WILD THEORIES WITH O-MINIMAL OPEN CORE

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Abstract. Let T be a consistent o-minimal theory extending the theory of densely ordered groups and let T' be a consistent theory. Then there is a complete theory T^* extending T such that T is an open core of T^*, but every model of T^* interprets a model of T'. If T' is NIP, T^* can be chosen to be NIP as well. From this we deduce the existence of an NIP expansion of the real field that has no distal expansion.

1. Introduction

Let $\mathcal{R}$ be an expansion of a dense linear order $(R, <)$ without endpoints. The **open core of** $\mathcal{R}$, denoted by $\mathcal{R}^\circ$, is the structure $(R, (U))$, where U ranges over all open sets of all arities definable in $\mathcal{R}$. Miller and Speissegger introduced this notion of an open core for expansions of $(\mathbb{R}, <)$ in [16], and established sufficient conditions on $\mathcal{R}$ such that its open core is o-minimal. Here we want to answer the following question:

Is there any restriction on what kind of structures can be interpreted in an expansion of $(R, <)$ with o-minimal open core?

This question, although formulated slightly differently, was already asked by Dolich, Miller and Steinhorn in a preprint version of [7]. Our answer is negative. To give a precise statement of our result, we need to recall the notion of an open core of a theory as introduced in [6]. Let T^* be a theory extending the theory of dense linear orders without endpoints in a language $\mathcal{L}^* \supseteq \{<\}$, and let T be another theory in a language $\mathcal{L}$. We say that T **is an open core of** T^* if for every $\mathcal{N} \models T^*$ there is $\mathcal{M} \models T$ such that $\mathcal{N}^\circ$ is interdefinable with $\mathcal{M}$.

Theorem A. Let T be a consistent o-minimal theory extending the theory of densely ordered groups and let T' be a consistent theory. Then there is a complete theory T^* extending T such that

(1) T^* interprets a model of T',
(2) T is an open core of T^*,
(3) T^* is NIP if T' is NIP,
(4) T^* is strongly dependent if T' is strongly dependent.

Statements (3) and (4) of Theorem A indicate that we can choose T^* in such a way that not only the open core of T^* is o-minimal, but also T^* remains tame in the sense of Shelah's combinatorial tameness notions. For definitions of NIP and

Date: June 19, 2018.
2010 *Mathematics Subject Classification.* Primary 03C64 Secondary 03C45.
Key words and phrases. O-minimal open core, Tameness, NIP, Distal, Expansions of the real line, Noiseless.
The first author was partially supported by NSF grant DMS-1654725.

strong dependence, we refer the reader to Simon [19].

We will deduce the following analogue for o-minimal expansions of the ordered real additive group from the proof of Theorem A.

Theorem B. Let $\mathcal{R}$ be an o-minimal expansion of $(\mathbb{R}, <, +)$ in a language $\mathcal{L}$ and let T' be a consistent theory such that $|\mathcal{L}| < |\mathbb{R}|$ and $|T'| \leq |\mathbb{R}|$. Then there exists an expansion $\mathcal{S}$ of $\mathcal{R}$ such that

(1) $\mathcal{S}$ interprets a model of T',
(2) the open core of $\mathcal{S}$ is interdefinable with $\mathcal{R}$,
(3) $\mathcal{S}$ is NIP if T' is NIP,
(4) $\mathcal{S}$ is strongly dependent if T' is strongly dependent.

We will deduce from work in [6] that an expansion of $(\mathbb{R}, <, +)$ has o-minimal open core if and only if it does not define a discrete linear order. Therefore in Theorem B the statement $\mathcal{S}$ *interprets a model of* T' cannot be replaced by the statement $\mathcal{S}$ *defines a model of* T'.

The outline of the proof of the above results is as follows. For simplicity, let $\mathcal{R}$ be $(\mathbb{R}, <, +)$ and let T' be a consistent theory in a countable language $\mathcal{L}'$ with an infinite model. Take a dense basis P of $\mathbb{R}$ as a $\mathbb{Q}$-vector space. By [7, 2.25] the open core of the structure $(\mathbb{R}, <, +, P)$ is $\mathcal{R}$. We further expand $(\mathbb{R}, <, +, P)$ by a binary predicate E such that E is an equivalence relation on P, has countably many equivalence classes and each equivalence class of E is dense in P. Now take a countable model M of T' and expand $(\mathbb{R}, <, +, P, E)$ to an expansion $\mathcal{S}$ such that the quotient P/E becomes an $\mathcal{L}'$-structure that is isomorphic to M. Since each equivalence class of E is dense in P and hence in $\mathbb{R}$, we can define this *fusion* $\mathcal{S}$ of $(\mathbb{R}, <, +, P, E)$ and M in a way that the open core of the resulting structure $\mathcal{S}$ is still $\mathcal{R}$. Indeed we use ideas and techniques from [7] to prove a quantifier-elimination result for $\mathcal{S}$ analogous to the one of $(\mathbb{R}, <, +, P)$ (see [7, 2.9]), and from that deduce that the open core of $\mathcal{S}$ is $\mathcal{R}$.

In the special case that $\mathcal{L}'$ is empty and T' is the theory of infinite sets, the construction we outlined above gives the following extension of the results from [7].

Theorem C. Let T be a complete o-minimal theory extending the theory of densely ordered groups in a language $\mathcal{L}$, and let $\mathcal{L}_e$ be the language $\mathcal{L}$ augmented by a unary predicate P and a binary predicate E. Let $T_{e,\infty}$ be the $\mathcal{L}_e$-theory containing T and axiom schemata expressing the following statements:

(1) P is dense and dcl_T-independent,
(2) $E \subseteq P^2$ is an equivalence relation on P,
(3) each equivalence class of E is dense in P,
(4) E has infinitely many equivalence classes.

Then $T_{e,\infty}$ is complete, and T is an open core of $T_{e,\infty}$.

Theorem B should be compared to Friedman and Miller [11, Theorem A]. Among other things, the latter result implies the existence of an expansion of the real field that defines a model of first-order arithmetic, but every subset of $\mathbb{R}$ definable in this expansion is a finite union of an open set and finitely many discrete sets. Therefore both our result and [11] describe situations in which topological tameness exists without model-theoretic tameness.

In general our results rule out that the property of having an o-minimal open core has any consequences in terms of model-theoretic tameness of the whole structure. At first glance this might look like a disappointing result. However, we do not share this viewpoint. We regard our results as further evidence that in model-theoretically wild situations geometric tameness can often prevail. In some of those situations the open core of a structure or theory seems to be the right tool that can capture precisely this tameness, making certain phenomena trackable by model-theoretic analysis.

Theorem B(3) has a few interesting corollaries about NIP expansions of $(\mathbb{R}, <, +)$. First of all, it states that for every NIP theory T' of cardinality at most continuum there is an NIP expansion of $(\mathbb{R}, <, +)$ that interprets a model of T'. Therefore the model theory of NIP expansions of $(\mathbb{R}, <, +)$ is in general as complicated as the model theory of arbitrary NIP theories. We use this observation to deduce a new result about the distality of NIP expansions of $(\mathbb{R}, <, +)$. The notion of distality was introduced by Simon in [18] to single out those NIP theories and structures that can be considered purely unstable. While every o-minimal expansions of $(\mathbb{R}, <, +)$ is distal, there are several natural examples of non-distal NIP expansions of $(\mathbb{R}, <, +)$ (see [13]). However, by Chernikov and Starchenko [4] even just having a distal expansion guarantees certain desirable combinatorial properties of definable sets (the strong Erdös-Hajnal property). Therefore it is interesting to know whether or not all NIP expansions of $(\mathbb{R}, <, +)$ have a distal expansion. Although we do not know it, we expect all examples of non-distal NIP expansions of $(\mathbb{R}, <, +)$ produced in [13] to have distal expansions. So far the only known NIP theory without an distal expansion is the theory of algebraically closed fields of characteristic p by [4, Proposition 6.2]. Combining this with Theorem B, we almost immediately obtain the following.

Theorem D. There is an NIP expansion of $(\mathbb{R}, <, +)$ that does not have a distal expansion.

This is also the first example of an NIP expansion of any densely ordered set that does not have a distal expansion.

While in general for every countable NIP theory there is an expansion of $(\mathbb{R}, <, +)$ that interprets a model of this theory, there is a natural class of expansions of $(\mathbb{R}, <, +)$ in which models of certain NIP theories in countable languages cannot be interpreted. A set $X \subseteq \mathbb{R}$ is somewhere dense and co-dense if there is an open interval I such that $X \cap I$ is dense and co-dense in I. We say an expansion of $(\mathbb{R}, <)$ is **noiseless** if it does not define a somewhere dense and co-dense subset of $\mathbb{R}$.[1] The expansion $\mathcal{S}$ we produce for Theorem B is not noiseless. It is therefore natural to ask whether in Theorem B we can require $\mathcal{S}$ to be noiseless. The answer to this question is negative.

[1]The name *noiseless* was suggested by Chris Miller. Being noiseless is equivalent to the statement that every definable subset of $\mathbb{R}$ either has interior or is nowhere dense. The latter condition has also been called *i-minimality* by Fornasiero [9].

Theorem E. Let $\mathcal{R}$ be a noiseless NIP expansion of $(\mathbb{R}, <, +, 1)$. Then $\mathcal{R}$ has definable choice, that is: for $A \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $\emptyset$-definable in $\mathcal{R}$ there is an $\emptyset$-definable function $f : \pi(A) \to \mathbb{R}^n$ such that

(1) $\mathrm{gr}(f) \subseteq A$,
(2) $f(a) = f(b)$ whenever $a, b \in \pi(A)$ and $A_a = A_b$,

where $\pi : \mathbb{R}^{m+n} \to \mathbb{R}^m$ is the projection onto the first m coordinates.

