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Heavy Flavor Wilson Coefficients in Deep-Inelastic Scattering: Recent Results*

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We present recent analytic results for the 3-loop corrections to the massive operator matrix element $A_{Qg}^{(3)}$ for further color factors. These results have been obtained using the method of arbitrarily large moments. We also give an overview on the results which were obtained solving all difference and differential equations for the corresponding master integrals that factorize at first order.

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1. Introduction

The precision determination of the strong coupling constant $\alpha_s(M_Z^2)$ [1], of the parton distribution functions [2], and of the heavy quark masses m_c and m_b [3] from the world deep inelastic data requires both the knowledge of the NNLO massless and massive QCD corrections to the parton densities. Theses quantities are fundamental parameters of the Standard Model or serve as an important input for other measurements at hadron colliders to determine further fundamental parameters, like those of the top-quark [4] and the Higgs sectors [5]. These are important places at present to find deviations from the Standard Model and therefore need the most precise possible description.

During the last years, essential progress has been made in calculating the massive 3-loop corrections to the deep-inelastic structure functions [6–21]. Different computational methods have been worked out in Refs. [22–24]. Progress has also been made in the 2-mass case, where several OMEs have been calculated, cf. [25–28]. For a recent summary on this see Ref. [29].

All but one of the massive operator matrix elements (OMEs) have been calculated. In this note we describe recent developments of the project concerning the calculation of the OME $A_{Qg}^{(3)}$, as well as some technical details of the calculation, the present results, and challenges for its completion.

The paper is organized as follows. In Section 2 we describe the method of arbitrarily large moments and calculation techniques for first order factorizing differential and difference equations. The contributions to $A_{Qg}^{(3)}$ that have been obtained already by using these methods are described in Section 3. There we also give an outline on the methods to be used to calculate the remaining terms, which are related to new functions, the iterative non–iterative integrals [30]. Section 4 contains the conclusions.

2. The method of arbitrarily large moments and calculation techniques for first order factorizing equations

In the calculation of the massive OMEs, we reduce the Feynman diagrams to master integrals using integration by parts (IBP) relations [31]. Depending on their complexity, different calculation methods exist to compute these integrals. The simplest integrals can be expressed in terms of (generalized) hypergeometric functions [32]. In more general cases, one seeks Mellin-Barnes representations [33] and uses the residue theorem to obtain convergent infinite sum representations. Such sums are also obtained in the case of the representation through hypergeometric functions after expanding in the dimensional parameter ε. In Mellin N space, also sum representations are obtained. For finite integrals, one may also use the method of hyperlogarithms [23, 34–36]. Having full control on the IBP reduction, one may map divergent integrals into convergent ones [37]. Another powerful tool consists in the method of differential equations, cf. Refs. [38] and [24]. In the present case, they can be obtained from the IBP relations by differentiating w.r.t. the resummation parameter x for the operator insertions, cf. [14]. Inserting the formal power series in x into the differential equations, one obtains recurrence relations, which, as well as the summation problems mentioned before, can be solved by difference field and ring methods [39–47] using the algorithms implemented in the packages `Sigma` [48, 49], `EvaluateMultiSums` and `SumProduction` [50]. In various places the package `HarmonicSums` [51–55] is used to operate on special functions of

different kinds emerging throughout the calculation. In some cases we also use multi-integration according to the Almkvist-Zeilberger theorem [56] as an integration method. This method is implemented in the package `MultiIntegrate` [51]. All these methods have been described in detail in Ref. [24]. In all cases where difference equations factorize at first order, `Sigma` will find the solution in Mellin N space. The transition to x-space is possible applying algorithms implemented in the package `HarmonicSums`.

