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# Moment representations of exceptional $X_1$ orthogonal polynomials ☆

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ABSTRACT

We generalize the representations of $X_1$ exceptional orthogonal polynomials through determinants of matrices that have certain adjusted moments as entries. We start out directly from the Darboux transformation, allowing for a universal perspective, rather than one dependent upon the particular system (Jacobi or Type of Laguerre polynomials). We include a recursion formula for the adjusted moments and provide the initial adjusted moments for each system. Throughout we relate to the various examples of $X_1$ exceptional orthogonal polynomials. We especially focus on and provide complete proofs for the Jacobi and the Type III Laguerre case, as they are less prevalent in literature. Lastly, we include a preliminary discussion explaining that the higher codimension setting becomes more involved. The number of possibilities and choices is exemplified, but only starts, with the lack of a canonical flag.



## 1. Introduction

The study of exceptional orthogonal polynomials arose as the result of extending exactly solvable and quasi-exactly solvable potentials in quantum mechanics beyond the Lie-algebraic setting [5,15,16,26,27]. The Laguerre and Jacobi exceptional polynomial systems of codimension one were first introduced in 2009 as an extension of Bochner's classification theorem for the classical orthogonal polynomials [8]. At that time, the approach to exceptional orthogonal polynomial systems was as state-preserving solutions to second-order Sturm–Liouville-type problems. Quesne initially identified the relationship between the exceptional orthogo-

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nal polynomials and the Darboux transforms [26]. This connection with the Darboux transform has allowed for exceptional polynomial systems of higher codimension to be obtained. Ultimately, a new Bochner-like classification theorem for the exceptional systems has been proven [9,10,12,6,25].

Of interest to mathematicians are the various properties of the exceptional orthogonal systems as they relate to classical orthogonal systems, as well as the asymptotic and interlacing properties of the zeros, recursion formulas, and the spectrum of the exceptional systems [1,4,12,11,14,17–19,22,21,23].

Here we use the Darboux transform and its relation to the exceptional Laguerre (Types I, II, and III) and Jacobi (Types I and II) polynomial systems to obtain the corresponding exceptional polynomials via the Gram–Schmidt method. In the case of classical orthogonal polynomials it is a well-established perspective to view the polynomials as the result of applying Gram–Schmidt to the *flag sequence* $\{1, x, x^2, \dots\}$, as is outlined in [3, Chapter 1.3]. Some of the authors [24] used these ideas to derive two representations for the Type I $X_1$-Laguerre polynomials in terms of their moments through using determinants. An adaptation and generalization of this method leads us to our main result, which relies solely on the moment functions and modified weights of the exceptional orthogonal polynomial systems. Our results are universal in the sense that they can be applied to $X_1$ orthogonal polynomial systems (independent of being "Laguerre" or "Jacobi" specific). As we are restricting ourselves to the $X_1$ systems, we refer to both the Type I and II Jacobi systems as "Jacobi" as in the codimension one case, they are equivalent. Due to structural reasons, there are no $X_1$-Hermite polynomials [7].

Other representations of exceptional orthogonal polynomial systems involve Wronskian, and sometimes pseudo-Wronskian, determinants of classical orthogonal polynomials, see e.g. [4,7].

The idea of this paper hinges on the following simple observations regarding the essential characteristics of exceptional orthogonal polynomials. First of all, the exclusion of eigenfunctions with certain degrees is caused by particular rational-function coefficients in the differential expression. When we apply the differential expression to eigenfunctions we must (at least) cancel any denominators introduced by these rational-function coefficients. This idea leads to what we call the *exceptional condition*, one of the central players in the theory. In this paper we primarily consider $X_1$ polynomials, so that this denominator consists of a linear polynomial. We call the root of this polynomial the *exceptional root*, and denote it by $\xi$. Our work is suspended on the ansatz that writing the Taylor expansion of the exceptional polynomials around $\xi$ inherits beneficial properties, see [24, Section 5]. There it was also noticed that adjusting moments by replacing the integrand $x^l$ by $(x - \xi)^l$ drastically simplifies matters.

We first gather some preliminary information about the Darboux transform and its relations to exceptional orthogonal polynomials (Section 2). For subsequent comparison, we introduce the Jacobi and the Type III Laguerre case in more detail. In Section 3, we focus on a universal expression for the exceptional condition as it was obtained recently in [6]. We write their condition in terms of the basic functions of the theory, and relate the condition to our examples. The flag sequence, which after the Gram–Schmidt algorithm returns the exceptional orthogonal polynomials, is the topic of Section 4. We state the determinantal representation in Theorem 5 of Section 5.1. Like in [24], we relate the degree $n$ exceptional polynomial in terms of a determinant of an $(n + 1) \times (n + 1)$-matrix. The first row of the matrix comes from the exceptional condition, while the second through last row contains certain adjusted moments. We then, in Section 6, present a recursion formula to compute these adjusted moments. We observe a curious fact: the moment representations for the Type I and II Laguerre polynomials only differ in the exceptional condition. The computation of the initial adjusted moments is deferred to the Appendix A. In Section 7, we include some preliminary observations on the flags and recursive moment formulas for $X_m$ orthogonal polynomials when $m > 1$, mainly so as to indicate some difficulties we expect to encounter when extending determinantal representations to the higher co-dimension setting. For example, flags have been explored for $m = 1$ see [11], whereas the Darboux transform is used when $m > 1$.

**Table 1**
Other notation.

| | |
|---|---|
| $\eta(x)$ | Polynomial function that occurs in the natural operator |
| $s(x)$ | Linear function that occurs in the natural operator |
| $\xi$ | The exceptional root; that is the root of $b(x)$ or, equivalently, the root of $\phi(x)$ or that of $\eta(x)$ after the $\alpha$-shift |
| $\mathcal{E}_n$ | Span of the first $n$ exceptional orthogonal polynomials |
| $\mathcal{F}_n$ | Set of polynomials of degree $\leq n$ satisfying the exceptional condition |
| $v_k$ | Flag elements $v_k(x) = (x-\xi)^k$ |
| $c_{n,i}$ | Coefficients of the Taylor expansion $\widehat{y}_n(x) = \sum_{i=0}^{n} c_{n,i}(x-\xi)^i$ of the exceptional polynomial around the exceptional root |
| $\widetilde{\mu}_m$ | Adjusted moments $\widetilde{\mu}_m = \int_I (x-\xi)^m \widehat{W}(x)dx$ |

### 1.1. Notation

Most of our notation leans on the standards used in the field of exceptional polynomials. The general trend is that the classical objects are denoted by letters without "$\widehat{\ }$". For example, $T^\alpha$ represents the classical Laguerre differential expression/operator; $L_n^\alpha(x)$, the classical Laguerre polynomial of degree $n$; and $W^{(\alpha,\beta)}(x)$, the classical Jacobi weight. Their exceptional counterparts are denoted by the respective letters but with the "$\widehat{\ }$". Here a subscript is used to refer to the codimension and superscripts I through III specify with which type of exceptional Laguerre we are dealing, e.g. $\widehat{T}_m^{I,\alpha}$ stands for the Type I $X_m$-Laguerre differential operator. When we talk more generally about a weight, differential expression or a general polynomial (and not specifically about either Laguerre or Jacobi) we do not include the parameters $\alpha$ and/or $\beta$. For example we use $W$, $T$ or $p_n$ for the classical objects, and $\widehat{W}$, $\widehat{T}$ or $\widehat{p}_n$ when we consider the exceptional counterparts. For the reader's convenience we include the other notation in Table 1.

## 2. Preliminaries

We first recall how the Darboux transform may be used to generate exceptional orthogonal polynomial systems. For further reading on the relationship of the Darboux transform and exceptional orthogonal polynomial systems, see [6,9,10,12], upon which the following exposition is based.

Suppose $T[y]$ is a second-order differential operator with rational coefficients; that is

$$T[y] = p(x)y''(x) + q(x)y'(x) + r(x)y(x)\,. \tag{2.1}$$

Define the following quasi-rational functions

$$P(x) = \exp\left(\int \frac{q(x)}{p(x)}\,dx\right)\,, \tag{2.2}$$

$$W(x) = \frac{P(x)}{p(x)}\,, \tag{2.3}$$

$$R(x) = r(x)W(x)\,. \tag{2.4}$$

By multiplying the eigenvalue equation $T[y] = \lambda y$ by $W(x)$, the Sturm–Liouville type equation

$$(Py')' + Ry = \lambda W y$$

is formed. Thus, we refer to $W(x)$ as the weight function associated with $T$.

For $T$ and a quasi-rational function $\phi(x)$, $\phi(x)$ is called a *quasi-rational eigenfunction* if

$$T[\phi] = \lambda\phi \quad \lambda \in \mathbb{C}\,. \tag{2.5}$$

In order to create the operator associated with the exceptional orthogonal polynomials, we first use the following decomposition proposition to rewrite $T$ as a composition of two first-order operators.