It follows from Theorem E that if a noiseless NIP expansion $\mathcal{R}$ of $(\mathbb{R}, <, +, 1)$ interprets a structure $\mathcal{M}$, then $\mathcal{R}$ defines an isomorphic copy of $\mathcal{M}$. We will prove Theorem E in greater generality. In particular, Theorem E not only holds for noiseless NIP expansions, but also for noiseless NTP_2 expansions (for a definition of NTP_2 see [19]).

We now show that Theorem B fails when we require $\mathcal{S}$ to be noiseless. Let p be a prime and $\mathbb{F}_p$ be the field with p elements. By Shelah and Simon [17, Theorem 2.1] if $\mathcal{V} = (V, +, \ldots)$ is an infinite $\mathbb{F}_p$-vector space and $\prec$ is a linear order on V, then $(\mathcal{V}, \prec)$ has IP. Suppose now that $\mathcal{M} = (M, <, \ldots)$ is an expansion of an infinite linear order $(M, <)$ and that $\mathcal{V}$ is an $\mathcal{M}$-definable infinite $\mathbb{F}_p$-vector space with underlying set $V \subseteq M^k$. The lexicographic order on M^k induced by $<$ is linear and induces a linear order on V. It follows that $\mathcal{M}$ has IP. Thus no NIP expansion of a linear order defines an infinite $\mathbb{F}_p$-vector space. By Theorem E no noiseless NIP expansion of $(\mathbb{R}, <, +, 1)$ interprets an infinite vector space over a finite field.

Open questions. We end the introduction with a few open questions.

1. We work here in the context of ordered structures and o-minimal open core. It is likely that our techniques can be used to extend our results to various other settings. In particular, by using the technology from Berenstein and Vassiliev [2] rather than from [7] one should be able produce analogues of Theorem A and B for other geometric structures such as the field of p-adic numbers.

2. Similar questions can be asked about NIP expansions of $(\mathbb{N}, <)$. Since every such expansion has definable Skolem functions, we again have some limitations on what kind of theories can be interpreted in such a structure. Can we say anything more? For example: can an NIP expansion of $(\mathbb{N}, <)$ interpret an infinite field? Is there an NIP expansion of $(\mathbb{N}, <)$ that does not admit a distal expansion?

3. Is there a noiseless NIP expansion of $(\mathbb{R}, <, +)$ that does not admit a distal expansion? Is every infinite field interpretable in a noiseless NIP expansion isomorphic to $(\mathbb{R}, +, \cdot)$ or $(\mathbb{C}, +, \cdot)$?

The previous question is even open for d-minimal NIP expansions, a subclass of the class of noiseless NIP expansions (see [15] for a definition of d-minimality). It follows from Fornasiero [10, Theorem 4.13] that any uncountable field interpretable in a d-minimal expansion is isomorphic to $(\mathbb{R}, +, \cdot)$ or $(\mathbb{C}, +, \cdot)$. Thus in this setting it suffices to show that no d-minimal NIP expansion interprets a countable field. It is not difficult to show that any countable set definable in a d-minimal expansion admits a definable order with order type ω. Thus, if the above question about the interpretability of infinite fields in NIP expansions of $(\mathbb{N}, <)$ has a negative answer, then any infinite field interpretable in a d-minimal NIP expansion of $(\mathbb{R}, <, +)$ is isomorphic to $(\mathbb{R}, +, \cdot)$ or $(\mathbb{C}, +, \cdot)$.

Is every noiseless NIP expansion of $(\mathbb{R}, <, +)$ d-minimal? We doubt that this statement is true, but it seems difficult to produce a counterexample.

Acknowledgments. The authors thank Antongiulio Fornasiero and Chris Miller for helpful conversations around the topic of this paper.

Notation. We will use m, n for natural numbers and κ for a cardinal. Let X, Y be sets. We denote the cardinality of X by $|X|$. For a function $f : X \to Y$, we denote the graph of f by $\mathrm{gr}(f)$. If $Z \subseteq X \times Y$ and $x \in X$, then Z_x denotes the set $\{y \in Y \ : \ (x, y) \in Z\}$. If $a = (a_1, \dots, a_n)$, we sometimes write Xa for $X \cup \{a_1, \dots, a_n\}$, and XY for $X \cup Y$.
Let $\mathcal{L}$ be a language and T an $\mathcal{L}$-theory. Let $M \models T$ and $A \subseteq M$. In this situation, $\mathcal{L}$-definable always means $\mathcal{L}$-definable with parameters. If we want to be precise about the parameters we write $\mathcal{L}$-A-definable to indicate $\mathcal{L}$-definability with parameters from A. Let $b \in M^n$. Then we write $\mathrm{tp}_{\mathcal{L}}(b|A)$ for the $\mathcal{L}$-type of b over A. Moreover, $\mathrm{dcl}_T(A)$ denotes the definable closure of A in M. Whenever T is o-minimal, dcl_T is a pregeometry.

2. The fusion

Let T be a consistent o-minimal theory extending the theory of densely ordered groups with a distinguished positive element, and let $\mathcal{L}$ be its language. Let $\mathcal{L}'$ be a relational language disjoint from $\mathcal{L}$, and let T' be a consistent $\mathcal{L}'$-theory. In this section we will construct a language $\mathcal{L}^* \supseteq \mathcal{L}$ and a complete $\mathcal{L}^*$-theory T^* extending T such that T is an open core of T^* and T^* interprets T'. In Section 3 we show that T^* is NIP whenever T is, and in Section 4 we prove that strong dependence of T' implies strong dependence of T^*.

By replacing T by a completion of T and T' by a completion of T', we can directly reduce to the case that both T and T' are complete. So from now, we assume that T and T' are complete.

Let $\mathcal{L}_e$ be $\mathcal{L}$ expanded by a unary predicate P and a binary predicate E such that neither P nor E are not in $\mathcal{L}'$. Let T_e be the extension of T by axiom schemata expressing the following statements:

(T1) P is dense and dcl_T-independent,
(T2) $E \subseteq P^2$ is an equivalence relation on P,
(T3) each equivalence class of E is dense in P.

Let $\mathcal{L}^* = \mathcal{L}_e \cup \mathcal{L}'$. For a given $\mathcal{L}'$-formula θ we define a $\mathcal{L}^*$-formula θ_e recursively as follow:

if θ is $x = y$, then define θ_e as Exy,

if θ is $Rx_1 \dots x_n$ where R is an n-ary predicate in $\mathcal{L}'$, then define θ_e as $Rx_1 \dots x_n$,

if θ is $\neg\theta'$, then define θ_e as $\neg\theta'_e$,

if θ is $\theta' \wedge \theta''$, then define θ_e as $\theta'_e \wedge \theta''_e$,

if θ is $\theta' \vee \theta''$, then define θ_e as $\theta'_e \vee \theta''_e$, if θ is $\exists x\theta'$, then define θ_e as $\exists x(Px \wedge \theta'_e)$,

if θ is $\forall x\theta'$, then define θ_e as $\forall x(Px \to \theta'_e)$.

Let T^* be the extension of T_e by the following axiom schemata:

(T4) $R \subseteq P^n$ and

$$\forall x_1 \forall y_1 \dots \forall x_n \forall y_n \left(\bigwedge_{i=1}^{n} Ex_i y_i \right) \to \Big(Rx_1 \dots x_n \leftrightarrow Ry_1 \dots y_n \Big)$$

for every $R \in \mathcal{L}'$ with $\mathrm{ar}(R) = n$,

(T5) φ_e for every $\varphi \in T'$.

We now fix some further notation. Given a model $\mathcal{M}$ of T^*, we will denote the underlying model of T by M, the interpretation of P and E by $P_{\mathcal{M}}$ and $E_{\mathcal{M}}$. For $b \in P^n_{\mathcal{M}}$ and $A \subseteq P_{\mathcal{M}}$ we denote by $\mathrm{tp}_{\mathcal{L}'}(b|A)$ the set of all $\mathcal{L}^*$-formulas of the form $\varphi_e(x, a)$ for some $\mathcal{L}'$-formula $\varphi(x, y)$ such that $a \in A^m$ and $\mathcal{M} \models \varphi_e(b, a)$.

A standard induction on $\mathcal{L}'$-formulas together with Axiom (T4) gives the following.

Lemma 2.1. *Let $\mathcal{M} \models T^*$, $a, b \in P_{\mathcal{M}}$ and $A \subseteq P_{\mathcal{M}}$. If $(a, b) \in E_{\mathcal{M}}$, then $\mathrm{tp}_{\mathcal{L}'}(a|A) = \mathrm{tp}_{\mathcal{L}'}(b|A)$.*

We now show that given a model of T with enough dcl_T-independent elements, this model can be expanded to a model of T^*. This result will be used to show consistency of T^*.

Lemma 2.2. *Let $M \models T$ and let $(A_b)_{b \in B}$ be a family of dense subsets of M such that*

- $\bigcup_{b \in B} A_b$ *is dcl_T-independent,*
- $A_b \cap A_{b'} = \emptyset$ *whenever $b \neq b'$,*
- *there is a model of T' with the same cardinality as B.*

Then M can be expanded to a model of T^*.