As it is well known, massive single-scale problems starting at 3-loop order are partly related to master integrals for which the corresponding difference and differential equations do not factorize in first order, [30]. Yet cases in which this applies to the master integrals, but not to their sum projected on some color-ζ factors, i.e. terms which can be distinguished by the corresponding product of Casimir operators and potential (multiple) zeta values [57], may be dealt with. Considering fixed moments N, cf. also [7], the results are given by rational numbers and multiple zeta values only [57]. The appearance of new higher functions like iterated non–iterative integrals manifests only in the corresponding rational sequences but not in new special numbers. These sequences can now be analyzed and the difference equations which describe them can be determined [58, 59] provided enough Mellin moments can be calculated. This is possible using the algorithm [60], implemented in the package `SolveCoupledSystem`, that can compute arbitrarily large moments. The corresponding difference equations can now be analyzed by the package `Sigma`, which will determine all its 1st order factors, and in the present case, a remainder term not factorizing in first order. We will illustrate this in the case of the unrenormalized OME $\hat{\hat{A}}_{Qg}^{(3)}$ in Section 3.

3. The contributions to $A_{Qg}^{(3)}$ through first order factorizing equations

In the following we calculate 3-loop contributions to the massive OME $A_{Qg}^{(3)}$ in the single heavy quark mass case. The 1358 Feynman diagrams are generated using `QGRAF` [61] and are brought into a form in which the operator insertions [7] are resummed into a propagator, cf. [14], using a formal resummation parameter x. The color algebra is performed using `Color` [62]. We use `Reduze2` [63][1] to map the problem to 340 master integrals. 224 master integrals were calculated using the methods described in [24]. We derive differential equations for all master integrals, which are turned into associated difference equations by the formal power series in x [66]. For these difference equations we use now the technique described in Ref. [24] and calculate the master integrals for fixed moments N, which are inserted into the expression of the unrenormalized OME, see Eq. (4.37) of Ref. [7]. Iteratively, we calculate higher and higher moments. For the complete expression we have calculated 2000 moments and for the contribution with the color factor $O(T_F^2 C_{A,F})$, 8000 moments. The method of guessing [58, 59] can now be used to find a difference equation for the different terms according to their contribution in the Laurent series in ε and their color and ζ-value factors.

We first have considered the $O(1/\varepsilon)$ term and extracted the 3-loop anomalous dimension $\gamma_{qg}^{(2)}(N)$ in Ref. [67] solving the difference equations obtained by guessing using the packages `Sigma` [48, 49] `EvaluatMultiSum` and `SumProduction` [50] and by simplifying the corre-

[1] The package uses the codes `GiNaC` [64] and `Fermat` [65].

sponding expressions using the package `HarmonicSums` [51–55]. This is the first independent recalculation of this anomalous dimension since it was first computed in [68].

We turn now to the $O(\varepsilon^0)$ terms. In the case of the $O(T_F^2 C_{A,F})$ contributions, the method of guessing has led to difference equations in all cases. The 2000 moments for the remaining color and zeta values were not enough, however, to determine difference equations for the pure color factors and those $\propto \zeta_3$, except for the N_F terms, for which the number of moments had been sufficient and which also had been calculated before in Ref. [8]. Analyzing the terms with pure color factors and those $\propto \zeta_3$ for the contributions $O(T_F^2 C_{A,F})$ using `Sigma`, it turned out that the corresponding difference equations do not factorize in first order completely. Already here we speak of difference equations of order $o = 45$ and degree $d \sim 1500$. They could be reduced to low order non-factorizing difference equations. The reason for this is that, as we know through other analyses, elliptic parts are contained in these cases and one expects iterative non-iterative integral solutions here, cf. [30].

