Gevrey regularity of the Navier-Stokes equations in a half-space

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ABSTRACT. We consider the Navier-Stokes equations posed on the half space, with Dirichlet boundary conditions. We give a direct energy-based proof for the instantaneous space-time analyticity and Gevrey-class regularity of the solution, uniformly up to the boundary of the half space.

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1. Introduction

In this paper, we consider the Navier-Stokes system

$$\begin{aligned} &\partial_t u - \Delta u + u\cdot\nabla u + \nabla p = f \\ &\nabla\cdot u = 0, \end{aligned} \tag{1.1}$$

in the half-space

$$\Omega = \left\{ x = (x_1,\ldots,x_d) \in \mathbb{R}^d : x_d > 0 \right\} \tag{1.2}$$

with the no-slip boundary condition

$$u\big|_{\partial\Omega} = 0 \tag{1.3}$$

and the initial condition

$$u(x,0) = u_0(x), \qquad x\in\Omega. \tag{1.4}$$

For simplicity, we let $d\in\{2,3\}$, but note that higher dimensions may be treated similarly. See e.g. [**7**,**30**,**34**] for the well-posedness and further properties of the solutions to (1.1)–(1.4).

In Theorem 2.3 below we prove that the solution to (1.1)–(1.4) immediately becomes space-time real analytic, with analyticity radius which is uniform up to the boundary $\partial\Omega$, under the hypothesis that the force is real analytic in space-time. The result only requires finite Sobolev regularity on the initial datum u_0.

Assuming that f is space-time analytic in $\Omega\times I$, where $\Omega\subseteq\mathbb{R}^3$ and I is a complex neighborhood of $(0,T)$, Masuda [**26**] proved that the interior analyticity of a solution u to the Navier-Stokes system follows from that of the external force f (see also [**15**]), answering a question posed by Serrin [**32**]. Furthermore, in the case that Ω is a bounded domain with analytic boundary $\partial\Omega$, assuming that f is analytic uniformly up to the boundary and that the solution (u,p) is C^∞, Komatsu [**17**,**18**] showed that $(u,\nabla p)$ is globally analytic in x up to the boundary $\partial\Omega$ and locally analytic in t. His technique is inspired by the previous work by Kinderlehrer and Nirenberg [**16**] for second order parabolic equations, and is based on an induction scheme on the number of derivatives (see also [**23**]). A semigroup approach for analyticity up to the boundary in (1.1)–(1.4) was later given by Giga [**11**] (see also [**29**]), and a complex variables-based proof was given by the second author and Grujić [**13**,**14**] (see also [**5**,**6**]). Establishing the analyticity of solutions to (1.1)–(1.4) on domains with boundaries is particularly important in the context of the vanishing viscosity limit [**31**], or equivalently, the infinite Grashof number limit in our context.

The proof of the instantaneous space-time analyticity uniformly up to the boundary of the half-space given in this paper is based solely on $L^2_{x,t}$ energy estimates of the solution and its derivatives (see also [**21**] for the non-homogeneous Stokes system). The main obstacle to energy-based proofs on domains with boundaries is that the normal derivatives of the solution do not obey good boundary conditions. We believe that our approach will be useful in establishing real-analytic and Gevrey-class regularization results for semilinear parabolic PDEs with different types of boundary conditions, by only appealing to energy estimates.

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We recall that in the case of no boundaries, Foias and Temam have developed in [10] a very efficient method to prove analyticity, or more generally Gevrey-type regularity, which has in turn inspired many works (cf. [1–4, 8, 9, 12, 19, 22, 25, 27, 28] and references therein). The technique in [10] is based on Fourier analysis, which is unavailable in the case of domains with boundaries. One of the main aims of this paper is to find a similarly direct approach for establishing analyticity, which is based on the summing Taylor coefficients, rather than on the Fourier techniques. Such methods were introduced in [20] for the propagation of analyticity in the Euler equation, and this in turn led to be efficient in estimating the size of the uniform radius in terms of the size of initial data. However, finding an analog in the case of the Navier-Stokes equations proved to be more difficult due to the Laplacian term. In [21], the last two authors of the present paper, inspired by Komatsu's work [18], have used classical energy inequalities for the heat and Laplace equations, to achieve normal, tangential and time derivative reductions on terms of the form $t^{i+j+k-3}\partial_t^i\partial_d^j\bar{\partial}^k u$. Here $\bar{\partial}$ and ∂_d denote the tangential derivative component and the normal derivative component, respectively. This derivative reduction method works for the heat equation and extends naturally to the inhomogeneous Stokes system, yielding the desired regularization result in [21].

In order to address the Navier-Stokes system, we use a Gevrey type norm

$$\phi_T(u) = \sum_{i+j+k\geq 3} \epsilon^i \tilde{\epsilon}^j \bar{\epsilon}^k N_{i,j,k} \|t^{i+j+k-3}\partial_t^i \partial_d^j \bar{\partial}^k u\|_{L^2([0,T]\times\Omega)} + \|u\|_{H^2([0,T]\times\Omega)}$$

where $N_{i,j,k}$ represent certain binomial expressions which account for the possible growth of the Taylor coefficients. Note that the finiteness of the norm $\phi_T(u)$ for some $T > 0$ implies that the function u is real-analytic in space-time on $(0,T)\times\Omega$ (see e.g. [28] and references therein). The main goal is to establish an inequality of the type

$$\phi_T(u) \lesssim K_{u_0} + \|f\| + K_{u_0} \sum_{j=1}^{N} T^{\alpha_j} (\phi_T(u))^{\beta_j}, \tag{1.5}$$

where $\alpha_j > 0$, $N \in \mathbb{N}$ is fixed, and $0 < \beta_j \leq 2$. Here $\|f\|$ represents a suitable analytic norm of f, and K_{u_0} is a constant that depends on the Sobolev norm of the initial datum. From (1.5) and a standard Grönwall-type barrier argument, we deduce that for short enough time $\phi_T(u)$ stays bounded from above by a constant which depends on a Sobolev norm of u_0 and a space-time analytic norm of f, establishing the desired joint space-time analytic regularization. Although establishing (1.5) comes with some computational difficulty due to the nonlinear term $\phi_T(u\cdot\nabla u)$, the logic behind the analyticity estimate remains as direct as observed in the case of the nonhomogeneous heat equation. We believe that this method directly generalizes to nonlinear Stokes systems with nonlinearity given by $N(x,t,u,\nabla u)$, a space-time analytic function in each of its variables.

The paper is organized as follows. In Section 2, we introduce some notation, define the Gevrey-class norm ϕ_T, and state the main result (cf. Theorem 2.3). Also in Section 2 we recall the derivative reduction estimates from [21]. In Section 3, we give the proof Theorem 2.3, assuming a suitable bound for the nonlinear term, given by terms on the right side of (1.5) (cf. Lemma 3.1). The proof of this nonlinear estimate is finally given in Section 4, and is split into three separate lemmas which each deal with one case of the derivative reduction estimates (cf. Lemmas 4.4, 4.5, and 4.6).

2. Main result

Before stating the main result of this paper, Theorem 2.3 below, we first introduce some notation. For $r \geq 1$ we define the index sets

$$B = \big\{(i,j,k) : i,j,k \in \mathbb{N}_0, i+j+k \geq r\big\} \quad \text{and} \quad B^c = \mathbb{N}_0^3 \backslash B.$$

For $m \in \mathbb{N}_0$ we define the real-analytic binomial coefficient

$$N_m = \frac{m^r}{m!}.$$

All the proofs and statements in this paper carry through for the Gevrey-class s binomial coefficients $N_m = m^r/(m!)^s$, for $s > 1$. For simplicity of notation we only discuss the stronger (analytic) case $s = 1$.

For a fixed time horizon $T \in (0, 1]$ and given small parameters $\tilde{\epsilon} < \bar{\epsilon} < \epsilon \in (0, 1]$, we consider the sum

$$\begin{aligned} \phi_T(u) &= \sum_B N_{i+j+k}\epsilon^i\tilde{\epsilon}^j\bar{\epsilon}^k\|t^{i+j+k-r}\partial_t^i\partial_d^j\bar{\partial}^k u\|_{L^2_{x,t}([0,T]\times\Omega)} + \sum_{B^c}\|\partial_t^i\partial_d^j\bar{\partial}^k u\|_{L^2_{x,t}([0,T]\times\Omega)} \\ &= \bar{\phi}_T(u) + \phi_{0,T}(u), \end{aligned} \tag{2.1}$$

where $\bar{\partial}$ denotes the vector of tangential derivatives $\bar{\partial} = (\partial_1, \ldots, \partial_{d-1})$. Above and in the sequel we use the notational agreement that for $k \in \mathbb{N}_0$ we use $\bar{\partial}^k$ to denote:

$$\|\partial_t^i\partial_d^j\bar{\partial}^k u\|_{L^2_{x,t}} = \sum_{\alpha\in\mathbb{N}_0^2, |\alpha|=k}\|\partial_t^i\partial_d^j\partial^\alpha u\|_{L^2_{x,t}}.$$

Moreover, if the domain in the Lebesgue/Sobolev space is not indicated, it is either Ω or $\Omega \times (0, T)$, and this will be clear from the context. Throughout the paper we use the symbol $a \lesssim b$ to mean that there exists a sufficiently large constant $C = C(\Omega, r, d) \geq 1$ such that $a \leq Cb$.

REMARK 2.1. In (2.1) we note that $\phi_{0,T}(u)$ is the $H^{r-1}([0, T] \times \Omega)$ norm of the solution u of (1.1)–(1.4). Under suitable smoothness and compatibility conditions on u_0 and f, and for sufficiently small T, it is known (cf. e.g. [**33**, Chapter III]) that $\phi_{0,T}(u)$ is a priori bounded in terms of Sobolev norms of u_0 and f.

REMARK 2.2. The finiteness of the norm $\phi_T(u)$ in (2.1), for some $T > 0$, implies that the function u is real-analytic in space-time on $(0, T) \times \Omega$ (see, e.g. [**28**]). Moreover, for any $t_0 \in (0, T)$, the finiteness of the sub-sums with $i = 0$ and $i = 1$ shows that $u(\cdot, t_0)$ is real-analytic in space, uniformly up to the boundary of the half space Ω. The radius of analyticity is bounded from below by a constant multiple of $t_0\tilde{\epsilon}$ and the analytic norm is bounded from above by $(1 + t_0^{-1/2})\phi_T(u)$. Note also that by changing the binomial weight N_m to $m^r/(m!)^s$, with $s > 1$, the finiteness of $\phi_T(u)$ implies the Gevrey-class s regularity of u.

