary 2.9] for the x_k's supported on reduced words with length equal to d with constants depending on d, independent of p.

3.3. *Length-independent estimates for Rosenthal's inequality*

We will apply Theorem 3.5 to obtain a length-free estimate for Rosenthal's inequality proved in [5, Theorem A]. In this section, we restrict $\varepsilon\in\{\pm1\}^{\mathbf{N}}$ and $\varepsilon_0=0$ in the form of $H_\varepsilon=\sum_{k\in\mathbf{N}}\varepsilon_kL_k$ and $H_\varepsilon^{\mathrm{op}}$. When there is no chance of confusion, we use the notation T instead of $T\otimes\mathrm{Id}$ for its ampliation.

Thanks to the previous results, we can define the following paraproduct for $x\in L^p(\mathcal{A})\otimes L^p(\mathcal{M})$ $(1<p<\infty)$ and $y\in L^q(\mathcal{A})\otimes L^q(\mathcal{M})$ with $\frac{1}{p}+\frac{1}{q}<1$ as

$$x\ddagger y=E_\varepsilon H_\varepsilon\big(H_\varepsilon(x)y\big)=\sum_{k\in\mathbf{N}}L_k\big((L_kx)y\big),$$

with E_ε the expectation with respect to the Haar measure on $\{\pm1\}^{\mathbf{N}}$. We also set

$$x\dagger y=xy-x\ddagger y-E(xy)=\sum_{k=0}^{\infty}\mathring{L}_k^{\perp}\big((L_kx)y\big).$$

Here $\mathring{L}_k^{\perp}=\sum_{j\neq k,j\in\mathbf{N}}L_j$ for any $k\ge0$.

If x and y are elementary tensors $(x\notin\mathcal{B})$, $x\ddagger y$ collects in the reduced form of xy all elements whose first letter is in the same algebra as x, while $x\dagger y$ collects the rest in the reduced form of xy, except the constant terms.

PROPOSITION 3.14

We have the following, for $1<p<\infty$, $1<q\le\infty$ *with* $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}<1$*:*

(i) $\|H_\varepsilon(x\ddagger y)\|_r\le c_rc_p\|x\|_p\|y\|_q$, $\|x\dagger y\|_r\le(2+c_rc_p)\|x\|_p\|y\|_q$;

(ii) $H_\varepsilon(x\ddagger y)=H_\varepsilon(x)\ddagger y$, $x\dagger H_\varepsilon^{\mathrm{op}}(y)=H_\varepsilon^{\mathrm{op}}(x\dagger y)$.
In particular, $x\ddagger y\in\mathcal{L}_k$ *if* $x\in\mathcal{L}_k$ *and* $x\dagger y\in\mathcal{R}_k$ *if* $y\in\mathcal{R}_k$.

Proof

Inequality (i) simply follows from Theorem 3.5 and the definitions. We now prove (ii). For $\ddagger$, this follows from its definition.

For †, we check the following formula from which the identity follows because of the translation invariance of the Haar measure on $\{-1,1\}^{\mathbf{N}}$:

$$x \dagger y = E_{\varepsilon'}\big(H^{\mathrm{op}}_{\varepsilon'}\big(x \dagger H^{\mathrm{op}}_{\varepsilon'}(y)\big)\big). \tag{14}$$

We first notice that the identity holds if $x \in \mathcal{B}$ as $x \ddagger y = 0$ and $x \dagger y = x(y - E(y))$ and similarly holds if $y \in \mathcal{B}$, $x \ddagger y = (x - E(x))y$ and $x \dagger y = 0$. Thus we can assume that $E(x) = E(y) = 0$. Apply the Cotlar identity (5) to $H_\varepsilon(x)$ and $H^{\mathrm{op}}_{\varepsilon'}(y^*)$, and note that $H^2_\varepsilon(x) = x$ and $H^{\mathrm{op}2}_{\varepsilon'}(y^*) = y^*$. We get

$$xy - Exy = H_\varepsilon\big(H_\varepsilon(x)y\big) + H^{\mathrm{op}}_{\varepsilon'}\big(xH^{\mathrm{op}}_{\varepsilon'}(y)\big) - H^{\mathrm{op}}_{\varepsilon'}H_\varepsilon\big(H_\varepsilon(x)H^{\mathrm{op}}_{\varepsilon'}(y)\big).$$

Taking expectations with respect to ε and ε' gives (14). One can also verify directly the identity for † in (ii) by its bilinearity, looking at elementary tensors $x, y \in \mathcal{W}$, and by using Proposition 3.2(iv)–(v). □

Remark 3.15
There are situations for which one can slightly improve those inequalities. For instance, if $r = 2$, then $\|x \dagger y\|_r \le (1 + c_p)\|x\|_p\|y\|_q$. Or in general $\|x \ddagger y\|_r \le c_r \sup_\varepsilon \|H_\varepsilon(x)\|_p\|y\|_q$ and $\|x \dagger y\|_r \le (2 + c_r)\sup_\varepsilon \|H_\varepsilon(x)\|_p\|y\|_q$.

LEMMA 3.16
For $2 \le p < \infty$ and $x \in L^p(\mathcal{A})$,

$$\Big\|\sum_{k\in\mathbf{N}} \overbrace{L_k(x)L_k(x)^*}^{\circ}\Big\|_{\frac{p}{2}} \le \gamma_p \Big\|\sum_{k\in\mathbb{N}} \big|(L_kx)^*\big|^2\Big\|^{\frac12}_{\frac{p}{2}}\Big(\sum_{k\in\mathbb{N}}\|L_kx\|^p_p\Big)^{\frac1p}, \tag{15}$$

$$\Big\|\sum_{k\in\mathbf{N}} \overbrace{R_k(x)^*R_k(x)}^{\circ}\Big\|_{\frac{p}{2}} \le \gamma_p \Big\|\sum_{k\in\mathbf{N}} \big|R_k(x)\big|^2\Big\|^{\frac12}_{\frac{p}{2}}\Big(\sum_{k\in\mathbb{N}}\|R_kx\|^p_p\Big)^{\frac1p}, \tag{16}$$

with $\gamma_p \le 3c_4^2$ for $2 < p \le 4$ and $\gamma_p \le 2\sqrt{2}(c^2_{\frac{p}{2}} + c_{\frac{p}{2}})$ for $p \ge 4$.

Proof
Let us assume that $p \ge 4$ first. We use the decomposition $L_k(x)L_k(x)^* - E(L_k(x)L_k(x)^*) = L_kx \ddagger (L_kx)^* + L_kx \dagger (L_kx)^*$. By Corollary 3.10 and Proposition 3.14, as $L_k(x) \ddagger L_k(x)^* \in \mathcal{L}_k$, we have

$$\Big\|\sum_{k\in\mathbf{N}} L_k(x)\ddagger L_k(x)^*\Big\|_{\frac{p}{2}}$$
$$\le \sqrt{2}c_{\frac{p}{2}} \max\Big[\Big\|\sum_{k\in\mathbf{N}} L_k(x)\ddagger L_k(x)^* \otimes e_{k1}\Big\|_{\frac{p}{2}}, \Big\|\sum_{k\in\mathbf{N}} L_k(x)\ddagger L_k(x)^* \otimes e_{1k}\Big\|_{\frac{p}{2}}\Big].$$

Using the bilinearity of ‡, we have

$$\begin{aligned}\sum_{k\in\mathbf{N}} L_k(x)\ddagger L_k(x)^*\otimes e_{1k} &= \sum_{k\in\mathbf{N}}\big(L_k(x)\otimes e_{1k}\big)\ddagger\big(L_k(x)^*\otimes e_{kk}\big)\\ &= E_\varepsilon\Big(\sum_k \varepsilon_k L_k(x)\otimes e_{1k}\Big)\ddagger\Big(\sum_k \varepsilon_k L_k(x)^*\otimes e_{kk}\Big)\\ &= E_\varepsilon\Big[H_\varepsilon\Big(\sum_k L_k(x)\otimes e_{1k}\Big)\ddagger H_\varepsilon^{\mathrm{op}}\Big(\sum_k L_k(x)^*\otimes e_{kk}\Big)\Big].\end{aligned}$$

