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Non–duality in three dimensions

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Abstract: We investigate M-theory and heterotic compactifications to 7 and 3 dimensions. In 7 dimensions we discuss a class of massive supergravities that arise from M-theory on K3 and point out obstructions to realizing these theories in a dual heterotic framework with a geometric description. Taking M-theory further down to 3 dimensions on K3 $\times$ K3 with a choice of flux leads to a rich landscape of theories with various amounts of supersymmetry, including those preserving 6 supercharges. We explore possible heterotic realizations of these vacua and prove a no–go theorem: every heterotic geometry that preserves 6 supercharges preserves 8 supercharges.

Contents

1 Introduction

Ever since its discovery [1], the seven-dimensional duality between M-theory compactified on K3 and the heterotic string compactified on T^3 has played an essential role in our understanding of string theory. In its most prosaic usage, this duality, together with its F-theory counterpart [2], have been used to produce and study many dual pairs of theories. The simplest way to generate such pairs is to compactify the resulting seven-dimensional theory on some common geometry; however, a much richer class of theories can be obtained by performing a fiber-wise duality. A classic example of this correspondence is between M-theory on K3 $\times$ K3 and heterotic flux vacua with target space a principal T^3 bundle over K3 [3]. Depending on the choice of G-flux on the M-theory side, these may have an F-theory lift with a corresponding heterotic vacuum that is a principal T^2 bundle over K3. The latter have been the focus of much attention, e.g. [4–7], and remain the sole compact examples of heterotic flux vacua.

The class of four- or three-dimensional vacua just discussed is particularly simple, and with a sufficient amount of supersymmetry, it should be possible to work out the duality map in some detail. However, these solutions are by no means exhaustive. Surprisingly, there are many M-theory vacua on K3 $\times$ K3 that are more challenging to describe from a heterotic perspective. The main aims of this work are to present these vacua, describe their basic features, and to point out the challenges in finding corresponding heterotic descriptions.

Given that the M-theory geometry $M_8 = \text{K3} \times \text{K3}$ is so simple, the reader will not be surprised that it is the choice of G-flux that is responsible for the extra complications. It is possible, while preserving $\mathcal{N} = 1, 2, 3$ three-dimensional super-Poincaré invariance[1], to choose G with a component that threads the volumes of the two K3 factors. Such a volume-threading flux automatically obstructs a lift to M-theory on K3 $\times \mathbb{R}^{1,6}$, so that we cannot apply a standard duality with heterotic string on $T^3 \times \mathbb{R}^{1,6}$.

On the other hand, given that we do have the $3d$ solutions with this sort of volume-threading flux, it is clear that there exists a supergravity theory in seven dimensions obtained by reducing M-theory on K3 surface X with a volume-threading flux $G \supset \text{const} \times \text{dVol}(X)$ that has supersymmetric vacua of the form $\mathbb{R}^{1,2} \times \widetilde{X}$, where $\widetilde{X}$ is a second K3. This seven-dimensional theory does not have Minkowski vacua, and, as we will argue, necessarily involves a spacetime potential and cosmological constant. The volume-threading flux will obstruct any lift to F-theory, and it will lead to a puzzle with any potential geometric heterotic dual: briefly stated, the Bianchi identity for the H-flux of the putative seven-dimensional heterotic dual theory would seem to involve a term of the form $dH = *H + \cdots$, where $\cdots$ refer to the familiar heterotic curvature terms; the $*H$ term is not present in standard formulations of the heterotic string.

This is a strong indication that there is no conventional dual description of M-theory on K3 with a volume-threading flux. We sharpen this statement as follows. First, we demonstrate in some detail that in M-theory on $X \times \widetilde{X}$ it is possible to have solutions that preserve three-dimensional $\mathcal{N} = 3$ super-Poincaré invariance. These sorts of vacua are interesting in themselves, since they lie between the challenging $\mathcal{N} = 2$ and the reasonably well-understood $\mathcal{N} = 4$ vacua. We show that these solutions necessarily involve volume-threading flux, and this is why they have not been previously encountered in the literature (for instance, they certainly do not have an F-theory lift). We next turn to the heterotic string, and we prove that every geometric compactification with $\mathcal{N} = 3$ invariance in fact preserves $\mathcal{N} = 4$.[2]

This work should be viewed as an exploration of the general structure of duality between M-theory and the heterotic string. Consider a compactification of M-theory based on an 8-dimensional Ricci-flat manifold M_8. Compactification geometries with extended supersymmetry are conveniently summarized by the following famous table.[3] The maximum number of supersymmetries $\mathcal{N}_{\max} = \widehat{A}(M_8)$, where $\widehat{A}(M_8)$ is the value of the Dirac index on M_8.