In [**21**], the last two authors of this paper have showed that the solution u of the Cauchy problem for the inhomogeneous Stokes system

$$\begin{aligned} \partial_t u - \Delta u + \nabla p &= f, &\quad &\text{in } \Omega, \\ \nabla \cdot u &= 0, & &\text{in } \Omega, \\ u &= 0, & &\text{on } \partial\Omega \end{aligned} \tag{2.2}$$

satisfies

$$\phi_T(u) \lesssim \phi_{0,T}(u) + M_T(f) \tag{2.3}$$

where

$$\begin{aligned} M_T(f) = &\sum_{i+j+k\geq(r-2)_+} N_{i+j+k+2}\epsilon^i\tilde{\epsilon}^{j+2}\bar{\epsilon}^k\|t^{i+j+k+2-r}\partial_t^i\partial_d^j\bar{\partial}^k f\|_{L^2_{x,t}((0,T)\times\Omega)} \\ &+ \sum_{i+k\geq(r-2)_+} N_{i+k+2}\epsilon^i\bar{\epsilon}^{k+2}\|t^{i+k+2-r}\partial_t^i\bar{\partial}^k f\|_{L^2_{x,t}((0,T)\times\Omega)} \\ &+ \sum_{i\geq r-1} N_{i+1}\epsilon^{i+1}\|t^{i+1-r}\partial_t^i f\|_{L^2_{x,t}((0,T)\times\Omega)}, \end{aligned} \tag{2.4}$$

provided that

$$0 < \tilde{\epsilon} \leq \bar{\epsilon} \leq \epsilon \leq 1 \tag{2.5}$$

are suitably chosen small constants depending on T, r, and d. For the sake of completeness, we recall from [**21**] that the constants $\tilde{\epsilon}, \bar{\epsilon}, \epsilon$ can be chosen as follows: There exists a constant $C = C(r, d) \geq 1$ such that in addition to (2.5) we require

$$\epsilon \leq \frac{1}{C}, \quad T\bar{\epsilon} \leq \frac{1}{C}, \quad (T\tilde{\epsilon})^{1/2} + \frac{\tilde{\epsilon}}{\epsilon} \leq \frac{1}{C}. \tag{2.6}$$

Since in this paper we consider $T \in (0, 1]$, the above condition is satisfied as soon as we impose

$$\epsilon \leq \frac{1}{C}, \quad \bar{\epsilon} \leq \frac{1}{C}, \quad \tilde{\epsilon}^{1/2} + \frac{\tilde{\epsilon}}{\epsilon} \leq \frac{1}{C}, \tag{2.7}$$

where as before $C = C(r, d) \geq 1$ is sufficiently large. Throughout this paper fix these values of $\epsilon, \bar{\epsilon}, \tilde{\epsilon}$ which obey (2.5) and (2.7), and emphasize that their values depend only on r and d. With this notation, the following is our main result.

THEOREM 2.3. *For $d \in \{2, 3\}$ and $r = 3$ there exist $\epsilon, \tilde{\epsilon}, \bar{\epsilon} \in (0, 1]$, such that the following statement holds: For any divergence-free $u_0 \in H_0^1(\Omega) \cap H^4(\Omega)$ which satisfies suitable compatibility conditions, and a space-time real-analytic $f \in L^\infty(0, 1; H^3(\Omega)) \cap \dot{W}^{1,\infty}(0, 1; H^1(\Omega)) \cap \dot{W}^{2,\infty}(0, 1; L^2(\Omega))$, for which $M_1(f) < \infty$, there exists $T_* \in (0, 1]$ such that the solution u of the Cauchy problem for (1.1)–(1.4) satisfies the estimate*

$$\phi_T(u) \lesssim 1 + M_T(f) \tag{2.8}$$

for any $T \in (0, T_*]$. The implicit constant only depends on Ω, r, and d.

REMARK 2.4. In Theorem 2.3 the time T_* depends on the datum through $\|u_0\|_{H^4}$, and on the force through $\|f\|_{L_t^\infty H^3} + \|\partial_t f\|_{L_t^\infty H_x^1} + \|\partial_t^2 f\|_{L_t^\infty L_x^2} + M_1(f)$, where $M_1(f)$ is as defined in (2.4).

REMARK 2.5. On the initial datum we have imposed, for simplicity, the requirement $u_0 \in H_0^1(\Omega) \cap H^4(\Omega)$, in addition to the usual compatibility conditions at the boundary of Ω. However, if we are only interested in the space-time analyticity of the solution on $(t_0, T] \times \Omega$, for an arbitrarily small $t_0 > 0$, we may simply take $u_0 \in H_0^1(\Omega)$. The local existence of the Cauchy problem to (1.1)–(1.4) with such initial datum is classical, and the H^4 regularity of $u(\cdot, t_0/2)$ follows from the Sobolev smoothing properties of the nonlinear Stokes equation [**33**], which allows us to apply Theorem 2.3 with initial datum $\tilde{u}_0 = u(\cdot, t_0/2)$.

The main idea in the proof of Theorem 2.3 is to apply the estimate (2.3) with f replaced by $f - u \cdot \nabla u$, and to perform a nonlinear estimate on $\phi_T(u \cdot \nabla u)$ in terms of $\phi_{0,T}(u)$ and $\phi_T(u)$. The goal is to arrive at an estimate like (1.5), which then concludes the proof of the theorem upon choosing a suitable T.

The main idea behind the estimate (2.3) in [**21**] is to split the sum ϕ_T in (2.1) into several sub-sums, and on each one perform a *derivative reduction estimate*. For convenience of the reader we recall from [**21**] these derivative reduction estimates, for a solution of the non homogeneous Stokes system (2.2) on the half-space. In all the below inequalities, we require $i + j + k \geq r$. As shown in [**21**, Section 5.1], we may achieve a normal derivative reduction for the Stokes operator

$$\begin{aligned}
&\|t^{i+j+k-r}\partial_t^i\partial_d^j\bar{\partial}^k u\|_{L^2_{x,t}} + \|t^{i+j+k-r}\partial_t^i\partial_d^{j-1}\bar{\partial}^k p\|_{L^2} \\
&\quad \lesssim \|t^{i+j+k-r}\partial_t^{i+1}\partial_d^{j-2}\bar{\partial}^k u\|_{L^2_{x,t}} + \|t^{i+j+k-r}\partial_t^i\partial_d^{j-1}\bar{\partial}^{k+1} u\|_{L^2_{x,t}} \\
&\qquad + \|t^{i+j+k-r}\partial_t^i\partial_d^{j-2}\bar{\partial}^{k+2} u\|_{L^2_{x,t}} + \|t^{i+j+k-r}\partial_t^i\partial_d^{j-2}\bar{\partial}^k u\|_{L^2_{x,t}} \\
&\qquad + \|t^{i+j+k-r}\partial_t^i\partial_d^{j-2}\bar{\partial}^k f\|_{L^2_{x,t}}, \qquad j \geq 2
\end{aligned} \tag{2.9}$$

which allows us to reduce the number of vertical derivatives (∂_d) in the Gevrey (analytic) norm. On the other hand, for $j = 1$, we have

$$\begin{aligned}
&\|t^{i+1+k-r}\partial_t^i\partial_d\bar{\partial}^k u\|_{L^2_{x,t}} + \|t^{i+1+k-r}\partial_t^i\bar{\partial}^k p\|_{L^2_{x,t}} \\
&\quad \lesssim \|t^{i+1+k-r}\partial_t^{i+1}\bar{\partial}^{k-1} u\|_{L^2_{x,t}} + \|t^{i+1+k-r}\partial_t^i\bar{\partial}^{k-1} f\|_{L^2_{x,t}}, \qquad k \geq 1.
\end{aligned} \tag{2.10}$$

For $j = 1$ and $k = 0$, we claim

$$\|t^{i+1-r}\partial_t^i \nabla u\|_{L^2_{x,t}} \lesssim \|t^{i+1-r}\partial_t^i u\|_{L^2_{x,t}}^{1/2} \|t^{i+1-r}\partial_t^{i+1} u\|_{L^2_{x,t}}^{1/2} + \|t^{i+1-r}\partial_t^i u\|_{L^2} + \|t^{i+1-r}\partial_t^i f\|_{L^2_{x,t}}, \qquad i \geq r. \tag{2.11}$$

In order to reduce the number of tangential derivatives, we apply the estimate

$$\|t^{i+k-r}\partial_t^i \bar{\partial}^k u\|_{L^2_{x,t}} + \|t^{i+k-r}\partial_t^i \bar{\partial}^{k-1} p\|_{L^2_{x,t}} \lesssim \|t^{i+k-r}\partial_t^{i+1} \bar{\partial}^{k-2} u\|_{L^2_{x,t}} + \|t^{i+k-r}\partial_t^i \bar{\partial}^{k-2} f\|_{L^2_{x,t}}, \qquad k \geq 2 \tag{2.12}$$

given in [**21**, Section 5.2]. For $k = 1$, we use a special case (replace ∇ with $\bar{\partial}$) of the inequality (2.11):

$$\|t^{i+1-r}\partial_t^i \bar{\partial} u\|_{L^2_{x,t}} \lesssim \|t^{i+1-r}\partial_t^i u\|_{L^2_{x,t}}^{1/2} \|t^{i+1-r}\partial_t^{i+1} u\|_{L^2_{x,t}}^{1/2} + \|t^{i+1-r}\partial_t^i u\|_{L^2} + \|t^{i+1-r}\partial_t^i f\|_{L^2_{x,t}}, \qquad i \geq r. \tag{2.13}$$

Lastly, for the pure time derivatives, we have

$$\|t^{i-r}\partial_t^i u\|_{L^2_{x,t}} \lesssim (i-r)\|t^{i-1-r}\partial_t^{i-1} u\|_{L^2_{x,t}} + \|t^{i-r}\partial_t^{i-1} f\|_{L^2_{x,t}}, \qquad i-1 \geq r \tag{2.14}$$

as obtained in [**21**, Section 5.3]. The proofs of these reductions are based on simple H^2 regularity considerations for the linear parabolic type equations. The estimate (2.3) is obtained by summing over $(i,j,k) \in B$ the estimates (2.9)–(2.14), and to absorb all the u-dependent terms into the left side of the inequality by choosing $\epsilon, \bar{\epsilon}, \tilde{\epsilon}$ such that (2.5) and (2.6) hold.

3. Proof of Theorem 2.3

We appeal to the results in [**21**] by rewriting the Navier-Stokes equation (1.1) as a forced Stokes equation

$$\begin{aligned} &\partial_t u - \Delta u + \nabla p = -u \cdot \nabla u + f \\ &\nabla \cdot u = 0 \end{aligned} \tag{3.1}$$

on the three-dimensional half-space $\Omega = \{x_3 > 0\}$, with the Dirichlet boundary condition for u on $\partial\Omega$. Choosing to work with $d = 3$ is nonessential, and is convenient only in fixing the Sobolev-embedding exponents in $L^\infty(\Omega) \subseteq H^2(\Omega)$ and $L^4(\Omega) \subseteq H^1(\Omega)$. With this choice of dimension, it is possible to set $r = 3$ in the definition of $\phi_{0,T}(u)$ and $\bar{\phi}_T(u)$. We note that replacing H^2 and H^1 with higher order Sobolev spaces, and increasing the value of r accordingly, we may treat (1.1) in any dimension $d \geq 2$.

3.1. Local existence. In order to establish the boundedness of $\phi_0(u)$ for a finite time horizon, we appeal to a local existence result for the Navier-Stokes equations [**33**, Chapter III] (see also [**24**] for more general local existence results for semi-linear parabolic problems with Dirichlet boundary conditions): Assume that $u_0 \in H_0^1(\Omega) \cap H^4(\Omega)$ is divergence free and obeys suitable compatibility conditions, and that the forcing f lies in $L^\infty_{\rm loc}([0,\infty); H^3(\Omega))$. Then there exists a time

$$T_* = T_*(\|u_0\|_{H^4(\Omega)}, \|f\|_{L^\infty_{\rm loc}([0,\infty);H^3(\Omega))}) > 0 \tag{3.2}$$

and a unique solution u to the Cauchy problem associated to (1.1)–(1.4) which obeys

$$\sup_{t\in[0,T_*]} \|u(t)\|_{H^4(\Omega)} \leq 2\|u_0\|_{H^4(\Omega)}. \tag{3.3}$$

Without loss of generality, in (3.2) we may take the $T_* \leq 1$. Furthermore, if we further assume that $\partial_t f \in L^\infty([0,1]; H^1(\Omega))$ and $\partial_t^2 f \in L^\infty([0,1]; L^2(\Omega))$ we conclude from (1.1) and (3.3) that there exists a constant $C = C(d,\Omega) \geq 1$ such that

$$\begin{aligned} &\sup_{t\in[0,T_*]} \Big(\|u(t)\|_{H^2(\Omega)} + \|\partial_t u(t)\|_{H^1(\Omega)} + \|\partial_t^2 u(t)\|_{L^2(\Omega)}\Big) \\ &\quad \leq C(1+\|u_0\|_{H^4(\Omega)})^3 + \|\partial_t f\|_{L^\infty([0,T_*];H^1(\Omega))} + \|\partial_t^2 f\|_{L^\infty([0,T_*];L^2(\Omega))} = C_*(u_0,f). \end{aligned} \tag{3.4}$$

The upshot of (3.4) is that for any $T \in (0, T_*] \subset (0, 1]$ we have

$$\phi_{0,T}(u) \le T^{1/2} C_*(u_0, f). \tag{3.5}$$

When T is sufficiently small, the above estimate implies the smallness of $\phi_{0,T}(u)$, which is essential in closing the nonlinear argument.