So we can conclude from Theorem 3.5 and Remark 3.15 that

$$\begin{aligned}&\Big\|\sum_{k\in\mathbf{N}} L_k(x)\ddagger L_k(x)^*\otimes e_{1k}\Big\|_{\frac{p}{2}}\\ &\quad\le c_{\frac{p}{2}}\sup_{\varepsilon,\varepsilon'}\Big\|H_{\varepsilon'}\sum_k L_k x\otimes e_{1k}\Big\|_p\Big\|H_\varepsilon^{\mathrm{op}}\sum_k L_k(x)^*\otimes e_{kk}\Big\|_p\\ &\quad= c_{\frac{p}{2}}\Big\|\sum_k L_k(x)\otimes e_{1k}\Big\|_p\Big(\sum_k\|L_k x\|_p^p\Big)^{\frac{1}{p}}.\end{aligned}\tag{17}$$

Similarly, we have

$$\begin{aligned}&\Big\|\sum_{k\in\mathbf{N}} L_k(x)\ddagger L_k(x)^*\otimes e_{k1}\Big\|_{\frac{p}{2}}\\ &\quad= \Big\|E_\varepsilon\Big[H_\varepsilon\Big(\sum_k L_k(x)\otimes e_{kk}\Big)\ddagger H_\varepsilon^{\mathrm{op}}\Big(\sum_k L_k(x)^*\otimes e_{k1}\Big)\Big]\Big\|_{\frac{p}{2}}\\ &\quad\le c_{\frac{p}{2}}\Big(\sum_k\|L_k x\|_p^p\Big)^{\frac{1}{p}}\Big\|\sum_k L_k(x)\otimes e_{1k}\Big\|_p.\end{aligned}\tag{18}$$

Combining these two estimates, we get

$$\Big\|\sum_{k\in\mathbf{N}} L_k(x)\ddagger L_k(x)^*\Big\|_{\frac{p}{2}}\le\sqrt{2}c_{\frac{p}{2}}^2\Big\|\sum_k L_k(x)\otimes e_{1k}\Big\|_p\Big(\sum_k\|L_k x\|_p^p\Big)^{\frac{1}{p}}$$

for $p\ge 4$. We can treat the † term similarly since $L_k x\dagger(L_k x)^*\in\mathcal{R}_k$ and get

$$\Big\|\sum_{k\in\mathbf{N}} L_k(x)\dagger L_k(x)^*\Big\|_{\frac{p}{2}}\le\sqrt{2}c_{\frac{p}{2}}(2+c_{\frac{p}{2}})\Big\|\sum_k L_k(x)\otimes e_{1k}\Big\|_p\Big(\sum_k\|L_k x\|_p^p\Big)^{\frac{1}{p}}.$$

We then get (15) for $p\ge 4$ with constant $2\sqrt{2}(c_{\frac{p}{2}}^2+c_{\frac{p}{2}})$.

To deal with the remaining cases, we will use interpolation by proving a better bilinear inequality for $2\le p\le 4$. We have

$$\Big\| \sum_{k\in\mathbf{N}} \overbrace{L_k(x)R_k(y)}^{\circ} \Big\|_{\frac{p}{2}} \le 3c_4^2 \Big\| \sum_k L_k x \otimes e_{1k} \Big\|_p \Big(\sum_k \|R_k y\|_p^p \Big)^{\frac{1}{p}}.$$

The spaces consisting of elements of the form $\sum_k L_k x \otimes e_{1k}$, and $\sum_k R_k y \otimes e_{kk}$ are c_p-complemented in $L^p(\mathcal{A}) \otimes S_p$ by Theorem 3.5. Hence, the norms on the right-hand side interpolate for $2 \le p \le 4$ (both with constant $c_4^{2(1-2/p)}$).

We just need to justify the endpoint inequalities. For $p = 2$, we have by Hölder's inequality that

$$\begin{aligned} \Big\| \sum_{k\in\mathbf{N}} \overbrace{L_k(x)R_k(y)}^{\circ} \Big\|_1 &\le 2 \Big\| \sum_k L_k(x) \otimes e_{1k} \Big\|_2 \Big\| \sum_k R_k(y) \otimes e_{k1} \Big\|_2 \\ &= 2 \Big\| \sum_k L_k(x) \otimes e_{1k} \Big\|_2 \Big(\sum_k \|L_k y\|_2^2 \Big)^{\frac{1}{2}}. \end{aligned}$$

For $p = 4$, by orthogonality and as in (17),

$$\begin{aligned} \Big\| \sum_{k\in\mathbf{N}} L_k(x) \ddagger R_k(y) \Big\|_2 &= \Big\| \sum_{k\in\mathbf{N}} L_k(x) \ddagger R_k(y) \otimes e_{1k} \Big\|_2 \\ &\le \Big\| \sum_k L_k x \otimes e_{1k} \Big\|_4 \Big(\sum_k \|R_k y\|_4^4 \Big)^{\frac{1}{4}}. \end{aligned}$$

Similarly, we get $\| \sum_{k\in\mathbf{N}} L_k(x) \dagger R_k(y)\|_2 \le 2 \| \sum_k L_k x \otimes e_{1k}\|_4 (\sum_k \|R_k y\|_4^4)^{\frac{1}{4}}$. Thus, by interpolation we get (15) for $2 < p < 4$ with a constant $3c_4^{4(1-2/p)}$. □

THEOREM 3.17
For $2 \le p < \infty$ and $x \in L^p(\mathcal{A})$,

$$\eta_p^{-1} \|x\|_p \le \max\Big\{ \Big\| \Big(\sum_{k=0}^{\infty} |L_k(x)|^2 \Big)^{\frac{1}{2}} \Big\|_p, \|E(xx^*)\|_{\frac{p}{2}}^{\frac{1}{2}} \Big\} \le c_p \|x\|_p$$

and

$$\eta_p^{-1} \|x\|_p \le \max\Big\{ \Big\| \Big(\sum_{j=0}^{\infty} |R_j(x)^*|^2 \Big)^{\frac{1}{2}} \Big\|_p, \|E(x^*x)\|_{\frac{p}{2}}^{\frac{1}{2}} \Big\} \le c_p \|x\|_p,$$

with $\eta_p \le \sqrt{2} c_p (1+\gamma_p) \lesssim c_p^3$.

Proof
For the first equivalence, the upper inequality follows from Corollary 3.10. For the

lower bound, by Lemma 3.16 as $E(xx^*) = \sum_{k\geq 0} E(L_k(x)L_k(x)^*)$ and $\overset{\circ}{\overbrace{L_0(x)L_0(x)^*}} = 0$,

$$\Big\|\sum_{k\geq 0} L_k(x)\otimes e_{1k}\Big\|_p^2 \leq \gamma_p \Big\|\sum_{k\in \mathbf{N}} L_k(x)\otimes e_{kk}\Big\|_p \Big\|\sum_{k\geq 0} L_k(x)\otimes e_{1k}\Big\|_p + \big\|E(xx^*)\big\|_{\frac{p}{2}}.$$

Hence, $\| \sum_{k\geq 0} L_k(x)\otimes e_{1k}\|_p \leq \gamma_p \| \sum_{k\geq 0} L_k(x)\otimes e_{kk}\|_p + \|E(xx^*)\|_{\frac{p}{2}}^{\frac{1}{2}}$. But as $p\geq 2$, the map $e_{k1}\mapsto e_{kk}$ is a contraction on L^p, so we deduce that

$$\Big\|\sum_{k\geq 0} L_k(x)\otimes e_{1k}\Big\|_p \leq \gamma_p \Big\|\sum_{k\geq 0} L_k(x)\otimes e_{k1}\Big\|_p + \big\|E(xx^*)\big\|_{\frac{p}{2}}^{\frac{1}{2}},$$

and we conclude the lower bound by Corollary 3.10 again. The other inequality follows by taking adjoints. □

We get the following Rosenthal-type inequality as a direct application.

COROLLARY 3.18

Let $2 < p < \infty$.