[1]The $\mathcal{N}$ counts the two-component Majorana spinors of $\mathbb{R}^{1,2}$.

[2]By a geometric compactification we mean a solution (perhaps with some formal α' expansion [7] with a smooth seven-dimensional geometry X_7 equipped with some gauge bundle, dilaton, and H-flux.

[3]Like Kodaira's list of singularities, the M-theory part of the table can be found in many string theory papers; we adapt it from [8].

M-theory	$\mathcal{N}_{\rm max}$	1	2	3	4
	M_8 holonomy	Spin(7) $\supset$	SU(4) $\supset$	Sp(2) $\supset$	Sp(1) $\times$ Sp(1)
Heterotic		$\cup$	$\cup$		$\cup\!\mid$
	X_7 structure	$G_2 \supset$	SU(3) $\supset$	???	Sp(1)

The lower line of the table can also be taken to indicate the holonomy of an internal eight-manifold in M-theory compactifications M_8, rather than the structure group of X_7 on the heterotic side. For all of these the $8d$ spinors are not chiral, and $\widehat{A}(M_8) = 0$. Moreover, the internal flux G vanishes. The amount of $3d$ supersymmetry changes as $\mathcal{N} = 2, 4, 8$ as one moves right along the line. $\mathcal{N} = 16$ corresponds to M_8 with trivial holonomy. All these theories have natural lifts to four dimensions, since M_8 will necessarily involve at least one trivial circle. In this work we will not consider such flux-free M-theory compactifications.

The table, while specifying the geometry, does not describe the conditions on the flux of M-theory or the choice of gauge bundle of heterotic compactifications. For each class of M_8 it may be possible to choose G to preserve the maximal $\mathcal{N}_{\rm max}$ supersymmetry; we know many examples of this form, and for a very large class of solutions (in particular those with an F-theory lift), we have conjectured (and in examples tested) dual pairs of M-theory/heterotic theories.

The solutions with $\mathcal{N} = 3$ are certainly the least familiar in the list, and we conclude our introduction by making two general points about them. First, we have a rather poor understanding of M-theory vacua based on M_8 with Sp(2) holonomy. This is in part due to a dearth of examples of hyper-Kähler manifolds; a primary example is a resolution of the symmetric product, $S^2(K3)$, of two K3 surfaces.[4] However, as our heterotic no–go result shows, as far as duality goes, the issue appears to be deeper: there are no candidates for dual heterotic geometries. Second, although general Sp(2) manifolds may be of our reach, there does not appear to be much of a difference from the perspective of the $\mathbb{R}^{1,2}$ spacetime theory between M_8 with Sp(2) holonomy or $M_8 = X \times \widetilde{X}$ with an appropriate choice of flux; so, even with existing geometric technology there are many $\mathcal{N} = 3$ vacua to be explored. We will discuss their most basic properties below.

The rest of this paper is organized according to the table of contents. Seven-dimensional dualities are discussed in section 2. Sections 3 and 4 are devoted to M-theory on K3 $\times$ K3 and heterotic three-dimensional compactifications respectively. An appendix contains some technical details.

[4] An analysis of flux choices for this case, using orbifold techniques, can be found in [3].

2 M–theory in seven dimensions

Let X be a K3 equipped with a hyper-Kähler metric. We denote the triplet of hyper-Kähler forms by j_a, $a = 1, 2, 3$, and we normalize them by

$$j_a \wedge j_b = 2\delta_{ab} v E \ , \tag{2.1}$$

where v is the volume of X and E is the generator of $H^4(X, \mathbb{Z})$. In addition, we have the 19 anti-self-dual forms ω_α, $\alpha = 1, \ldots, 19$ that satisfy

$$\omega_\alpha \wedge j_a = 0 \ , \qquad \omega_\alpha \wedge \omega_\beta = -2\delta_{\alpha\beta} v E \ . \tag{2.2}$$

In what follows we will often suppress the explicit $\wedge$ when there is no possibility of confusion.

These conditions are invariant with respect to $\mathrm{SO}(3) \times \mathrm{SO}(19)$ rotations that act on the j_a and ω_α in the obvious fashion. The triplet j_a defines an SU(2) structure; in particular, the j_a determine the metric in the following way: a combination of two of them, say $j_2 + i j_3$, determines an integrable complex structure, and then the orthogonal complement, in this case j_1, becomes a corresponding Kähler form. There are $\mathrm{SO}(3)/\mathrm{U}(1) = S^2$ ways of picking a complex structure and, evidently, every SO(3) rotation of j_a yields exactly the same Einstein metric. The double cover SU(2) of this SO(3) turns out to be the SU(2) R-symmetry of the 7–dimensional theory.