3.2. The Stokes estimate. From the result in [**21**], namely the estimates (2.3)–(2.4) for the nonlinear Stokes equation (3.1), we obtain

$$\begin{aligned} \phi_T(u) &\lesssim \phi_{0,T}(u) + M_T(f) + M_T(u \cdot \nabla u) \\ &= \phi_{0,T}(u) + M_T(f) + \sum_{i+j+k \ge (r-2)_+} N_{i+j+k+2} \epsilon^i \tilde{\epsilon}^{j+2} \bar{\epsilon}^k \| t^{i+j+k+2-r} \partial_t^i \partial_d^j \bar{\partial}^k (u \cdot \nabla u) \|_{L^2_{x,t}} \\ &\quad + \sum_{i+k \ge (r-2)_+} N_{i+k+2} \epsilon^i \bar{\epsilon}^{k+2} \| t^{i+k+2-r} \partial_t^i \bar{\partial}^k (u \cdot \nabla u) \|_{L^2_{x,t}} \\ &\quad + \sum_{i \ge (r-1)_+} N_{i+1} \epsilon^{i+1} \| t^{i+1-r} \partial_t^i (u \cdot \nabla u) \|_{L^2_{x,t}} \\ &= \phi_{0,T}(u) + M_T(f) + M_1 + M_2 + M_3 \end{aligned} \tag{3.6}$$

where $M_T(f)$ is as defined in (2.4), and the parameters $\epsilon, \tilde{\epsilon}, \bar{\epsilon}$ are fixed as in (2.5)–(2.7), so that they depend only on $d = 3$ and $r = 3$. The bulk of the proof of Theorem 2.3 is to bound the sums M_1, M_2, and M_3 appearing on the right side of (3.6), in terms of $\phi_{0,T}(u)$ and $\bar{\phi}_T(u)$.

3.3. Bounds for the nonlinear term. These estimates for M_1, M_2, and M_3 are performed in detail in Section 4 below (cf. Lemmas 4.4, 4.5, and 4.6), and may be summarized as follows:

LEMMA 3.1. *Fix $T \in (0, 1]$, $d \in \{2, 3\}$ and $r = 3$. Then we have*

$$M_1 + M_2 + M_3 \lesssim \phi_{0,T}(u)^{1/2} \phi_T(u)^{3/2} + T^{1/2} \phi_T(u)^2, \tag{3.7}$$

where the implicit constant depends only on r, d, and Ω, and is in particular independent of T.

From (3.6) and (3.7) we conclude that there exists $\bar{C} = \bar{C}(r, d, \Omega) \ge 1$ such that

$$\phi_T(u) \le \bar{C} \phi_{0,T}(u)^{1/2} (\phi_{0,T}(u)^{1/2} + \phi_T(u)^{3/2}) + \bar{C} M_T(f) + \bar{C} T^{1/2} \phi_T(u)^2 \tag{3.8}$$

for any $T \in (0, T_*]$.

3.4. Conclusion of the proof of Theorem 2.3. In order to complete the proof of Theorem 2.3, it only remains to combine (3.5) with (3.8). This is a standard barrier argument, which we sketch briefly. The goal is to prove that for T sufficiently small, we have

$$\phi_T(u) \le 4\bar{C} + 4\bar{C} M_1(f) = M \tag{3.9}$$

where the constant $\bar{C}$ is the one given in (3.8). Note that $M_T(f) \le M_1(f)$. In order to prove (3.9) for T sufficiently small, first use (3.5) and take $T_* \le 1/C_*(u_0, f)^2$, which ensures that

$$\phi_{0,T}(u) \le 1$$

for $0 < T \le T_*$. Therefore, letting $T_* \le 1$, from (3.5) and (3.8) we obtain

$$\phi_T(u) \le \bar{C} + \bar{C}(C_*(u_0, f))^{1/2} T^{1/4} \phi_T(u)^{3/2} + \bar{C} M_1(f) + \bar{C} T^{1/2} \phi_T(u)^2. \tag{3.10}$$

If our assertion (3.9) does not hold, there exists $\bar{T} \le T_*$ such that $\phi(T) < M$ for $T < \bar{T}$ and $\phi(\bar{T}) = M$. Then, using (3.10) with $T = \bar{T}$, we get

$$\begin{aligned} M &\le \bar{C} + \bar{C}(C_*(u_0, f))^{1/2} \bar{T}^{1/4} \phi_{\bar{T}}(u)^{3/2} + \bar{C} M_1(f) + \bar{C} \bar{T}^{1/2} \phi_{\bar{T}}(u)^2 \\ &\le \frac{M}{4} + \bar{C}(C_*(u_0, f))^{1/2} T_*^{1/4} M^{3/2} + \bar{C} T_*^{1/2} M^2. \end{aligned} \tag{3.11}$$

Restricting T_* so that the last two terms are both less than or equal to $M/4$ gives $M = 0$, which leads to a contradiction and thus proves Theorem 2.3.

4. Space-time analytic estimates for the nonlinear term

For the remainder of the proof, we omit the T-subindex in the quantities $\phi_T(u)$, $\bar\phi_T(u)$, and $\phi_{0,T}(u)$, and simply denote them as $\phi(u)$, $\bar\phi(u)$, and $\phi_0(u)$.

It is convenient to use the notation $|(i,j,k)| = i+j+k$, which indicates the length of the multi-index, and to denote

$$U_{i,j,k} := \begin{cases} N_{i+j+k}\epsilon^i \tilde\epsilon^j \bar\epsilon^k \|t^{i+j+k-r}\partial_t^i \partial_d^j \bar\partial^k u\|_{L^2_{x,t}}, & i+j+k \geq r, \\ \|\partial_t^i \partial_d^j \bar\partial^k u\|_{L^2_{x,t}}, & 0 \leq i+j+k \leq r-1, \end{cases} \tag{4.1}$$

where we recall that $N_{i+j+k} = |(i,j,k)|^r/|(i,j,k)|!$. Here the parameters $\epsilon, \tilde\epsilon, \bar\epsilon$ are fixed as in (2.5)–(2.7). With this notation we have

$$\bar\phi(u) = \sum_{i+j+k\geq r} U_{i,j,k} \quad \text{and} \quad \phi_0(u) = \sum_{0\leq i+j+k\leq r-1} U_{i,j,k}. \tag{4.2}$$

It shall be convenient to denote $\nabla = (\bar\partial, \partial_d)$ and $u = (\bar u, u_d)$, so that $u\cdot\nabla u = \bar u\cdot\bar\partial u + u_d\,\partial_d u$.

REMARK 4.1. We emphasize that throughout this last section the implicit constants in the $\lesssim$ symbols are allowed to depend on ϵ^m, $\tilde\epsilon^m$, and $\bar\epsilon^m$, where $m\in\mathbb{Z}$ is such that $|m|\leq 100$. Indeed, since the $\tilde\epsilon, \bar\epsilon, \epsilon$ have been fixed solely in terms of Ω, d, and r, cf. (2.5)–(2.7), they are independent of time and thus any a-priori finite power of these parameters may be hidden in the $\lesssim$ symbol.

4.1. Gagliardo-Nirenberg inequalities. We use a number of well-known space-time Gagliardo-Nirenberg inequalities that we summarize next. For $u\in H^2(\Omega)$, we shall frequently use the following estimates:

$$\|u\|_{L^\infty(\Omega)} \lesssim \|u\|_{\dot H^2(\Omega)}^{d/4}\|u\|_{L^2(\Omega)}^{1-d/4} + \|u\|_{L^2(\Omega)}, \qquad u\in H^2(\Omega), \tag{4.3}$$

$$\|u\|_{L^\infty(\Omega)} \lesssim \|u\|_{\dot H^2(\Omega)}^{d/4}\|u\|_{L^2(\Omega)}^{1-d/4}, \qquad u\in H^2(\Omega),\ \text{with } u\big|_{\partial\Omega} = 0, \tag{4.4}$$

$$\|u\|_{L^4(\Omega)} \lesssim \|u\|_{\dot H^1(\Omega)}^{d/4}\|u\|_{L^2(\Omega)}^{1-d/4} + \|u\|_{L^2(\Omega)}, \qquad u\in H^1(\Omega), \tag{4.5}$$

$$\|u\|_{L^4(\Omega)} \lesssim \|u\|_{\dot H^1(\Omega)}^{d/4}\|u\|_{L^2(\Omega)}^{1-d/4}, \qquad u\in H^1(\Omega),\ \text{with } u\big|_{\partial\Omega} = 0. \tag{4.6}$$

For $v\in H^1(0,T)$ such that $v|_{t=0} = 0$, we use Agmon's inequality

$$\|v\|_{L^\infty(0,T)} \lesssim \|v\|_{L^2(0,T)}^{1/2}\|\partial_t v\|_{L^2(0,T)}^{1/2}, \tag{4.7}$$

while in the case $v|_{t=0}\neq 0$, a lower order term is needed in the above estimate, namely,

$$\|v\|_{L^\infty(0,T)} \lesssim \|v\|_{L^2(0,T)}^{1/2}\|\partial_t v\|_{L^2(0,T)}^{1/2} + \|v\|_{L^2(0,T)}. \tag{4.8}$$

Together, the estimates (4.3)–(4.8) imply that for $u\in H^1(0,T;H^2(\Omega))$, we have

$$\|u\|_{L^\infty_{x,t}} \lesssim \|\partial_t u\|_{L^2_t\dot H^2_x}^{1/2}\|u\|_{L^2_t\dot H^2_x}^{1/2} + \|u\|_{L^2_t\dot H^2_x} + \|\partial_t u\|_{L^2_{x,t}} + \|u\|_{L^2_{x,t}}. \tag{4.9}$$

Similarly, for $u\in H^1(0,T;H^1(\Omega))$, we may bound

$$\|u\|_{L^\infty_t L^4_x} \lesssim \|\partial_t u\|_{L^2_t\dot H^1_x}^{1/2}\|u\|_{L^2_t\dot H^1_x}^{1/2} + \|u\|_{L^2_t\dot H^1_x} + \|\partial_t u\|_{L^2_{x,t}} + \|u\|_{L^2_{x,t}}. \tag{4.10}$$