(i) *For* $x = \sum_{k=0}^{\infty} x_k \in L^p(\mathcal{A})$ *with* $x_k \in \mathcal{L}_k$, *we have*

$$\eta_p^{-2}\|x\|_p \leq \max\Big\{\big\|E(xx^*)\big\|_{\frac{p}{2}}^{\frac{1}{2}}, \big\|E(x^*x)\big\|_{\frac{p}{2}}^{\frac{1}{2}}, \Big\|\sum_{k,j} R_j(x_k)\otimes e_{kj}\Big\|_p\Big\}$$
$$\leq c_p^2\|x\|_p.$$

(ii) *For* $x = \sum_{k=0}^{\infty} x_k$ *with* $x_k \in \mathcal{L}_k \cap \mathcal{R}_k$, *we have*

$$\eta_p^{-2}\|x\|_p \leq \max\Big\{\big\|E(xx^*)\big\|_{\frac{p}{2}}^{\frac{1}{2}}, \big\|E(x^*x)\big\|_{\frac{p}{2}}^{\frac{1}{2}}, \Big(\sum_k \|x_k\|_p^p\Big)^{\frac{1}{p}}\Big\} \leq c_p^2\|x\|_p.$$

Proof

Apply Theorem 3.17 twice, and note that the $(e_{1k}\otimes e_{k1})$'s generate the canonical basis of ℓ_p in S_p. We get (i), and (ii) follows immediately. □

Remark 3.19

We point out that Corollary 3.18(ii) was proved in [5, Theorem A] when the x_k's are supported on reduced words with a fixed length with constants independent of p but dependent on the length. Noting that by the Khintchine inequalities from [14], H_ε and $H_\varepsilon^{\mathrm{op}}$ are bounded on words of length at most d (in L^∞) with a constant that depends only on d, we see that, by interpolation, the above argument also implies Corollary 3.18(ii) with constants independent of p but dependent on the length.

Remark 3.20
All the results of this section also hold in the completely bounded setting.

4. Free groups

We can apply the previous results to the free group as it is naturally a free product. Let g_i, $i \in \mathbf{N}$ be the set of generators of $\mathbf{F}_\infty$. We let $\mathcal{L}_{g_i}$ and $\mathcal{L}_{g_i^{-1}}$ be the set of reduced words starting by g_i and g_i^{-1}, respectively, and $\mathcal{L}_{g_i^\pm} = \mathcal{L}_{g_i} \cup \mathcal{L}_{g_i^{-1}}$. We denote by L_{g_i}, $L_{g_i^{-1}}$, and $L_{g_i^\pm}$ the associated projections. We use the notation $\mathcal{R}_{g_i}$ and $\mathcal{R}_{g_i^{-1}}, \dots$ for the right analogues. We will often use the convention $g_0 = e, L_{g_0} = \tau$, $g_i = g_{-i}^{-1}$ for $i < 0$ so that $L_{g_i^{-1}} = L_{g_{-i}}$ for any $i \in \mathbf{Z}^*$. Finally, S will denote the set $\{g_i; i \in \mathbf{Z}^*\}$. Given g, h reduced words of $\mathbf{F}_\infty$, we write $g \leq h$ (or $h \geq g$) if $h = gk$ with g, h, k reduced words; that is, $|g^{-1}h| = |h| - |g|$. We write $g \nleq h$ otherwise. More generally, we set

$$\mathcal{L}_h := \{g \in \mathbf{F}_\infty; h \leq g\},$$

and we let L_h be the associated L^2-projection. Let $\mathcal{M}$ be a finite von Neumann algebra. We will consider $x \in L^p(\mathcal{L}\mathbf{F} \otimes \mathcal{M})$. When there is no possibility for confusion, we use the notation $x = \sum_g a_g \lambda_g$ instead of $x = \sum_g \lambda_g \otimes a_g, a_g \in L^p(\mathcal{M})$ for its ampliation.

Theorem 3.5 immediately gives that, for any $1 < p < \infty$ and sequences of unitaries $\varepsilon_i \in \mathcal{Z}(\mathcal{M})$, $\|\varepsilon_k\| \leq 1$,

$$\Big\|(\tau \otimes \mathrm{Id})x + \sum_i \varepsilon_i (L_{g_i^\pm} \otimes \mathrm{Id})(x)\Big\|_p \simeq^{c_p} \|x\|_p. \tag{19}$$

We slightly extend it as follows.

THEOREM 4.1
Let $(\varepsilon_k)_{k\in\mathbf{Z}}$ be a sequence in $\mathcal{Z}(\mathcal{M})$, $\|\varepsilon_k\| \leq 1$. Then for any $x \in L^p(\mathcal{L}(\mathbf{F}_\infty) \otimes \mathcal{M})$ and $1 < p < \infty$,

$$\Big\|\varepsilon_0(\tau \otimes \mathrm{Id})x + \sum_{k\in\mathbf{Z}^*}^{\infty} \varepsilon_k (L_{g_k} \otimes \mathrm{Id})(x)\Big\|_p \leq c_p \|x\|_p.$$

The equivalence holds if we assume further that the ε_k's are unitaries in $\mathcal{Z}(\mathcal{M})$.

Proof
We may assume that $\varepsilon_0 = 1$. We consider the following group embedding $\pi : \mathbf{F}_\infty \to \mathbf{F}_\infty * \mathbf{F}_\infty$ defined by $\pi(g_i) = g_i h_i$, where the (h_i)'s are the free generators of the second copy of $\mathbf{F}_\infty$. This extends to a complete isometry for L^p-spaces, and one

checks directly that

$$\Big(\sum_{k=0}^{\infty}\varepsilon_{-k}L_{h_k^{\pm}}+\tau+\sum_{k=0}^{\infty}\varepsilon_k L_{g_k^{\pm}}\Big)\circ\pi=\pi\circ\Big(\sum_{k=0}^{\infty}\varepsilon_{-k}L_{g_k^{-1}}+\tau+\sum_{k=0}^{\infty}\varepsilon_k L_{g_k}\Big).$$

The statement follows from the amalgamated version of (19). □

The proof of Lemma 3.16 and Theorem 3.17 can easily be adapted to the free group, where $H_\varepsilon=\varepsilon_e L_e+\sum_{h\in S}\varepsilon_h L_h$ with $|\varepsilon_h|=1$ and the convention $L_e x=\tau x$. We simply give the result as follows.

THEOREM 4.2
For $2<p<\infty$, $x\in L^p(\mathcal{L}(\mathbf{F}_\infty)\otimes\mathcal{M})$,

$$\eta_p^{-1}\|x\|_p\le\max\Big\{\Big\|\Big(\sum_{|h|\le 1}\big|L_h(x)\big|^2\Big)^{\frac12}\Big\|_p,\ \big\|(\tau\otimes\mathrm{Id})(xx^*)\big\|_{\frac{p}{2}}^{\frac12}\Big\}\le c_p\|x\|_p$$

and

$$\eta_p^{-1}\|x\|_p\le\max\Big\{\Big\|\Big(\sum_{|h|\le 1}\big|R_h(x)^*\big|^2\Big)^{\frac12}\Big\|_p,\ \big\|(\tau\otimes\mathrm{Id})(x^*x)\big\|_{\frac{p}{2}}^{\frac12}\Big\}\le c_p\|x\|_p.$$

Remark 4.3
All results before this section hold for free groups with L_k, R_k replaced by L_{g_k} (resp., $L_{g_k^{-1}}$ or $L_{g_k^{\pm}}$) and R_{g_k} (resp., $R_{g_k^{-1}}$ or $R_{g_k^{\pm}}$). We can strengthen some of them. These will be recorded in the following.

COROLLARY 4.4
For any $1<p<\infty$, $h\in\mathbf{F}_\infty$ and $x\in L^p(\mathcal{L}(\mathbf{F}_\infty)\otimes\mathcal{M})$,

$$\|L_h x\|_p\le\frac{c_p+1}{2}\|x\|_p.$$

Moreover, $\lim_{|h|\to\infty}\|L_h x\|_p\to 0$.