2.1 Dualities between massive theories

We are interested in the physics of M-theory compactified on X with volume v. In the absence of any flux, this background is famously dual to the heterotic or type I string compactified on T^3. This is a strong-weak duality with

$$e^{\phi_7} = v^{3/4}, \tag{2.3}$$

where e^{ϕ_7} is the 7-dimensional heterotic string coupling. For elliptic X with section, this background has an 8-dimensional F-theory limit, corresponding to decompactifying a circle of the heterotic T^3.

We would like to ask whether a volume threading flux, $G \supset \mathrm{const} \times \mathrm{dVol}(X)$, which is compatible with Lorentz invariance in 7 dimensions, admits a dual description. Kaluza-Klein reduction on this background was first studied in [9, 10]. At first sight, the question itself might appear strange. The background with flux is not a solution of the equations of motion with an $\mathbb{R}^{1,6}$ Minkowski spacetime. This is easy to see for an unwarped spacetime metric from the M-theory equations of motion,

$$\mathcal{R}_{\mu\nu} = -\frac{1}{6}\eta_{\mu\nu}|G|^2, \tag{2.4}$$

since the spacetime Ricci tensor, which should vanish, is sourced by the flux. For a warped background, we consider the metric Ansatz

$$ds^2 = e^{2\omega}\eta + ds_X^2 \ , \tag{2.5}$$

where η denotes the usual Minkowski metric, and ω is the warp factor. For warp factors that do not depend on the spacetime coordinates, the Ricci tensor takes the form

$$\mathcal{R}_{\mu\nu} = -\frac{1}{5}\eta_{\mu\nu}e^{-5\omega}\nabla^2 e^{5\omega}. \tag{2.6}$$

However on a compact space like X, the equation

$$\nabla^2 e^{5\omega} = \frac{5}{6}e^{5\omega}|G|^2 \tag{2.7}$$

has no solution. Therefore there are no Minkowski vacua for M-theory reduced on X with volume threading G. Indeed, it is easy to extend this argument to see that there are no solutions for any maximally symmetric spacetime: this background cannot be realized as an on-shell solution in string theory without breaking maximal spacetime symmetry. Alternately, it can appear as an intermediate massive theory en route to a static Minkowski or AdS solution in lower dimension.

Regardless, we can still perform a Kaluza-Klein reduction on such a background and some theory must determine the set of quantum corrections to the classical spacetime effective action. From this latter perspective, it is still reasonable to ask whether a weakly coupled description might control the physics of small volume v, while eleven-dimensional supergravity together with higher derivative corrections controls the perturbative physics of large v. A natural guess based on the flux-free duality might be that the heterotic string on T^3 with a volume threading flux $H \supset \text{const} \times \text{dVol}(T^3)$ provides such a description. The gauged massive supergravities that arise from toroidal compactifications of the heterotic string with backgrounds fluxes have been studied in [11]. In both cases, decompactification to 8 dimensions is obstructed by the quantized flux.

However, there are immediate issues with such a proposed duality. To construct a macroscopic heterotic string, we usually wrap an M5-brane on X. The M5-brane world-volume supports a self-dual 3-form field strength h_3, which obeys a Bianchi identity:

$$dh_3 = G. \tag{2.8}$$

This obstructs wrapping X without some added ingredient to satisfy the Gauss law charge constraint; for example, stretched M2-branes which realize self-dual strings on the world-volume of the M5-brane. However, any such ingredient breaks additional Lorentz invariance beyond the breaking introduced by the stretched macroscopic string.

Another immediate issue is seen by examining the form of the Bianchi identity for the heterotic H-flux, derived from M-theory. In the flux-free duality, we identify

$$H = *_7 G, \tag{2.9}$$

where the Hodge dual is taken in 7 dimensions; the Bianchi identity for H follows from the equation of motion for G. Since these are the only propagating 3-forms in 7 dimensions, any proposed duality would need some identification between them. However the case with flux threading X produces a new coupling in the heterotic Bianchi identity:

$$\int_X d * G = -\frac{1}{2}\int_X G \wedge G \quad \implies \quad dH \sim *_7 H + \ldots. \tag{2.10}$$

The omitted terms involve both gauge-fields from the Kaluza-Klein reduction of G on 2-forms of X, and gravitational couplings from higher derivative interactions in M-theory. The new coupling involving $*_7 H$ in (2.10), which is proportional to the amount of flux through X, has no obvious realization in geometric heterotic compactifications.