Proof. Let $\mathcal{N}$ be a model of T' with the same cardinality as B. Without loss of generality, we can assume that B is the universe of $\mathcal{N}$. We now expand M to an $\mathcal{L}^*$-structure $\mathcal{M}$. We interpret the relation symbol P as $P_{\mathcal{M}} := \bigcup_{b \in B} A_b$. For $a, a' \in P_{\mathcal{M}}$ we say $aE_{\mathcal{M}}a'$ if and only if there is $b \in B$ such that $a, a' \in A_b$. It is clear that $E_{\mathcal{M}}$ is an equivalence relation on $P_{\mathcal{M}}$ and that every equivalence class of $E_{\mathcal{M}}$ is dense in M. Thus $(M, P_{\mathcal{M}}, E_{\mathcal{M}})$ is an $\mathcal{L}_e$-structure that models T_e. It is left to interpret the elements of $\mathcal{L}'$. Let R be an n-ary relation symbol in $\mathcal{L}'$. We define its interpretation $R_{\mathcal{M}}$ by

$$\{(a_1, \dots, a_n) \in P^n_{\mathcal{M}} \ : \ \exists b_1, \dots, b_n \in B \ \bigwedge_{i=1}^{n} a_i \in A_{b_i} \wedge \mathcal{N} \models R(b_1, \dots b_n)\}.$$

Let $\mathcal{M} := (M, P_{\mathcal{M}}, E_{\mathcal{M}}, \big(R_{\mathcal{M}}\big)_{R \in \mathcal{L}'})$. It is clear from the definition of $E_{\mathcal{M}}$ and $R_{\mathcal{M}}$ that $\mathcal{M}$ satisfies (T4). By a straightforward induction on formulas we see that for every $\mathcal{L}'$-formula $\varphi(x)$ and for every $a_1, \dots, a_n \in P_{\mathcal{M}}$ and $b_1, \dots, b_n \in B$ with $a_i \in A_{b_i}$ we have

$$\mathcal{M} \models \varphi_e(a_1, \dots, a_n) \text{ if and only if } \mathcal{N} \models \varphi(b_1, \dots, b_n).$$

Thus $\mathcal{M}$ satisfies (T5), and therefore $\mathcal{M} \models T^*$.

$\square$

Proposition 2.3. *The theory T^* is consistent.*

Proof. By [7, 1.11] there is a model M of T and a family $(A_b)_{b\in B}$ of dense subsets of M such that the family $(A_b)_{b\in B}$ satisfies the assumptions of Lemma 2.2. The statement of the proposition then follows from Lemma 2.2. $\square$

Proposition 2.4. *Every model of T^* interprets a model of T'.*

Proof. Let $\mathcal{M} := (M, P_{\mathcal{M}}, E_{\mathcal{M}}, (R_{\mathcal{M}})_{R\in\mathcal{L}'}) \models T^*$. Let N be the set of equivalence classes of $E_{\mathcal{M}}$. For an r-ary relation symbol R, let

$$R_{\mathcal{N}} := \{([a_1]_{E_{\mathcal{M}}}, \dots, [a_n]_{E_{\mathcal{M}}}) \in N^n \ : \ (a_1, \dots, a_n) \in R_{\mathcal{M}}\}.$$

Note that $R_{\mathcal{N}}$ is well-defined by Lemma 2.1. Let $\mathcal{N} = (N, (R_{\mathcal{N}})_{R\in\mathcal{L}'})$. Since $\mathcal{N}$ is interpretable in $\mathcal{M}$, it is only left to show that $\mathcal{N} \models T'$. Using a straightforward induction on $\mathcal{L}'$-formulas and Axiom (T4) the reader can check that for every $\mathcal{L}'$-formula $\varphi(x)$ and for every $a_1, \dots, a_n \in P_{\mathcal{M}}$

$$\mathcal{M} \models \varphi_e(a_1, \dots, a_n) \text{ if and only if } \mathcal{N} \models \varphi([a_1]_{E_{\mathcal{M}}}, \dots, [a_n]_{E_{\mathcal{M}}}).$$

Thus $\mathcal{N} \models T'$, since $\mathcal{M}$ satisfies (T5). $\square$

Proposition 2.4 shows that T^* satisfies condition (1) of Theorem A. In the rest of this section we will show that T^* also satisfies condition (2). In order to do so we have to carefully analyse the definable sets in models of T^*.

2.1. **Back-and-forth system.** To better understand definable sets and types in models of T^*, we follow the general strategy of the proofs of [7, 2.8] and van den Dries [8, Theorem 2.5] by constructing a back-and-forth system between models of T^*. Let κ be a cardinal larger than $|T^*|$. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two κ-saturated models of T^*. Let $\mathcal{I}$ be the set of all partial $\mathcal{L}$-isomorphisms $\iota : X \to Y$ between $\mathcal{M}_1$ and $\mathcal{M}_2$ such that there are

- finite $A \subseteq P_{\mathcal{M}_1}$ and $A' \subseteq P_{\mathcal{M}_2}$,
- finite $Z \subseteq \mathcal{M}_1$ and $Z' \subseteq \mathcal{M}_2$

with

(i) $\iota(A) = A'$ and $\iota(Z) = Z'$.
(ii) Z and Z' are dcl-independent over $P_{\mathcal{M}_1}$ and $P_{\mathcal{M}_2}$ respectively,
(iii) $X = \mathrm{dcl}_T(AZ)$ and $Y = \mathrm{dcl}_T(A'Z')$,
(iv) for all $a_1, \dots, a_n \in A$,

$$\mathcal{M}_1 \models \varphi_e(a_1, \dots, a_n) \text{ if and only if } \mathcal{M}_2 \models \varphi_e(\iota(a_1), \dots, \iota(a_n)).$$

In the following we will show that $\mathcal{I}$ is back-and-forth system of partial $\mathcal{L}^*$-isomorphisms.

Lemma 2.5. *Let $\iota : X \to Y \in \mathcal{I}$ and let $A, Z \subseteq \mathcal{M}_1$ and $A', Z' \subseteq \mathcal{M}_2$ be such that A, A, Z, Z' satisfy conditions (i)-(iv) above. Then ι is a partial $\mathcal{L}^*$-isomorphism and $X \cap P_{\mathcal{M}_1} = A$.*

Proof. We first show that $X \cap P_{\mathcal{M}_1} = A$. Suppose there is $z \in (X \cap P_{\mathcal{M}_1}) \setminus A$. Since $P_{\mathcal{M}_1}$ is dcl_T-independent and $A \subseteq P_{\mathcal{M}_1}$, we have that $z \notin \mathrm{dcl}_T(A)$. Thus $z \in \mathrm{dcl}_T(AZ) \setminus \mathrm{dcl}_T(A)$. Since dcl_T is a pregeometry and $z \in P_{\mathcal{M}_1}$, this contradicts the dcl_T-independence of Z over $P_{\mathcal{M}_1}$. Similarly we can show that $Y \cap P_{\mathcal{M}_2} = A'$. Since $\iota(A) = A'$, it follows that $\iota(X \cap P_{\mathcal{M}_1}) = Y \cap P_{\mathcal{M}_2}$. Since $(x = y)_e$ is Exy, we can easily deduce from (iv) that ι is an $\mathcal{L}_e$-isomorphism. Applying (iv) once more, we see that ι is also an $\mathcal{L}^*$-isomorphism. $\square$

Lemma 2.6. *The set $\mathcal{I}$ is a back-and-forth system.*

Proof. Let $\iota : X \to Y \in \mathcal{I}$ and $b \in \mathcal{M}_1$. By symmetry it is enough to show that if $b \notin X$, then we can find $\iota' \in \mathcal{I}$ extending ι such that b is in the domain of ι'. From now on, assume that $b \notin X$.

Case I: $b \in P_{\mathcal{M}_1}$. Let p be the collection of all $\mathcal{L}'$-formulas $\varphi(x, \iota(a))$ such that $a \in A^n$ and $\mathcal{M}_1 \models \varphi_e(b, a)$. By saturation of $\mathcal{M}_2$ and since $\iota \in \mathcal{I}$, there is $b' \in P_{\mathcal{M}_2}$ such that $\mathcal{M}_2 \models \varphi_e(b', \iota(a))$ for every $\varphi(x, \iota(a)) \in p$. By density of the equivalence classes of E, we can take an element $b'' \in P_{\mathcal{M}_2}$ such that $(b', b'') \in E_{\mathcal{M}_2}$ and the cuts realized by b in X and by b'' in Y correspond via ι. Thus ι extends to an $\mathcal{L}$-isomorphism $\iota' : \mathrm{dcl}_T(ZAb) \to \mathrm{dcl}_T(ZA'b'')$ with $\iota'(b) = b''$. Since $(b', b'') \in E_{\mathcal{M}_2}$, we have by Lemma 2.1 that $\mathcal{M}_2 \models \varphi_e(b'', \iota(a))$ for every $\varphi(x, \iota(a)) \in p$. It is clear from our choice of b'' that $\iota' \in \mathcal{I}$.

Case II: $b \in \mathrm{dcl}_T(ZAP_{\mathcal{M}_1})$. Let $a_1, \dots, a_m \in P_{\mathcal{M}_1}$ be such that $b \in \mathrm{dcl}_T(ZAa_1 \dots a_m)$. By applying Case I m times, we can find an element $\iota' \in \mathcal{I}$ extending ι such that $a_1, \dots, a_m$ are in the domain of ι'. Since the domain of ι' contains $\mathrm{dcl}_T(ZAa_1 \dots a_m)$, it also contains b.