For quite a series of color and ζ-factors, namely all but 10, the constant part of the unrenormalized OME $\hat{A}_{Qg}^{(3)}$, $a_{Qg}^{(3)}(N)$, could be calculated using the above methods. It is given by

$$
\begin{aligned}
a_{Qg}^{(3)}(N) \;=\; & C_A^2 T_F \Bigg\{ t_{C_A^2 T_F}(N) + t_{C_A^2 T_F \zeta_3}(N)\zeta_3 \\
& + \frac{72(-2+N)(3+N)P_2}{(N-1)N^2(1+N)^2(2+N)^2}\zeta_4 - \frac{4P_{17}}{(N-1)N^2(1+N)^2(2+N)^2}\mathsf{B}_4 \\
& + p_{qg}^{(0)}\Bigg[-16\mathsf{B}_4 S_1 + 144\zeta_4 S_1 + \Bigg(-16S_1^3 - \frac{4P_7}{3(N-1)N(1+N)(2+N)}S_2 - 32S_1 S_2 \\
& - 8S_3 + \Bigg(-\frac{8P_8}{3(N-1)N(1+N)(2+N)} - 48S_1 \Bigg) S_{-2} - 8S_{-3} + 16S_{-2,1} \Bigg)\zeta_2 \Bigg] \\
& + \Bigg[\frac{2P_{35}}{9(N-1)^2N^4(1+N)^4(2+N)^4} + \frac{4P_{31}}{9(N-1)^2N^3(1+N)^3(2+N)^3}S_1 \\
& - \frac{4P_{18}}{3(N-1)N^2(1+N)^2(2+N)^2}S_1^2 \Bigg]\zeta_2 \Bigg\} + C_A T_F^2 \Bigg\{ t_{C_A T_F^2}(N) + t_{C_A T_F^2 \zeta_3}(N)\zeta_3 \\
& + N_F \Bigg\{ -\frac{8P_{36}}{243(N-1)N^5(1+N)^5(2+N)^5} + p_{qg}^{(0)}\Bigg[\Bigg(\frac{1888}{27}S_3 + \frac{224}{9}S_{2,1} \Bigg) S_1 \\
& + \frac{32}{27}S_1^4 + \frac{176}{9}S_1^2 S_2 + \frac{80}{9}S_2^2 + \frac{640}{9}S_4 + \Bigg(-\frac{64(2N-1)}{(N-1)N}S_1 + \frac{128}{3}S_2 \Bigg) S_{-2} + \frac{64}{9}S_{-4} \\
& - \frac{32}{3}S_{1,-3} - \frac{64}{3}S_{2,-2} - \frac{32}{9}S_{3,1} + \frac{64(2N-1)}{(N-1)N}S_{-2,1} + 64S_{1,1,-2} - \frac{416}{9}S_{2,1,1} \\
& + \Bigg(\frac{16}{3}S_1^2 + \frac{16}{3}S_2 + \frac{32}{3}S_{-2} \Bigg)\zeta_2 + \Bigg(\frac{448\left(1+N+N^2\right)}{9(N-1)N(1+N)(2+N)} - \frac{224}{9}S_1 \Bigg)\zeta_3 \Bigg] \\
& + \Bigg(\frac{16P_{32}}{243(N-1)N^2(1+N)^4(2+N)^4} - \frac{16P_{10}}{27N(1+N)^2(2+N)^2}S_2 \Bigg) S_1 \\
& + \frac{8P_{20}}{81N(1+N)^3(2+N)^3}S_1^2 - \frac{16P_9}{81N(1+N)^2(2+N)^2}S_1^3
\end{aligned}
$$