The estimates (4.9)–(4.10) are used repeatedly throughout the next sections. We note that in view of the three derivative loss in the first term on right side of (4.9), one in time and two in space, the smallest value we may take for r in the definition of $\phi(u)$ is 3, which justifies our choice $r = 3$. In order to simplify the computations, we rewrite (4.9) and (4.10) for a function of the form $t^{\ell+n+m}\partial_t^\ell\partial_d^n\bar\partial^m u$. The former inequality becomes

$$
\begin{aligned}
&\|t^{\ell+n+m}\partial_t^\ell\partial_d^n\bar\partial^m u\|_{L^\infty_{x,t}}\\
&\lesssim \|\partial_t(t^{\ell+n+m}\partial_t^\ell\partial_d^n\bar\partial^m u)\|_{L^2_t\dot H^2_x}^{1/2}\|t^{\ell+n+m}\partial_t^\ell\partial_d^n\bar\partial^m u\|_{L^2_t\dot H^2_x}^{1/2} + \|t^{\ell+n+m}\partial_t^\ell\partial_d^n\bar\partial^m u\|_{L^2_t\dot H^2_x}\\
&\quad + \|\partial_t(t^{\ell+n+m}\partial_t^\ell\partial_d^n\bar\partial^m u)\|_{L^2_{x,t}} + \|t^{\ell+n+m}\partial_t^\ell\partial_d^n\bar\partial^m u\|_{L^2_{x,t}}\\
&\lesssim \left(\|t^{\ell+n+m}\partial_t^{\ell+1}\partial_d^n\bar\partial^m u\|_{L^2_t\dot H^2_x}^{1/2} + |(\ell,n,m)|^{1/2}\|t^{\ell+n+m-1}\partial_t^\ell\partial_d^n\bar\partial^m u\|_{L^2_t\dot H^2_x}^{1/2}\right)\\
&\qquad\times \|t^{\ell+n+m}\partial_t^\ell\partial_d^n\bar\partial^m u\|_{L^2_t\dot H^2_x}^{1/2}\\
&\quad + \|t^{\ell+n+m}\partial_t^\ell\partial_d^n\bar\partial^m u\|_{L^2_t\dot H^2_x} + \|t^{\ell+n+m}\partial_t^\ell\partial_d^n\bar\partial^m u\|_{L^2_{x,t}}\\
&\quad + \left(\|t^{\ell+n+m}\partial_t^{\ell+1}\partial_d^n\bar\partial^m u\|_{L^2_{x,t}} + |(\ell,n,m)|\|t^{\ell+n+m-1}\partial_t^\ell\partial_d^n\bar\partial^m u\|_{L^2_{x,t}}\right)
\end{aligned}
\tag{4.11}
$$

when $\ell+n+m\ge 1$. Note that since $T\le 1$, the second and third term in (4.11) is dominated by the first and the last term, respectively.

In the following lemma, we express (4.11) using the notation $U_{\ell,n,m}$. Also, we denote

$$
V_{\ell,n,m} = U_{\ell,n,m} + U_{\ell,n-1,m+1} + U_{\ell,n-2,m+2}. \tag{4.12}
$$

LEMMA 4.2. *For $u\in H^1(0,T;H^2(\Omega))$ and all multi-indices $|(\ell,n,m)|\ge 1$, we have*

$$
\begin{aligned}
&N_{\ell+n+m}\epsilon^\ell\tilde\epsilon^n\bar\epsilon^m\|t^{\ell+n+m}\partial_t^\ell\partial_d^n\bar\partial^m u\|_{L^\infty_{x,t}}\\
&\lesssim V_{\ell+1,n+2,m}^{1/2}V_{\ell,n+2,m}^{1/2}T^{1/2}|(\ell,n,m)|^{5/2} + V_{\ell,n+2,m}T^{1/2}|(\ell,n,m)|^{5/2}\\
&\quad + U_{\ell+1,n,m}\left(T\mathbb{1}_{\ell+n+m=1} + T^2\mathbb{1}_{\ell+n+m\ge 2}\right)|(\ell,n,m)|\\
&\quad + U_{\ell,n,m}\left(T^{\ell+n+m-1}\mathbb{1}_{\ell+n+m\le 2} + T^2\mathbb{1}_{\ell+n+m\ge 3}\right)|(\ell,n,m)|.
\end{aligned}
$$

Similarly, we write the inequality (4.10) for the function $t^{\ell+n+m-1}\partial_t^\ell\partial_d^n\bar\partial^m u$ with $\ell+n+m\ge 2$,

$$
\begin{aligned}
&\|t^{\ell+n+m-1}\partial_t^\ell\partial_d^n\bar\partial^m u\|_{L^\infty_t L^4_x}\\
&\lesssim \|\partial_t(t^{\ell+n+m-1}\partial_t^\ell\partial_d^n\bar\partial^m u)\|_{L^2_t\dot H^1_x}^{1/2}\|t^{\ell+n+m-1}\partial_t^\ell\partial_d^n\bar\partial^m u\|_{L^2_t\dot H^1_x}^{1/2} + \|t^{\ell+n+m-1}\partial_t^\ell\partial_d^n\bar\partial^m u\|_{L^2_t\dot H^1_x}\\
&\quad + \|\partial_t(t^{\ell+n+m-1}\partial_t^\ell\partial_d^n\bar\partial^m u)\|_{L^2_{x,t}} + \|t^{\ell+n+m-1}\partial_t^\ell\partial_d^n\bar\partial^m u\|_{L^2_{x,t}}\\
&\lesssim \left(\|t^{\ell+n+m-1}\partial_t^{\ell+1}\partial_d^n\bar\partial^m u\|_{L^2_t\dot H^1_x}^{1/2} + |(\ell,n,m)|^{1/2}\|t^{\ell+n+m-2}\partial_t^\ell\partial_d^n\bar\partial^m u\|_{L^2_t\dot H^1_x}^{1/2}\right)\\
&\quad\times \|t^{\ell+n+m-1}\partial_t^\ell\partial_d^n\bar\partial^m u\|_{L^2_t\dot H^1_x}^{1/2}\\
&\quad + \|t^{\ell+n+m-1}\partial_t^\ell\partial_d^n\bar\partial^m u\|_{L^2_t\dot H^1_x} + \|t^{\ell+n+m-1}\partial_t^\ell\partial_d^n\bar\partial^m u\|_{L^2_{x,t}}\\
&\quad + \left(\|t^{\ell+n+m-1}\partial_t^{\ell+1}\partial_d^n\bar\partial^m u\|_{L^2_{x,t}} + |(\ell,n,m)|\|t^{\ell+n+m-2}\partial_t^\ell\partial_d^n\bar\partial^m u\|_{L^2_{x,t}}\right).
\end{aligned}
\tag{4.13}
$$

Using the notation $U_{\ell,n,m}$, we rewrite (4.13) as follows.

LEMMA 4.3. *For $u\in H^1(0,T;H^1(\Omega))$ and all multi-indices $|(\ell,n,m)|\ge 2$, we have*

$$
\begin{aligned}
&N_{\ell+n+m}\epsilon^\ell\tilde\epsilon^n\bar\epsilon^m\|t^{\ell+n+m-1}\partial_t^\ell\partial_d^n\bar\partial^m u\|_{L^\infty_t L^4_x}\\
&\lesssim \left(U_{\ell+1,n+1,m} + U_{\ell+1,n,m+1}\right)^{1/2}\left(U_{\ell,n+1,m} + U_{\ell,n,m+1}\right)^{1/2}T^{1/2}|(\ell,n,m)|^{3/2}\\
&\quad + \left(U_{\ell,n+1,m} + U_{\ell,n,m+1}\right)T^{1/2}|(\ell,n,m)|^{3/2} + U_{\ell+1,n,m}T|(\ell,n,m)|\\
&\quad + U_{\ell,n,m}\left(\mathbb{1}_{\ell+n+m=2} + T\mathbb{1}_{\ell+n+m\ge 3}\right)|(\ell,n,m)|.
\end{aligned}
\tag{4.14}
$$

We have used once again $T \le 1$ to have the second and third term in (4.13) dominated by the first and the last term, respectively.

4.2. Terms with only time derivatives. In this section we estimate M_3 in (3.6). From the Leibniz rule we obtain that

$$M_3 \le \sum_{i\ge 2}\sum_{\ell=0}^{i}\binom{i}{\ell}N_{i+1}\epsilon^i\left(\|t^{i-2}\partial_t^\ell \bar u\cdot\partial_t^{i-\ell}\bar\partial u\|_{L^2_{x,t}}+\|t^{i-2}\partial_t^\ell u_d\,\partial_t^{i-\ell}\partial_d u\|_{L^2_{x,t}}\right).$$

Recalling the notation (4.2) we assert that:

LEMMA 4.4. *For solution u of the Cauchy problem (1.1)–(1.3), we have*

$$M_3 \lesssim \phi_0(u)^{1/2}\phi(u)^{3/2}+T^{1/2}\phi(u)^2 \tag{4.15}$$

for $0<T\le 1$.

PROOF OF LEMMA 4.4. We split M_3 into sums M_{31} and M_{32} corresponding to $i=2$ or $i\ge 3$, respectively. Then

$$\begin{aligned}M_{31}\lesssim{}& \|u\|_{L^\infty_{x,t}}\left(\|\partial_t^2\bar\partial u\|_{L^2_{x,t}}+\|\partial_t^2\partial_d u\|_{L^2_{x,t}}\right)+\|\partial_t u\|_{L^\infty_t L^4_x}\left(\|\partial_t\bar\partial u\|_{L^2_tL^4_x}+\|\partial_t\partial_d u\|_{L^2_tL^4_x}\right)\\ &+\|\partial_t^2 u\|_{L^2_tL^4_x}\left(\|\bar\partial u\|_{L^\infty_tL^4_x}+\|\partial_d u\|_{L^\infty_tL^4_x}\right)\end{aligned}$$

where the three terms correspond to $\ell=0,1,2$. Using the notation $\lfloor x\rfloor=[x]$ and $\lceil x\rceil=[x]+1$, we have

$$\begin{aligned}M_{32}\le{}&\sum_{i\ge3}\sum_{\ell=1}^{\lfloor i/2\rfloor}\binom{i}{\ell}N_{i+1}\epsilon^i\|t^\ell\partial_t^\ell u\|_{L^\infty_{x,t}}\left(\|t^{i-\ell-2}\partial_t^{i-\ell}\bar\partial u\|_{L^2_{x,t}}+\|t^{i-\ell-2}\partial_t^{i-\ell}\partial_d u\|_{L^2_{x,t}}\right)\\ &+\sum_{i\ge3}N_{i+1}\epsilon^i\|u\|_{L^\infty_{x,t}}\left(\|t^{i-2}\partial_t^i\bar\partial u\|_{L^2_{x,t}}+\|t^{i-2}\partial_t^i\partial_d u\|_{L^2_{x,t}}\right)\\ &+\sum_{i\ge3}\sum_{\ell=\lceil i/2\rceil}^{i-1}\binom{i}{\ell}N_{i+1}\epsilon^i\|t^{\ell-2}\partial_t^\ell u\|_{L^2_tL^4_x}\left(\|t^{i-\ell}\partial_t^{i-\ell}\bar\partial u\|_{L^\infty_tL^4_x}+\|t^{i-\ell}\partial_t^{i-\ell}\partial_d u\|_{L^\infty_tL^4_x}\right)\\ &+\sum_{i\ge3}N_{i+1}\epsilon^i\|t^{i-2}\partial_t^i u\|_{L^2_tL^4_x}\left(\|\bar\partial u\|_{L^\infty_tL^4_x}+\|\partial_d u\|_{L^\infty_tL^4_x}\right)\\ ={}&M_{321}+M_{322}+M_{323}+M_{324},\end{aligned}\tag{4.16}$$

where we separated away $\ell=0$ and $\ell=i$ from the main sums. We start by bounding M_{31}. Since $\partial_t^\ell u\big|_{\partial\Omega}=0$ for $\ell\ge1$, we may apply (4.9) and (4.10) to conclude

$$\begin{aligned}M_{31}\lesssim{}&\left(\|\partial_t u\|^{1/2}_{L^2_t\dot H^2_x}\|u\|^{1/2}_{L^2_t\dot H^2_x}+\|u\|_{L^2_t\dot H^2_x}+\|\partial_t u\|_{L^2_{x,t}}+\|u\|_{L^2_{x,t}}\right)\left(\|\partial_t^2\bar\partial u\|_{L^2_{x,t}}+\|\partial_t^2\partial_d u\|_{L^2_{x,t}}\right)\\ &+\left(\|\partial_t^2 u\|^{1/2}_{L^2_t\dot H^1_x}\|\partial_t u\|^{1/2}_{L^2_t\dot H^1_x}+\|\partial_t u\|_{L^2_t\dot H^1_x}+\|\partial_t^2 u\|_{L^2_{x,t}}+\|\partial_t u\|_{L^2_{x,t}}\right)\\ &\quad\times\left(\|\partial_t\bar\partial u\|^{d/4}_{L^2_t\dot H^1_x}\|\partial_t\bar\partial u\|^{1-d/4}_{L^2_{x,t}}+\|\partial_t\partial_d u\|^{d/4}_{L^2_t\dot H^1_x}\|\partial_t\partial_d u\|^{1-d/4}_{L^2_{x,t}}+\|\partial_t\partial_d u\|_{L^2_{x,t}}\right)\\ &+\left(\|\partial_t^2 u\|^{d/4}_{L^2_t\dot H^1_x}\|\partial_t^2 u\|^{1-d/4}_{L^2_{x,t}}\right)\\ &\quad\times\Bigg(\|\partial_t\bar\partial u\|^{1/2}_{L^2_t\dot H^1_x}\|\bar\partial u\|^{1/2}_{L^2_t\dot H^1_x}+\|\bar\partial u\|_{L^2_t\dot H^1_x}+\|\partial_t\bar\partial u\|_{L^2_{x,t}}+\|\bar\partial u\|_{L^2_{x,t}}\\ &\qquad+\|\partial_t\partial_d u\|^{1/2}_{L^2_t\dot H^1_x}\|\partial_d u\|^{1/2}_{L^2_t\dot H^1_x}+\|\partial_d u\|_{L^2_t\dot H^1_x}+\|\partial_t\partial_d u\|_{L^2_{x,t}}+\|\partial_d u\|_{L^2_{x,t}}\Bigg).\end{aligned}$$