Proof
Without loss of generality, we may assume that $h\in\mathcal{R}_{g_1}$ and that $h=h'g_1$. Then $L_h x=\lambda_{h'}L_{g_1}(\lambda_{h'^{-1}}x)$. The L^p-bound follows from Theorem 4.1. Note that the L^p-space is defined as the closure of $C_c(\mathbf{F}_\infty)$, so we get the convergence by the uniform boundedness of L_h on L^p. □

COROLLARY 4.5
For any $1 < p < \infty$, any sequences $(h_k) \in \mathbf{F}_\infty \setminus \{e\}$, and $(x_k) \in L^p(\ell_2^c)$, we have

$$\Big\| \Big(\sum_{k=1}^{\infty} |L_{h_k} x_k|^2 \Big)^{\frac{1}{2}} \Big\|_p \leq c_p \Big\| \Big(\sum_{k=1}^{\infty} |x_k|^2 \Big)^{\frac{1}{2}} \Big\|_p. \tag{20}$$

Proof
Let us assume such that $h_k \in \mathcal{R}_{g_{i_k}}, i_k \in \mathbf{Z}$. Assume that $h_k = h'_k g_{i_k}$. Then

$$L_{h_k} x_k = \lambda_{h'_k} L_{g_{i_k}} (\lambda_{h'^{-1}_k} x_k).$$

So

$$\sum_{k=1}^{\infty} |L_{h_k} x_k|^2 = \sum_{k=1}^{\infty} \big| L_{g_{i_k}} (\lambda_{h'^{-1}_k} x_k) \big|^2.$$

We get the result by the free group version of Corollary 3.12. ☐

4.1. A length reduction formula
In this section, we use standard notation from operator space theory. We denote the p-row and p-column spaces over some index set I by $R_p = \overline{\mathrm{span}}\{e_{1k}\} \subset S_p(B(\ell_2(I))$ and $C_p = \overline{\mathrm{span}}\{e_{k1}\} \subset S_p(B(\ell_2(I))$. To lighten notation, we set $r_k = e_{1k}$ and $c_k = e_{k1}$. The reader should not be confused with the previous notation as the objects are of very different nature.

Let $\mathcal{W}_{\geq d}$ be the set of words in $\mathbf{F}_\infty$ of length greater than or equal to d. Also denote by $W_{\geq d}$ the subspace in L^p generated by $\lambda_w, w \in \mathcal{W}_{\geq d}$. For $w \in \mathbf{F}_\infty$, we let w_l denote its lth letter (if it exists) and $\partial w = w_1^{-1} w$.

Take any $x = \sum_{w \in \mathcal{W}_{\geq 1}} x_w \lambda_w \in W_{\geq 1}$. We have

$$\Big\| \Big(\sum_{h \in S} \big| L_h(x) \big|^2 \Big)^{\frac{1}{2}} \Big\|_p = \Big\| \sum_{w \in \mathcal{W}_{\geq 1}} x_w \lambda_w \otimes c_{w_1} \Big\|_p = \Big\| \sum_{w \in \mathcal{W}_{\geq 1}} x_w \lambda_{\partial w} \otimes c_{w_1} \Big\|_p.$$

At the operator space level, Theorem 4.2 means that the map $\iota : W_{\geq d} \to C_p \otimes W_{\geq d-1} \oplus R_p$ given by $\iota(\lambda_w) = \lambda_{\partial w} \otimes c_{w_1} \oplus r_w$ is a complete isomorphism. By iterating, we obtain a complete isomorphism for $2 < p < \infty$

$$\iota_d : \begin{cases} W_{\geq d} \to C_p^{\otimes d} \otimes L^p(\hat{\mathbf{F}}_\infty) \oplus C_p^{\otimes d-1} \otimes R_p \oplus \cdots \oplus C_p \otimes R_p \oplus R_p, \\ \lambda_w \quad \mapsto c_{w_1,\ldots,w_d} \otimes \lambda_{\partial^d w} \oplus c_{w_1,\ldots,w_{d-1}} \otimes r_{\partial^{d-1} w} \oplus \cdots \oplus c_{w_1} \otimes r_{\partial w} \oplus r_w. \end{cases}$$

Let us state this as a corollary, which generalizes the result of [9].

COROLLARY 4.6 (Length reduction formula)
For any $d \geq 1$, ι_d *extends to a completely bounded isomorphism such that for* $x \in W_{\geq d}$, $2 \leq p < \infty$,

$$\eta_p^{-d} \|x\|_p \leq \|\iota_d x\| \leq c_p^d \|x\|_p,$$

for all $x \in L^p(\hat{\mathbf{F}}_\infty)$.

Fix some $d \in \mathbf{N}$, and let P_d be the projection onto $W_{d+1}^{\perp} \in L^p(\hat{\mathbf{F}}_\infty)$. Recall that by [14] or [5], P_d is completely bounded on L^p (this also follows from Theorem 4.2). For any reduced word $w = w_1 \cdots w_n$ in the generators, we define

$$L_h^{(d)}(\lambda_w) = \delta_{w_d = h} \lambda_w \qquad \text{and} \qquad H_\varepsilon^{(d)} = \varepsilon_e P_{d-1} + \sum_{h \in S} \varepsilon_h L_h^{(d)},$$

for any choice of $\varepsilon_h, |h| \leq 1$ with $|\varepsilon_h| \leq 1$. Note that

$$\|\iota_{d-1} H_{\varepsilon(1)}^{(1)} H_{\varepsilon(2)}^{(2)} \cdots H_{\varepsilon(d)}^{(d)} x\| = \|H_{\varepsilon(d)} \iota H_{\varepsilon(d-1)} \iota \cdots \iota H_{\varepsilon(1)} x\|.$$

We immediately get the following.

THEOREM 4.7
For any $d \geq 1$ *and* $x \in L^p(\hat{\mathbf{F}}_\infty) \otimes L^p(\mathcal{M})$, $1 < p < \infty$,

$$\|H_{\varepsilon(1)}^{(1)} H_{\varepsilon(2)}^{(2)} \cdots H_{\varepsilon(d)}^{(d)} x\|_p \simeq^{C_{p,d}} \|x\|_p$$

with $C_{p,d} \leq c_p^{2d-1} \eta_p^{d-1} \lesssim c_p^{5d-4}$ *and* $\|H_\varepsilon^{(d)} x\|_p \simeq^{c_p^d \eta_p^{d-1}} \|x\|_p$ *for any choice of* $|\varepsilon_k| = 1$.

We give a faster argument for the boundedness of $H_\varepsilon^{(d)}$. Consider $\varepsilon_h = \pm 1$ for $h \in \mathbf{F}_\infty$. Let

$$H_\varepsilon^{Ld} = \varepsilon_e P_{d-1} + \sum_{h \in \mathbf{F}_\infty, |h| = d} \varepsilon_h L_h, \qquad H_\varepsilon^{Rd} = \varepsilon_e P_{d-1} + \sum_{h \in \mathbf{F}_\infty, |h| = d} \varepsilon_{h^{-1}} R_h.$$

Recall that L_h (resp., R_h) is defined as the projection onto the set of all reduced words starting (resp., ending) with h. We get $H_\varepsilon^{(d)}$ from H_ε^{Ld} if ε_h depends only on the dth letter of h.

COROLLARY 4.8
For any $1 < p < \infty$, *we have for any* $x \in L^p(\hat{\mathbf{F}}_\infty)$,

$$\|x\|_p \simeq \|H_\varepsilon^{Ld} x\|_p \simeq \left\|(L_h x)_{|h| = d}\right\|_{L^p(\hat{\mathbf{F}}_\infty, \ell_2^{cr})}. \tag{21}$$

Proof
Note that a similar identity to (5) holds for free groups with H_ε^{Ld} and any g, h with $|g^{-1}h| \geq 2d-1$. We then have

$$P_{2d-2}^{\perp}\big[H_\varepsilon^{Ld} x (H_\varepsilon^{Ld} x)^*\big] = P_{2d-2}^{\perp}\big[H_\varepsilon^{Ld}\big(x H_\varepsilon^{Ld}(x^*)\big) + H_\varepsilon^{Ld}\big(H_\varepsilon^{Ld}(x) x^*\big) - H_\varepsilon^{Rd} H_\varepsilon^{Ld}(x x^*)\big]. \quad (22)$$

Let $c_{p,d}$, $p \geq 2$ be the best constant c so that $\|H_\varepsilon^{Ld} x\|_p \leq c\|x\|_p$. Recall that by the Haagerup inequality, the L^1- and L^p-norms are equivalent on the range of P_{2d-2}:

$$\begin{aligned}\big\|P_{2d-2}\big[H_\varepsilon^{Ld} x (H_\varepsilon^{Ld} x)^*\big]\big\|_p &\leq (2d-1)^{2-\frac{2}{p}} \big\|H_\varepsilon^{Ld} x (H_\varepsilon^{Ld} x)^*\big\|_1 \\ &\leq (2d-1)^{2-\frac{2}{p}} \|x\|_{2p}^2, \\ \big\|P_{2d-2}\big[H_\varepsilon^{Ld}\big(x H_\varepsilon^{Ld}(x^*)\big)\big]\big\|_p &\leq (2d-1)^{1-\frac{2}{p}} \big\|H_\varepsilon^{Ld}\big(x H_\varepsilon^{Ld}(x^*)\big)\big\|_2 \\ &\leq (2d-1)^{1-\frac{2}{p}} c_{2p} \|x\|_{2p}^2\end{aligned}$$

for any $p > 2$. Therefore,

$$c_{2p,d}^2 \leq 2(2d-1)^{2-\frac{2}{p}} + 2c_{2p,d}(2d-1)^{1-\frac{2}{p}} + 2c_{p,d}c_{2p,d} + c_{p,d}^2.$$