**Proposition 2.1.** [6, Proposition 3.5] *For a second-order differential operator $T[y]$ having rational coefficients, let $\phi(x)$ be a quasi-rational eigenfunction of $T$ with eigenvalue $\lambda$, and let $b(x)$ be an arbitrary, non-zero rational function. Define rational functions*

$$w = \frac{\phi'}{\phi}\,, \tag{2.6}$$

$$\widehat{b} = \frac{p}{b}\,, \tag{2.7}$$

$$\widehat{w} = -w - \frac{q}{p} + \frac{b'}{b} \tag{2.8}$$

*and first-order operators $A$ and $B$ by*

$$A[y] = b(y' - wy) \quad \text{and} \quad B = \widehat{b}(y' - \widehat{w}y)\,. \tag{2.9}$$

*With $A$ and $B$ as above, $T$ has a rational factorization of the form $T = BA + \lambda$.*

Using the factorization of $T$ and the definitions of $A$ and $B$ given in Proposition 2.1, we will define a new operator, called the *partner operator*, to be $\widehat{T} := AB + \lambda$. The rational Darboux transformation maps $T$ to $\widehat{T}$. Operator $\widehat{T}$ will also be a second-order differential operator with rational coefficients; that is

$$\widehat{T}[y] = p(x)y''(x) + \widehat{q}(x)y'(x) + \widehat{r}(x)y(x)\,. \tag{2.10}$$

Note that the coefficient of the second-order term is the same for both the original and partner operators. Additionally, another Sturm–Liouville type equation is induced.

**Proposition 2.2.** [6, Proposition 3.6] *Suppose that $T$ and $\widehat{T}$ are second-order differential operators with rational coefficients which are related via a rational Darboux transformation. Then $T$ and $\widehat{T}$ have the same second-order coefficients, while first- and zero-order coefficients $q, \widehat{q}, r, \widehat{r} \in \mathbb{Q}$, and the quasi-rational weights $W(x)$ and $\widehat{W}(x)$ are related by*

$$\widehat{q} = q + p' - \frac{2pb'}{b}\,, \tag{2.11}$$

$$\widehat{r} = -p(\widehat{w}' + \widehat{w}^2) - \widehat{q}\widehat{w} \tag{2.12}$$

$$= r + q' + wp' - \frac{b'}{b}(q + p') + \left(2\left(\frac{b'}{b}\right)^2 - \frac{b''}{b} + 2w'\right)p\,, \text{ and} \tag{2.13}$$

$$\widehat{W} = \frac{pW}{b^2} = \frac{P(x)}{b^2}\,. \tag{2.14}$$

We are interested in the Darboux transform applied to classical Bochner systems; in particular, the systems of Laguerre and Jacobi. When $T$ is defined to be one of these classical systems and $\phi$ is carefully chosen, the partner operator will be an exceptional polynomial system operator.

### *2.1. General $X_m$ expression*

The moment representations will be approached in the general case (without reliance on the specifics of the Laguerre or Jacobi setting) and will rely solely on the information provided via the Darboux transform and Bochner systems. Therefore, the differential expression for an $X_m$ orthogonal polynomial system will be defined by $\widehat{T}[\widehat{y}]$, with the weight function given by equation (2.14). Recall that $\xi$ is the exceptional root. (That is, $\xi$ is the root of the denominator function $b(x)$ of the weight $\widehat{W}(x)$.) Later, when we restrict ourselves to $m = 1$, $b(x)$ will be of degree two and have one repeated root.

Since we assume that $T$ is a classical Bochner operator and $\phi$ has been carefully chosen to produce an $X_m$ orthogonal polynomial operator, the differential equation $\widehat{\ell}[\widehat{y}] = \widehat{\lambda}\widehat{y}$ will be satisfied by a sequence of polynomials $\widehat{y} = \{\widehat{y}_n\}_{n\in\mathbb{N}_0\setminus A}$, where $\widehat{y}_n$ is of degree $n$ and $A$ is a finite set of dimension $m$, and corresponding sequence of eigenvalues $\widehat{\lambda} = \{\widehat{\lambda}_n\}_{n\in\mathbb{N}_0\setminus A}$. We mention that the eigenvalues for the classical orthogonal systems and exceptional orthogonal systems are *not* necessarily equal as shifting may occur under the Darboux transform. On an open interval, $I = (a, b)$, this sequence of polynomials will satisfy the orthogonality relation

$$\langle \widehat{y}_n, \widehat{y}_k\rangle_{\widehat{W}} = \int_I \widehat{y}_n \widehat{y}_k \widehat{W}\, dx = K_n \delta_{n,k} \tag{2.15}$$

where $\delta_{n,k}$ is the Kronecker-$\delta$ symbol (equal to 1 when $n = k$ and 0 when $n \neq k$). Of course, the interval $I = (0, \infty)$ for Laguerre and $I = (-1, 1)$ when we work with Jacobi systems.

### *2.2. $X_m$-Laguerre expression*

Here we provide a brief insight into the Type I, II, and III exceptional Laguerre orthogonal polynomial systems. A more rigorous look at the Darboux transform applied to the classical Laguerre expression may be found in [9]. Recall the classical Laguerre differential operator

$$T^\alpha[y] = xy'' + (-x + \alpha + 1)y', \tag{2.16}$$

and corresponding weight function

$$W^\alpha(x) = x^\alpha e^{-x}.$$

The classical Laguerre polynomials are shape-invariant under the factorizations:

$$\begin{aligned} T^\alpha &= B^\alpha A^\alpha \\ \widehat{T}^{\alpha+1} &= A^\alpha B^\alpha + 1, \end{aligned}$$

where

$$\begin{aligned} A^\alpha(y) &= y' \quad \text{and} \\ B^\alpha(y) &= xy' + (\alpha + 1 - x)y. \end{aligned}$$

The quasi-rational eigenfunctions and eigenvalues of $T^\alpha[y]$ are:

$$\begin{aligned} &\phi_1(x) = L_m^\alpha(x), && \lambda = -m, \\ &\phi_2(x) = x^{-\alpha} L_m^{-\alpha}(x), && \lambda = \alpha - m, \\ &\phi_3(x) = e^x L_m^\alpha(-x), && \lambda = \alpha + 1 + m, \text{ and} \\ &\phi_4(x) = x^{-\alpha} e^x L_m^\alpha(-x), && \lambda = m + 1, \end{aligned}$$

where $m \in \mathbb{N}_0$. The factorizations corresponding to each of these eigenfunctions have been studied, see [9,14]. It has been shown that $\phi_1$ with $m = 0$ corresponds to a state-deleting transformation and corresponds to the classical Laguerre polynomials. For $m > 0$, the eigenfunctions corresponding to $\phi_1$ yield singular operators, which means that no new families of orthogonal polynomials arise. The family associated with $\phi_4$ is state-adding and therefore, the resulting orthogonal polynomials are not of codimension $m$. The factorizations of $\phi_2$ and $\phi_3$ result in new orthogonal polynomials—in fact, these factorizations respectively produce the Type I and Type II exceptional orthogonal polynomials of codimension $m$. For further reading regarding the properties of the Type I and Type II exceptional Laguerre polynomial systems, see [22].

The Type III exceptional Laguerre polynomials $\{\widehat{L}^{III,\alpha}_{m,n}(x)\}_{n=0 \text{ or } n>m}$ are a class of Laguerre-type orthogonal polynomials which were extensively studied in [22]. We will focus on the Type III exceptional operator for several of our examples in this paper.

There is another rational factorization of the classical Laguerre expression $T^\alpha[\cdot]$, which yields the Type III second-order expression. Let

$$A^{III,\alpha}_m(y) = xL^{-\alpha}_m(-x)y' - (m+1)L^{-\alpha-1}_{m+1}(-x)y\,, \text{ and}$$

$$B^{III,\alpha}_m(y) = \frac{y'}{L^{-\alpha}_m(-x)}\,.$$

The classical Laguerre operator may be written as

$$T^\alpha = B^{III,\alpha}_m A^{III,\alpha}_m + m + 1\,,$$

and the Darboux transformation associated with the above factorizations yields the Type III operator

$$\widehat{T}^{III,\alpha}_m = A^{III,\alpha+1}_m B^{III,\alpha+1}_m + m - \alpha.$$

That is, explicitly, we have

$$\widehat{T}^{III,\alpha}_m[y] = -xy'' + \left(-1 + \alpha + x + 2x\frac{(L^{-\alpha-1}_m(-x))'}{L^{-\alpha-1}_m(-x)}\right) y' - \alpha y\,. \tag{2.17}$$

The corresponding weight for the Type III case is

$$\widehat{W}^{III,\alpha}_m(x) = \frac{x^\alpha e^{-x}}{\left(L^{-\alpha-1}_m(-x)\right)^2}\,. \tag{2.18}$$

The Type III eigenvalue equation will have orthogonal polynomial solutions on $(0,\infty)$ if and only if $-1 < \alpha < 0$. In fact, the associated Type III differential equation will have a polynomial solution $y(x) = \widehat{L}^{III,\alpha}_{m,n}(x)$ of degree $n$ for $n = 0$ and for $n \geq m+1$; that is, solutions of degrees $\{1, 2, \ldots, m\}$ are missing.