Therefore,

$$
\begin{aligned}
M_{31} &\lesssim \Big(\phi_0(u)^{1/2}\bar\phi(u)^{1/2}+\phi_0(u)\Big)\,\bar\phi(u)\\
&\quad+\Big(\phi_0(u)^{1/2}\bar\phi(u)^{1/2}+\phi_0(u)\Big)\Big(\bar\phi(u)^{d/4}\phi_0(u)^{1-d/4}+\phi_0(u)\Big)\\
&\quad+\bar\phi(u)^{d/4}\phi_0(u)^{1-d/4}\Big(\bar\phi(u)^{1/2}\phi_0(u)^{1/2}+\phi_0(u)\Big)\\
&\lesssim \phi_0(u)^2+\phi_0(u)^{1/2}\phi(u)^{3/2}.
\end{aligned}
\tag{4.17}
$$

For M_{32}, we note that $i\geq 3$ for each of the sums. We start with the boundary sums M_{322} and M_{324}, and treat M_{321} and M_{323} further below. Using (4.9), we have

$$
\begin{aligned}
M_{322} &\lesssim \sum_{i\geq 3} N_{i+1}\epsilon^i\Big(\|\partial_t u\|_{L_t^2\dot H_x^2}^{1/2}\|u\|_{L_t^2\dot H_x^2}^{1/2}+\|u\|_{L_t^2\dot H_x^2}+\|\partial_t u\|_{L^2_{x,t}}+\|u\|_{L^2_{x,t}}\Big)\\
&\qquad\times\Big(\|t^{i-2}\partial_t^i\bar\partial u\|_{L^2_{x,t}}+\|t^{i-2}\partial_t^i\partial_d u\|_{L^2_{x,t}}\Big)\\
&\lesssim (\phi_0(u)+\phi_0(u)^{1/2}\bar\phi(u)^{1/2})\sum_{i\geq 3}N_{i+1}\epsilon^i\Big(\|t^{i-2}\partial_t^i\bar\partial u\|_{L^2_{x,t}}+\|t^{i-2}\partial_t^i\partial_d u\|_{L^2_{x,t}}\Big)\\
&\lesssim (\phi_0(u)+\phi_0(u)^{1/2}\bar\phi(u)^{1/2})\bar\phi(u)\lesssim \phi_0(u)\phi(u)+\phi_0(u)^{1/2}\phi(u)^{3/2}.
\end{aligned}
\tag{4.18}
$$

For M_{324} we proceed with the Gagliardo-Nirenberg inequalities (4.5)–(4.10) and write

$$
\begin{aligned}
M_{324} &\lesssim \sum_{i\geq 3}N_{i+1}\epsilon^i\Big(\|t^{i-2}\partial_t^i u\|_{L_t^2\dot H_x^1}^{d/4}\|t^{i-2}\partial_t^i u\|_{L^2_{x,t}}^{1-d/4}\Big)\\
&\qquad\times\Big(\|\partial_t\bar\partial u\|_{L_t^2\dot H_x^1}^{1/2}\|\bar\partial u\|_{L_t^2\dot H_x^1}^{1/2}+\|\bar\partial u\|_{L_t^2\dot H_x^1}+\|\partial_t\bar\partial u\|_{L^2_{x,t}}+\|\bar\partial u\|_{L^2_{x,t}}\Big)\\
&\quad+\sum_{i\geq 3}N_{i+1}\epsilon^i\Big(\|t^{i-2}\partial_t^i u\|_{L_t^2\dot H_x^1}^{d/4}\|t^{i-2}\partial_t^i u\|_{L^2_{x,t}}^{1-d/4}\Big)\\
&\qquad\times\Big(\|\partial_t\partial_d u\|_{L_t^2\dot H_x^1}^{1/2}\|\partial_d u\|_{L_t^2\dot H_x^1}^{1/2}+\|\partial_d u\|_{L_t^2\dot H_x^1}+\|\partial_t\partial_d u\|_{L^2_{x,t}}+\|\partial_d u\|_{L^2_{x,t}}\Big).
\end{aligned}
$$

Expressing the estimates in terms of $U_{i,j,k}$ we get

$$
\begin{aligned}
M_{324} &\lesssim \sum_{i\geq 3}N_{i+1}\epsilon^i\,(U_{i,1,0}+U_{i,0,1})^{d/4}\,U_{i,0,0}^{1-d/4}\\
&\qquad\times(\bar\phi(u)^{1/2}\phi_0(u)^{1/2}+\phi_0(u))\left(\frac{(i+1)!}{(i+1)^3\epsilon^i}\right)^{d/4}\left(\frac{i!T}{i^3\epsilon^i}\right)^{1-d/4}\\
&\lesssim T^{1-d/4}(\bar\phi(u)^{1/2}\phi_0(u)^{1/2}+\phi_0(u))\sum_{i\geq 3}(U_{i,1,0}+U_{i,0,1})^{d/4}\,U_{i,0,0}^{1-d/4}\\
&\lesssim T^{1-d/4}(\bar\phi(u)^{1/2}\phi_0(u)^{1/2}+\phi_0(u))\phi(u)\lesssim T^{1-d/4}\phi_0(u)\phi(u)+T^{1-d/4}\phi_0(u)^{1/2}\phi(u)^{3/2}
\end{aligned}
\tag{4.19}
$$

where we have used $1/C\leq\tilde\epsilon\leq\bar\epsilon\leq\epsilon\leq 1$.

For M_{321}, we express $\|t^{i-\ell-2}\partial_t^{i-\ell}\bar\partial u\|_{L^2_{x,t}}$ and $\|t^{i-\ell-2}\partial_t^{i-\ell}\partial_d u\|_{L^2_{x,t}}$ in terms of $U_{i-\ell,1,0}$ and $U_{i-\ell,0,1}$ and write, using (4.11),

$$
M_{321}\lesssim\sum_{i\geq 3}\sum_{\ell=1}^{\lfloor i/2\rfloor}\|t^\ell\partial_t^\ell u\|_{L^\infty_{x,t}}\,(U_{i-\ell,1,0}+U_{i-\ell,0,1})\,\frac{(i+1)^2\epsilon^\ell}{\ell!(i-\ell+1)^2}.
\tag{4.20}
$$

Then, we utilize Lemma 4.2 (which we have derived from (4.9)) and $\ell \leq \lfloor i/2 \rfloor$ to obtain

$$\begin{aligned} M_{321} \lesssim T^{1/2} \sum_{i\geq 3} \sum_{\ell=1}^{\lfloor i/2 \rfloor} V_{\ell+1,2,0}^{1/2} V_{\ell,2,0}^{1/2} \left(U_{i-\ell,0,1} + U_{i-\ell,1,0}\right) \\ + T^{1/2} \sum_{i\geq 3} \sum_{\ell=1}^{\lfloor i/2 \rfloor} V_{\ell,2,0} \left(U_{i-\ell,0,1} + U_{i-\ell,1,0}\right) \\ + \sum_{i\geq 3} \sum_{\ell=1}^{\lfloor i/2 \rfloor} U_{\ell+1,0,0} \left(U_{i-\ell,0,1} + U_{i-\ell,1,0}\right) \left(T \mathbb{1}_{\ell=1} + T^2 \mathbb{1}_{\ell\geq 2}\right) \\ + \sum_{i\geq 3} \sum_{\ell=1}^{\lfloor i/2 \rfloor} U_{\ell,0,0} \left(U_{i-\ell,0,1} + U_{i-\ell,1,0}\right) \left(T^{\ell-1} \mathbb{1}_{\ell\leq 2} + T^2 \mathbb{1}_{\ell\geq 3}\right). \end{aligned}$$

By appealing to the discrete Young's inequality and the definition of $\phi_0(u)$, we get

$$M_{321} \lesssim T^{1/2} \bar{\phi}(u)^2 + T\bar{\phi}(u)^2 + T\phi_0(u)\bar{\phi}(u) + T^2\phi_0(u)\bar{\phi}(u) + T^3\bar{\phi}(u)^2 + \phi_0(u)\bar{\phi}(u) + T^2\bar{\phi}(u)^2.$$

Once again using $T \leq 1$ and keeping the dominant terms , we obtain:

$$M_{321} \lesssim \phi_0(u)\phi(u) + T^{1/2}\phi(u)^2. \tag{4.21}$$

Lastly, we treat M_{323} in a similar manner. We split the sum into two parts, as done above for M_{321}, by appealing to (4.6):

$$\begin{aligned} M_{323} &\lesssim \sum_{i\geq 3} \sum_{\ell=\lceil i/2 \rceil}^{i-1} \binom{i}{\ell} N_{i+1}\epsilon^i \left(\|t^{\ell-2}\partial_t^\ell u\|_{L_t^2 \dot{H}_x^1}^{d/4} \|t^{\ell-2}\partial_t^\ell u\|_{L_{x,t}^2}^{1-d/4} \right) \|t^{i-\ell}\partial_t^{i-\ell}\bar{\partial} u\|_{L_t^\infty L_x^4} \\ &\quad + \sum_{i\geq 3} \sum_{\ell=\lceil i/2 \rceil}^{i-1} \binom{i}{\ell} N_{i+1}\epsilon^i \left(\|t^{\ell-2}\partial_t^\ell u\|_{L_t^2 \dot{H}_x^1}^{d/4} \|t^{\ell-2}\partial_t^\ell u\|_{L_{x,t}^2}^{1-d/4} \right) \|t^{i-\ell}\partial_t^{i-\ell}\partial_d u\|_{L_t^\infty L_x^4} \\ &= M_{3231} + M_{3232}. \end{aligned}$$