On its own, these pieces of evidence might not be convincing. It might be the case that heterotic compactified on T^3 with H-flux simply admits no macroscopic string solutions, and perhaps there is some subtle modification of the Bianchi identity to evade (2.10). There is a more direct approach. To dualize M-theory on X to a type I/heterotic background, start with an orbifold limit where $X = T^4/\mathbb{Z}_2$. The first step in the duality chain is to reduce on a circle of X to a type IIA orientifold:

$$T^3/\Omega\mathbb{Z}_2. \tag{2.11}$$

The G-flux through X implies a volume threading H-flux in this type IIA background. Arriving at a type I background requires T-dualizing all three directions of the T^3. While one or two T-dualities can be performed in this background without great difficulty, dualizing all three directions is difficult to understand. Each T-duality requires a potential for H, but any trivialization of H breaks one isometry of T^3. For more discussion of what such a resulting heterotic theory might possibly look like, see, for example, [18]. This obstruction looks difficult to evade, and each known duality chain that leads to a heterotic or type I dual description meets this same issue in one guise or the other.

On the other hand, heterotic on T^3 with volume threading H-flux does admit a dual description, which can be seen as follows: consider heterotic on T^2. There are 2 periodic scalars (τ_1, ρ_1) associated to the complex structure of T^2 and to the volume threading B_2-flux. Let us focus on this latter scalar. Under the duality to the type IIB orientifold $T^2/\Omega(-1)^{F_L}\mathbb{Z}_2$ described in [12], this scalar maps to the type IIB axion C_0. Now compactify on a further S^1 and permit both ρ_1 of heterotic and C_0 of type IIB to depend linearly on the circle coordinate. On the one side, we have heterotic on T^3 with volume threading H. The dual description is type IIB compactified on

$$T^2/\Omega(-1)^{F_L}\mathbb{Z}_2 \times S^1 \tag{2.12}$$

with constant quantized RR F_1 field strength in the S^1 direction. This is a dual pair. The usual route to find a lift to M-theory involves T-dualizing on the S^1 direction. This maps F_1 to F_0 and we arrive at a massive type IIA background [13].[5] The Romans mass obstructs a lift to M-theory. The dual description is actually given by either the type IIA or type IIB 7-dimensional orientifold background, depending on the size of S^1. This story can be extended to a more general F-theory setting by identifying both periodic scalars (τ_1, ρ_1) in the geometry of an elliptic X [15, 16], and allowing them to depend on the S^1 coordinate, along the lines described in [17].

The bigger picture suggested by this example is a collection of dualities between lower-dimensional massive supergravity theories, induced from higher-dimensional more conventional dual pairs. Interestingly, there is no obvious candidate for a dual description of M-theory on X with volume threading G. Like the Romans theory in ten dimensions, it might simply exist without a relation to other known massive backgrounds.

3 M–theory vacua on K3 $\times$ K3

In the previous section we discussed M–theory compactifications to seven dimensions with a G–flux threading the K3. We saw that such theories do not have $\mathbb{R}^{1,6}$ vacua; on the other hand, it is possible to compactify further and obtain $\mathbb{R}^{1,2}$ vacua. In this section we explore the resulting theories in some detail. We will find familiar examples of three–dimensional $\mathcal{N}=1$, $\mathcal{N}=2$ and $\mathcal{N}=4$ supersymmetric theories, but we will also find solutions that are a bit more exotic and realize $\mathcal{N}=3$.

The interest in these solutions is two–fold. First, we will be able sharpen the puzzles of "non-duality," because, as we will see in the section that follows, there are no geometric $\mathcal{N}=3$ heterotic compactifications. Second, we will see that the choice of flux allows the rather simple geometry of K3$\times$K3 to realize the features of more sophisticated solutions with Spin(7), SU(4) or Sp(2) structures.

3.1 M–theory on M_8

We begin with a brief review of M–theory compactification of the form $\mathbb{R}^{1,2} \times M_8$. This is determined by a choice of a warped metric g and flux G on M_8. The latter needs to obey two topological conditions: the flux is quantized according to [19]

$$\tfrac{1}{2\pi}G - \tfrac{1}{4}p_1(M_8) \in H^4(M_8, \mathbb{Z}) \ ,$$

and it satisfies the tadpole equation for C_3 [20, 21]. A necessary and sufficient condition for that is

$$\frac{1}{2}\int_{M_8} \frac{G}{2\pi} \wedge \frac{G}{2\pi} = \frac{\chi(M_8)}{24} - N(M_2) \ . \tag{3.1}$$

[5]The interpretation of massive theories from the perspective of M-theory and F-theory has been described in [14].

Here $\chi(M_8)$ is the Euler number of M_8, while $N(M_2)$ is the number of space-filling $M2$-branes. In this paper we will be interested in solutions with $N(M_2) = 0$.

Minimal supersymmetry requires M_8 to admit a Spin(7) holonomy metric and also imposes a condition on G. $H^4(M_8, \mathbb{R})$ can be decomposed according to representations of Spin(7) [8], and we must have $G \in H^4_{+\mathbf{27}}(M_8, \mathbb{R})$. That is, the flux is self-dual and in the **27** [22].