Case III: $b \notin \mathrm{dcl}_T(AP_{\mathcal{M}_1})$. By [7, 2.1] and saturation of $\mathcal{M}_2$, there exists an element $b' \in \mathcal{M}_2 \setminus \mathrm{dcl}_T(A'P_{\mathcal{M}_2})$ such that the cuts realized by b over X and b' over Y correspond via ι. Therefore we can find an $\mathcal{L}$-isomorphism $\iota' : \mathrm{dcl}_T(AZb) \to \mathrm{dcl}_T(A'Z'b')$ extending ι and mapping b to b'. It is easy to check that $\iota' \in \mathcal{I}$. $\square$

2.2. **Completeness and quantifier-reduction.** We now use the back-and-forth system $\mathcal{I}$ to deduce certain desirable properties of T^*. In particular, we show completeness of T^* and a quantifier-reduction result.

Theorem 2.7. *The theory T^* is complete.*

Proof. By Proposition 2.3, T^* is consistent. In the previous section we constructed a back-and-forth system between any two κ-saturated models of T^*. This implies that two such models are elementary equivalent. Completeness of T^* follows. $\square$

Definition 2.8. We call an $\mathcal{L}^*$-formula $\chi(y)$ **special** if it is of the form

$$\exists x \ Px \wedge \psi_e(x) \wedge \varphi(x, y),$$

where ψ is an $\mathcal{L}'$-formula and φ is an $\mathcal{L}$-formula.

We now establish that T^* has quantifier-elimination up to boolean combinations of special formulas (compare this result and its proof to [7, 2.9]) and [8, Theorem 1]).

Theorem 2.9. *Each $\mathcal{L}^*$-formula is T^*-equivalent to a boolean combination of special formulas.*

Proof. Let $\mathcal{M}$ be a κ-saturated model of T^*. Let $\mathcal{M}_1 := \mathcal{M}_2 := \mathcal{M}$ and let $\mathcal{I}$ be the back-and-forth system between $\mathcal{M}_1$ and $\mathcal{M}_2$ constructed in the previous section. Let $a = (a_1, \dots, a_n), b = (b_1, \dots, b_n) \in M^n$ be such that a and b satisfy the same special formulas. To establish the theorem it suffices to show that $\mathrm{tp}_{\mathcal{L}^*}(a) = \mathrm{tp}_{\mathcal{L}^*}(b)$. To prove the latter statement, it is enough to find $\iota \in I$ that maps a to b. By permuting the coordinates we can assume there is $r \in \{0, \dots, n\}$ such that $a_1, \dots, a_r$ are dcl_T-independent over $P_{\mathcal{M}}$ and $a_{r+1}, \dots, a_n \in \mathrm{dcl}_T(a_1 \dots a_r P_{\mathcal{M}})$.

Since a and b satisfy the same special formulas, the reader can easily verify that $b_1,\dots,b_r$ are dcl_T-independent over $P_{\mathcal{M}}$. Let $m\in\mathbb{N}$ and $g=(g_1,\dots,g_m)\in P_{\mathcal{M}}^m$ be such that $a_{r+1},\dots,a_n\in \mathrm{dcl}_T(a_1\dots a_r g)$. For $i=r+1,\dots,n$, let $f_i: M^{r+m}\to M$ be an $\mathcal{L}$-$\emptyset$-definable function such that $f_i(a_1,\dots,a_r,g)=a_i$. We will now find $h=(h_1,\dots,h_m)\in P_{\mathcal{M}}^m$ such that

(1) $\mathrm{tp}_{\mathcal{L}'}(h)=\mathrm{tp}_{\mathcal{L}'}(g)$,
(2) $f_i(b_1,\dots,b_r,h)=b_i$ for each $i=r+1,\dots,n$.

If we have such h, we can find an $\mathcal{L}$-isomorphism $\iota:\mathrm{dcl}(a_1\dots a_r g)\to \mathrm{dcl}(b_1\dots b_r h)$ such that $\iota(g)=h$ and $\iota(a_i)=b_i$ for each $i=1,\dots,r$. Since h satisfies (1) and each of the sets $\{a_1\dots a_r\}$ and $\{b_1\dots b_r\}$ is dcl_T-independent over $P_{\mathcal{M}}$, it is easy to check that $\iota\in\mathcal{I}$. Because h also satisfies (2), we get that $\iota(a_i)=b_i$ for $i=r+1,\dots,n$. Thus ι is the desired element of $\mathcal{I}$.

We now prove the existence of an $h\in P_{\mathcal{M}}^m$ satisfying (1) and (2). Observe that there is an $\mathcal{L}$-formula $\psi(x,y)$ such that an element $h\in M^m$ satisfies (2) if and only if $\mathcal{M}\models\psi(h,b)$. By saturation, in order to find h satisfying (1) and (2), it is enough to find for every $\mathcal{L}'$-formula $\varphi(x)$ with $\mathcal{M}\models\varphi_e(g)$ an $h\in P_{\mathcal{M}}^m$ such that $\mathcal{M}\models\varphi_e(h)\wedge\psi(h,b)$. So let $\varphi(x)$ be an $\mathcal{L}'$-formula with $\mathcal{M}\models\varphi_e(g)$. Consider the special formula $\chi(y)$ given by

$$\exists x\ Px\wedge\varphi_e(x)\wedge\psi(x,y).$$

Since $\mathcal{M}\models\chi(a)$ and a and b satisfy the same special formulas, we get that $\mathcal{M}\models\chi(b)$. Thus there exists $h\in P_{\mathcal{M}}^m$ such that $\mathcal{M}\models\varphi_e(h)\wedge\psi(h,b)$. □

2.3. **Types.** In order to show statements (2)-(4) of Theorem A we need better control over the $\mathcal{L}^*$-types in models of T^*. We establish the necessary results in this section. Throughout let $\mathcal{M}$ be a κ-saturated model of T^*. We first introduce the following notation: for $C\subseteq\mathcal{M}$ and $n\in\mathbb{N}$ we denote by $D_n(C)$ the set

$$\{z\in\mathcal{M}^n\ :\ z \text{ is } \mathrm{dcl}_T\text{-independent from } CP_{\mathcal{M}}\}.$$

Proposition 2.10. *Let $a\in P_{\mathcal{M}}^k$, $z\in D_l(\emptyset)$, $b\in P_{\mathcal{M}}^m$ and $y\in D_n(z)$. Then $\mathrm{tp}_{\mathcal{L}^*}(by|az)$ is implied by the conjunction of*

- $\mathrm{tp}_{\mathcal{L}}(by|az)$,
- *"$b\in P_{\mathcal{M}}^m$" and* $\mathrm{tp}_{\mathcal{L}'}(b|a)$,
- *"$y\in D_n(z)$".*

Proof. Set $\mathcal{M}_1:=\mathcal{M}_2:=\mathcal{M}$ and let $\mathcal{I}$ be the back-and-forth system between $\mathcal{M}_1$ and $\mathcal{M}_2$ constructed in the previous section. Let $b_1,b_2\in P_{\mathcal{M}}^m$ and $y_1,y_2\in D_n(z)$ be such that $\mathrm{tp}_{\mathcal{L}}(b_1y_1|az)=\mathrm{tp}_{\mathcal{L}}(b_2y_2|az)$ and $\mathrm{tp}_{\mathcal{L}'}(b_1|a)=\mathrm{tp}_{\mathcal{L}'}(b_2|a)$. In order to show that $\mathrm{tp}_{\mathcal{L}^*}(b_1y_1|az)=\mathrm{tp}_{\mathcal{L}^*}(b_2y_2|az)$, we only need to find $\iota\in\mathcal{I}$ such that $\iota(b_1y_1)=b_2y_2$ and the coordinates of a and z are in the domain of ι. It is immediate that the identity on $\mathrm{dcl}_T(az)$ is in $\mathcal{I}$. Since $\mathrm{tp}_{\mathcal{L}}(b_1y_1|az)=\mathrm{tp}_{\mathcal{L}}(b_2y_2|az)$, there is a partial $\mathcal{L}$-isomorphism from $\mathrm{dcl}_T(azb_1y_1)$ to $\mathrm{dcl}_T(azb_2y_2)$ mapping b_1y_1 to b_2y_2. Because $\mathrm{tp}_{\mathcal{L}'}(b_1|a)=\mathrm{tp}_{\mathcal{L}'}(b_2|a)$ and $y_1,y_2\in D_n(z)$, it is immediate that $\iota\in\mathcal{I}$. □

We immediately obtain the following three corollaries from Proposition 2.10.

Corollary 2.11. *Let $C\subseteq\mathcal{M}$ be finite and $y\in D_n(C)$. Then $\mathrm{tp}_{\mathcal{L}^*}(y|C)$ is implied by $\mathrm{tp}_{\mathcal{L}}(y|C)$ in conjunction with "$y\in D_n(C)$".*

Corollary 2.12. *Let $a \in P^k_{\mathcal{M}}$, $z \in D_l(\emptyset)$ and $b \in P^n_{\mathcal{M}}$. Then $\mathrm{tp}_{\mathcal{L}^*}(b|az)$ is implied by $\mathrm{tp}_{\mathcal{L}}(b|az)$, "$b \in P^n_{\mathcal{M}}$" and $\mathrm{tp}_{\mathcal{L}'}(b|a)$.*

Corollary 2.13. *Let $Z \subseteq P^n_{\mathcal{M}}$ be $\mathcal{L}^*$-definable. Then there is an $\mathcal{L}$-definable set $Y \subseteq \mathcal{M}^n$ and an $\mathcal{L}'$-formula $\varphi(x)$ such that*

$$Z = Y \cap \{a \in P^n_{\mathcal{M}} \ : \ \mathcal{M} \models \varphi_e(a)\}.$$

Combining Proposition 2.10 with a result of Boxall and Hieronymi [3], we are now able to deduce statement (2) of Theorem A.