For M_{3231} we use Lemma 4.3 for the triple $(i-\ell, 0, 1)$ and obtain

$$\begin{aligned} M_{3231} \lesssim T^{1/2} \sum_{i\geq 3} \sum_{\ell=\lceil i/2 \rceil}^{i-1} \left(U_{\ell,1,0} + U_{\ell,0,1}\right)^{d/4} U_{\ell,0,0}^{1-d/4} \\ \times \left(U_{i-\ell+1,1,1} + U_{i-\ell+1,0,2}\right)^{1/2} \left(U_{i-\ell,1,1} + U_{i-\ell,0,2}\right)^{1/2} \\ + T^{1/2} \sum_{i\geq 3} \sum_{\ell=\lceil i/2 \rceil}^{i-1} \left(U_{\ell,1,0} + U_{\ell,0,1}\right)^{d/4} U_{\ell,0,0}^{1-d/4} \left(U_{i-\ell,1,1} + U_{i-\ell,0,2}\right) \\ + T \sum_{i\geq 3} \sum_{\ell=\lceil i/2 \rceil}^{i-1} \left(U_{\ell,1,0} + U_{\ell,0,1}\right)^{d/4} U_{\ell,0,0}^{1-d/4} U_{i-\ell+1,0,1} \\ + \sum_{i\geq 3} \sum_{\ell=\lceil i/2 \rceil}^{i-1} \left(U_{\ell,1,0} + U_{\ell,0,1}\right)^{d/4} U_{\ell,0,0}^{1-d/4} U_{i-\ell,0,1} (\mathbb{1}_{i-\ell=1} + T\mathbb{1}_{i-\ell\geq 2}). \end{aligned} \tag{4.22}$$

Applying the discrete Young's inequality and using the definition of $\phi_0(u)$, we get

$$\begin{aligned} M_{3231} \lesssim\, & T^{1/2}\phi_0(u)^{1-d/4}\bar\phi(u)^{1+d/4} + T^{1/2}\bar\phi(u)^2 + T\phi_0(u)^{2-d/4}\bar\phi(u)^{d/4} + T\phi_0(u)\bar\phi(u) \\ & + T^2\phi_0(u)^{1-d/4}\bar\phi(u)^{1+d/4} + T^2\bar\phi(u)^2 + T\phi_0(u)^{1-d/4}\bar\phi(u)^{1+d/4} + T\bar\phi(u)^2 \\ & + \phi_0(u)^{2-d/4}\bar\phi(u)^{d/4} + \phi_0(u)\bar\phi(u) + T\phi_0(u)^{1-d/4}\bar\phi(u)^{1+d/4} + T\bar\phi(u)^2. \end{aligned} \tag{4.23}$$

In estimating M_{3232}, we follow the same steps as in (4.22), as the only difference is due to having the differential ∂_d in M_{3232} instead of $\bar\partial$. We obtain that M_{3232} obeys the same exact estimate as M_{3231} in (4.23), from which we obtain the desired bound for M_{323}, namely

$$M_{323} \lesssim \phi_0(u)\phi(u) + T^{1/2}\phi(u)^2. \tag{4.24}$$

Combining all the terms in (4.17), (4.18), (4.19), (4.21), (4.24), and selecting the maximal prefactors in T we obtain the estimate (4.15). ☐

4.3. Terms with no normal derivatives.

In this section we estimate M_2.

LEMMA 4.5. *For solutions u of the Cauchy problem* (1.1)–(1.3)*, we get*

$$M_2 \lesssim \phi_0(u)^{3/2}\phi(u)^{1/2} + T\phi_0(u)^{1/2}\phi(u)^{3/2} + T^{3/4}\phi(u)^2 \tag{4.25}$$

for $0 < T \le 1$.

PROOF OF LEMMA 4.5. Writing $u\cdot\nabla u = \bar u\cdot\bar\partial u + u_d\cdot\partial_d u$ and separating the terms with $i+k=1$, we obtain

$$\begin{aligned} M_2 \le & \sum_{i+k\ge 2}\sum_{|(\ell,m)|=0}^{i+k}\binom{i}{\ell}\binom{k}{m} N_{i+k+2}\epsilon^i\bar\epsilon^k \\ & \times\Big(\|t^{i+k-1}\partial_t^\ell\bar\partial^m\bar u\cdot\partial_t^{i-\ell}\bar\partial^{k+1-m}u\|_{L^2_{x,t}} + \|t^{i+k-1}\partial_t^\ell\bar\partial^m u_d\,\partial_t^{i-\ell}\bar\partial^{k-m}\partial_d u\|_{L^2_{x,t}}\Big) \\ & + \|\partial_t(u_d\cdot\partial_d u) + \partial_t(\bar u\cdot\bar\partial u)\|_{L^2_{x,t}} + \|\bar\partial(u_d\cdot\partial_d u) + \bar\partial(\bar u\cdot\bar\partial u)\|_{L^2_{x,t}} \\ = & M_{21} + M_{22} + M_{23}. \end{aligned}$$

We start with the lower order terms. Using Hölder's inequality, we get

$$\begin{aligned} M_{22}+M_{23} \lesssim\, & \|\partial_t u\|_{L^\infty_t L^4_x}\Big(\|\bar\partial u\|_{L^2_t L^4_x} + \|\partial_d u\|_{L^2_t L^4_x}\Big) + \|u\|_{L^\infty_{x,t}}\Big(\|\partial_t\bar\partial u\|_{L^2_{x,t}} + \|\partial_t\partial_d u\|_{L^2_{x,t}}\Big) \\ & + \|\bar\partial u\|_{L^\infty_t L^4_x}\Big(\|\bar\partial u\|_{L^2_t L^4_x} + \|\partial_d u\|_{L^2_t L^4_x}\Big) + \|u\|_{L^\infty_{x,t}}\Big(\|\bar\partial^2 u\|_{L^2_{x,t}} + \|\bar\partial\partial_d u\|_{L^2_{x,t}}\Big) \end{aligned}$$

and recalling the definition of $\bar\phi(u)$ and $\phi_0(u)$, we obtain

$$M_{22}+M_{23} \lesssim \bar\phi(u)^{1/2}\phi_0(u)^{3/2} + \phi_0(u)^2. \tag{4.26}$$

Now, we split M_{21} into two parts as

$$\begin{aligned} M_{21} \lesssim & \sum_{i+k\ge 2}\sum_{|(\ell,m)|=0}^{\lfloor (i+k)/2\rfloor}\binom{i}{\ell}\binom{k}{m} N_{i+k+2}\epsilon^i\bar\epsilon^k\|t^{\ell+m}\partial_t^\ell\bar\partial^m u\|_{L^\infty_{x,t}} \\ & \times\Big(\|t^{(i+k)-(\ell+m)-1}\partial_t^{i-\ell}\bar\partial^{k+1-m}u\|_{L^2_{x,t}} + \|t^{(i+k)-(\ell+m)-1}\partial_t^{i-\ell}\bar\partial^{k-m}\partial_d u\|_{L^2_{x,t}}\Big) \\ & + \sum_{i+k\ge 2}\sum_{|(\ell,m)|\ge\lceil (i+k)/2\rceil}^{i+k}\binom{i}{\ell}\binom{k}{m} N_{i+k+2}\epsilon^i\bar\epsilon^k\|t^{\ell+m-1}\partial_t^\ell\bar\partial^m u\|_{L^\infty_t L^4_x} \\ & \times\Big(\|t^{(i+k)-(\ell+m)}\partial_t^{i-\ell}\bar\partial^{k+1-m}u\|_{L^2_t L^4_x} + \|t^{(i+k)-(\ell+m)}\partial_t^{i-\ell}\bar\partial^{k-m}\partial_d u\|_{L^2_t L^4_x}\Big) \\ = & M_{211} + M_{212}. \end{aligned} \tag{4.27}$$

We start with the first sum in (4.27), namely M_{211}. We apply Lemma 4.2 on $\|t^{\ell+m}\partial_t^\ell \bar\partial^m u\|_{L^\infty_{x,t}}$ except when $|(\ell,0,m)|=0$. Singling out $|(\ell,0,m)|=0$, we have

$$\begin{aligned}
M_{211} \lesssim& \sum_{i+k\geq 2} \epsilon^i \bar\epsilon^k N_{i+k+2} \left(U_{i,0,k+1}+U_{i,1,k}\right)\left(\frac{T}{N_{i+k+1}\epsilon^i\bar\epsilon^k}\right)\\
&\times\left(\|\partial_t u\|^{1/2}_{L^2_t\dot H^2_x}\|u\|^{1/2}_{L^2_t\dot H^2_x}+\|u\|_{L^2_t\dot H^2_x}+\|\partial_t u\|_{L^2_{x,t}}+\|u\|_{L^2_{x,t}}\right)\\
&+\sum_{i+k\geq 2}\sum_{\ell+m=1}^{\lfloor (i+k)/2\rfloor} \epsilon^i\bar\epsilon^k \binom{i}{\ell}\binom{k}{m} N_{i+k+2}\left(U_{i-\ell,0,k+1-m}+U_{i-\ell,1,k-m}\right)\\
&\times\left(\frac{T}{N_{i+k-\ell-m+1}\epsilon^{i-\ell}\bar\epsilon^{k-m}}\right)\frac{(\ell+m)!}{\epsilon^\ell\bar\epsilon^m}\\
&\times\Bigg(\frac{T^{1/2}}{(\ell+m)^{1/2}}V^{1/2}_{\ell+1,2,m}V^{1/2}_{\ell,2,m}+\frac{T^{1/2}}{(\ell+m)^{1/2}}V_{\ell,2,m}\\
&\quad+\frac{1}{(\ell+m)^2}\Big(U_{\ell+1,0,m}(T\mathbb{1}_{\ell+m=1}+T^2\mathbb{1}_{\ell+m\geq 2})\\
&\qquad+U_{\ell,0,m}(T^{\ell+m-1}\mathbb{1}_{\ell+m\leq 2}+T^2\mathbb{1}_{\ell+m\geq 3})\Big)\Bigg).
\end{aligned}$$

Note that when $|(\ell,m)|\leq\lfloor (i+k)/2\rfloor$, we may bound

$$\frac{N_{i+k+2}(\ell+m)!}{N_{i+k-\ell-m+1)}(\ell+m)^{1/2}}\binom{i}{\ell}\binom{k}{m}\lesssim\frac{\binom{i}{\ell}\binom{k}{m}}{\binom{i+k}{\ell+m}}\lesssim 1. \tag{4.28}$$

Then we get

$$M_{211}\lesssim T\phi_0(u)\phi(u)+T\phi_0(u)^{1/2}\phi(u)^{3/2}+T^{3/2}\phi(u)^2. \tag{4.29}$$

Next, we split M_{212} into two parts as

$$\begin{aligned}
M_{212} =& \sum_{i+k\geq 2}\sum_{|(\ell,m)|=\lceil (i+k)/2\rceil}^{i+k}\binom{i}{\ell}\binom{k}{m}N_{i+k+2}\epsilon^i\bar\epsilon^k\|t^{\ell+m-1}\partial_t^\ell\bar\partial^m u\|_{L^\infty_t L^4_x}\\
&\times\|t^{(i+k)-(\ell+m)}\partial_t^{i-\ell}\bar\partial^{k+1-m}u\|_{L^2_tL^4_x}\\
&+\sum_{i+k\geq 2}\sum_{|(\ell,m)|=\lceil (i+k)/2\rceil}^{i+k}\binom{i}{\ell}\binom{k}{m}N_{i+k+2}\epsilon^i\bar\epsilon^k\|t^{\ell+m-1}\partial_t^\ell\bar\partial^m u\|_{L^\infty_t L^4_x}\\
&\times\|t^{(i+k)-(\ell+m)}\partial_t^{i-\ell}\bar\partial^{k-m}\partial_d u\|_{L^2_tL^4_x}\\
=& M_{2121}+M_{2122}.
\end{aligned}$$

Both terms are treated analogously, and thus we only bound the first one.