We may write the Type III exceptional Laguerre polynomials using the Darboux transformation

$$\widehat{L}^{III,\alpha}_{m,n}(x) = \begin{cases} -A^{III,\alpha+1}_m[L^{\alpha+1}_{n-m-1}(x)], & n = m+1, m+2, \ldots \\ 1 & n = 0\,. \end{cases} \tag{2.19}$$

### 2.3. $X_m$-Jacobi expression

We provide a brief outline of the Type I exceptional Jacobi orthogonal polynomial systems. A more rigorous look at the Darboux transform applied to the classical Jacobi expression may be found in [14]. For $\alpha, \beta > -1$,

$$T^{\alpha,\beta}[y] = (1-x^2)y'' + (\beta - \alpha - (\alpha+\beta+2)x)y' \tag{2.20}$$

is the classical Jacobi differential operator. For

$$\alpha,\beta > -1, \quad \alpha+1-m-\beta \notin \{0,1,\ldots,m-1\}, \quad \text{and} \quad \operatorname{sgn}(\alpha+1-m) = \operatorname{sgn}(\beta),$$

define

$$A_m^{\alpha,\beta}[y] = (1-x)P_m^{(-\alpha,\beta)}y' + (m-\alpha)P_m^{(-\alpha,\beta)}y\,,$$
$$B_m^{\alpha,\beta}[y] = \frac{(1+x)y' + (1+\beta)y}{P_m^{(-\alpha,\beta)}}\,,$$

where $P_m^{(\alpha,\beta)}$ is the classical Jacobi polynomial of degree $m$. It follows that

$$T^{\alpha,\beta}[y] = B_m^{\alpha,\beta}A_m^{\alpha,\beta}y - (m-\alpha)(m+\beta+1)y$$

and the exceptional Jacobi operator is defined as

$$\begin{aligned}\widehat{T}_m^{\alpha,\beta}[y] &= A_m^{\alpha+1,\beta-1}B_m^{\alpha+1,\beta-1}y - (m-\alpha-1)(m+\beta)y \\ &= T^{\alpha,\beta}[y] + (\alpha-\beta-m+1)my - (\log P_m^{(-\alpha-1,\beta-1)})'\left(\beta(1-x)y + (1-x^2)y'\right) \qquad (2.21)\end{aligned}$$

For $\alpha,\beta > -1$ and $n \geq m$, the $X_m$-Jacobi polynomial of degree $n$ can be written as

$$\widehat{P}_{m,n}^{(\alpha,\beta)}(x) = \frac{(-1)^{m+1}}{\alpha+1+j}A_m^{\alpha+1,\beta-1}\left[P_j^{(\alpha+1,\beta-1)}(x)\right], \quad j = n-m \geq 0\,.$$

Note that the exceptional operator extends the classical Jacobi operator; that is

$$\widehat{T}_0^{\alpha,\beta}[y] = T^{\alpha,\beta}[y]\,.$$

The $X_m$-Jacobi polynomials $\{\widehat{P}_{m,n}^{(\alpha,\beta)}\}_{n\geq m}$ are orthogonal on $(-1,1)$ with respect to the weight function

$$\widehat{W}_m^{\alpha,\beta}(x) = \frac{(1-x)^{\alpha}(1+x)^{\beta}}{\left(P_m^{(-\alpha-1,\beta-1)}(x)\right)^2}\,.$$

## 3. Exceptional condition

The most noticeable difference between a classical orthogonal polynomial expression, as classified by Bochner, and the exceptional orthogonal polynomial expressions is that the coefficient functions for the first- and zero-order terms are no longer polynomial. From Proposition 2.2, we see that the coefficient functions for any second-order partner operator formed via a rational Darboux transformation will have denominators containing powers of $b(x)$ and $\phi(x)$.

We turn our attention to our particular case, where we are working with $\widehat{T}$, an exceptional polynomial operator. By [6, Definition 7.1, ii-c], $f(x)p(x)\widehat{W}(x) \to 0$ at the endpoints of $(a,b)$ for every polynomial $f(x)$. Consequently, the operator $\widehat{T}$ will be polynomially regular and thus, semi-simple [6, Remark 4.10]. In other words, since we seek polynomial solutions to the associated eigenvalue problem, these non-polynomial coefficients require a specific structure condition for the remaining polynomials; that is, "cancellation" must occur in order to have polynomial eigenfunctions. The coefficient functions which are non-polynomial form the *exceptional term*, and the specific structure induced on solutions is referred to as the *exceptional condition*. Polynomials of every degree cannot satisfy both the exceptional condition and form a maximal invariant subspace under $\widehat{T}$. Therefore, we do not have a full sequence of polynomial eigenfunctions for the exceptional operators—that is $A$ is ensured to be non-empty.

In order to find the exceptional condition which characterizes the exceptional polynomial systems we introduce an additional setting in which we can consider the exceptional orthogonal polynomial operators. We say that for any two second-order operators having rational coefficients, $T$ and $\widehat{T}$ are *gauge-equivalent* if there exists a rational function $\sigma$ such that

$$\sigma T = \widehat{T}\sigma\,.$$

By [6, Proposition 2.5, Theorem 5.4], every exceptional operator will be gauge equivalent to a natural operator. A natural operator is a second-order operator of the form

$$py'' + \left(\frac{p'}{2} + s - \frac{2p\eta'}{\eta}\right)y' + \left(\frac{p\eta''}{\eta} + \left(\frac{p'}{2} - s\right)\frac{\eta'}{\eta}\right)y\,, \tag{3.1}$$

where $p$ is a second-degree polynomial, $\eta$ is a polynomial, and $s$ is a linear function. It is the case that the polynomial $p$ is the same regardless of whether $\widehat{T}$ is written in standard form (2.10) or in natural gauge form (3.1). Obviously, if the standard form of $\widehat{T}$ is known (that is, $b$ is known), by setting the coefficients to be equal, one can find the polynomial $s$. Rather, we would like to approach the task of finding $s$ without knowing $b$. To do this, we will utilize [6, Corollary 5.25] which states (in paraphrased form) that the exceptional term for $\widehat{T}$ in the natural gauge is given by

$$\frac{2p\eta' y' - \left(p\eta'' + \dfrac{p'\eta'}{2} - s\eta'\right)y}{\eta}\,. \tag{3.2}$$

Finding the linear polynomial $s$ given the exceptional terms from the two representations of $\widehat{T}$ will be the focus of Subsection 3.1.

### 3.1. Finding the linear polynomial $s$

We now turn our attention to finding the linear polynomial $s$, which is found in (3.1).

While it is clear that $\eta$ should be the polynomial part of the quasi-rational eigenfunction $\phi$ (that initiates the Darboux transform), it seems less obvious what the function $s \in \mathcal{P}_1$ would be. Here $\mathcal{P}_1$ denotes the polynomials of degree less than or equal to one. Of course, one could extract $s$ by comparing the coefficient of $y'$ in $\widehat{T}$ to $\widehat{q}$ in (2.11) after having computed everything. But we found it beneficial to express $s$ directly from $\eta$ and the factorization gauge.

**Lemma 3.1.** *The linear polynomial* $s = q + \dfrac{p'}{2} + 2p\left(\dfrac{\eta'}{\eta} - \dfrac{b'}{b}\right)$.

From this formula, it is not immediately clear that $s \in \mathcal{P}_1$ in general. However, this is exactly the topic of [6, Section 5]. Below we include the values for $s$ for the Type I, II, and III Laguerre and for the Jacobi case.

**Proof.** We set the coefficient of $y'$ in (3.1) equal to $\widehat{q}$ in Proposition 2.2

$$-\frac{2\eta' p}{\eta} + \frac{p'}{2} + s = q + p' - \frac{2b' p}{b}.$$

Solving for $s$ yields the lemma. □

**Table 2**
Choices of $\eta$ and $s$ for the natural operator in equation (3.1).

| | $\phi(x)$ | $\eta(x)$ | $b(x)$ | Shift | $s(x)$ |
|---|---|---|---|---|---|
| Type I Lag. | $e^x L_m^{\alpha}(-x)$ | $L_m^{\alpha}(-x)$ | $L_m^{\alpha}(-x)$ | $\alpha \mapsto \alpha - 1$ | $x - \alpha - 1/2$ |
| Type II Lag. | $x^{-\alpha} L_m^{-\alpha}(x)$ | $L_m^{-\alpha}(x)$ | $xL_m^{-\alpha}(x)$ | $\alpha \mapsto \alpha + 1$ | $x - \alpha - 1/2$ |
| Type III Lag. | $x^{-\alpha} e^x L_m^{-\alpha}(-x)$ | $L_m^{-\alpha}(-x)$ | $xL_m^{-\alpha}(-x)$ | $\alpha \mapsto \alpha + 1$ | $x - \alpha - 1/2$ |
| Jacobi | $(1-x)^{-\alpha} P_m^{(-\alpha,\beta)}(x)$ | $P_m^{(-\alpha,\beta)}(x)$ | $(1-x)P_m^{(-\alpha,\beta)}(x)$ | $\alpha \mapsto \alpha + 1$ $\beta \mapsto \beta - 1$ | $\beta - \alpha - (\alpha + \beta + 1)x$ |

Table 2 includes the choices of $\eta$ and $s$ for the three types of Laguerre systems. We took the liberty of already including all the shifts. As required, we observe that $s \in \mathcal{P}_1$ in all cases.