3.2 M-theory on K3×K3 and $\mathcal{N} = 1, 2, 3, 4$ examples

We consider M-theory on $M_8 = X \times \widetilde{X}$, where X and $\widetilde{X}$ are both K3 surfaces. Minkowski $\mathbb{R}^{1,2}$ vacua with this compactification geometry are labeled by a choice of Einstein metric on M_8 and a choice of G obeying the integrality and supersymmetry conditions. In our case $p_1(M_8) = p_1(X) + p_1(\widetilde{X})$ is divisible by 4, so the integrality condition on G is simply that $\frac{1}{2\pi} G \in H^4(X, \mathbb{Z})$. We have the identification

$$H^4(M_8, \mathbb{Z}) = H^2(X, \mathbb{Z}) \otimes H^2(\widetilde{X}, \mathbb{Z}) \oplus H^4(X, \mathbb{Z}) \oplus H^4(\widetilde{X}, \mathbb{Z}) \ . \tag{3.2}$$

In much of the work on this compactification, e.g. [3, 23], the flux does not involve any components in the last two terms. However, as has been observed more recently [24], minimal supersymmetry allows a more general flux.

Consider now $X \times \widetilde{X}$. The two components X and $\widetilde{X}$ have self-dual forms j_a and $\widetilde{j}_{\dot{a}}$ and anti-self-dual forms ω_α, $\widetilde{\omega}_{\dot{\alpha}}$ in an obvious extension of the notation from section 2 . We denote the generators of $H^4(X, \mathbb{Z})$ and $H^4(\widetilde{X}, \mathbb{Z})$ by, respectively, E and $\widetilde{E}$; similarly X and $\widetilde{X}$ have volumes v and $\widetilde{v}$. With this notation, we can state the general result [24]: up to an $\mathrm{SO}(3) \times \mathrm{SO}(3)$ rotation, the most general form of the G-flux on $X \times \widetilde{X}$ consistent with $\mathcal{N} = 1$ supersymmetry in $\mathbb{R}^{1,2}$ is

$$G = j_a M_{a\dot{a}} \widetilde{j}_{\dot{a}} + (4A - 2C) \left[vE + \widetilde{v}\widetilde{E} \right] + f_{\alpha\dot{\alpha}} \omega_\alpha \widetilde{\omega}_{\dot{\alpha}} \ , \tag{3.3}$$

and the 3×3 constant matrix M has the form

$$M = \begin{pmatrix} C & D_1 & D_2 \\ D_1 & A + B_1 & B_2 \\ -D_2 & -B_2 & B_1 - A \end{pmatrix} \ . \tag{3.4}$$

The last term in G with the 19×19 constant matrix f just involves the anti-self-dual forms ω and $\widetilde{\omega}$. G should also satisfy the integrality and tadpole constraints.

We can now give some examples of solutions that preserve different amounts of supersymmetry.

1. The most familiar way to satisfy (3.3) is to set $M = 0$; this also eliminates the "volume–threading" term. In this case G is invariant under $\mathrm{SO}(3) \times \mathrm{SO}(3)$ rotations and corresponds to an $\mathcal{N} = 4$ vacuum. We can think of this as two statements: the underlying

manifold has SU(2) × SU(2) structure, and the flux respects this:

$$j_a \wedge G = 0 \ , \qquad\qquad \widetilde{\jmath}_{\dot{a}} \wedge G = 0 \ . \tag{3.5}$$

2. We can reduce supersymmetry by taking $M = \lambda \mathbb{1}_3$, so that

$$G = \lambda(j_1\widetilde{\jmath}_1 + j_2\widetilde{\jmath}_2 + j_3\widetilde{\jmath}_3) - 2\lambda\left[vE + \widetilde{v}\widetilde{E}\right] + f_{\alpha\dot{\alpha}}\omega_\alpha\widetilde{\omega}_{\dot{\alpha}} \ . \tag{3.6}$$

   G is invariant under a diagonal SO(3) ⊂ SO(3) × SO(3) action; in fact it respects an Sp(2) structure on the underlying manifold and therefore leads to $\mathcal{N} = 3$ in $\mathbb{R}^{1,2}$. The Sp(2) structure is generated by the three 2–forms[6]

$$\mathcal{J}_a = j_a + \widetilde{\jmath}_a \ , \tag{3.7}$$

   and for all a

$$\mathcal{J}_a \wedge G = 0 \ . \tag{3.8}$$

3. To obtain $\mathcal{N} = 2$ symmetry we demand that G only preserved by U(1) ⊂ SO(3) ⊂ SO(3) × SO(3). For instance, following [3] we can take

$$G = \lambda(j_1\widetilde{\jmath}_1 + j_2\widetilde{\jmath}_2) + f_{\alpha\dot{\alpha}}\omega_\alpha\widetilde{\omega}_{\dot{\alpha}} \ . \tag{3.9}$$