Theorem 2.14. *The theory T is an open core of T^*.*

Proof. We will use [3, Corollary 3.1] to show that every $\mathcal{L}^*$-definable open set in $\mathcal{M}$ is also $\mathcal{L}$-definable. Let $X \subseteq \mathcal{M}^n$ be open and $\mathcal{L}^*$-definable over some finite parameter set C. We will now apply [3, Corollary 3.1], using $D_n(C)$ as $D_{S_1\dots S_n}$. Therefore it is left to check that conditions (1)-(3) of [3, Corollary 3.1] hold for $D_n(C)$. These three conditions are

(1) $D_n(C)$ is dense in $\mathcal{M}$,
(2) for every $y \in D_n(C)$ and every open set $U \subseteq \mathcal{M}^n$, if $\mathrm{tp}_{\mathcal{L}}(y|C)$ is realized in U, then $\mathrm{tp}_{\mathcal{L}}(y|C)$ is realized in $U \cap D_n(C)$,
(3) for every $y \in D_n(z)$, $\mathrm{tp}_{\mathcal{L}^*}(y|C)$ is implied by $\mathrm{tp}_{\mathcal{L}}(y|C)$ in conjunction with "$y \in D_n(C)$".

Condition (1) follows easily from saturation of $\mathcal{M}$ and [7, 2.1]. Using o-minimality of T, it is easy to deduce Condition (2) from Condition (1). Finally, Condition (3) holds by Corollary 2.11. ☐

2.4. **Completions of T_e.** Using results from the previous sections we will now give a characterizations of all complete $\mathcal{L}_e$-theories containing T_e.

Definition 2.15. Let $T_{e,\infty}$ be the $\mathcal{L}_e$-theory consisting of T_e and an axiom schema expressing the following statement:

(T6) E has infinitely many equivalence classes.

Similarly, for every $n \in \mathbb{N}_{>0}$ define $T_{e,n}$ to be the $\mathcal{L}_e$-theory consisting of T_e and a sentence stating that E has exactly n equivalence classes.

Theorem 2.16. *Let $p \in \mathbb{N}_{>0} \cup \{\infty\}$. The theory $T_{e,p}$ is complete.*

Proof. We first consider the case that $p = \infty$. Let $\mathcal{L}'$ be empty and T' be the (complete) $\mathcal{L}'$-theory of infinite sets. Let $\mathcal{L}^*$ and $\mathcal{T}^*$ be constructed as above. Since $\mathcal{L}' = \emptyset$, we have that $\mathcal{L}^* = \mathcal{L}_e$. Since T^* is complete, it is enough to show that every model of $T_{e,\infty}$ is also a model of T^*. Let $\mathcal{M} \models T_{e,\infty}$. Since $\mathcal{L}' = \emptyset$, we immediately get that $\mathcal{M}$ satisfies (T4). It is left to show that $\mathcal{M}$ satisfies (T5). Let $\varphi \in T'$. Since T' is the theory of infinite sets, there is $n \in \mathbb{N}$ such that φ is the following formula

$$\exists x_1 \dots \exists x_n \bigwedge_{1 \le i < j \le n} x_i \neq x_j.$$

It is easy to check that φ_e is the $\mathcal{L}_e$-formula

$$\exists x_1 \dots \exists x_n \bigwedge_{1 \le i < j \le n} Px_i \wedge \neg E x_i x_j.$$

Since $\mathcal{M}$ satisfies (T6), we get that $\mathcal{M} \models \varphi_e$. Thus $\mathcal{M}$ satisfies (T5).
The proof of the case $p \in \mathbb{N}_{>0}$ can be done similarly by replacing the $\mathcal{L}'$-theory of infinite sets by the $\mathcal{L}'$-theory of a set with exactly p elements. ☐

From Theorem 2.16 we can directly deduce the following characterization of completions of T_e.

Corollary 2.17. *Let $\tilde{T}$ be a complete $\mathcal{L}_e$-theory such that $T_e \subseteq \tilde{T}$. Then there is $p \in \mathbb{N}_{>0} \cup \{\infty\}$ such that $T_{e,p} \models \tilde{T}$.*

We obtain the following corollary as an immediate consequence of Theorem 2.14 and the proof of Theorem 2.16.

Corollary 2.18. *Let $p \in \mathbb{N}_{>0} \cup \{\infty\}$. The theory T is an open core of $T_{e,p}$.*

3. Preservation of NIP

Let T be a complete o-minimal extension of the theory of densely ordered groups with a distinguished positive element, and let $\mathcal{L}$ be its language. As before, let $\mathcal{L}'$ be a relational language disjoint from $\mathcal{L}$, and let T' be a complete $\mathcal{L}'$-theory. Furthermore, let T^* be the $\mathcal{L}^*$-theory constructed in the previous section. We will now show that T^* is NIP if T' is NIP. As we will see, this can be deduced rather directly from Corollaries 2.11 and 2.12 and the following result of Günaydın and Hieronymi [12].

Fact 3.1. [12, Proposition 2.4] *Let $\mathcal{L}_0$ be a first-order language and let $\mathcal{L}_1$ be a language containing $\mathcal{L}_0$ and a unary predicate symbol U not in $\mathcal{L}_0$. Let T_0 be a complete $\mathcal{L}_0$-theory and let T_1 be a complete $\mathcal{L}_1$-theory extending T_0. Let $\mathbb{M}$ be a monster model of T_1. Suppose that*

(i) dcl_{T_0} *is a pregeometry,*

(ii) *for every $\mathcal{L}_1$-formula $\varphi(x, y)$, indiscernible sequence $(g_i)_{i\in\omega}$ from $U^p_{\mathbb{M}}$ and $b \in \mathbb{M}^q$, the set $\{i \in \omega : \mathbb{M} \models \varphi(g_i, b)\}$ is either finite or co-finite (in ω),*

(iii) *for every formula $\varphi(x, y)$, indiscernible sequence $(a_i)_{i\in\omega}$ from $\mathbb{M}$ and $b \in \mathbb{M}^q$ with $a_i \notin \mathrm{dcl}_{\mathcal{L}_0}(U_{\mathbb{M}} b)$ for every $i \in \omega$, the set $\{i \in \omega : \mathbb{M} \models \varphi(a_i, b)\}$ is either finite or co-finite (in ω).*

Then T_1 is NIP.

Theorem 3.2. *If T' is NIP, so is T^*.*

Proof. We apply Fact 3.1 with $T_0 := T$ and $T_1 := T^*$. Since T is o-minimal, dcl_T is a pregeometry.
For (ii), let $\varphi(x, y)$ be an $\mathcal{L}^*$-formula, $(g_i)_{i\in\omega}$ an indiscernible sequence from $P^p_{\mathbb{M}}$ and $b \in \mathbb{M}^q$. Without loss of generality, we can assume that there are $b_1 \in P^{q_1}_{\mathbb{M}}$ and $b_2 \in \mathbb{M}^{q_2}$ such that b_2 is dcl_T-independent over $P_{\mathbb{M}}$ and $b = (b_1, b_2)$. By Corollary 2.12 there is an $\mathcal{L}$-formula $\psi(x, u, v)$ and an $\mathcal{L}'$-formula $\theta(x, u)$ such that for all $a \in P^p_{\mathbb{M}}$

$$(*) \qquad \mathbb{M} \models \varphi(a, b) \leftrightarrow \big(\psi(a, b_1, b_2) \wedge \theta_e(a, b_1)\big).$$

Since both T and T' are NIP, it follows immediately from $(*)$ that $\{i \in \omega : \mathbb{M} \models \varphi(g_i, b)\}$ is either finite or co-finite.
For (iii), let $\varphi(x, y)$ be an $\mathcal{L}^*$-formula, $(a_i)_{i\in\omega}$ an indiscernible sequence from $\mathbb{M}$ and $b \in \mathbb{M}^q$ with $a_i \notin \mathrm{dcl}_T(P_{\mathbb{M}} b)$ for every $i \in \omega$. By Corollary 2.11 there is an $\mathcal{L}$-formula $\psi(x, b)$ such that for all $a \in \mathbb{M} \setminus \mathrm{dcl}_T(P_{\mathbb{M}} b)$

$$\mathbb{M} \models \varphi(a, b) \leftrightarrow \psi(a, b).$$

Since T is NIP, $\{i \in \omega : \mathbb{M} \models \varphi(a_i, b)\}$ is either finite or co-finite (in ω). □

We can now give a proof of Theorem D that there is an NIP expansion of T without a distal expansion.

Proof of Theorem D. Fix a prime p. Let T' be ACF_p. Since T' is stable, T' is NIP. Suppose T^* has a distal expansion $\tilde{T}$. Then $\tilde{T}^{eq}$ is distal by [19, Remark after Definition 9.17]. However, by Proposition 2.4 every model of $\tilde{T}^{eq}$ defines a model of T'. By [4, Proposition 6.2] $\tilde{T}^{eq}$ cannot be distal. A contradiction. □

4. Preservation of Strong Dependence

In this section, we will show that T^* (as constructed in Section 2) is strongly dependent if T' is. We essentially follow the proof of Berenstein, Dolich and Onshuus [1, Theorem 2.11].

Let $\mathcal{L}_0$ be a first-order language containing $<$ and let $\mathcal{L}_1$ be a language containing $\mathcal{L}_0$ and a unary predicate symbol U not in $\mathcal{L}_0$. Let T_0 be a complete $\mathcal{L}_0$-theory extending the theory of linear ordered sets such that dcl_{T_0} is a pregeometry. Let T_1 be a complete $\mathcal{L}_1$-theory extending T_0, and let $\mathbb{M}$ be a monster model of T_1. If X, Y are subsets of $\mathbb{M}$, we say X is **U-independent** over Y if $X \setminus U_{\mathbb{M}}$ is dcl_{T_0}-independent over $U_{\mathbb{M}}Y$. If $Y = \emptyset$, we simply say that X is U-independent. We say an indiscernible sequence $(a_i)_{i\in I}$ of tuples of elements of $\mathbb{M}$ is **U-independent** if each a_i is U-independent.