We then have

$$c_{2p,d} \leq (2d-1)^{1-\frac{2}{p}} + c_{p,d} + \sqrt{2}\big(c_{p,d} + 3(2d-1)^{1-\frac{1}{p}}\big).$$

Asymptotically, $c_{p,d} \simeq p^{\frac{\ln(1+\sqrt{2})}{\ln 2}}$ for d given and $c_{p,d} \simeq d^{1-\frac{2}{p}}$ for p given. So

$$\|H_\varepsilon^{Ld} x\|_p \leq c_{p,d}\|x\|_p.$$

Since $H_\varepsilon^{Ld} H_\varepsilon^{Ld} = \mathrm{id}$, we get the equivalence. The $1 < p < 2$ case follows by duality. □

Remark 4.9
A straightforward completely bounded version of Corollary 4.8 is false for $\mathbf{F}_\infty$ (true for $\mathbf{F}_n$, although with a constant depending on n). This is because the operator-valued Haagerup inequality is an equivalence between the L^p-norm and the more complicated norm given by Corollary 4.6. For instance, it yields that the set $\{\lambda(g_i g_j)\}$ is not completely unconditional; this would be a direct consequence of Corollary 4.8.

For any $x \in \mathcal{L}(\mathbf{F}_n), n < \infty$, and any choice of signs, $H_\varepsilon x$ can be viewed as an unbounded operator on $L^2(\hat{\mathbf{F}}_n)$ with domain $C_c(\mathbf{F}_n)$. As usual, $\mathbb{K}$ stands for the compact operators. Ozawa [8] asked whether the commutator $[R_h, x]$ sends the unit ball of $\mathcal{L}(\mathbf{F}_n)$ into a compact set of $L^2(\hat{\mathbf{F}}_n)$ for any $h \in \mathbf{F}_n$ and $x \in \mathcal{L}(\mathbf{F}_n)$ and pointed out that the L^p-boundedness of R_h implies a positive answer. We record a general result in the following corollary.

COROLLARY 4.10

We have for $d \in \mathbf{N}$ and any choice of signs ε that

(i) $[H_\varepsilon^{Rd}, x] \in B(L^2(\hat{\mathbf{F}}_n))$ *if* $x = x_1 + H_{\varepsilon'}^{Ld} x_2$ *for some* $|\varepsilon'| \le 1, x_1, x_2 \in \mathcal{L}(\mathbf{F}_n)$;

(ii) $[H_\varepsilon^{Rd}, x] \in \mathbb{K}(L^2(\hat{\mathbf{F}}_n))$ *for all* $x = x_1 + H_{\varepsilon'}^{Ld} x_2$ *for some* $|\varepsilon'| \le 1, x_1, x_2 \in C_\lambda^*(\mathbf{F}_n)$;

(iii) $[H_\varepsilon^{Rd}, x]$ *maps the closed unit ball of* $\mathcal{L}(\mathbf{F}_n)$ *into a compact subset of* $L^2(\hat{\mathbf{F}}_n)$ *if* $x \in L^p(\hat{\mathbf{F}}_n)$ *for some* $p > 2$ *(in particular, if* $x \in \mathcal{L}(\mathbf{F}_n)$*).*

Proof

Similar to (22), we have

$$\begin{aligned} &P_{2d-2}^{\perp}\big[(H_{\varepsilon'}^{Ld} x)(H_\varepsilon^{Rd} y)\big] \\ &\quad = P_{2d-2}^{\perp}\big[H_{\varepsilon'}^{Ld}\big(x(H_\varepsilon^{Rd} y)\big) + H_\varepsilon^{Rd}\big((H_{\varepsilon'}^{Ld} x)y\big) - H_{\varepsilon'}^{Ld} H_\varepsilon^{Rd}(xy)\big].\end{aligned}$$

So, up to a finite-rank perturbation, for $y \in L^2(\hat{\mathbf{F}}_n)$,

$$[H_\varepsilon^{Rd}, H_{\varepsilon'}^{Ld} x]y = -H_{\varepsilon'}^{Ld}\big(x(H_\varepsilon^{Rd} y)\big) + H_{\varepsilon'}^{Ld} H_\varepsilon^{Rd}(xy) = H_{\varepsilon'}^{Ld}\big([H_\varepsilon^{Rd}, x]y\big).$$

Therefore, for $x = x_1 + H_{\varepsilon'}^{Ld} x_2$, up to a finite-rank perturbation

$$[H_\varepsilon^{Rd}, x] = [H_\varepsilon^{Rd}, x_1] + H_{\varepsilon'}^{Ld}\big([H_\varepsilon^{Rd}, x_2]\big).$$

This implies (i). Note that $[R_h, \lambda_g]$ is finite rank for each h, g. We have $[H_\varepsilon^{Rd}, x] \in \mathbb{K}(\ell_2(\mathbf{F}_n))$ for all $x \in C_\lambda^*(\mathbf{F}_n)$. So (ii) is true. For (iii), following the argument of Ozawa, we have, by Hölder's inequality and Theorem 4.1,

$$\big\|[H_\varepsilon^{Rd}, x]y\big\|_{L^2(\hat{\mathbf{F}}_n)} \lesssim \|x\|_{L^p(\hat{\mathbf{F}}_n)} \|y\|_{L^q(\hat{\mathbf{F}}_n)}$$

for any $y \in L^q(\hat{\mathbf{F}}_n)$, $\frac{1}{q} + \frac{1}{p} = \frac{1}{2}$. By density of $C_\lambda^*(\mathbf{F}_n)$ in $L^p(\hat{\mathbf{F}}_n)$, $p < \infty$ and since $\mathcal{L}(\mathbf{F}_n) \subset L^p(\hat{\mathbf{F}}_n)$ contractively, we get the desired result. □

Remark 4.11

When $n = 1$, the space of functions x in Corollary 4.10(i) (resp., (ii)) is called BMO (resp. VMO). It characterizes the class of x such that the commutator $[H, x]$ is bounded (resp., compact).

Remark 4.12

The content of this remark is from a communication with Ozawa. Let $\mathcal{M}$ be a finite von Neumann algebra with a finite normal faithful trace τ. Let $L^p(\mathcal{M}), 1 \le p < \infty$ be the associated noncommutative L^p-spaces (see [12]). Recall that we set $L^\infty(\mathcal{M}) = \mathcal{M}$. For the operators $X \in B(L^2(\mathcal{M})), p \ge 2$, define a seminorm

$$\|X\|_{p\to 2} = \sup\big\{\|Xy\|_2; y \in L^p(\mathcal{M}) \subset L^2(\mathcal{M}), \|y\|_p \le 1\big\}.$$

Note that $\|X\|_{2\to 2}$ is just the operator norm $\|X\|$. Identify $\mathcal{M}$ as subalgebra of $B(L^2(\mathcal{M}))$ by the left multiplication on $L^2(\mathcal{M})$. Then $\mathcal{M}' \subset B(L^2(\mathcal{M}))$ corresponds to right multiplications by elements of $\mathcal{M}$ on $L^2(\mathcal{M})$. We write y' for the right multiplication by $y \in \mathcal{M}$ on $L^2(\mathcal{M})$. For $x \in \mathcal{M}$, we have by Hölder's inequality that

$$\|x\|_{p\to 2} = \|x'\|_{p\to 2} = \|x\|_q$$

for $\frac{1}{q} + \frac{1}{p} = \frac{1}{2}$. The lemma of [8, Section 3] says that, for $X \in B(L^2(\mathcal{M}))$,

$$\|X\|_{\infty\to 2} \le \inf\big\{\|Y\|\|x\|_2 + \|Z\|\|y\|_2\big\} \le 4\|X\|_{\infty\to 2}. \tag{23}$$