$$+\frac{8P_{26}}{81(N-1)N^3(1+N)^3(2+N)^3}S_2-\frac{32P_{22}}{81(N-1)N^2(1+N)^2(2+N)^2}S_3$$
$$+\frac{32P_{21}}{81N(1+N)^3(2+N)^3}S_{-2}+\frac{32P_{15}}{27(N-1)N^2(1+N)^2(2+N)^2}S_{-3}$$
$$-\frac{64P_{16}}{9(N-1)N^2(1+N)^2(2+N)^2}S_{1,-2}-\frac{64P_4}{27N(1+N)^2(2+N)^2}S_{2,1}$$
$$+\left(-\frac{4P_{27}}{9(N-1)N^3(1+N)^3(2+N)^3}-\frac{16P_5}{9N(1+N)^2(2+N)^2}S_1\right)\zeta_2\Bigg\}$$
$$+\left(-\frac{4P_{28}}{9(N-1)N^3(1+N)^3(2+N)^3}+\frac{160\left(4-N+N^2+4N^3+N^4\right)}{9N(1+N)^2(2+N)^2}S_1\right)\zeta_2$$
$$+p_{qg}^{(0)}\left(\frac{40}{3}S_1^2+\frac{40}{3}S_2+\frac{80}{3}S_{-2}\right)\zeta_2\Bigg\}+C_F^2T_F\Bigg\{t_{C_F^2T_F}(N)+t_{C_F^2T_F\zeta_3}(N)\zeta_3$$
$$-\frac{16(N-1)\left(-2+3N+3N^2\right)}{N^2(1+N)^2}\mathsf{B}_4+\frac{72(N-1)\left(-2+3N+3N^2\right)}{N^2(1+N)^2}\zeta_4$$
$$+\left[\frac{P_{23}}{2N^4(1+N)^4(2+N)}+\frac{8P_{19}}{N^3(1+N)^3(2+N)}S_1+\frac{4P_3}{N^2(1+N)^2(2+N)}S_1^2\right]\zeta_2$$
$$+p_{qg}^{(0)}\Bigg[-16S_1^3-\frac{8\left(2+3N+3N^2\right)}{N(1+N)}S_2+32S_1S_2+16S_3$$
$$+\left(-\frac{16}{N(1+N)}+32S_1\right)S_{-2}+16S_{-3}-32S_{-2,1}\Bigg]\zeta_2\Bigg\}+C_F\Bigg\{C_AT_F\Bigg\{t_{C_FC_AT_F}(N)+$$
$$t_{C_FC_AT_F\zeta_3}(N)\zeta_3+\frac{32P_{13}}{(N-1)N^2(1+N)^2(2+N)^2}\left(\mathsf{B}_4-\frac{9}{2}\zeta_4\right)$$
$$+p_{qg}^{(0)}\Bigg[32\mathsf{B}_4S_1-144\zeta_4S_1+\Bigg(32S_1^3-\frac{12P_1}{(N-1)N(1+N)(2+N)}S_2-8S_3$$
$$+\left(-\frac{8\left(1+3N+3N^2\right)}{N(1+N)}+16S_1\right)S_{-2}-8S_{-3}+16S_{-2,1}\Bigg)\zeta_2\Bigg]$$
$$+\Bigg(\frac{P_{25}}{18(N-1)N^3(1+N)^3(2+N)^3}-\frac{4P_{29}}{9(N-1)N^3(1+N)^3(2+N)^3}S_1$$
$$+\frac{8P_{12}}{3(N-1)N^2(1+N)^2(2+N)^2}S_1^2\Bigg)\zeta_2\Bigg\}+T_F^2\Bigg\{t_{C_FT_F^2}(N)+t_{C_FT_F^2\zeta_3}(N)\zeta_3$$
$$+N_F\Bigg\{\frac{P_{37}}{243(-1+N)N^6(1+N)^6(2+N)^5}+p_{qg}^{(0)}\Bigg[\left(-\frac{256}{27}S_3-\frac{128}{3}S_{2,1}\right)S_1$$
$$-\frac{32}{27}S_1^4-\frac{64}{9}S_1^2S_2-\frac{128}{9}S_2^2+\frac{256}{9}S_4-\frac{128}{3}S_{3,1}+\frac{256}{3}S_{2,1,1}-\frac{16}{3}S_1^2\zeta_2$$
$$+\left(-\frac{56P_{14}}{9(-1+N)N^2(1+N)^2(2+N)}+\frac{224}{9}S_1\right)\zeta_3\Bigg]+\left(-\frac{16P_{11}}{243N^2(1+N)^3(2+N)}\right.$$