$$\begin{aligned} M_{2121} \lesssim \sum_{i+k\geq 2} \sum_{|(\ell,m)|=\lceil (i+k)/2\rceil}^{i+k} &\binom{i}{\ell}\binom{k}{m} N_{i+k+2}\epsilon^i\bar{\epsilon}^k \|t^{\ell+m-1}\partial_t^\ell\bar{\partial}^m u\|_{L_t^\infty L_x^4} \\ &\times (U_{i-\ell,0,k+2-m}+U_{i-\ell,1,k+1-m})^{d/4}(U_{i-\ell,0,k+1-m})^{1-d/4} \\ &\times \left(\frac{T\mathbb{1}_{\ell+m\leq i+k-1}+\mathbb{1}_{\ell+m=i+k}}{N_{i+k+2-\ell-m}\epsilon^{i-\ell}\bar{\epsilon}^{k-m}}\right)^{d/4} \\ &\times \left(\frac{T^2\mathbb{1}_{\ell+m\leq i+k-2}+T^{i+k-\ell-m}\mathbb{1}_{\ell+m\geq i+k-1}}{N_{i+k+1-\ell-m}\epsilon^{i-\ell}\bar{\epsilon}^{k-m}}\right)^{1-d/4}, \end{aligned} \tag{4.30}$$

where we used $1/C \leq \tilde{\epsilon} \leq \bar{\epsilon} \leq \epsilon \leq 1$. Note that the last two factors in (4.30) are bounded from above by

$$\left(\frac{T^{2-d/4}\mathbb{1}_{\ell+m\leq i+k-2}+T^{i+k-\ell-m}\mathbb{1}_{\ell+m\geq i+k-1}}{\epsilon^{i-\ell}\bar{\epsilon}^{k-m}}\right)\frac{(i+k-\ell-m)!}{(i+k-\ell-m)^{2-d/4}}. \tag{4.31}$$

Denote

$$\begin{aligned} A_{\ell,m}^{i,k}(u) = {}&(U_{i-\ell,0,k+2-m}+U_{i-\ell,1,k+1-m})^{d/4}(U_{i-\ell,0,k+1-m})^{1-d/4} \\ &\times \left(\frac{T^{2-d/4}\mathbb{1}_{\ell+m\leq i+k-2}+T^{i+k-\ell-m}\mathbb{1}_{\ell+m\geq i+k-1}}{\epsilon^{i-\ell}\bar{\epsilon}^{k-m}}\right)\frac{(i+k-\ell-m)!}{(i+k-\ell-m)^{2-d/4}}. \end{aligned}$$

Applying Lemma 4.3 on the term $\|t^{\ell+m-1}\partial_t^\ell\bar{\partial}^m u\|_{L_t^\infty L_x^4}$, we obtain

$$\begin{aligned} M_{2121} \lesssim \sum_{i+k\geq 2} \sum_{|(\ell,m)|=\lceil (i+k)/2\rceil}^{i+k} &\binom{i}{\ell}\binom{k}{m} N_{i+k+2}\epsilon^i\bar{\epsilon}^k A_{\ell,m}^{i,k}(u) \\ &\times \left(\frac{T^{1/2}}{\epsilon^\ell\bar{\epsilon}^m}\frac{(\ell+m)!}{(\ell+m)^{3/2}}(U_{\ell+1,1,m}+U_{\ell+1,0,m+1})^{1/2}(U_{\ell,1,m}+U_{\ell,0,m+1})^{1/2}\right. \\ &+\frac{T^{1/2}}{\epsilon^\ell\bar{\epsilon}^m}\frac{(\ell+m)!}{(\ell+m)^{3/2}}(U_{\ell,1,m}+U_{\ell,0,m+1})+\frac{(\ell+m)!}{(\ell+m)^2}U_{\ell+1,0,m}\frac{T}{\epsilon^{\ell+1}\bar{\epsilon}^m} \\ &\left.+\frac{(\ell+m)!}{(\ell+m)^2}U_{\ell,0,m}\frac{\mathbb{1}_{\ell+m=2}+T\mathbb{1}_{\ell+m\geq 3}}{\epsilon^\ell\bar{\epsilon}^m}\right). \end{aligned} \tag{4.32}$$

For $i+k-\ell-m\geq 2$, we have

$$\binom{i}{\ell}\binom{k}{m}\frac{N_{i+k+2}(i+k-\ell-m)!}{(i+k-\ell-m)^{2-d/4}}\frac{(\ell+m)!}{(\ell+m)^{3/2}} \lesssim \frac{\binom{i}{\ell}\binom{k}{m}}{\binom{i+k}{\ell+m}} \lesssim 1.$$

Then, using Young's inequality in (4.32), we deduce

$$\begin{aligned} M_{2121} &\lesssim T^{1/2}\phi_0(u)\bar{\phi}(u)+\phi_0(u)^2+T\phi_0(u)^{1-d/4}\bar{\phi}(u)^{d/4}\left(T^{1/2}\bar{\phi}(u)+\phi_0(u)\right) \\ &\quad+T^{1/2}\bar{\phi}(u)\left(T^{2-d/4}\bar{\phi}(u)+T\phi_0(u)^{1-d/4}\bar{\phi}(u)^{d/4}+\phi_0(u)\right) \\ &\lesssim \phi_0(u)^2+T^{1/2}\phi_0(u)\phi(u)+T^{3/2}\phi_0(u)^{1-d/4}\phi(u)^{1+d/4} \\ &\quad+T\phi_0(u)^{2-d/4}\phi(u)^{d/4}+T^{3/2-d/4}\phi(u)^2. \end{aligned} \tag{4.33}$$

Since M_{2122} is nearly identical to M_{2121}, the right side of (4.33) gives us an estimate for M_{212}. Finally, using that $d\in\{2,3\}$, we add the estimates (4.26), (4.29), and (4.33) to get (4.25) in Lemma 4.5. □

4.4. Terms with all mixed derivatives. In this section we estimate M_1.

LEMMA 4.6. *For solutions u of the Cauchy problem (1.1)–(1.3), we have*

$$M_1 \lesssim \phi_0(u)^{3/2}\phi(u)^{1/2} + T^{1/2}\phi_0(u)\phi(u) + T^{3/2}\phi(u)^2 \tag{4.34}$$

for all $0 < T \le 1$.

PROOF OF LEMMA 4.6. Using the Leibniz rule we obtain

$$\begin{aligned}M_1 \le &\sum_{i+j+k\ge 1}\sum_{\ell=0}^{i}\sum_{n=0}^{j}\sum_{m=0}^{k}\binom{i}{\ell}\binom{j}{n}\binom{k}{m}N_{i+j+k+2}\epsilon^i\tilde{\epsilon}^j\bar{\epsilon}^k\\ &\times\Big(\|t^{i+j+k-1}\partial_t^\ell\bar{\partial}_d^n\partial^m\bar{u}\cdot\partial_t^{i-\ell}\partial_d^{j-n}\bar{\partial}^{k-m+1}u\|_{L^2_{x,t}} + \|t^{i+j+k-1}\partial_t^\ell\partial_d^n\bar{\partial}^m u_d\,\partial_t^{i-\ell}\partial_d^{j-n+1}\bar{\partial}^{k-m}u\|_{L^2_{x,t}}\Big).\end{aligned}$$

We separate the case $|(i,j,k)| = 1$ from the sum and then split the rest into two parts, leading to

$$\begin{aligned}M_1 \lesssim &\sum_{i+j+k=1}\Big(\|u\|_{L^\infty_{x,t}}\|\nabla\partial_t^i\partial_d^j\bar{\partial}^k u\|_{L^2_{x,t}} + \|\partial_t^i\partial_d^j\bar{\partial}^k u\|_{L^\infty_t L^4_x}\|\nabla u\|_{L^2_t L^4_x}\Big)\\ &+\sum_{i+j+k\ge 2}\sum_{|(\ell,n,m)|=0}^{\lfloor (i+j+k)/2\rfloor}\binom{i}{\ell}\binom{j}{n}\binom{k}{m}N_{i+j+k+2}\epsilon^i\tilde{\epsilon}^j\bar{\epsilon}^k\|t^{\ell+n+m}\partial_t^\ell\partial_d^n\bar{\partial}^m u\|_{L^\infty_{x,t}}\\ &\quad\times\Big(\|t^{(i+j+k)-(\ell+n+m)-1}\partial_t^{i-\ell}\partial_d^{j-n}\bar{\partial}^{k-m+1}u\|_{L^2_{x,t}}\\ &\qquad + \|t^{(i+j+k)-(\ell+n+m)-1}\partial_t^{i-\ell}\partial_d^{j-n+1}\bar{\partial}^{k-m}u\|_{L^2_{x,t}}\Big)\\ &+\sum_{i+j+k\ge 2}\sum_{|(\ell,n,m)|=\lceil (i+j+k)/2\rceil}^{i+j+k}\binom{i}{\ell}\binom{j}{n}\binom{k}{m}N_{i+j+k+2}\epsilon^i\tilde{\epsilon}^j\bar{\epsilon}^k\|t^{\ell+n+m-1}\partial_t^\ell\partial_d^n\bar{\partial}^m u\|_{L^\infty_t L^4_x}\\ &\quad\times\Big(\|t^{(i+j+k)-(\ell+n+m)}\partial_t^{i-\ell}\partial_d^{j-n}\bar{\partial}^{k-m+1}u\|_{L^2_t L^4_x}\\ &\qquad + \|t^{(i+j+k)-(\ell+n+m)}\partial_t^{i-\ell}\partial_d^{j-n+1}\bar{\partial}^{k-m}u\|_{L^2_t L^4_x}\Big)\\ &= M_{11} + M_{12} + M_{13}.\end{aligned}$$

The contribution from $|(i,j,k)| = 1$ is bounded as

$$M_{11} \lesssim \phi_0(u)^2 + \phi_0(u)^{3/2}\phi(u)^{1/2}. \tag{4.35}$$

For M_{12} and M_{13}, we follow the same strategy as in the last section. Starting with M_{12}, we apply Lemma 4.2 to estimate $\|t^{\ell+n+m}\partial_t^\ell\partial_d^n\bar{\partial}^m u\|_{L^\infty_{x,t}}$ in terms of the analyticity norm (4.2). Denote

$$\begin{aligned}B^{i,j,k}_{\ell,n,m}(u) = &\|t^{(i+j+k)-(\ell+n+m)-1}\partial_t^{i-\ell}\partial_d^{j-n}\bar{\partial}^{k-m+1}u\|_{L^2_{x,t}}\\ &+ \|t^{(i+j+k)-(\ell+n+m)-1}\partial_t^{i-\ell}\partial_d^{j-n+1}\bar{\partial}^{k-m}u\|_{L^2_{x,t}}.\end{aligned}$$

Next, using the notation (4.1)

$$\begin{aligned}B^{i,j,k}_{\ell,n,m}(u) \lesssim &(T\mathbb{1}_{\ell+n+m\le i+j+k-2} + \mathbb{1}_{\ell+n+m=i+j+k-1})\frac{(i+j+k-\ell-n-m+1)!}{(i+j+k-\ell-n-m+1)^3}\\ &\times\Big(\frac{U_{i-\ell,j-n,k-m+1}}{\epsilon^{i-\ell}\tilde{\epsilon}^{j-n}\bar{\epsilon}^{k-m}} + \frac{U_{i-\ell,j-n+1,k-m}}{\epsilon^{i-\ell}\tilde{\epsilon}^{j-n}\bar{\epsilon}^{k-m}}\Big).\end{aligned}$$

Estimating $\|t^{\ell+n+m}\partial_t^\ell\bar\partial_d^n\bar\partial^m u\|_{L^\infty_{x,t}}$, the factorial terms obey

$$\binom{i}{\ell}\binom{j}{n}\binom{k}{m}\frac{(\ell+n+m)!}{(i+j+k)!}\frac{(i+j+k-\ell-n-m)!}{(i+j+k-\ell-n-m)^2}\frac{1}{(\ell+n+m)^{1/2}}\leq 1.$$