We exemplify how these expressions play out in the case of the Type III Laguerre system.

**Example.** First recall that the coefficients of the classical Laguerre expression (2.16) are $p(x) = -x$, $q(x) = x - \alpha - 1$ and $r(x) \equiv 0$. The Type III Laguerre orthogonal polynomial system is derived from the classical Laguerre system by using the quasi-rational eigenfunction $\phi_4(x) = x^{-\alpha} e^x L_m^{-\alpha}(-x)$, so that $\widetilde{\eta}(x) := L_m^{-\alpha}(-x)$. In [22] we used the factorization gauge $b(x) = xL_m^{-\alpha}(-x)$. We also use the shift $\alpha \mapsto \alpha+1$. This shift can be seen from the $\alpha + 1$ superscript in the partner operator

$$-\widehat{T}_m^{III,\alpha} = A_m^{III,\alpha+1} \circ B_m^{III,\alpha+1} + m - \alpha.$$

We compute

$$\frac{\widetilde{\eta}'(x)}{\widetilde{\eta}(x)} = \frac{(L_m^{-\alpha}(-x))'}{L_m^{-\alpha}(-x)} \qquad \text{and} \qquad \frac{b'(x)}{b(x)} = \frac{1}{x} + \frac{(L_m^{-\alpha}(-x))'}{L_m^{-\alpha}(-x)},$$

so that the expression in Lemma 3.1

$$q + \frac{p'}{2} + 2p\left(\frac{\widetilde{\eta}'}{\widetilde{\eta}} - \frac{b'}{b}\right) = x - \alpha - 1 - \frac{1}{2} - 2x\left(-\frac{1}{x}\right) = x - \alpha + \frac{1}{2}.$$

And after the shift $\alpha \mapsto \alpha + 1$ we obtain

$$\eta(x) = L_m^{-\alpha-1}(-x) \qquad \text{and} \qquad s(x) = x - \alpha - \frac{1}{2}.$$

We verify that the coefficient of $y'$ is correct by comparing

$$\begin{aligned}\left(\frac{p'}{2} + s - \frac{2p\eta'}{\eta}\right) y' &= \left(-\frac{1}{2} + x - \alpha - \frac{1}{2} - 2x\frac{(L_m^{-\alpha-1}(-x))'}{L_m^{-\alpha-1}(-x)}\right) y' \\ &= \left(x - \alpha - 1 - 2x\frac{(L_m^{-\alpha-1}(-x))'}{L_m^{-\alpha-1}(-x)}\right) y'\end{aligned}$$

with the $X_m$ expression (2.17) for Type III Laguerre.

We also compute the coefficient of the zero-order term to ensure that this choice for $s$ produces an equivalent expression in standard form. Using $\eta = L_m^{-\alpha-1}(-x)$ and the coefficient for $y$ from (3.1), we have the following calculation, which relies on the fact that $\eta = L_m^{-\alpha-1}(-x)$ is a solution to the classical Laguerre differential equation $T[\eta] = m\eta$, where $T$ is defined in (2.16):

$$\frac{\left(p\eta''+\dfrac{p'\eta'}{2}-s\eta'\right)}{\eta}=\frac{-x\left(L_m^{-\alpha-1}(-x)\right)''+\left(-\dfrac{1}{2}-x+\alpha+\dfrac{1}{2}\right)\left(L_m^{-\alpha-1}(-x)\right)'}{L_m^{-\alpha-1}(-x)}$$
$$=\frac{-x\left(L_m^{-\alpha-1}(-x)\right)''+(-x+\alpha)\left(L_m^{-\alpha-1}(-x)\right)'}{L_m^{-\alpha-1}(-x)}$$
$$=\frac{mL_m^{-\alpha-1}(-x)}{L_m^{-\alpha-1}(-x)}$$
$$=m.$$

Note that the coefficient of the zero-order term we found using the natural operator and $s$ differs from the coefficient in the standard form given in (2.17) by a constant. This is a result of the shifting of eigenvalues which occurs when the Darboux transformation is applied. We are not concerned by this discrepancy as the structure of the operator remains the same.

**Example.** The Jacobi expression, $\phi(x)=(1-x)^{-\alpha}P_m^{(-\alpha,\beta)}(x)$, is the quasi-rational eigenfunction. Using Lemma 3.1 together with the function $\eta(x)=P_m^{(-\alpha,\beta)}(x)$, the gauge $b(x)=(1-x)P_m^{(-\alpha,\beta)}(x)$ and the shifts $\alpha\mapsto\alpha+1$ and $\beta\mapsto\beta-1$ yields

$$s(x)=\beta-\alpha-(\alpha+\beta+1)x.$$

We verify that the coefficient of $y'$ is correct by computing

$$\left(\frac{p'}{2}+s-\frac{2p\eta'}{\eta}\right)y'=\left(-x+\beta-\alpha-(\alpha+\beta+1)x-2(1-x^2)\frac{(P_m^{(-\alpha-1,\beta-1)}(x))'}{P_m^{(-\alpha-1,\beta-1)}(x)}\right)y'$$
$$=\left(\beta-\alpha-(\alpha+\beta+2)x-2x\frac{(P_m^{(-\alpha-1,\beta-1)}(x))'}{P_m^{(-\alpha-1,\beta-1)}(x)}\right)y'$$

with the $X_m$ expression for Jacobi given by (2.20) and (2.21).

## 4. The flag

In this section, we characterize the subspace spanned by the first $n$ of the $X_1$-exceptional orthogonal polynomials as those polynomials satisfying the exceptional condition

$$\left[2p\eta'y'-\left(p\eta''+\frac{p'\eta'}{2}-s\eta'\right)y\right]\Bigg|_{x=\xi}=0\,. \tag{4.1}$$

Recall that for the exceptional orthogonal systems of codimension one, $\xi$ represents the root of $\eta$ (that is, the exceptional root). The first $m$ $X_1$-exceptional orthogonal polynomials will be those of degree less than or equal to $m$ excluding degree 0 in the case of the Type I and Type II $X_1$-Laguerre or $X_1$-Jacobi polynomial systems. For the Type III $X_1$-Laguerre polynomials system, the first $m$ polynomials will be those of degree 0 or between 2 and $m$, inclusively.

In preparation for Lemma 4.1 below, we present two definitions. The definitions, as given, are directly applicable for the Type I and II Laguerre and Jacobi systems. We abuse notation slightly for the Type III Laguerre case by using $\widehat{y}_1$ to represent $\widehat{y}_0$. Recall that $\widehat{y}_n$ is defined to have degree $n$ and for the Type III Laguerre polynomials, we have a polynomial $\widehat{y}_0$, but not $\widehat{y}_1$. Let $\mathcal{P}_n$ denote the set of polynomials of degree less than or equal to $n$ and define the span of the first $n$ exceptional orthogonal polynomials

**Table 3**
In the definition of $\mathcal{E}_n = \text{span}\{\widehat{y}_j : j = 1, \ldots, n\}$ as the first $n$ exceptional polynomials we have $\widehat{y}_j$.

| | | | |
|---|---|---|---|
| Exceptional Laguerre | Type I | $\widehat{y}_j = \widehat{L}^{I,\alpha}_{1,j}$ | |
| | Type II | $\widehat{y}_j = \widehat{L}^{II,\alpha}_{1,j}$ | |
| | Type III | $\widehat{y}_j = \begin{cases} \widehat{L}^{III,\alpha}_{1,j} & j \geq 2 \\ 1 & j = 1 \end{cases}$ | |
| Exceptional Jacobi | | $\widehat{y}_j = \widehat{P}^{(\alpha,\beta)}_{1,j}$ | |

$$\mathcal{E}_n := \text{span}\{\widehat{y}_j : j = 1, \ldots, n\},$$

where $\widehat{y}_j$ is defined as in Table 3.

Further we define

$$\mathcal{F}_n := \{p \in \mathcal{P}_n : p \text{ satisfies } (4.1)\}.$$

**Lemma 4.1.** *The sets $\mathcal{E}_n = \mathcal{F}_n$ for all $n \in \mathbb{N}$.*

The proof is analogous to the proof for [24, Lemma 2.1], but as in [13, Proposition 5.3], the exceptional condition is now replaced by a universal one. We choose to include the argument for the convenience of the reader.