Lemma 4.1. *Let κ be an infinite cardinal and let $(a_i)_{i\in I}$ be an indiscernible sequence of tuples of elements of $\mathbb{M}$ of length κ. Then there is an U-independent indiscernible sequence $J = (b_i)_{i\in I}$ of tuples of elements of $\mathbb{M}$ of length κ such that for every $j < \kappa$ there is an $\mathcal{L}_0$-$\emptyset$-definable function $f : \mathbb{M}^n \to \mathbb{M}$, and $j_1, \dots, j_n < \kappa$ such that for every $i \in I$*

$$a_{i,j} = f(b_{i,j_1}, \dots, b_{i,j_n}).$$

Proof. We inductively construct a sequence $(b_i)_{i\in I}$ from the sequence $(a_i)_{i\in I}$ by removing U-dependencies. Let $\alpha < \kappa$ be minimal such that there is $i \in I$ such that $\{a_{i,j} : j \le \alpha\}$ is not U-independent. By minimality of α there are $j_1 < \dots < j_m < \alpha$ and an $\mathcal{L}_0$-$\emptyset$-definable $f : M^{m+\ell+1} \to M$ such that

$$(*) \qquad \exists u_{i,0}, \dots, u_{i,\ell} \in U_{\mathbb{M}}\ f(a_{i,j_1}, \dots, a_{i,j_m}, u_{i,0}, \dots, u_{i,\ell}) = a_{i,\alpha}.$$

By indiscernibility of $(a_i)_{i\in I}$, $(*)$ holds for every $i \in I$. For each $i \in I$, define a set

$$S_i = \{(u_0, \dots, u_\ell) \in U^{l+1} \ :\ f(a_{i,j_1}, \dots, a_{i,j_m}, u_0, \dots, u_\ell) = a_{i,\alpha}\}.$$

By indiscernibility of $(a_i)_{i\in I}$, we have that S_i is finite for some i if and and only S_i is finite for every $i \in I$.

We first consider the case that S_i is finite for every $i \in I$. Then for each $i \in I$ we may choose $u_i = (u_{i,0}, \dots u_{i,\ell})$ to be the lexicographically least member of S_i. Let b_i be the tuple where $a_{i,\alpha}$ is replaced by u_i. As $a_{i,\alpha}$ and u_i are interdefinable over $\{a_{i,j_1}, \dots, a_{i,j_m}\}$, $(b_i)_{i\in I}$ is indiscernible. Furthermore, the set $\{b_{i,1}, \dots, b_{i,\alpha+\ell}\}$ is U-independent for each $i \in I$.

Now suppose that S_i is infinite. Consider the collection of formulas in variables $(x_{i,j})_{i\in I}$ for $j < \kappa$ stating:

1. $x_{i,j} = a_{i,j}$ for $j < \alpha$
2. $f(a_{i,j_1}, \dots a_{i,j_m}, x_{i,\alpha}, \dots, x_{i,\alpha+\ell}) = a_{i,\alpha}$
3. $x_{i,\alpha}, \dots, x_{i,\alpha+\ell} \in U$

(4) $x_{i,\alpha+\ell+j} = a_{i,\alpha+j}$ for $j \geq 1$
(5) The sequence $(x_i)_{i\in I}$ is indiscernible.

As S_i is infinite, it can be shown by a standard argument using Ramsey's theorem that this collection is finitely satisfiable. Therefore, by saturation there is a realization $(b_i)_{i\in I}$ of this collection. By construction, we have for every $i \in I$ that $\{b_{i,1}, \dots, b_{i,\alpha+\ell}\}$ is U-independent and $a_{i,\alpha} = f(b_{i,j_1}, \dots b_{i,j_m}, b_{i,\alpha}, \dots, b_{i,\alpha+\ell})$. Inductively continuing, we arrange the sequence $(b_i)_{i\in I}$ as desired. $\square$

We will use Lemma 4.1 to show a criterion for strong dependence for T_1. Before we do so, we recall the definition of strong dependence. If I is a linear order, we denote its completion by $\operatorname{compl}(I)$. If I is a linear order, $c = (c_1, \dots, c_n) \in \operatorname{compl}(I)^n$ and $i, i' \in I$, we write $i \sim_c i'$ if

$$\bigwedge_{j=1}^{n} \big((i < c_j \leftrightarrow i' < c_j) \wedge (i = c_j \leftrightarrow i' = c_j)\big)$$

Note that $\sim_c$ defines an equivalence relation $\sim_c$ on I.

Definition 4.2. A theory $\tilde{T}$ in a language $\tilde{\mathcal{L}}$ is **strongly dependent** if for every $\mathcal{M} \models \tilde{T}$, every $b \in \mathcal{M}^m$ and every indiscernible sequence $(a_i)_{i\in I}$, there is $n \in \mathbb{N}$ and $c \in \operatorname{compl}(I)^n$ such that $i \sim_c j \Rightarrow \operatorname{tp}_{\tilde{\mathcal{L}}}(a_i|b) = \operatorname{tp}_{\tilde{\mathcal{L}}}(a_j|b)$.

For more details and other equivalent definitions of strong dependence, we refer the reader to [19, Chapter 4].

Lemma 4.3. *The following are equivalent:*

(i) *For every $b \in \mathbb{M}$, and every U-independent indiscernible sequence $(a_i)_{i\in I}$, if b is U-independent over $\{a_i : i \in I\}$, then there is $n \in \mathbb{N}$ and $c \in \operatorname{compl}(I)^n$ such that $i \sim_c j \Rightarrow \operatorname{tp}_{\mathcal{L}_1}(a_i|b) = \operatorname{tp}_{\mathcal{L}_1}(a_j|b)$.*
(ii) *For every $b \in \mathbb{M}$, and indiscernible sequence $(a_i)_{i\in I}$, if b is U-independent over $\{a_i : i \in I\}$), then there is $n \in \mathbb{N}$ and $c \in \operatorname{compl}(I)^n$ such that $i \sim_c j \Rightarrow \operatorname{tp}_{\mathcal{L}_1}(a_i|b) = \operatorname{tp}_{\mathcal{L}_1}(a_j|b)$.*
(iii) *T_1 is strongly dependent.*

Proof. It is clear that (iii)⇒(ii)⇒(i). Observe that (i)⇒(ii) follows easily from Lemma 4.1. So we only need to show that (ii) implies (iii). Let $b \in \mathbb{M}^m$ and $(a_i)_{i\in I}$ be an indiscernible sequence of possibly infinite tuples from $\mathbb{M}$. It is enough to consider the case $m = 1$ (see for example [19, Proposition 4.26]). Suppose $b \in \operatorname{dcl}_{T_0}(U_{\mathbb{M}}\{a_i : i \in I\})$. Then there are $g \in U^l_{\mathbb{M}}$, $e = (e_1, \dots, e_k) \in I^k$ and an $\mathcal{L}_0$-$\emptyset$-definable function f such that

$$b = f(g, a_{e_1}, \dots, a_{e_k}). \tag{†}$$

Without loss of generality assume that $e_1 < \dots < e_k$. Set $a_e = (a_{e_1}, \dots, a_{e_k})$, $e_0 = -\infty$ and $e_{k+1} = +\infty$. Let $t \in \{0, \dots, k\}$. Now observe that $(a_e a_j)_{j\in(e_t,e_{t+1})\cap I}$ is an indiscernible sequence. By (ii) there is $d_t \in (\operatorname{compl}(I) \cap (e_t, e_{t+1}))^{n_t}$ such that for all $i, j \in (e_t, e_{t+1})$ we have $i \sim_{d_t} j \Rightarrow \operatorname{tp}_{\mathcal{L}_1}(a_e a_i|g) = \operatorname{tp}_{\mathcal{L}_1}(a_e a_j|g)$. By (†) we get that for all such i, j

$$i \sim_{d_t} j \Rightarrow \operatorname{tp}_{\mathcal{L}_1}(a_i|b) = \operatorname{tp}_{\mathcal{L}_1}(a_j|b).$$

Set $c := (d_0 e_1 d_1 \dots e_k d_{k+1})$. It can be checked easily that this is the desired $c \in \operatorname{compl}(I)^n$. $\square$

Let us now recall the setting of Section 2. Let T be a complete o-minimal extension of the theory of densely ordered groups with a distinguished positive element, and let $\mathcal{L}$ be its language. As before, let $\mathcal{L}'$ be a language disjoint from $\mathcal{L}$, and let T' be a complete $\mathcal{L}'$-theory. Furthermore, let T^* be the $\mathcal{L}^*$-theory constructed in Section 2.

Theorem 4.4. *If T' is strongly dependent, so is T^*.*

Proof. We now apply Lemma 4.3 with $T_0 := T$, $T_1 := T^*$ and $U := P$. As before, note that dcl_T is a pregeometry, since T is o-minimal.
Let $b \in \mathbb{M}$ and $(a_i)_{i\in I}$ be an P-independent sequence such that b is P-independent over $\{a_i : i \in I\}$. Since each a_i is P-independent, we have (after possibly changing the order of entries of the a_i's) that for each $i \in I$ there are tuples u_i, v_i of elements of $\mathbb{M}$ such that for each $i \in i$

- $a_i = u_i v_i$,
- u_i is a tuple of elements in $P_{\mathbb{M}}$,
- v_i is dcl_T-independent over $P_{\mathbb{M}}$.