   This flux respects an SU(4) structure of $X \times \widetilde{X}$: we set

$$J = \mathcal{J}_1 \ , \qquad\qquad \Omega = \tfrac{1}{2}(\mathcal{J}_2 + i\mathcal{J}_3)^2 \ , \tag{3.10}$$

   and G is (2,2) and primitive with respect to this SU(4) structure. That is the familiar condition for preserving $\mathcal{N} = 2$ supersymmetry [21].

3.3 Structures and extended supersymmetry

As we have seen, for particular choices of flux we obtain vacua with various amounts of extended supersymmetry. In this section we will make a more systematic study of the constraints that lead to $\mathcal{N} = 2, 3, 4$, and we will also explore the massless spectrum of these theories. To start, we note that every Sp(2) structure on $X \times \widetilde{X}$ compatible with the product metric takes the form

$$\mathcal{J}_A = R_{Aa} j_a + \widetilde{R}_{A\dot{a}}\widetilde{\jmath}_{\dot{a}} \ , \tag{3.11}$$

where R and $\widetilde{R}$ are 3 × 3 SO(3) matrices; it is an easy matter to check that these satisfy the defining relations of Sp(2) structure (see footnote 6). To show that every Sp(2) structure

[6] That is, the $\mathcal{J}_a$ are three non-degenerate two-forms that satisfy the defining cubic and quartic relations $3\mathcal{J}_a\mathcal{J}_b\mathcal{J}_c = \delta_{ab}\mathcal{J}_c^3 + \delta_{ca}\mathcal{J}_b^3 + \delta_{bc}\mathcal{J}_a^3$ and $\mathcal{J}_a\mathcal{J}_b\mathcal{J}_c\mathcal{J}_d = 8\,\mathrm{dVol}_8\,[\delta_{ab}\delta_{cd} + \delta_{ca}\delta_{bd} + \delta_{bc}\delta_{ad}]$ [25].

takes this form, we just note that by raising an index on the $\mathcal{J}_A$ with the metric we should obtain the triplet of complex structures satisfying the familiar quaternionic algebra; that fixes the $\mathcal{J}_A$ in the form shown.

Similarly, the most general SU(4) structure on $X \times \widetilde{X}$ takes the form

$$J = y_A \mathcal{J}_A \ , \qquad\qquad \Omega = \tfrac{1}{2}(u_A \mathcal{J}_A)^2 \ . \tag{3.12}$$

Here y_A is a real vector and u_A is a complex vector such that the 3×3 matrix

$$\begin{pmatrix} \mathrm{Re}(u_1) & \mathrm{Im}(u_1) & y_1 \\ \mathrm{Re}(u_2) & \mathrm{Im}(u_2) & y_2 \\ \mathrm{Re}(u_3) & \mathrm{Im}(u_3) & y_3 \end{pmatrix} \in \mathrm{SO}(3) \ .$$

In particular, $y^T u = 0$, $2y^T y = u^\dagger u = 2$, and $u^T u = 0$. Setting $x = y^T R$, $t = u^T R$, and similarly for $\widetilde{x}$ and $\widetilde{t}$, and using (3.11), we can write the most general SU(4) structure on $X \times \widetilde{X}$ as

$$J = x_a j_a + \widetilde{x}_{\dot{a}} \widetilde{j}_{\dot{a}} \ , \qquad\qquad \Omega = t_a \widetilde{t}_{\dot{a}} j_a \widetilde{j}_{\dot{a}} \ , \tag{3.13}$$

where x and t, as well as $\widetilde{x}$ and $\widetilde{t}$ satisfy the same conditions as y, u.

$\mathcal{N} \geq 2$ supersymmetry

Having taken care of the preliminaries, the analysis of the supersymmetry conditions is now straightforward. To preserve at least $\mathcal{N} = 2$ we know that G must be a primitive (2,2) form [21] with respect to some SU(4) structure. In other words,

$$J \wedge G = 0 \ , \qquad\qquad \Omega \wedge G = 0 \ , \tag{3.14}$$

and G has no (1,3) or (3,1) components. Applying the first two conditions to the general flux in (3.3), we obtain

$$\begin{aligned} J \wedge G = 0 \quad &\Longleftrightarrow \quad M\widetilde{x} + (2A - C)x = 0 \quad \text{and} \quad x^T M + (2A - C)\widetilde{x}^T = 0 \ , \\ \Omega \wedge G = 0 \quad &\Longleftrightarrow \quad t^T M \widetilde{t} = 0 \ . \end{aligned} \tag{3.15}$$