**Proof.** Since $\mathcal{E}_n$ and $\mathcal{F}_n$ are clearly vector spaces of equal dimension $n$, it suffices to show that $\mathcal{E}_n \subseteq \mathcal{F}_n$.

To see this, take $f \in \mathcal{E}_n$. Then $f$ is a linear combination of the first $n$ exceptional polynomials. So $f \in \mathcal{P}_n$. Further, $\widehat{T}[f]$ is polynomial, and so is the expression (4.1) for $y = f$. It follows that $f \in \mathcal{F}_n$. □

With the exceptional root $\xi$, we define degree $k$ polynomials

$$v_k(x) := \begin{cases} \widehat{y}_1(x) & k = 1 \\ (x-\xi)^k & k \geq 2 \end{cases} \tag{4.2}$$

where $\widehat{y}_1(x)$ is given in Table 3.

**Lemma 4.2.** *The sequence of polynomials $\{v_1, v_2, v_3, \ldots\}$ forms a flag for $\widehat{T}$.*

**Proof.** Our definition of $v_k$ ensures that the polynomials have the appropriate degrees. That is, for Types I and II $X_1$-Laguerre and $X_1$-Jacobi systems, the constant polynomial is missing, and for Type III Laguerre the linear polynomial is excluded.

In virtue of the previous lemma, it suffices to prove that all $v_k$ satisfy the exceptional condition (4.1). This follows immediately for the first flag element, as $\widehat{y}_1 = v_1$ is the first exceptional polynomial and hence an eigenfunction of $\widehat{T}$.

For $k \geq 2$, we recall that $v_k(x) := (x-\xi)^k$. In particular, we have $v_k(\xi) = 0$ so that the second term of (4.1) vanishes. On the left-hand side of (4.1) we are left with $2p(\xi)\eta'(\xi)v_k'(\xi)$. But we also have $v_k'(\xi) = k(x-\xi)^{k-1}\big|_{x=\xi} = 0$, because $k \geq 2$. So the exceptional condition (4.1) is satisfied and $\mathcal{F}_n = \text{span}\{v_1, v_2, v_3, \ldots, v_n\}$. □

## 5. Determinantal representations

In this section, we provide the details for finding the determinantal representations for the Type II and III $X_1$-Laguerre and $X_1$-Jacobi orthogonal polynomial systems. In the case of the Type I $X_1$-Laguerre

**Table 4**
Square of the norms of the $X_1$ polynomials.

| | | |
|---|---|---|
| Exceptional Laguerre | Type I | $K_n = \frac{(\alpha+n)\Gamma(\alpha+n-1)}{(n-1)!}$ for $n \geq 1$ |
| | Type II | $K_n = \frac{(\alpha+n-1)\Gamma(\alpha+n+1)}{(n-1)!}$ for $n \geq 1$ |
| | Type III | $K_n = \begin{cases} \frac{n\Gamma(n+\alpha)}{(n-2)!} & n \geq 2 \\ \frac{\Gamma(\alpha+1)\Gamma(-\alpha)}{\Gamma(1-\alpha)} & n = 0 \end{cases}$ |
| Exceptional Jacobi | | $K_n = \frac{2^{\alpha+\beta+1}(\alpha+n)(\beta+n)\Gamma(\alpha+n)\Gamma(\beta+n)}{4(\alpha+n-1)(\beta+n-1)(\alpha+\beta+2n-1)\Gamma(n)\Gamma(\alpha+\beta+n)}$ |

polynomials, we do confirm that our results agree with [24] and refer the reader to [24] for the details of that particular case. It is the case that $\eta$ will have one and only one exceptional root, $\xi$. The exceptional condition as discussed in Sections 3 and 4 and in [6, Corollary 5.25] reduces to (4.1).

To find the first row of entries in the determinantal representation, we follow the methods outlined in [24] and use the ansatz for degree $n$, $n \geq 2$, to write the exceptional polynomial

$$\widehat{y}_n(x) := \sum_{i=0}^{n} c_{n,i}(x-\xi)^i. \tag{5.1}$$

Note that $\widehat{y}_n(x)$ may be any $X_1$-Laguerre-type or $X_1$-Jacobi polynomial of degree $n$. We fill in specific details for each case below.

Then

$$\widehat{y}_n{}'(x) := \sum_{i=1}^{n} i c_{n,i}(x-\xi)^{i-1}, \tag{5.2}$$

and $\widehat{y}_n(\xi) = c_{n,0}$ and $\widehat{y}_n{}'(\xi) = c_{n,1}$.

In order to obtain the determinantal representation we notice that the coefficients $c_{n,i}$, $i = 0, 1, \ldots, n$, are given as the unique solution of a system of $n+1$ linear equations with matrix form $Ac = b$. The objects $A$, $c$ and $b$ are given below.

To this end we define the *adjusted moments*

$$\widetilde{\mu}_m := \int_I (x-\xi)^m \widehat{W}(x)dx \tag{5.3}$$

and the vectors

$$c := (c_{n,0}, c_{n,1}, \ldots, c_{n,n})^\top \in \mathbb{R}^{n+1} \quad \text{and} \quad b := (0, \ldots, 0, K_n)^\top \in \mathbb{R}^{n+1}.$$

The constant $K_n := \langle \widehat{p}_n, \widehat{p}_n \rangle$ determines the normalization of the exceptional polynomials.

**Remark.** For the reader's reference we also include the square, $K_n$, of the norms for each of the exceptional sequences in Table 4. The norms for the $X_m$-Laguerre polynomials are found in [22]; for Jacobi, see [8].

**Theorem 5.1.** *The $X_1$ orthogonal polynomials have the determinantal representation formula*

$$\widehat{y}_n(x) = \frac{1}{\det A} \sum_{i=0}^{n} (\det A_i)(x-\xi)^i \tag{5.4}$$

$$= \frac{K_n}{\det A} \begin{vmatrix} & & \textit{First } n \textit{ rows of the matrix } A & & \\ 1 & (x-\xi) & (x-\xi)^2 & \ldots & (x-\xi)^n \end{vmatrix} \tag{5.5}$$

**Table 5**
The values of the constants $c_{1,0}$ and $c_{1,1}$ in the matrix $A$. At the same time $c_{1,0}$ and $c_{1,1}$ are also the coefficients of $\widehat{y}_1(x)$, which is defined in (5.1).

| | | $c_{1,0}$ | $c_{1,1}$ |
|---|---|---|---|
| Exceptional Laguerre | Type I | 1 | 1 |
| | Type II | 1 | 1 |
| | Type III | 1 | 0 |
| Exceptional Jacobi | | $\frac{\alpha+\beta}{\beta-\alpha}$ | $\frac{1}{2}$ |

*where the* $(n+1)\times(n+1)$*-matrix* $A$ *is given by*

$$A = \begin{bmatrix} p(\xi)\eta''(\xi) + \frac{p'(\xi)\eta'(\xi)}{2} - s(\xi)\eta'(\xi) & 2p(\xi)\eta'(\xi) & 0 & \dots & 0 \\ c_{1,0}\widetilde{\mu}_0 + c_{1,1}\widetilde{\mu}_1 & c_{1,0}\widetilde{\mu}_1 + c_{1,1}\widetilde{\mu}_2 & c_{1,0}\widetilde{\mu}_2 + c_{1,1}\widetilde{\mu}_3 & \dots & c_{1,0}\widetilde{\mu} + c_{1,1}\widetilde{\mu}_{n+1} \\ \widetilde{\mu}_2 & \widetilde{\mu}_3 & \widetilde{\mu}_4 & \dots & \widetilde{\mu}_{n+2} \\ \vdots & \vdots & \vdots & & \vdots \\ \widetilde{\mu}_n & \widetilde{\mu}_{n+1} & \widetilde{\mu}_{n+2} & \dots & \widetilde{\mu}_{2n} \end{bmatrix},$$

*the adjusted moments are defined to be*

$$\widetilde{\mu}_m = \int_I (x-\xi)^m \widehat{W}(x)\,dx,$$

*and where the matrix* $A_k$ *is obtained from* $A$ *by replacing the* $(k+1)$*-st column with the vector* $b$.

In Section 6, we work out recursion formulas for the adjusted moments, $\widetilde{\mu}_k$, of each $X_1$ family. In addition, we compute $L^{III,\alpha}_{1,2}(x)$ as an example.

In Table 5 we present the specifics for the matrix corresponding to each of the exceptional cases, noting that the constants $c_{1,0}$ and $c_{1,1}$ are easily obtained by comparing the formula (5.1) with the table of $v_1 = \widehat{y}_1$ from Section 4.