Since b is P-independent over $\{a_i : i \in I\}$, we get that either $b \in P_{\mathbb{M}}$ or $b \notin \mathrm{dcl}_T(\{a_i : i \in I\}P_{\mathbb{M}})$. We consider the two different cases.

Let $b \in P_{\mathbb{M}}$. By Proposition 2.10 the type $\mathrm{tp}_{\mathcal{L}^*}(u_i v_i|b)$ is determined by

- $\mathrm{tp}_{\mathcal{L}}(u_i v_i|b)$,
- the statement "u_i is a tuple of elements of $P_{\mathbb{M}}$" and $\mathrm{tp}_{\mathcal{L}'}(u_i|b)$,
- the statement "v_i is dcl_T-independent over $P_{\mathbb{M}}$".

Since both T and T' are strongly dependent, we can find $c \in \mathrm{compl}(I)^n$ such that for every $i, j \in I$

$$i \sim_c j \Rightarrow \Big(\mathrm{tp}_{\mathcal{L}}(u_i v_i|b) = \mathrm{tp}_{\mathcal{L}}(u_j v_j|b) \text{ and } \mathrm{tp}_{\mathcal{L}'}(u_i|b) = \mathrm{tp}_{\mathcal{L}'}(u_j|b)\Big).$$

Thus for every $i, j \in I$ with $i \sim_c j$ we get $\mathrm{tp}_{\mathcal{L}^*}(a_i|b) = \mathrm{tp}_{\mathcal{L}^*}(a_j|b)$.

Now suppose that $b \notin \mathrm{dcl}_T(\{a_i : i \in I\}P_{\mathbb{M}})$. In particular, $b \notin \mathrm{dcl}_T(P_{\mathbb{M}})$. Since dcl_T is a pregeometry, v_i is dcl_T-independent over $P_{\mathbb{M}}b$ for each $i \in I$. By Proposition 2.10, for each $i \in I$ the type $\mathrm{tp}_{\mathcal{L}^*}(u_i v_i|b)$ is determined by

- $\mathrm{tp}_{\mathcal{L}}(u_i v_i|b)$,
- the statement "u_i is a tuple of elements of $P_{\mathbb{M}}$" and $\mathrm{tp}_{\mathcal{L}'}(u_i)$,
- the statement "v_i is dcl_T-independent over $P_{\mathbb{M}}b$".

As before using strong dependence of T and T', we can find $c \in \mathrm{compl}(I)^n$ such that for every $i, j \in I$

$$i \sim_c j \Rightarrow \Big(\mathrm{tp}_{\mathcal{L}}(u_i v_i|b) = \mathrm{tp}_{\mathcal{L}}(u_j v_j|b) \text{ and } \mathrm{tp}_{\mathcal{L}'}(u_i) = \mathrm{tp}_{\mathcal{L}'}(u_j)\Big).$$

Thus for every $i, j \in I$ with $i \sim_c j$ we get $\mathrm{tp}_{\mathcal{L}^*}(a_i|b) = \mathrm{tp}_{\mathcal{L}^*}(a_j|b)$. □

This completes the proof of Theorem A. In the next section we will deduce Theorem B from Theorem A.

It is worth pointing out in this section on strong dependence that by Dolich and Goodrick [5, Corollary 2.4] every strongly dependent expansion of the real field has o-minimal open core. In contrast to this restriction, our Theorem B(4) shows that there is a large variety of such expansions of the real field.

5. Proof of Theorem B

The purpose of this section is twofold. We first deduce Theorem B from our proof of Theorem A. Then we show that in Theorem B the statement "$\mathcal{S}$ interprets a model of T'" cannot be replaced by the statement "$\mathcal{S}$ defines a model of T'".

Proof of Theorem B. Let $\mathcal{R} = (\mathbb{R}, <, +, \dots)$ be an o-minimal expansion of the real ordered additive group in a language $\mathcal{L}$ and let T' be a theory such that $|\mathcal{L}| < |\mathbb{R}|$ and $|T'| \leq |\mathbb{R}|$. Let T^* be the theory as constructed in Section 2. Since T^* satisfies the statements (1)-(4) of Theorem A, it is only left to show that $\mathcal{R}$ can be expanded to a model of T^*. Since $|\mathcal{L}| < |\mathbb{R}|$, we can find a dcl_T-basis of cardinality at least $|T'|$. Since $\mathrm{dcl}_T(\emptyset)$ is dense in $\mathbb{R}$, we are able to choose this basis such that it is dense in $\mathbb{R}$. Now apply Lemma 2.2. $\square$

Proposition 5.1. *Let $\mathcal{S}$ be an expansion of $(\mathbb{R}, <, +)$. The following are equivalent*

(1) *$\mathcal{S}$ defines an infinite discrete linear order.*
(2) *$\mathcal{S}$ defines an order with order type ω.*
(3) *The open core of $\mathcal{S}$ is not o-minimal.*

Proof. We show that (1) implies (2). Suppose $\mathcal{S}$ defines an infinite discrete linear order $(D, \prec)$. Fix $d \in D$. Either $D_{\prec d}$ or $D_{\succ d}$ is infinite. After replacing $\prec$ with the reverse order if necessary, we may suppose that $D_{\succ d}$ is infinite. After replacing $(D, \prec)$ with $(D_{\succeq d}, \prec)$ if necessary we suppose that $(D, \prec)$ has a minimal element. Let $E \subseteq D$ be the set of e such that $D_{\prec e}$ is finite. Recall that a subset of $\mathbb{R}^n$ is finite if and only if it is closed, bounded and discrete. It follows that E is definable. Note that $E_{\prec e}$ is finite for all $e \in D$. Then $(E, \prec)$ is a discrete linear order with minimal element and finite initial segments. Thus it has order type ω.
We now show that (2) implies (3). Suppose that $(D, \prec)$ is a definable order with order type ω and $D \subseteq \mathbb{R}^n$. First suppose that there is no coordinate projection $\pi : \mathbb{R}^n \to \mathbb{R}$ such that $\pi(D)$ is somewhere dense. Since D is infinite, there is a coordinate projection $\rho : \mathbb{R}^n \to \mathbb{R}$ such that $\rho(D)$ is infinite. Then $\rho(D)$ is an infinite, nowhere dense, subset of $\mathbb{R}$. Thus the open core of $\mathcal{S}$ is not o-minimal.
Now let $\pi : \mathbb{R}^n \to \mathbb{R}$ be a coordinate projection such that $\pi(D)$ is somewhere dense. Let $a, b \in \mathbb{R}$ such that (a, b) is an interval in in which $\pi(D)$ is dense. We now reduce to the case when D is a dense subset of an open interval. Note that $D' = \{e \in D : a < \pi(e) < b\}$ is an infinite, and hence $\prec$-cofinal, subset of D. It follows that $(D', \prec)$ has order type ω. After replacing D with D' if necessary we suppose that $\pi(D)$ is a subset of (a, b). We put an order $\prec_\pi$ on $\pi(D)$ by declaring $x \prec_\pi y$ if there is a $e \in D$ such that $\pi(e) = x$ and $\pi(e') \neq y$ for all $e' \prec e$. It is easy to see that $(\pi(D), \prec_\pi)$ has order type ω. After replacing $(D, \prec)$ with $(\pi(D), \prec_\pi)$ we suppose that D is a dense subset of (a, b). We declare

$$Y := \{x \in D \ : \ \forall z \in D(z \prec x) \to (z < x)\}.$$

That is, Y is the set of $e \in D$ such that e the $<$-maximal element of $D_{\preceq e}$. By density of D in (a, b), it is easy to see that Y is infinite and that $(Y, <)$ is order-isomorphic to $(\mathbb{N}, <)$. Thus Y is an infinite discrete definable subset of $\mathbb{R}$. Hence the closure of Y does not have interior, but infinitely many connected components. Therefore $\mathcal{S}$ does not have o-minimal open core.

Since (2) trivially implies (1), it is enough to show that (3) implies (2). Suppose that the open core of $\mathcal{S}$ is not o-minimal. By [6, 2.14 (2)] there is an infinite discrete subset $D \subseteq \mathbb{R}$ definable in $\mathcal{S}$. First consider the case that $D \cap [-a, a]$ is a finite set for every $a \in \mathbb{R}_{>0}$. Then either $((-D) \cap [0, \infty), <)$ or $(D \cap [0, \infty), <)$ has order type ω. From now on we can assume that there is $a \in \mathbb{R}_{>0}$ such that the cardinality of $D \cap [-a, a]$ is infinite. Thus without loss of generality we can assume that D is bounded. For $\varepsilon \in \mathbb{R}_{>0}$ set

$$D_\varepsilon := \{d \in D \ : \ (d - \varepsilon, d + \varepsilon) \cap D = \{d\}\}.$$

Since D is bounded, each D_ε is finite. Moreover, since D is discrete and infinite, there is a function $f : D \to \mathbb{R}_{>0}$ definable in $\mathcal{S}$ mapping $d \in D$ to the supremum of all $\varepsilon \in \mathbb{R}_{>0}$ with $d \in D_\varepsilon$. We now define the following order on D: let $d_1, d_2 \in D$, we set $d_1 \prec d_2$ whenever one of the following conditions holds:

- $f(d_1) > f(d_2)$,
- $f(d_1) = f(d_2)$ and $d_1 < d_2$.