To ensure no (1,3) or (3,1) components in G we note that the harmonic $(3,1)$ forms on $X \times \widetilde{X}$ with respect to the chosen SU(4) structure are all linear combinations of

$$t_a j_a \wedge \widetilde{x}_{\dot{a}} \widetilde{j}_{\dot{a}} \ , \qquad t_a j_a \wedge \omega_{\dot{\alpha}} \ , \qquad x_a j_a \wedge \widetilde{t}_{\dot{a}} \widetilde{j}_{\dot{a}} \ , \qquad \omega_\alpha \wedge \widetilde{t}_{\dot{a}} \widetilde{j}_{\dot{a}} \ . \tag{3.16}$$

So, our final requirement is that G is annihilated by each of these terms. This leads to the conditions

$$t^T M \widetilde{x} = 0 \ , \qquad\qquad x^T M \widetilde{t} = 0 \ . \tag{3.17}$$

The vectors $x, \mathrm{Re}(t), \mathrm{Im}(t)$ form an orthonormal basis, as do $\widetilde{x}, \mathrm{Re}(\widetilde{t}), \mathrm{Im}(\widetilde{t})$, and by taking real and imaginary parts of (3.15, 3.17), we find that in order for G to be compatible with some SU(4) structure the matrix M must be expressible as

$$M = S^T \begin{pmatrix} C-2A & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\beta & \alpha \end{pmatrix} \widetilde{S} \ , \tag{3.18}$$

where S and $\widetilde{S}$ are SO(3) matrices. This implies

$$MM^T = S^T \begin{pmatrix} (C-2A)^2 & 0 & 0 \\ 0 & \alpha^2+\beta^2 & 0 \\ 0 & 0 & \alpha^2+\beta^2 \end{pmatrix} S \ , \tag{3.19}$$

so that MM^T has $(C-2A)^2$ as an eigenvalue, and MM^T has at least two equal eigenvalues. These requirements can be easily translated into polynomial conditions on the parameters A, B_1, B_2, C, D_1, D_2 that appear in (3.3) .

$\mathcal{N} \geq 3$ supersymmetry

The flux will be compatible with an Sp(2) structure if and only if $\mathcal{J}_A \wedge G = 0$ for $A = 1,2,3$. Using the $\mathcal{J}_A$ in (3.11), this leads to

$$M = (C-2A) R^T \widetilde{R} \ , \tag{3.20}$$

so that $MM^T = (C-2A)^2 \mathbb{1}_3$. Finally, to be compatible with $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ structure and therefore $\mathcal{N} = 4$ supersymmetry, the condition on M has to be true for all $R, \widetilde{R}$ in $\mathrm{SO}(3) \times \mathrm{SO}(3)$. This forces $M = 0$.

Note that the volume-threading term in the G-flux is proportional to $(C-2A)$. This means that every $\mathcal{N} \geq 3$ vacuum without a volume-threading term necessarily has $M = 0$, so that it is actually preserving $\mathcal{N} = 4$.

It is not obvious that we can choose an integral G-flux that both takes the $\mathcal{N} = 3$ form and satisfies the Bianchi identity without M2-branes. Appendix B shows this to be the case.

3.4 Massless spectrum

Like the existence of the vacuum, the massless spectrum also correlates nicely with the structure preserved by the flux. We will not go into a detailed study of the interactions and, for example, explicit expressions for the moduli space metric; this has been carried out at the supergravity level for the most general flux compatible with minimal supersymmetry in [26]. Instead, we will just point out how the counting of massless degrees of freedom correlates with the structure.

Metric moduli

The 58–dimensional space of first-order deformations of an Einstein metric on X can be parametrized in terms of a scalar parameter x that corresponds to rescaling the total volume, as well as a 3×19 matrix $\mathcal{X}_{a\alpha}$:

$$\delta j_a = x j_a + \mathcal{X}_{a\alpha}\omega_\alpha \ , \qquad \delta\omega_\alpha = x\omega_\alpha - j_a \mathcal{X}_{a\alpha} \ , \qquad \delta v = 2xv \ . \tag{3.21}$$

It is easy to see that this preserves the defining conditions:

$$\delta\left(j_a j_b - 2\delta_{ab} vE\right) = 0 \ , \qquad \delta\left(j_a\omega_\alpha\right) = 0 \ , \qquad \delta\left(\omega_\alpha\omega_\beta + 2\delta_{\alpha\beta} vE\right) = 0 \ .$$

We have analogous expressions for the other K3 $\widetilde{X}$, except for tildes and dots.