Here the infimum is taken over all possible decomposition $X = Yx + Zy'$ with $Y, Z \in B(L^2(\mathcal{M})), x, y \in \mathcal{M}$. One can easily see that an analogue of the first inequality of (23) holds for all $p > 2$; that is,

$$\|X\|_{p\to 2} \le \inf\big\{\|Y\|\|x\|_q + \|Z\|\|y\|_q\big\}, \tag{24}$$

for $\frac{1}{q} + \frac{1}{p} = \frac{1}{2}$. Since $\|x\|_q^q \le \|x\|_2^2\|x\|_{\mathcal{M}}^{q-2} = \|x\|_2^2\|x\|_{\mathcal{M}}^{\frac{2}{p}}$, we get the following Hölder-type inequality for $X \in B(L^2(\mathcal{M}))$:

$$\|X\|_{p\to 2} \le 4\|X\|_{\infty\to 2}^{\frac{p-2}{p}}\|X\|^{\frac{2}{p}}. \tag{25}$$

Suppose that $Y \in B(L^2(\mathcal{M}))$ satisfies that, for some $p > 2$,

$$\|Y\|_{\infty\to p} = \sup\big\{\|Yx\|_p; x \in L^\infty(\mathcal{M}) \subset L^2(\mathcal{M}), \|x\|_{\mathcal{M}} \le 1\big\} < \infty.$$

Inequality (25) implies that

$$\|XY\|_{\infty\to 2} \le \|X\|_{p\to 2}\|Y\|_{\infty\to p} \le 4\|X\|_{\infty\to 2}^{\frac{p-2}{p}}\|X\|^{\frac{2}{p}}\|Y\|_{\infty\to p}. \tag{26}$$

Let $\mathbb{K}_{\mathcal{M}}^L \in B(L^2(\mathcal{M}))$ be the collection of all operators sending the unit ball of $\mathcal{M}$ into a compact subset of $L^2(\mathcal{M})$. Let $\mathbb{K}_{\mathcal{M}} = (\mathbb{K}_{\mathcal{M}}^L)^* \cap \mathbb{K}_{\mathcal{M}}^L$ be the associated C^*-algebra. Let $M(\mathbb{K}_{\mathcal{M}})$ be the multiplier algebra of $\mathbb{K}_{\mathcal{M}}$, that is, the algebra of all operators $X \in B(L^2(\mathcal{M}))$ such that both $X\mathbb{K}_{\mathcal{M}}$ and $\mathbb{K}_{\mathcal{M}}X$ still belong to $\mathbb{K}_{\mathcal{M}}$. The proposition of [8, Section 2] says that $X \in \mathbb{K}_{\mathcal{M}}$ if and only if for every sequence of finite-rank projections Q_n strongly converging to the identity of $B(L^2(\mathcal{M}))$, $\|X - Q_nX\|_{\infty\to 2} \to 0$. Combining this with (26), we see that Y above belongs to $M(\mathbb{K}_{\mathcal{M}})$.

This applies to the particular case when Y is the free Hilbert transform H_ε or $H_\varepsilon^{\mathrm{op}}$ and $\mathcal{M}$ is an amalgamated free product. Ozawa promoted the study of the C^*-algebra

$$B_{\mathcal{M}} = \{X \in M(\mathbb{K}_{\mathcal{M}}); [X, y] \in \mathbb{K}_{\mathcal{M}}, \forall y \in \mathcal{M} \subset B(L^2(\mathcal{M}))\}.$$

Theorem 3.5 and Corollary 4.10(iii) imply that $H_\varepsilon^{Rd} \in B_{\mathcal{L}(\mathbf{F}_n)}$, and similarly that $H_\varepsilon^{Ld} \in B_{\mathcal{L}'(\mathbf{F}_n)}$. Here $\mathcal{L}'(\mathbf{F}_n)$ is the von Neumann algebra generated by the right regular representation ρ_g's.

Let $\bar{\mathbf{F}}_n = \mathbf{F}_n \cup \partial \mathbf{F}_n$, and let $C(\bar{\mathbf{F}}_n)$ be the C^*-algebra of continuous functions on $\bar{\mathbf{F}}_n$. Note that $C(\bar{\mathbf{F}}_n)$ is isomorphic to the sub-C^*-algebra of $B(\ell^2(\mathbf{F}_n))$ generated by $\rho_g L_h \rho_{g^{-1}}, g, h \in \mathbf{F}_n$. We then obtain

$$C(\bar{\mathbf{F}}_n) \subset B_{\mathcal{L}'(\mathbf{F}_n)}.$$

4.2. *Connections to carré du champ*

We use the same notation to denote elements of $\mathbf{F}_\infty$ and points on its Cayley graph. The *Gromov product* for g^{-1}, g' (on the Cayley graph) is defined as

$$\langle g, g' \rangle = \frac{|g| + |g'| - |gg'|}{2}.$$

A closely related object is the so-called *carré du champ* of Meyer,

$$\Gamma(\lambda_g, \lambda_{g'}) = \frac{A(\lambda_g^*)\lambda_{g'} + \lambda_g^* A(\lambda_{g'}) - A(\lambda_g^* \lambda_{g'})}{2} = \langle g^{-1}, g' \rangle \lambda_{g^{-1}g'}$$

associated to the conditionally negative operator $A : \lambda_g \mapsto |g| \lambda_g$.

The following identity is a key connection to the operator L_h in previous sections:

$$2\Gamma(\lambda_g, \lambda_{g'}) = \sum_{h \in \mathbf{F}_\infty} (L_h(\lambda_g))^* L_h(\lambda_{g'}). \tag{27}$$

Let us extend this notation to $x = \sum_g a_g \lambda_g \in L^2(\hat{\mathbf{F}}_\infty) \otimes L^2(\mathcal{M})$, and we set

$$A^r(x) = \sum_g a_g |g|^r \lambda_g,$$

$$\Gamma(x, x) = \langle x, x \rangle = \sum a_g^* a_{g'} \langle g^{-1}, g' \rangle \lambda_{g^{-1}g'}.$$

We then have

$$2E_\varepsilon \langle H_\varepsilon x, H_\varepsilon x \rangle = \sum_{h \in \mathbf{F}_\infty} |L_h x|^2 = A(x^*)x + x^* A(x) - A(|x|^2). \tag{28}$$

The following square function estimate was proved in [4]. One direction of the inequality has been proved in [2] and [3] in a more general setting.

LEMMA 4.13 ([4, Theorem A.1, Example (c)])
For any $2 \le p < \infty$, $x \in L^p(\hat{\mathbf{F}}_\infty) \otimes L^p(\mathcal{M})$,

$$\|A^{\frac{1}{2}}x\|_p \simeq_{\frac{p^4}{(p-1)^2}} \Big\|\Big(\sum_{h\in\mathbf{F}_\infty} |L_h x|^2\Big)^{\frac{1}{2}}\Big\|_p + \Big\|\Big(\sum_{h\in\mathbf{F}_\infty} \big|L_h(x^*)\big|^2\Big)^{\frac{1}{2}}\Big\|_p.$$

Remark 4.14
The above equivalence may fail if one replaces $L_h(x^*)$ by $(L_h x)^*$ on the right-hand side. Corollary 4.9 of [3] gives constants $\simeq p$ for the $\lesssim$ direction.

4.3. Littlewood–Paley inequalities
In the case of the free group, we adapt the definitions of paraproducts studied in Section 3.3. Assume $x = \sum_g a_g \lambda_g \in L^p$, $y = \sum_h d_h \lambda_h \in L^q$. We then find that

$$x \ddagger y = \sum_{k\in\mathbf{Z}^*} L_{g_k}\big((L_{g_k}x)y\big) = \sum_{g^{-1}\not\leq h} a_g d_h \lambda_{gh},$$

$$x \dagger y = \sum_{k\in\mathbf{Z}} \overset{\circ}{L}{}^{\perp}_{g_k}\big((L_{g_k}x)y\big) = \sum_{g^{-1}<h} a_g d_h \lambda_{gh}.$$

Here $L_{g_0} = \tau$ and $\overset{\circ}{L}{}^{\perp}_{g_k} = \sum_{j\neq k, j\in\mathbf{Z}^*} L_{g_j}$. Recall that we write $g \le h$ (or $h \ge g$) if $h = gk$ with g, h, k reduced words and $g < h$ if $g \le h$ and $g \neq h$.