**Proof of Theorem 5.1.** We fill the matrix $A$ row-wise.

Substituting the ansatz (5.1) and its derivative (5.2) into the exceptional equation (4.1), leads to

$$\left[2p\eta' c_{n,1} - \left(p\eta'' + \frac{p'\eta'}{2} - s\eta'\right) c_{n,0}\right]\Big|_{x=\xi} = 0\,.$$

We collect this information in the first row of the determinantal representation. Since our matrix equation has the form $Ac = b$, the first row of the matrix $A$ now reads

$$\left[p(\xi)\eta''(\xi) + \frac{p'(\xi)\eta'(\xi)}{2} - s(\xi)\eta'(\xi) \qquad 2p(\xi)\eta'(\xi) \qquad 0 \qquad \dots \qquad 0\right].$$

The other $n$ rows come from the orthogonality relations (2.15). These conditions inform us that not only $\widehat{y}_n \perp \widehat{y}_k$ for $n \neq k$, but also that $\widehat{y}_n \perp \mathcal{E}_m$ for $m < n$. Since $v_m \in \mathcal{E}_m$, the orthogonality conditions imply

$$\langle \widehat{y}_n, v_k\rangle_{\widehat{W}} = 0 \quad \text{for } 1 \le k < n, \text{ and}$$
$$\langle \widehat{y}_n, \widehat{y}_n\rangle_{\widehat{W}} = K_n.$$

The second row of the matrix $A$ is obtained by substitution of

$$v_1(x) = \widehat{y}_1 = c_{1,0} + c_{1,1}(x - \xi)$$

and the ansatz (5.1) into this orthogonality relation. Using $v_1$ and the adjusted moments (5.3), we compute

$$\begin{aligned} 0 = \langle \widehat{y}_n, v_1 \rangle_{\widehat{W}} &= \sum_{i=0}^{n} c_{n,i} \langle (x-\xi)^i, v_1 \rangle \\ &= \sum_{i=0}^{n} c_{n,i} \left[ \langle (x-\xi)^i, c_{1,0} \rangle + c_{1,1} \langle (x-\xi)^i, (x-\xi) \rangle \right] \\ &= \sum_{i=0}^{n} c_{n,i} \left[ c_{1,0} \widetilde{\mu}_i + c_{1,1} \widetilde{\mu}_{i+1} \right] . \end{aligned}$$

Thus, the second row of the determinantal representation is

$$\begin{bmatrix} c_{1,0}\widetilde{\mu}_0 + c_{1,1}\widetilde{\mu}_1 & c_{1,0}\widetilde{\mu}_1 + c_{1,1}\widetilde{\mu}_2 & \dots & c_{1,0}\widetilde{\mu}_n + c_{1,1}\widetilde{\mu}_{n+1} \end{bmatrix}$$

For $2 \le k \le n$ the $(k+1)$-st row of matrix $A$ is found in analogy. Namely, recalling the definition of $v_k = (x - \xi)^k$ for $k \ge 2$, we compute

$$\begin{aligned} 0 = \langle \widehat{y}_n, v_k \rangle_{\widehat{W}} &= \sum_{i=0}^{n} c_{n,i} \langle (x-\xi)^i, v_k \rangle_{\widehat{W}} \\ &= \sum_{i=0}^{n} c_{n,i} \widetilde{\mu}_{i+k} . \end{aligned}$$

Thus, row $l = k + 1$, $3 \le l \le n + 1$, is given by

$$\begin{aligned} &\begin{bmatrix} \widetilde{\mu}_k & \widetilde{\mu}_{k+1} & \dots & \widetilde{\mu}_{n+k} \end{bmatrix} \quad \text{or} \\ &\begin{bmatrix} \widetilde{\mu}_{l-1} & \widetilde{\mu}_l & \dots & \widetilde{\mu}_{n+l-1} \end{bmatrix} . \end{aligned}$$

Equation (5.4) now follows from Cramer's rule, and (5.5) comes about from the co-factor definition of determinants, upon expanding (5.5) along the last row. □

## 6. Recursion relations for the adjusted moments

The exceptional polynomials $\widehat{y}_n$ may be represented using the adjusted moments $\widetilde{\mu}_k$. Thus, we can develop a recursive formula for the moments via the three-term recursion relation associated with the polynomial sequence. It is remarkable that these adjusted moments follow three-term recurrence relations as the exceptional polynomials themselves follow a five-term recurrence relation at best. In Table 6 we present the recursive formulas for all $X_1$ orthogonal polynomial systems. We include the proofs for both the Type III Laguerre and Jacobi moments. The proofs follow in analogy to the recursive formula of the Type I moments found in [24].

Prior to stating Theorem 6.1 we note that to simplify notation, we allow $\widehat{W}$ and $I$ to respectively represent the appropriate weight function and interval of orthogonality pertaining to each exceptional system.

**Table 6**
Recursion formulas for $X_1$ moments.

| | | Recursion formula |
|---|---|---|
| Exceptional Laguerre | Type I | $\widetilde{\mu}_{k+2} = (2\alpha + k)\widetilde{\mu}_{k+1} + \alpha(1-k)\widetilde{\mu}_k$ |
| | Type II | $\widetilde{\mu}_{k+2} = (2\alpha + k)\widetilde{\mu}_{k+1} + \alpha(1-k)\widetilde{\mu}_k$ |
| | Type III | $\widetilde{\mu}_{k+2} = k\widetilde{\mu}_{k+1} - \alpha(1-k)\widetilde{\mu}_k$ |
| Exceptional Jacobi | | $\widetilde{\mu}_{k+2} = \left[\frac{(2-\alpha-\beta-2k)\xi+\beta-\alpha}{\alpha+\beta+k}\right]\widetilde{\mu}_{k+1} + \left[\frac{(k-2)(1-\xi^2)}{\alpha+\beta+k}\right]\widetilde{\mu}_k$ |

**Theorem 6.1.** *For an $X_1$ orthogonal polynomial sequence satisfying*

$$a_2y'' + a_1y' + a_0y = \lambda y$$

*the adjusted moments $\widetilde{\mu}_k = \int_I (x-\xi)^k \widehat{W}(x)\,dx$ satisfy the recursion formulas for $k \in \mathbb{N}_0$:*

$$\widetilde{\mu}_{k+1} = -(r_2 + s_1)^{-1}\left[(kr_0 + s_{-1})\widetilde{\mu}_{k-1} + (kr_1 + s_0)\widetilde{\mu}_k\right],$$

*where $a_2(x) = \sum_{\ell=0}^{2} r_\ell\,(x-\xi)^\ell$ and $a_1(x) = \sum_{m=-1}^{1} s_m\,(x-\xi)^m$.*

Specifically, for each of the exceptional Laguerre and Jacobi families, the recursion formulas are provided in Table 6. The initial moments with which to begin the recursion are provided in Appendix A.

**Remark.** The details of the recursion formula of Theorem 6.1 are provided below and the details of the initial moments are given in Appendix A. Before the proof, we provide two critical observations. First, note that for functions $f$ and $g$, which are smooth on $I$, the associated moment functionals satisfy:

$$\left\langle \widehat{W}', f\right\rangle = -\left\langle \widehat{W}, f'\right\rangle \text{ and } \left\langle g\widehat{W}, f\right\rangle = \left\langle \widehat{W}, fg\right\rangle,$$

where $\langle\cdot,\cdot\rangle$ represents the inner product with respect to the Lebesgue measure on $I$.

Second, the second order linear operator given by

$$\ell[y] = a_2y'' + a_1y' + a_0y$$

may be written as a symmetry equation

$$a_2y' + (a_2' - a_1)y = 0$$

that is solvable by the weight function $\widehat{W}$ to which the associated eigenfunctions are orthogonal.

Combining these two notes, we observe that

$$\begin{aligned}\left\langle a_2\widehat{W}', (x-\xi)^k\right\rangle &= \left\langle \widehat{W}', a_2(x-\xi)^k\right\rangle = -\left\langle \widehat{W}, \left(a_2(x-\xi)^k\right)'\right\rangle \\ &= -k\left\langle \widehat{W}, a_2(x-\xi)^{k-1}\right\rangle - \left\langle \widehat{W}, a_2'(x-\xi)^k\right\rangle.\end{aligned}$$

Therefore, for $k \in \mathbb{N}$,

$$\begin{aligned}0 &= \left\langle a_2\widehat{W}' + (a_2' - a_1)W, (x-\xi)^k\right\rangle \\ &= \left\langle a_2\widehat{W}', (x-\xi)^k\right\rangle + \left\langle a_2'\widehat{W}, (x-\xi)^k\right\rangle - \left\langle a_1\widehat{W}, (x-\xi)^k\right\rangle \\ &= -k\left\langle \widehat{W}, a_2(x-\xi)^{k-1}\right\rangle - \left\langle a_1\widehat{W}, (x-\xi)^k\right\rangle. \qquad (6.1)\end{aligned}$$

**Proof.** Using equation (6.1), we aim to prove a general recursion formula. First rewrite the coefficients $a_2$ and $a_1$ from the differential expression to be in terms of powers of $(x-\xi)$. In the case of the $X_1$-Laguerre families, $a_2$ is of degree 1; for the $X_1$-Jacobi family, $a_2$ is of degree 2. Therefore, we have

$$a_2(x) = \sum_{\ell=0}^{2} r_\ell (x-\xi)^\ell,$$

where the coefficients $r_\ell$ are appropriately chosen as indicated below:

$$a_2(x) = \begin{cases} -(x-\xi)-\xi & \text{Types I, II, III } X_1\text{-Laguerre} \\ -(x-\xi)^2 - 2\xi(x-\xi) + (1-\xi^2) & X_1\text{-Jacobi}\,. \end{cases} \tag{6.2}$$