Therefore,

$$\begin{aligned}M_{12}&\lesssim T\phi_0(u)\bar\phi(u)+T\phi_0(u)^{1/2}\bar\phi(u)^{3/2}+T^{1/2}\phi_0(u)\bar\phi(u)+\phi_0(u)^2\\&\quad+T^{3/2}\bar\phi(u)^2+T\phi_0^{1/2}\bar\phi(u)^{3/2}+T\phi_0(u)\bar\phi(u)\\&\lesssim\phi_0(u)^2+T^{1/2}\phi_0(u)\phi(u)+T^{3/2}\phi(u)^2.\end{aligned}\tag{4.36}$$

Lastly, we deal with M_{13}. Similarly to M_{212}, we split M_{13} into two parts

$$\begin{aligned}M_{13}\lesssim&\sum_{i+j+k\geq 2}\sum_{|(\ell,n,m)|=\lceil(i+j+k)/2\rceil}^{i+j+k}\binom{i}{\ell}\binom{j}{n}\binom{k}{m}N_{i+j+k+2}\epsilon^i\tilde\epsilon^j\bar\epsilon^k\|t^{\ell+n+m-1}\partial_t^\ell\bar\partial_d^n\bar\partial^m u\|_{L_t^\infty L_x^4}\\&\quad\times\left(\|t^{(i+j+k)-(\ell+n+m)}\partial_t^{i-\ell}\partial_d^{j-n}\bar\partial^{k-m+1}u\|_{L_t^2L_x^4}\right)\\&\quad+\sum_{i+j+k\geq 2}\sum_{|(\ell,n,m)|=\lceil(i+j+k)/2\rceil}^{i+j+k}\binom{i}{\ell}\binom{j}{n}\binom{k}{m}N_{i+j+k+2}\epsilon^i\tilde\epsilon^j\bar\epsilon^k\|t^{\ell+n+m-1}\partial_t^\ell\partial_d^n\bar\partial^m u\|_{L_t^\infty L_x^4}\\&\quad\times\left(\|t^{(i+j+k)-(\ell+n+m)}\partial_t^{i-\ell}\partial_d^{j-n+1}\bar\partial^{k-m}u\|_{L_t^2L_x^4}\right)\\&\quad=M_{131}+M_{132},\end{aligned}\tag{4.37}$$

and consider M_{131}. Denote

$$A^{i,j,k}_{\ell,n,m}(u)=\left(\|t^{(i+j+k)-(\ell+n+m)}\partial_t^{i-\ell}\partial_d^{j-n}\bar\partial^{k-m+1}u\|_{L_t^2L_x^4}\right).$$

Using (4.5)–(4.6), and (4.2) we bound $A^{i,j,k}_{\ell,n,m}(u)$ from above by

$$\begin{aligned}A^{i,j,k}_{\ell,n,m}(u)&\lesssim\left(U_{i-\ell,j-n+1,k-m+1}+U_{i-\ell,j-n,k-m+2}\right)^{d/4}\left(U_{i-\ell,j-n,k-m+2}\right)^{1-d/4}\\&\quad\times\left(\frac{(i+j+k-\ell-n-m)!}{(i+j+k-\ell-n-m+1)^{2-d/4}}\right)\\&\quad\times\frac{1}{\epsilon^{i-\ell}\tilde\epsilon^{j-n}\bar\epsilon^k}\left(T^{2-d/4}\mathbb{1}_{\ell+n+m\leq i+j+k-2}+T^{i+j+k-\ell-n-m}\mathbb{1}_{\ell+n+m\geq i+j+k-1}\right)\\&\quad+U_{i-\ell,j-n,k-m+1}\frac{(i+j+k-\ell-n-m)!}{(i+j+k-\ell-n-m+1)^2}\frac{1}{\epsilon^{i-\ell}\tilde\epsilon^{j-n}\bar\epsilon^k}\\&\quad\times\left(T^2\mathbb{1}_{\ell+n+m\leq i+j+k-2}+T^{i+j+k-\ell-n-m}\mathbb{1}_{\ell+n+m\geq i+j+k-1}\right).\end{aligned}$$

Applying Lemma 4.3 to $\|t^{\ell+n+m-1}\partial_t^\ell\partial_d^n\bar\partial^m u\|_{L_t^\infty L_x^4}$ and using Young's inequality, we obtain

$$\begin{aligned}M_{131}&\lesssim(\phi_0(u)+T^{1/2}\bar\phi(u))\left(\phi_0(u)+T\phi_0(u)^{1-d/4}\bar\phi(u)^{d/4}\right)\\&\quad+T^{1/2}\bar\phi(u)\left(\phi_0(u)+T\phi_0(u)^{1-d/4}\bar\phi(u)^{d/4}+T^{2-d/4}\bar\phi(u)\right).\end{aligned}\tag{4.38}$$

Note that in the above line we use $T\leq 1$. Comparing the two sums M_{131} and M_{132} in (4.37), we observe that they have the same prefactor $t^{i+j+k-\ell-n-m}$ and the same total number of derivatives. As a result, M_{132}

too is dominated by the right hand side of (4.38). Selecting the maximal coefficients in T, we write

$$
\begin{aligned}
M_{13} \lesssim \phi_0(u)^2 + T\phi_0(u)^{2-d/4}\phi(u)^{d/4} + T^{1/2}\phi_0(u)\phi(u) \\
+ T^{3/2}\phi_0(u)^{1-d/4}\phi(u)^{1+d/4} + T^{5/2-d/4}\phi(u)^2.
\end{aligned}
\tag{4.39}
$$

Adding (4.35), (4.36), and (4.39), we arrive at the conclusion in Lemma 4.6. □

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References

[1] A. Biswas. Local existence and Gevrey regularity of 3-D Navier-Stokes equations with l_p initial data. *J. Differential Equations*, 215(2):429–447, 2005.

[2] A. Biswas and C. Foias. On the maximal space analyticity radius for the 3D Navier-Stokes equations and energy cascades. *Ann. Mat. Pura Appl. (4)*, 193(3):739–777, 2014.

[3] A. Biswas and D. Swanson. Gevrey regularity of solutions to the 3-D Navier-Stokes equations with weighted l_p initial data. *Indiana Univ. Math. J.*, 56(3):1157–1188, 2007.

[4] A. Biswas and E. Tadmor. Dissipation versus quadratic nonlinearity: from a priori energy bound to higher order regularizing effect. *Nonlinearity*, 27(3):545–562, 2014.

[5] J.L. Bona, Z. Grujić, and H. Kalisch. Algebraic lower bounds for the uniform radius of spatial analyticity for the generalized KdV equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 22(6):783–797, 2005.

[6] Z. Bradshaw, Z. Grujić, and I. Kukavica. Local analyticity radii of solutions to the 3D Navier-Stokes equations with locally analytic forcing. *J. Differential Equations*, 259(8):3955–3975, 2015.

[7] P. Constantin and C. Foias. *Navier-Stokes equations*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1988.

[8] A.B. Ferrari and E.S. Titi. Gevrey regularity for nonlinear analytic parabolic equations. *Comm. Partial Differential Equations*, 23(1-2):1–16, 1998.

[9] C. Foias, M.S. Jolly, R. Lan, R. Rupam, Y. Yang, and B. Zhang. Time analyticity with higher norm estimates for the 2D Navier-Stokes equations. *IMA J. Appl. Math.*, 80(3):766–810, 2015.

[10] C. Foias and R. Temam. Gevrey class regularity for the solutions of the Navier-Stokes equations. *J. Funct. Anal.*, 87(2):359–369, 1989.

[11] Y. Giga. Time and spatial analyticity of solutions of the Navier-Stokes equations. *Comm. Partial Differential Equations*, 8(8):929–948, 1983.

[12] Z. Grujić and H. Kalisch. Local well-posedness of the generalized Korteweg-de Vries equation in spaces of analytic functions. *Differential Integral Equations*, 15(11):1325–1334, 2002.

[13] Z. Grujić and I. Kukavica. Space analyticity for the Navier-Stokes and related equations with initial data in L^p. *J. Funct. Anal.*, 152(2):447–466, 1998.

[14] Z. Grujić and I. Kukavica. Space analyticity for the nonlinear heat equation in a bounded domain. *J. Differential Equations*, 154(1):42–54, 1999.

[15] T. Kato and K. Masuda. Nonlinear evolution equations and analyticity. I. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 3(6):455–467, 1986.

[16] D. Kinderlehrer and L. Nirenberg. Analyticity at the boundary of solutions of nonlinear second-order parabolic equations. *Comm. Pure Appl. Math.*, 31(3):283–338, 1978.

[17] G. Komatsu. Analyticity up to the boundary of solutions of nonlinear parabolic equations. *Comm. Pure Appl. Math.*, 32(5):669–720, 1979.

[18] G. Komatsu. Global analyticity up to the boundary of solutions of the Navier-Stokes equation. *Comm. Pure Appl. Math.*, 33(4):545–566, 1980.

[19] I. Kukavica and V. Vicol. On the radius of analyticity of solutions to the three-dimensional Euler equations. *Proc. Amer. Math. Soc.*, 137(2):669–677, 2009.

[20] I. Kukavica and V. Vicol. The domain of analyticity of solutions to the three-dimensional Euler equations in a half space. *Discrete Contin. Dyn. Syst.*, 29(1):285–303, 2011.

[21] I. Kukavica and V. Vicol. Direct approach to Gevrey regularity on the half-space. *Partial differential equations in fluid mechanics*, edited by C. L. Fefferman, J. C. Robinson, and J. L. Rodrigo, to appear, 2018.

[22] P.G. Lemarié-Rieusset. Une remarque sur l'analyticité des solutions milds des équations de Navier-Stokes dans $\mathbf{R}^3$. *C. R. Acad. Sci. Paris Sér. I Math.*, 330(3):183–186, 2000.

[23] P.-L. Lee and Y.-J. Lin Guo. Gevrey class regularity for parabolic equations. *Differential Integral Equations*, 14(8):989–1004, 2001.

[24] J.-L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications. Vol. II*. Springer-Verlag, New York, 1972. Translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 182.

[25] C.D. Levermore and M. Oliver. Analyticity of solutions for a generalized Euler equation. *J. Differential Equations*, 133(2):321–339, 1997.

[26] K. Masuda. On the analyticity and the unique continuation theorem for solutions of the Navier-Stokes equation. *Proc. Japan Acad.*, 43:827–832, 1967.

[27] M. Oliver and E.S. Titi. On the domain of analyticity of solutions of second order analytic nonlinear differential equations. *J. Differential Equations*, 174(1):55–74, 2001.

[28] M. Oliver and E.S. Titi. Remark on the rate of decay of higher order derivatives for solutions to the Navier-Stokes equations in R^n. *J. Funct. Anal.*, 172(1):1–18, 2000.

[29] J. Prüss, J. Saal, and G. Simonett. Existence of analytic solutions for the classical Stefan problem. *Math. Ann.*, 338(3):703–755, 2007.

[30] J.C. Robinson, J. L. Rodrigo, and W. Sadowski. *The three-dimensional Navier-Stokes equations*, volume 157 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2016. Classical theory.

[31] M. Sammartino and R.E. Caflisch. Zero viscosity limit for analytic solutions, of the Navier-Stokes equation on a half-space. I. Existence for Euler and Prandtl equations. *Comm. Math. Phys.*, 192(2):433–461, 1998.

[32] J. Serrin. On the interior regularity of weak solutions of the Navier-Stokes equations. *Arch. Rational Mech. Anal.*, 9:187–195, 1962.

[33] V. A. Solonnikov. Estimates for solutions of nonstationary Navier-Stokes equations. *Journal of Mathematical Sciences*, 8(4): 467-529, 1977.

[34] R. Temam. *Navier-Stokes equations*. AMS Chelsea Publishing, Providence, RI, 2001. Theory and numerical analysis, Reprint of the 1984 edition.

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