We consider a decomposition of $\mathbf{F}_\infty$ into disjoint geodesic paths. To get one, first pick a (randomly decided) geodesic path $\mathbb{P}_0$ starting at the unit element e. Then for any length 1 elements not in $\mathbb{P}_0$, pick a (randomly decided) geodesic path starting at each of them. We then go to length 2 elements which are not contained in any of the previous picked paths, and we pick a (randomly decided) geodesic path starting at each of them. We repeat this procedure and get countably many disjoint geodesic paths $\mathbb{P}_n$ such that $\bigcup_n \mathbb{P}_n = \mathbf{F}_\infty$.

Let T_n be the L^2-projection onto the span of $\mathbb{P}_n$. Let $h_1(n)$ be the root of $\mathbb{P}_n$, that is, the first element in $\mathbb{P}_n$. Let S_n be the projection to the collection of words smaller than $h_1(n)$ (note that $S_0 = 0$).

COROLLARY 4.15
For any $1 < p < \infty$, *the maps* T_n *are completely bounded on* L^p *with*

$$\|T_n\|_{p\to p} \lesssim c_p^2. \tag{29}$$

Moreover, for any $p > 2$

$$\Big\|\sum_n \big|T_n(x) + S_n(x)\big|^2 - \big|S_n(x)\big|^2\Big\|_{\frac{p}{2}} \lesssim c_p^2 \|x\|_p^2. \tag{30}$$

Proof
We write $x = \sum a_g \lambda_g$ and $T_n(x) = \sum_{g\in\mathbb{P}_n} a_g \lambda_g$. Then

$$\big(T_n(x)\big)^* T_n(x) - \sum_{g\in\mathbb{P}_n} |a_g|^2 \lambda_e = \sum_{g<h\in\mathbb{P}_n} a_g^* a_h \lambda_{g^{-1}h} + \sum_{h<g\in\mathbb{P}_n} a_g^* a_h \lambda_{g^{-1}h}$$
$$= \big(T_n(x)\big)^* \dagger T_n(x) + \Big(\big(T_n(x)\big)^* \dagger T_n(x)\Big)^*.$$

Since $(T_n(x) + S_n(x))^* \dagger T_n(x) = x^* \dagger T_n(x)$, we have

$$\big(T_n(x)\big)^* \dagger T_n(x) = x^* \dagger T_n(x) - \big(S_n(x)\big)^* T_n(x).$$

Therefore,

$$\big(T_n(x)\big)^* T_n(x) - \sum_{g\in\mathbb{P}_n} |a_g|^2 \lambda_e$$
$$= x^* \dagger T_n(x) + \big(x^* \dagger T_n(x)\big)^* - \big(S_n(x)\big)^* T_n(x) - \big(T_n(x)\big)^* S_n(x). \tag{31}$$

In particular, for $n = 0$, we have actually

$$\big(T_0(x)\big)^* T_0(x) - \sum_{g\in\mathbb{P}_0} |a_g|^2 \lambda_e = x^* \dagger T_0(x) + \big(x^* \dagger T_0(x)\big)^*.$$

Assume $p > 2$. By Proposition 3.14, we have

$$\big\|T_0(x)\big\|_p^2 \le (4 + 2c_p c_{\frac{p}{2}})\|x\|_p \big\|T_0(x)\big\|_p + \|x\|_p^2.$$

So $\|T_0(x)\|_p \le (5 + 2c_{\frac{p}{2}} c_p)\|x\|_p$ for $p > 2$. One concludes that T_0 is (completely) bounded on L^p. One can improve the bound on $\|T_0\|_{p\to p}$ when p is close to 2 by using interpolation. The case $p < 2$ follows by duality. Thus we have obtained (29) for an arbitrary $\mathbb{P}_0$ starting at e; for general $\mathbb{P}_n$ this follows by using translations.

Summing (31) over n, we get

$$\sum_{n\ge 0} \big|T_n(x) + S_n(x)\big|^2 - \big|S_n(x)\big|^2$$
$$= \sum_{n\ge 0} \Big[\big(T_n(x)\big)^* T_n(x) + \big(S_n(x)\big)^* T_n(x) + \big(T_n(x)\big)^* S_n(x)\Big]$$
$$= \sum_{n\ge 0} x^* \dagger T_n(x) + \big(x^* \dagger T_n(x)\big)^* + \sum_g |a_g|^2 \lambda_e$$
$$= x^* \dagger x + (x^* \dagger x)^* + \tau |x|^2 \lambda_e.$$

Then (30) follows from Proposition 3.14. □

We now consider a concrete partition given by geodesic paths. For any $h_0 \notin \mathcal{R}_{g^\pm}$ and $g \in S$, let $\mathbb{P}_{h_0,g} = \{h_0 g^k; k \in \mathbf{N}\}$. They form a countable partition of $\mathbf{F}_\infty \setminus \{e\}$, and we may index it with $\mathbf{Z}^* = \mathbf{Z} \setminus \{0\}$. We still denote the root of $\mathbb{P}_n$ by $h_1(n)$ and put $h_0(n) = h_0$, $k_n = k$ if $\mathbb{P}_n = \mathbb{P}_{h_0,g_k}$. By definition $h_0(n) \in \mathcal{R}^\perp_{g^\pm_{k_n}}$ if $h_1(n) \in \mathcal{R}^\pm_{g_{k_n}}$.

LEMMA 4.16
Let T_n be the L^2-projection onto $\mathbb{P}_n$ described above. We have for any $p \geq 2$, $x \in L^p(\hat{\mathbf{F}}_\infty)$,

$$\Big\| \Big(\sum_{n \in \mathbf{Z}^*} \big|T_n(x)\big|^2 \Big)^{\frac{1}{2}} \Big\|_p \lesssim c_p \|x\|_p. \tag{32}$$

Proof
We may assume that $\tau x = 0$. Let E_k be the projection from the group von Neumann algebra $\mathcal{L}(\mathbf{F}_\infty)$ onto the von Neumann algebra generated by λ_{g_k}. We can easily verify that, for $k \in \mathbf{Z}^*$,

$$E_{|k|} |R_{g_k} x|^2 = E_{|k|} \Big| \sum_{h_1(n) \in R_{g_k}} T_n(x) \Big|^2 = \sum_{h_1(n) \in R_{g_k}} \big|T_n(x)\big|^2,$$

because, if $h_1(n), h_1(n') \in \mathcal{R}_{g_k}$, then $h_0(n), h_0(n') \in \mathcal{R}^\perp_{g_k}$ and $h_0^{-1}(n) h_0(n') \in E_{|k|}(\mathbf{F}_\infty)$ if and only if $n = n'$. Therefore,

$$\begin{aligned} \sum_{n \in \mathbf{Z}^*} \big|T_n(x)\big|^2 &= \sum_{k=1}^{\infty} E_k\big(|R_{g_k} x|^2 + |R_{g_{-k}} x|^2\big) \\ &= \tau |x|^2 + \sum_{k=1}^{\infty} \overbrace{E_k\big(|R_{g_k} x|^2 + |R_{g_{-k}} x|^2\big)}^{\circ}. \end{aligned}$$

By the free Rosenthal-type inequality ([5, Theorem A] or Corollary 3.18) for length 1 polynomials, we get for $p \geq 4$, with $X_k = \overbrace{E_k(|R_{g_k} x|^2 + |R_{g_{-k}} x|^2)}^{\circ}$,

$$\begin{aligned} \Big\| \Big(\sum_n \big|T_n(x)\big|^2 \Big)^{\frac{1}{2}} \Big\|_p^2 &\lesssim \tau |x|^2 + \Big(\sum_{k \in \mathbf{N}} \|X_k\|_{\frac{p}{2}}^{\frac{p}{2}} \Big)^{\frac{2}{p}} + \Big(\sum_{k \in \mathbf{N}} \|X_k\|_2^2 \Big)^{\frac{1}{2}} \\ &\lesssim \Big(\sum_{k \in \mathbf{Z}^*} \|R_{g_k} x\|_p^p \Big)^{\frac{2}{p}} + \Big(\sum_{k \in \mathbf{Z}^*} \|R_{g_k} x\|_4^4 \Big)^{\frac{1}{2}} \\ &\lesssim \Big\| \Big(\sum_{k \in \mathbf{Z}^*} |R_{g_k} x|^2 \Big)^{\frac{1}{2}} \Big\|_p^2 \lesssim^{c_p^2} \|x\|_p^2, \end{aligned}$$

where we used the obvious facts by interpolation that $L^p(\ell_2^c) \to \ell^p(L^p)$ and $L^p(\ell_2^c) \to \ell_4(L^4)$ are contractions. The case of $p = 2$ is obvious. We then get the estimate for all $2 \leq p < \infty$ by interpolation. $\square$