Since $a_1$ may be written as $p' + q + \frac{2b'p}{b}$, using a degree argument, $a_1$ may be written as

$$a_1(x) = \sum_{m=-1}^{1} s_m (x-\xi)^m,$$

where the coefficients $s_m$ are appropriately chosen as indicated below:

$$a_1(x) = \begin{cases} (x-\xi) + (1+2\xi) + 2\xi(x-\xi)^{-1} & \text{Type I } X_1\text{-Lag.} \\ (x-\xi) - (3+2\xi) + 2\xi(x-\xi)^{-1} & \text{Type II } X_1\text{-Lag.} \\ (x-\xi) + 2\xi(x-\xi)^{-1} & \text{Type III } X_1\text{-Lag.} \\ -(\alpha+\beta)(x-\xi) + (2-\alpha-\beta)\xi + \beta\alpha + 2(\xi^2-1)(x-\xi)^{-1} & X_1\text{-Jacobi.} \end{cases} \tag{6.3}$$

Substituting $a_2$ and $a_1$ into (6.1), rearranging, and collecting coefficients produces

$$0 = -(kr_0 + s_{-1})\widetilde{\mu}_{k-1} - (kr_1 + s_0)\widetilde{\mu}_k - (kr_2 + s_1)\widetilde{\mu}_{k+1} \quad (\text{for } k \in \mathbb{N})\,.$$

In other words, for $k \in \mathbb{N}$,

$$\widetilde{\mu}_{k+1} = -(kr_2 + s)^{-1}\left((kr_0 + s_{-1})\widetilde{\mu}_k - (kr_1 + s_0)\widetilde{\mu}_{k+1}\right)\,. \tag{6.4}$$

Shifting $k \mapsto k+1$, we obtain the result. □

**Example.** It is a short exercise, using $\xi = \alpha$ and equations (6.2) and (6.3), for the reader to see that (6.4) simplifies to the formulas given in Table 6.

We will verify for $n = 2$, that the polynomials in Theorem 5.1 indeed agree with (2.19).

For $n = 2$, recall that $c_{1,0} = 1$ and $c_{1,1} = 0$ as in Table 5. Therefore,

$$\begin{aligned} L_{1,2}^{III,\alpha}(x) &= \begin{vmatrix} 0 & 2\alpha & 0 \\ \widetilde{\mu}_0 & \widetilde{\mu}_1 & \widetilde{\mu}_2 \\ 1 & (x-\alpha) & (x-\alpha)^2 \end{vmatrix} \\ &= 2\alpha\widetilde{\mu}_2 - 2\alpha\widetilde{\mu}_0(x-\alpha)^2. \end{aligned}$$

Using the recursion formula in part (a) of Theorem 6.1 with $k = 0$,

$$\widetilde{\mu}_0 = \frac{\widetilde{\mu}_2}{-\alpha}\,,$$

we have, up to normalization,

$$L_{1,2}^{III,\alpha}(x) = 2\widetilde{\mu}_2 \left(x^2 - 2\alpha x + \alpha(\alpha+1)\right).$$

This is in agreement with the polynomial given in (2.19) since they both span the same eigenspace.

**Remark 6.2.** Although different in many ways, the moment representations for the Type I and II Laguerre polynomials *only* differ in the exceptional condition (3) (up to normalization, see Table 4). The all rows in the matrix of Theorem 5.1 except the first, the moment recursion formulas in Table 6, and the initial moments $\widetilde{\mu}_0$ and $\widetilde{\mu}_1$ in Table 7 turned out completely identical for the Type I and II Laguerre polynomials.

**Remark 6.3.** There are additional ways to compute the moments of the $X_1$ polynomial families. In particular, generating functions may be used. This method may be seen for the Type I Laguerre family in [24].

## 7. Some observations regarding higher co-dimensions

The introduction of the Darboux approach to the field [26,27] allowed the community to study $X_m$ orthogonal polynomials when $m > 1$. The approach towards Sections 2 and 3 was taken from the perspective of allowing higher codimensional sequences. Results regarding codimension two case can be found in [20], but as can be seen in the discussion below, generalized and higher order results should not be expected for $m > 1$.

Further, it is not hard to see that Lemma 4.1 can be generalized to the case of $m > 1$ by simply replacing the exceptional condition by a set of $m$ exceptional conditions. We must assume that condition (4.1) holds for $x = \xi_i$ where $\xi_i$ $(i = 1, \dots, m)$ denote the $m$ roots of the function $\eta$. Namely, we just replace (4.1) by the $m$ exceptional conditions

$$\left[2p\eta' y' - \left(p\eta'' + \frac{p'\eta'}{2} - s\eta'\right) y\right]\Big|_{x=\xi_i} = 0 \qquad \text{for } i = 1, \dots, m. \tag{7.1}$$

### *7.1. Possible flags*

As the codimension increases, there become increasingly more possibilities for the organization of the flag; that is that there are more choices in which degrees are removed from the sequence. Certain choices simplify subsequent computations like moment recursion formulas and the set-up of matrix $A$ in Theorem 5.1. The importance of making "good" choices that simplify the computations has already become clear in [24]. Here we do not go into the details of how to simplify the computations, but rather provide some preliminary discussions on what kind of choices are allowed in the case of the Type I $X_2$-Laguerre polynomials.

For example, consider the case when $m = 2$. Of course, we must take the first flag element to be

$$v_2(x) = L_2^{\alpha}(-x).$$

After some consideration, it appears that for $m = 2$ the choice

$$v_3(x) = (x - \xi_1)^2 (x - \xi_2 + 1)$$

is appropriate. To prove that a set $\{v_2, v_3, \dots\}$ indeed forms a flag, it suffices to take polynomials that satisfy the exceptional conditions (7.1) and are of the appropriate degree. As a result, there are now several choices for higher degree flag elements. For $n \geq 4$ we set

$$v_n(x) = (x - \xi_1)^k (x - \xi_2)^{n-k} \qquad \text{where we choose } 2 \leq k \leq n - 2.$$

In general, for $m \geq 2$, we can take

$$v_m(x) = L_m^\alpha(-x),$$
$$v_n(x) = \prod_{i=1}^{m}(x-\xi_i)^{k_i} \qquad \text{for } n \geq 2m, k_i \geq 2 \text{ and } \sum k_i = n.$$

We expect that the flag elements of degree $m+1, \dots, 2m-1$ are more complicated. But in principle the only requirements are that they have the correct degree and that they satisfy the exceptional conditions (7.1).

*7.2. Recursion type formulas*

The general method of finding recursion type formulas for moments of the exceptional weights applies to the higher co-dimension setting. In [24], using an adjusted moment greatly simplifies the computations. As in the discussion of possible flags above, there is again much more freedom, and it is not expected that all choices will yield favorable results.

Currently, it is clear that a "good" adjustment for the moments is given by

$$\widetilde{\mu}_{(l_1,\dots,l_m)} := \int_I \prod_{i=1}^{m}(x-\xi_i)^{l_i}\widehat{W}(x)dx \qquad \text{where } l_i \in \mathbb{N}_0.$$

For example for $m = 2$, the adjusted moments of interest will take the form

$$\widetilde{\mu}_{(l_1,l_2)} := \int_I (x-\xi_1)^{l_1}(x-\xi_2)^{l_2}\widehat{W}(x)dx,$$

so that recursion formulas will also need to generate $\widetilde{\mu}_{(l_1,l_2)}$ for all $(l_1, l_2) \in \mathbb{N}_0 \times \mathbb{N}_0$.

The choices made for the flag and for the ansatz used to generalize (5.1) will determine which moments will occur in the generalization of matrix $A$. This in turn tells us which of the adjusted moments we need to generate.

**Acknowledgments**

We would like to acknowledge the anonymous referee for drawing our attention to [13] as it relates to Lemma 4.1.