$$+\frac{32\big(24+83N+49N^2+10N^3\big)}{27N^2(1+N)(2+N)}S_2\Bigg)S_1-\frac{32P_6}{81N^2(1+N)^2(2+N)}S_1^2$$
$$+\frac{32\big(24+83N+49N^2+10N^3\big)}{81N^2(1+N)(2+N)}S_1^3+\frac{8P_{33}}{27(-1+N)N^4(1+N)^4(2+N)^3}S_2$$
$$-\frac{16P_{24}}{81(-1+N)N^3(1+N)^3(2+N)^2}S_3-\frac{128\big(-2-3N+N^2\big)}{3N^2(1+N)(2+N)}S_{2,1}$$
$$+\left(\frac{2(-2+N)P_{30}}{9(-1+N)N^4(1+N)^4(2+N)^3}+\frac{16\big(12+28N+11N^2+5N^3\big)}{9N^2(1+N)(2+N)}S_1\right)\zeta_2\Bigg\}$$
$$+\left(\frac{2P_{34}}{9(-1+N)N^4(1+N)^4(2+N)^3}+\frac{80\big(6+11N+4N^2+N^3\big)}{9N^2(1+N)(2+N)}S_1\right)\zeta_2$$
$$+p_{qg}^{(0)}\left(-\frac{40}{3}S_1^2+8S_2\right)\zeta_2\Bigg\}\Bigg\}-\frac{64}{9}p_{qg}^{(0)}T_F^3\zeta_3. \tag{3.1}$$

Here we used the shorthand notation

$$p_{qg}^{(0)}(N)=\frac{N^2+N+2}{(N+2)(N+1)N}, \tag{3.2}$$

$S_{\vec{a}}$ denote the nested harmonic sums [69]

$$S_{b,\vec{a}}\equiv S_{b,\vec{a}}(N)=\sum_{k=1}^{N}\frac{(\mathrm{sign}(b))^k}{k^{|b|}}S_{\vec{a}}(k),\quad S_{\emptyset}=1,\ \ b,a_i\in\mathbb{Z}\backslash\{0\}, \tag{3.3}$$

the constant B_4 is

$$\mathsf{B}_4=-4\zeta_2\ln^2(2)+\frac{2}{3}\ln^4(2)-\frac{13}{2}\zeta_4+16\mathrm{Li}_4\left(\frac{1}{2}\right), \tag{3.4}$$

and P_i denote polynomials in N (which have be computed explicitly), and the functions $t_j(N)$ have still to be calculated. Using the methods of Ref. [24], we have calculated a lot more Feynman diagrams for the color factors contributing to $\hat{\hat{A}}_{Qg}^{(3)}$, which could not yet been gotten using the method of arbitrarily large moments, so that 1122 of 1358 diagrams have been calculated by now.

4. Conclusions

Advanced methods in calculating Feynman and master integrals for massive 3-loop OMEs allowed us already to compute a significant part of the contributions to $a_{Qg}^{(3)}$ analytically. Our toolbox employs various methods [24], and relies in particular on very efficient solvers based on difference field and ring theory [39–47]. These algorithms have been implemented in the packages `Sigma` [48,49], `EvaluateMultiSums` and `SumProduction` [50]. A very efficient treatment of special functions has been possible using the package `HarmonicSums` [51–55]. The method of arbitrarily large moments [60] could now be used to calculate the 3-loop anomalous dimension $\gamma_{qg}^{(2)}(N)$ automatically from first principles even in the more involved massive environment [67]. The same method allowed to calculate the contributions to the $O(\varepsilon^0)$ term $a_{Qg}^{(3)}$ for 18 out of 28 of the color-ζ terms in analytic form.

Including the use of other methods described in [24], 1122 of 1358 Feynman diagrams contributing to $A_{Qg}^{(3)}$ at $O(\varepsilon^0)$ have been calculated. The remaining terms contain iterative non-iterative integrals over also elliptic letters and are currently being computed.

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