Let $\mathbb{P}_j = \{h_1(j) < h_2(j) < \cdots h_k(j) < \cdots\}$ be arbitrary geodesic paths of $\mathbf{F}_\infty$. For $x_j = \sum_{k \in \mathbf{N}} a_k \lambda_{h_k(j)}$ supported on $\mathbb{P}_j$, we consider its dyadic parts

$$M_{n,j} x_j = \sum_{2^n \leq k < 2^{n+1}} a_k \lambda_{h_k(j)}. \tag{33}$$

Dealing with $\mathbf{F}_1 = \mathbf{Z}$ with the $\mathbf{N} \cup \{0\}$ and $-\mathbf{N}$ as geodesic paths, the classical Littlewood–Paley theory says that

$$\Big\| \Big(\sum_{n=1}^{\infty} |M_{n,1} x|^2 + |M_{n,2} x|^2 \Big)^{\frac{1}{2}} \Big\|_p \simeq^{c_p} \|x\|_p \tag{34}$$

for all $1 < p < \infty$ and $x \in C_c(\mathbf{Z})$.

THEOREM 4.17
Suppose that $(x_j)_{j \in \mathbf{N}} \in L^p(\mathcal{L}(\mathbf{F}_\infty), \ell_2^c)$ is a sequence such that every x_j is supported on a geodesic path $\mathbb{P}_j$. We have

$$\Big\| \Big(\sum_{n,j=1}^{\infty} |M_{n,j} x_j|^2 \Big)^{\frac{1}{2}} \Big\|_p \leq C p^2 c_p^2 \Big\| \Big(\sum_j |x_j|^2 \Big)^{\frac{1}{2}} \Big\|_p \tag{35}$$

for all $2 \leq p < \infty$.

Proof
As usual, $g_1, g_2, \ldots$ are the free generators of $\mathbf{F}_\infty$. We embed $\mathbf{F}_\infty$ into the free product $\mathbf{F}_\infty * \mathbf{F}_\infty$, and we denote by $g'_1, g'_2, \ldots$ the generators of the second copy of $\mathbf{F}_\infty$. Let $y_j = \lambda_{g'_j h_1^{-1}(j)} x_j$. The y_j's are supported on disjoint paths $\mathbb{P}'_j \subset \mathbf{F}_\infty * \mathbf{F}_\infty$ with roots of distinct generators g'_j. Note that $|x_j|^2 = |y_j|^2$ and $|M_n x_j|^2 = |M_n y_j|^2$. By considering y_j instead, we may assume that $\mathbb{P}_j = \{h_1(j) < h_2(j) < \cdots h_k(j) \cdots\}$ with $|h_k(j)| = k$ and $L_{h_k(j)} x_m = 0$ for $j \neq m$.

Let

$$\begin{aligned} M_{\varphi_n, j} = {} & 2^{1-\frac{n}{2}} \sum_{2^{n-1} < k \leq 2^n} L_{h_k(j)} A^{-\frac{1}{2}} + \sum_{2^n < k \leq 2^{n+1}} (\sqrt{k} - \sqrt{k-1}) L_{h_k(j)} A^{-\frac{1}{2}} \\ & - 2^{-\frac{n+1}{2}} \sum_{2^{n+1} < k \leq 2^{n+2}} L_{h_k(j)} A^{-\frac{1}{2}}. \end{aligned}$$

Then $M_{\varphi_n, j}(\lambda_{h_l(m)}) = 0$ unless $m = j$ and $l \in (2^{n-1}, 2^{n+2}]$, and one can check that

$$M_{\varphi_n,j}(\lambda_{h_l(j)}) = \varphi_n(l)\lambda_{h_l(j)},$$

for some $\varphi_n : \mathbf{N} \to \mathbf{R}$ with $\chi_{[2^n,2^{n+1})} \leq \varphi_n \leq \chi_{(2^{n-1},2^{n+2})}$. Note that

$$M_{\varphi(n),j} = \sum_{2^{n-1}<k\leq 2^{n+2}} a_{k,j} L_{h_k(j)} A^{-\frac{1}{2}}$$

with $\sum_k a_{k,j}^2 \leq c$. By the convexity of the operator-valued function $|\cdot|^2$, we have

$$|M_{\varphi_n,j} x_j|^2 \leq c \sum_{2^{n-1}<k\leq 2^{n+2}} |L_{h_k(j)} A^{-\frac{1}{2}} x_j|^2,$$

and $M_{\varphi_n,j} x_m = 0$ for $m \neq j$. Note that the $M_{\varphi_n,j}, M_{\varphi_{n'},j}$'s are disjoint for $|n-n'| \geq 2$. Applying Lemma 4.13 to $x = \sum_j x_j \otimes c_j$, we obtain

$$\Big\|\Big(\sum_{n=0}^{\infty} |M_{\varphi_n,j} x_j|^2\Big)^{\frac{1}{2}}\Big\|_p \leq c \Big\|\Big(\sum_{k,j} |L_{h_k(j)} A^{-\frac{1}{2}} x_j|^2\Big)^{\frac{1}{2}}\Big\|_p \leq cp^2 \Big\|\Big(\sum_j |x_j|^2\Big)^{\frac{1}{2}}\Big\|_p.$$

Assume that $h_{2^n}(j) \in \mathcal{R}_{g_{n,j}}, h_{2^{n+1}}(j) \in \mathcal{R}_{g_{n',j}}$. We have

$$\lambda_{h_{2^{n+1}(j)}} L_{g_{n',j}^{-1}} \lambda_{h_{2^{n+1}}(j)^{-1}} (\lambda_{h_{2^n-1}(j)} L_{g_{n,j}} \lambda_{h_{2^n-1}(j)^{-1}} M_{\varphi_n} x_j) = M_n x_j \quad (36)$$

because $h_{2^{n+1}+1}(j) \in \mathcal{R}^{\perp}_{g_{n',j}^{-1}}$. By (12),

$$\Big\|\Big(\sum_{n=1}^{\infty} |M_{n,j} x_j|^2\Big)^{\frac{1}{2}}\Big\|_p \leq cc_p^2 \Big\|\Big(\sum_{n=1}^{\infty} |M_{\varphi_n,j} x_j|^2\Big)^{\frac{1}{2}}\Big\|_p \leq cp^2 c_p^2 \Big\|\Big(\sum_j |x_j|^2\Big)^{\frac{1}{2}}\Big\|_p$$

for all $2 \leq p < \infty$. □

Remark 4.18
A "smoothed" version of Theorem 4.17 for a fixed path $\mathbb{P}$ was proved in [4].

Let $\mathbb{P}_k$ be the paths described in the paragraph preceding Lemma 4.16. Let T_k be the L^2-projections onto $\mathbb{P}_k$. Denote $M_{n,k} x = M_{n,k}(T_k x)$ for $x \in L^p(\hat{\mathbf{F}}_\infty)$. We obtain the following corollary from Theorem 4.17 and duality with the fact that $\|x^*\|_p = \|x\|_p$.

COROLLARY 4.19
For all $2 \leq p < \infty$, and $x \in L^p(\hat{\mathbf{F}}_\infty)$,

$$\max\Big\{\Big\|\Big(\sum_{n,k} |M_{n,k} x|^2\Big)^{\frac{1}{2}}\Big\|_p, \Big\|\Big(\sum_{n,k} \big|M_{n,k}(x^*)\big|^2\Big)^{\frac{1}{2}}\Big\|_p\Big\} \leq Cp^2 c_p^3 \|x\|_p, \quad (37)$$

and for all $1 \le p < 2$, *and* $x \in L^p$

$$\|x\|_p \le Cp'^2 c_{p'}^3 \inf\Big\{ \Big\| \Big(\sum_{n,k} |M_{n,k} y|^2 \Big)^{\frac{1}{2}} \Big\|_p + \Big\| \Big(\sum_{n,k} \big| M_{n,k}(z^*) \big|^2 \Big)^{\frac{1}{2}} \Big\|_p ; x = y + z \Big\}.$$

Acknowledgments. The first author would like to thank Marius Junge and Gilles Pisier for helpful discussions.

Mei's research was partially supported by National Science Foundation grants DMS-1266042 and DMS-1632435.

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