## Appendix A. Initial moments $\widetilde{\mu}_0$ and $\widetilde{\mu}_1$

**Lemma A.1.** *The initial moments for the $X_1$-exceptional orthogonal polynomial systems of Laguerre and Jacobi are given in Table 7, where the Gamma function is given by*

$$\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}\,dt$$

*and the incomplete Gamma function by*

$$\Gamma(a,x) := \int_x^\infty t^{a-1}e^{-t}\,dt \text{ for } x > 0\,.$$

**Table 7**
The values of the initial moments $\widetilde{\mu}_0$ and $\widetilde{\mu}_1$.

| | | $\widetilde{\mu}_0$ | $\widetilde{\mu}_1$ |
|---|---|---|---|
| Exceptional Laguerre | I | $\Gamma(\alpha) - 2e^{\alpha}\alpha^{\alpha}\Gamma(1+\alpha)\Gamma(-\alpha,\alpha)$ | $e^{\alpha}\alpha^{\alpha}\Gamma(1+\alpha)\Gamma(-\alpha,\alpha)$ |
| | II | $\Gamma(\alpha) - 2e^{\alpha}\alpha^{\alpha}\Gamma(1+\alpha)\Gamma(-\alpha,\alpha)$ | $e^{\alpha}\alpha^{\alpha}\Gamma(1+\alpha)\Gamma(-\alpha,\alpha)$ |
| | III | $\frac{-\Gamma(\alpha+1)}{\alpha}$ | $e^{-\alpha}(-\alpha)^{\alpha}\Gamma(1+\alpha)\Gamma(-\alpha,-\alpha)$ |
| Exceptional Jacobi | | $\frac{\Gamma(\alpha+1)\Gamma(\beta+1)(\alpha+\beta)}{2\alpha\beta\Gamma(\alpha+\beta+2)} + \frac{(J_1+J_2)(2\alpha\beta-\alpha-\beta)}{\alpha\beta}$ | $\frac{4}{\beta-\alpha}(J_1+J_2)$ |

*Further we use the notation*

$$J_1 = \frac{-1}{(\alpha+1)(\alpha+\beta)}\, F_1\left(1,-\beta,1,\alpha+2;-1,\frac{\beta-\alpha}{\alpha+\beta}\right)$$

*as well as*

$$J_2 = \frac{-1}{(\beta+1)(\alpha+\beta)}\, F_1\left(1,-\alpha,1,\beta+2;-1,\frac{\alpha-\beta}{\alpha+\beta}\right),$$

*where $F_1(\cdot)$ denotes the first Appell hypergeometric series.*

In the case of the Type I Laguerre polynomial family, the proof has been published in [24, Theorem 4.1]. We will prove that the moments given in Table 7 are correct for the Type III $X_1$-Laguerre and $X_1$-Jacobi polynomial families. The proof for the Type II $X_1$-Laguerre family follows in analogy to the other Laguerre types.

**Type III $X_1$-Laguerre Proof.** We prove that the expressions given in Table 7 are the moments $\widetilde{\mu}_0$ and $\widetilde{\mu}_1$ associated with the Type III $X_1$-Laguerre expression. Set

$$E_a(x) = \int_1^{\infty} e^{-xt}t^{-a}\,dt\,, \qquad \text{where} \qquad x > 0\,.$$

Recalling the definitions of $\Gamma(x)$ and $\Gamma(a,x)$, these functions and the exponential integral function, $E_a$, are related via

$$E_a(x) = x^{a-1}\Gamma(1-a,x)\,.$$

In addition,

$$(a-1)E_a(x) = e^{-x} - xE_{a-1}(x)\,.$$

Following a chain of manipulations, which may be found in [24], we note the following relation:

$$\int_0^{\infty} \frac{e^{-x}x^{\beta}}{x-\alpha}dx = e^{-\alpha}E_{1+\beta}(-\alpha)\Gamma(1+\beta)\,, \qquad \text{for } \alpha > 0\,,\ \beta > -1\,.$$

In consequence, we obtain the following expression for the first adjusted moment, $\widetilde{\mu}_1 = \int_0^{\infty} \frac{x^{\alpha}e^{-x}}{x-\alpha}\,dx$:

$$\begin{aligned}\widetilde{\mu}_1 &= e^{-\alpha}E_{1+\alpha}(-\alpha)\Gamma(1+\alpha)\\ &= e^{-\alpha}(-\alpha)^{\alpha}\Gamma(-\alpha,-\alpha)\Gamma(1+\alpha)\,.\end{aligned}$$

Notice that

$$\widetilde{\mu}_2 = \int_0^\infty (x-\alpha)^2 \frac{x^\alpha e^{-x}}{(x-\alpha)^2}\, dx = \int_0^\infty x^\alpha e^{-x}\, dx = \Gamma(\alpha+1)\, .$$

To finish the proof, we set $k = 0$, use the recursion formula in Table 6, and solve for $\widetilde{\mu}_0$ to show $\widetilde{\mu}_0 = \frac{-\Gamma(\alpha+1)}{\alpha}$. □

**$X_1$-Jacobi Proof.** We prove that the expressions given in Table 7 are the moments $\widetilde{\mu}_0$ and $\widetilde{\mu}_1$ associated with the $X_1$-Jacobi expression. Per (5.3),

$$\widetilde{\mu}_1 := \int_I (x-\xi)\widehat{W}(x)dx.$$

For the $X_1$-Jacobi system, $\xi = \frac{\alpha+\beta}{\beta-\alpha}$, and

$$\widehat{W}_1^{\alpha,\beta}(x) = \frac{(1-x)^\alpha(1+x)^\beta}{\left(P_1^{(-\alpha-1,\beta-1)}(x)\right)^2}\, .$$

Substituting these into the equation for $\widetilde{\mu}_1$, along with the classical Jacobi polynomial in the denominator of the weight function, we find that

$$\widetilde{\mu}_1 = \frac{4}{\beta-\alpha} \int_{-1}^1 \frac{(1-x)^\alpha(1+x)^\beta}{(\beta-\alpha)x-\alpha-\beta} dx.$$

We recast this integral as the sum of two integrals, so that now

$$\widetilde{\mu}_1 = \frac{4}{\beta-\alpha}[J_1+J_2],$$

where

$$J_1 = \int_0^1 \frac{(1-x)^\alpha(1+x)^\beta}{(\beta-\alpha)x-\alpha-\beta} dx,$$

and

$$J_2 = \int_0^1 \frac{(1-x)^\beta(1+x)^\alpha}{(\alpha-\beta)x-\alpha-\beta} dx.$$

To obtain the value of these integrals, we note that the first Appell hypergeometric series has an integral representation if two of its parameters meet certain restrictions. Namely, we find

$$F_1(a,b_1,b_2,c;x,y) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b_1}(1-yt)^{-b_2} dt$$

for Re $c >$ Re $a > 0$, see [2, Chapter 9].

Observe that setting $a = 1$, $b_1 = -\beta$, $b_2 = 1$, $c = \alpha + 2$, $x = -1$, and $y = \frac{\beta-\alpha}{\alpha+\beta}$ allows us to equate $J_1$ with the Appell series. These values also satisfy the restrictions for $a$ and $c$. Similarly, we set $a = 1$, $b_1 = -\alpha$, $b_2 = 1$, $c = \beta + 2$, $x = -1$, and $y = \frac{\alpha-\beta}{\alpha+\beta}$ to equate $J_2$ with the Appell series.

This results in the following values:

$$J_1 = \frac{-1}{(\alpha+1)(\alpha+\beta)} F_1 \left(1, -\beta, 1, \alpha+2; -1, \frac{\beta-\alpha}{\alpha+\beta}\right),$$

and

$$J_2 = \frac{-1}{(\beta+1)(\alpha+\beta)} F_1 \left(1, -\alpha, 1, \beta+2; -1, \frac{\alpha-\beta}{\alpha+\beta}\right).$$

As mentioned above, $\widetilde{\mu}_1 = \frac{4}{\beta-\alpha}[J_1 + J_2]$.

We calculate $\widetilde{\mu}_0$ indirectly by first finding $\widetilde{\mu}_2$, which is an easier computation. Again, per equation (5.3),

$$\widetilde{\mu}_2 := \int_I (x-\xi)^2 \widehat{W}(x) dx.$$

Substituting the expressions for $\xi$ and the exceptional weight, we find that

$$\widetilde{\mu}_2 = \frac{4}{(\beta-\alpha)^2} \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta dx.$$

A standard table of integrals informs us that for $m, n > -1$, and $b > a$,

$$\int_a^b (x-a)^m (b-x)^n dx = (b-a)^{m+n+1} \, \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)}.$$

Setting $a = -1$, $b = 1$, $m = \beta$, and $n = \alpha$ allows us to compute the value of $\widetilde{\mu}_2$. The restriction on $m$ and $n$ matches those for the Jacobi parameters $\alpha$ and $\beta$:

$$\widetilde{\mu}_2 = \frac{4\,\Gamma(\alpha+1)\Gamma(\beta+1)}{(\beta-\alpha)^2\,\Gamma(\alpha+\beta+2)}.$$

Finally, using the recursion formula given in Table 6 for the $X_1$-Jacobi system, and setting $k = 0$, we calculate $\widetilde{\mu}_0$. After substituting in the expression for $\xi$ in terms of $\alpha$ and $\beta$, and further simplifications, we obtain the following value:

$$\widetilde{\mu}_0 = \frac{(\alpha+\beta)\Gamma(\alpha+1)\Gamma(\beta+1)}{2\alpha\beta\Gamma(\alpha+\beta+2)} + \frac{(2\alpha\beta-\alpha-\beta)(J_1+J_2)}{\alpha\beta}.$$

This concludes the proof of Lemma A.1. □

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