It can be checked easily that $(D, \prec)$ has order type ω. □

Let T' be the theory of an infinite discrete order. By Theorem B there exists an expansion of $(\mathbb{R}, <, +)$ that has o-minimal open core and interprets a model of T'. However, by Proposition 5.1 there is no expansion of $(\mathbb{R}, <, +)$ that has o-minimal open core and defines a model of T'. Therefore in Theorem B the statement "$\mathcal{S}$ interprets a model of T'" cannot be replace by the statement "$\mathcal{S}$ defines a model of T'".

6. Noiseless NIP expansions of $(\mathbb{R}, <, +)$

Recall that an expansion of $(\mathbb{R}, <)$ is noiseless if it does not define a somewhere dense and co-dense subset of $\mathbb{R}$. In this section we show that every noiseless NIP expansion of $(\mathbb{R}, <, +, 1)$ has definable choice and hence eliminates imaginaries. This statement will be established for the slightly larger class of noiseless expansions of $(\mathbb{R}, <, +, 1)$ that do not define a Cantor set. A **Cantor set** is a non-empty compact subset of $\mathbb{R}$ that neither has interior nor isolated points. By [14, Theorem B] every NTP_2 (and hence every NIP) expansion of $(\mathbb{R}, <, +)$ does not define a Cantor set.

Fix a noiseless expansion $\mathcal{R}$ of $(\mathbb{R}, <, +, 1)$ that does not define a Cantor set. Throughout this section, *definable* will mean *definable in* $\mathcal{R}$. For a subset $X \subseteq \mathbb{R}^n$, we denote the (topological) closure of X by $\mathrm{Cl}(X)$ and the interior of X by $\mathrm{Int}(X)$.

Lemma 6.1. *Let $X \subseteq \mathbb{R}$ be a non-empty definable set with empty interior. Then X contains an isolated point.*

Proof. Since $\mathcal{R}$ is noiseless, the closure $\mathrm{Cl}(X)$ of X has empty interior. Because $\mathcal{R}$ does not define a Cantor set, $\mathrm{Cl}(X)$ has an isolated point. It follows directly that X has an isolated point. □

Therefore in an expansion of $(\mathbb{R}, <, +)$ that does not define a Cantor set, every definable subset of $\mathbb{R}$ contains a locally closed point. For expansions of the real field, the existence of definable Skolem functions in expansions satisfying the latter condition was shown in [9, Lemma 9.1].

Lemma 6.2. *Let $C \subseteq \mathbb{R}^{n+1}$ be $\emptyset$-definable such that C_x has empty interior for every $x \in \mathbb{R}^n$. Then there is an $\emptyset$-definable function $f : \pi(C) \to \mathbb{R}$ such that $\mathrm{gr}(f) \subseteq C$, where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection onto the first n coordinates.*

Proof. By Lemma 6.1 we have that for all $x \in \pi(C)$ the set C_x has an isolated point whenever C_x is non-empty. Let $g : \pi(C) \to \mathbb{R}$ map $x \in \pi(C)$ to

$$\sup\{r \in \mathbb{R}_{>0} \ : \ \exists y \in C_x \ (y - r, y + r) \cap C_x = \{y\}\}$$

if such supremum exists, and to 1 otherwise. Define

$$D := \{(x, y) \in C \ : \ (y - \tfrac{g(x)}{2}, y + \tfrac{g(x)}{2}) \cap C_x = \{y\}\}.$$

It is easy to check that D_x is non-empty if and if C_x is non-empty. For each $x \in \pi(D)$ and $y_1, y_2 \in D_x$, we have $|y_1 - y_2| \geq \frac{g(x)}{2}$. Therefore the set D_x is closed and discrete for each $x \in \pi(D)$. Let $f : \pi(C) \to \mathbb{R}$ be the function defined by

$$x \mapsto \begin{cases} \min D_x \cap [0, \infty), & \text{if } D_x \cap [0, \infty) \text{ is non-empty} \\ \max D_x \cap (-\infty, 0), & \text{otherwise.} \end{cases}$$

Observe that f is well-defined, because D_x is closed and discrete. From the definition of f we obtain directly that $\mathrm{gr}(f) \subseteq C$. □

Proposition 6.3. *Let $A \subseteq \mathbb{R}^m \times \mathbb{R}^n$ be $\emptyset$-definable. Then there is an $\emptyset$-definable function $f : \pi(A) \to \mathbb{R}^n$ such that*

(1) $\mathrm{gr}(f) \subseteq A$,
(2) $f(a) = f(b)$ *whenever* $a, b \in \pi(A)$ *and* $A_a = A_b$,

where $\pi : \mathbb{R}^{m+n} \to \mathbb{R}^m$ is the projection onto the first m coordinates.

Proof. Using induction it is easy to reduce to the case that $n = 1$. We can split A into $B, C \subseteq \mathbb{R}^{m+n}$ such that $A = B \cup C$ and

$$B := \{(x, y) \in A \ : \ y \in \mathrm{Int}(A_x)\}, \quad C := \{(x, y) \in A \ : \ y \in A_x \setminus \mathrm{Int}(A_x)\}.$$

Observe that C_x has empty interior for each $x \in \pi(C)$. Thus by Lemma 6.2 there is a definable function $f_1 : \pi(C) \to \mathbb{R}$ such that $\mathrm{gr}(f_1) \subseteq C$. Now define a subset $D \subseteq \mathbb{R}^{m+1}$ such that $(x, y) \in D$ whenever one the following conditions holds:

- y is a midpoint of a connected component of B_x,
- $y = 1 + \sup(\mathbb{R} \setminus B_x)$ and $\mathbb{R} \setminus B_x$ is bounded from above,
- $y = -1 + \inf(\mathbb{R} \setminus B_x)$ and $\mathbb{R} \setminus B_x$ is bounded from below,
- $y = 0$ and $B_x = \mathbb{R}$.

It is easy to see that D is definable, $D \subseteq B$ and $\pi(B) = \pi(D)$. Moreover, D_x has empty interior for each $x \in \pi(D)$. By Lemma 6.2 there is a definable function $f_2 : \pi(D) \to \mathbb{R}$ such that $\mathrm{gr}(f_2) \subseteq D$. We now define $f : \pi(A) \to \mathbb{R}$ by

$$x \mapsto \begin{cases} f_1(x), & \text{if } B_x = \emptyset; \\ f_2(x), & \text{otherwise.} \end{cases}$$

It follows directly that $\mathrm{gr}(f) \subseteq A$. Furthermore, the reader can easily check that for $a \in \pi(A)$ the value of $f(a)$ only depends on A_a and not on a. Therefore condition (2) holds for f as well. □

Theorem E follows immediately from Proposition 6.3. Note that Theorem E fails for NIP expansions of $(\mathbb{R}, <, +, 1)$ in general. For example, the structure $(\mathbb{R}, <, +, 1, \mathbb{Q})$ is NIP (see for example [12, Corollary 3.2]), does not have definable Skolem functions ([6, 5.4]) and does not eliminate imaginaries ([6, 5.5]).

References

[1] A. Berenstein, A. Dolich, and A. Onshuus. The independence property in generalized dense pairs of structures. *J. Symbolic Logic*, 76(2):391–404, 2011.

[2] A. Berenstein and E. Vassiliev. Geometric structures with a dense independent subset. *Selecta Math. (N.S.)*, 22(1):191–225, 2016.

[3] G. Boxall and P. Hieronymi. Expansions which introduce no new open sets. *J. Symbolic Logic*, 77(1):111–121, 2012.

[4] A. Chernikov and S. Starchenko. Regularity lemma for distal structures. *J. European Math. Soc. to appear*, 2016.

[5] A. Dolich and J. Goodrick. Strong theories of ordered abelian groups. *arXiv:1511.08274*, 2015.

[6] A. Dolich, C. Miller, and C. Steinhorn. Structures having o-minimal open core. *Trans. Amer. Math. Soc.*, 362(3):1371–1411, 2010.

[7] A. Dolich, C. Miller, and C. Steinhorn. Expansions of o-minimal structures by dense independent sets. *Ann. Pure Appl. Logic*, 167(8):684–706, 2016.

[8] L. v. d. Dries. Dense pairs of o-minimal structures. *Fund. Math.*, 157(1):61–78, 1998.

[9] A. Fornasiero. Tame structures and open cores. *arXiv:1003.3557, unpublished note*, 2010.

[10] A. Fornasiero. Groups and rings definable in d-minimal structure. *arXiv:1205.4177, unpublished note*, 2012.

[11] H. Friedman and C. Miller. Expansions of o-minimal structures by sparse sets. *Fund. Math.*, 167(1):55–64, 2001.

[12] A. Günaydin and P. Hieronymi. Dependent pairs. *J. Symbolic Logic*, 76(2):377–390, 2011.

[13] P. Hieronymi and T. Nell. Distal and non-distal pairs. *J. Symb. Log.*, 82(1):375–383, 2017.

[14] P. Hieronymi and E. Walsberg. Interpreting the monadic second order theory of one successor in expansions of the real line. *Israel J. Math.*, to appear, arXiv:1601.04555, 2016.

[15] C. Miller. Tameness in expansions of the real field. In *Logic Colloquium '01*, volume 20 of *Lect. Notes Log.*, pages 281–316. Assoc. Symbol. Logic, Urbana, IL, 2005.

[16] C. Miller and P. Speissegger. Expansions of the real line by open sets: o-minimality and open cores. *Fund. Math.*, 162(3):193–208, 1999.

[17] S. Shelah and P. Simon. Adding linear orders. *J. Symbolic Logic*, 77(2):717–725, 2012.

[18] P. Simon. Distal and non-distal NIP theories. *Ann. Pure Appl. Logic*, 164(3):294–318, 2013.

[19] P. Simon. *A guide to NIP theories*, volume 44 of *Lecture Notes in Logic*. Cambridge University Press, 2015.

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