Not all of these geometric deformation parameters correspond to three–dimensional massless modes: a necessary condition is that the integral (and therefore rigid) flux G satisfies the same conditions with respect to the deformed and undeformed j_a and $\widetilde{\jmath}_{\dot a}$. Since we parametrized G in terms of the basis of self-dual and anti-self-dual forms on X and $\widetilde{X}$, this amounts to finding δM and δf in (3.3) such that under (3.21) $\delta G = 0$. Plugging all of the variations into G and demanding $\delta G = 0$, we obtain the following conditions:

$$\delta M = -(x + \widetilde{x})M \ , \qquad \delta f = -(x + \widetilde{x})f \ , \tag{3.22}$$

and

$$(2A - C)(x - \widetilde{x}) = 0 \ , \qquad \mathcal{X}^T M = f\widetilde{\mathcal{X}}^T \ , \qquad M\widetilde{\mathcal{X}} = \mathcal{X} f \ . \tag{3.23}$$

The first two equations merely determine δM and δf and do not lead to interesting constraints. On the other hand, the remaining three are interesting. First, we see that if $2A \neq C$ then $x = \widetilde{x}$, so that while the overall volume modulus of $X \times \widetilde{X}$ remains massless, it is not possible to tune the volumes of X and $\widetilde{X}$ separately. Thus, $2A = C$ is a necessary condition to be able to lift the vacuum to 7 dimensions.

The remaining conditions are covariant with respect to the obvious $\mathrm{O}(3)\times\mathrm{O}(3)\times\mathrm{O}(19)\times\mathrm{O}(19)$ action on the $j_a, \widetilde{\jmath}_{\dot a}$ and $\omega_\alpha, \widetilde{\omega}_{\dot\alpha}$. This means we can use singular value decomposition to bring M and f to canonical form:

$$M = \mathrm{diag}(\mu_1, \mu_2, \mu_3) \ , \qquad f = \mathrm{diag}(\phi_1, \phi_2, \ldots, \phi_{19}) \ , \tag{3.24}$$

where the μ_a and ϕ_a are all non-negative (they are positive square roots of the eigenvalues of, respectively, MM^T and ff^T.). In this form the conditions on $\mathcal{X}$ and $\widetilde{\mathcal{X}}$ are written as

$$\mu_a \mathcal{X}_{a\alpha} = \widetilde{\mathcal{X}}_{a\alpha}\phi_\alpha \ , \qquad \mu_a \widetilde{\mathcal{X}}_{a\alpha} = \mathcal{X}_{a\alpha}\phi_\alpha \ , \qquad \text{no sum on } a \text{ or } \alpha. \tag{3.25}$$

Generically these require $\mathcal{X} = \widetilde{\mathcal{X}} = 0$, but if some of the eigenvalues of M match those of f, there are more solutions. The number of independent parameters is given by

$$
\begin{aligned}
n(\mathcal{X},\widetilde{\mathcal{X}}) &= 2\dim\ker M \dim\ker f + \sum_{a,\alpha\ |\mu_a\neq 0} \delta(\mu_a-\phi_a)\ , \\
&= \dim\ker M \dim\ker f + \sum_{a,\alpha} \delta(\mu_a-\phi_a)\ ,
\end{aligned} \tag{3.26}
$$

where $\delta(\mu_a-\phi_\alpha)=1$ if $\mu_a=\phi_\alpha$ and is zero otherwise. To understand this, note that the second line follows trivially from the first. The first line merely says that if $\mu_a=0$ then the equations require that $\mathcal{X}_a$ and $\widetilde{\mathcal{X}}_{\dot a}$ both belong to $\ker f$; if for a fixed a $\mu_a\neq 0$, then $\widetilde{\mathcal{X}}_{a\alpha}$ is determined by $\mathcal{X}_{a\alpha}$, and the latter satisfies $(\mu_a-\phi_\alpha)\mathcal{X}_{a\alpha}=0$.

Including the constraints on the $x,\widetilde{x}$, we find that the massless metric moduli are counted by

$$
\#(\text{metric moduli}) = \begin{cases} 2+n(\mathcal{X},\widetilde{\mathcal{X}})\ , & 2A=C\ , \\ 1+n(\mathcal{X},\widetilde{\mathcal{X}})\ , & 2A\neq C\ . \end{cases} \tag{3.27}
$$

Massless vectors

Fluctuations of C give rise to massless vectors in three dimensions: $C=\cdots+\boldsymbol{V}^I\Omega_I$, where the $\boldsymbol{V}^I$ are three-dimensional vectors with field strengths $\boldsymbol{F}^I=d\boldsymbol{V}^I$, and the Ω_I are harmonic forms on $X\times\widetilde{X}$. Inserting this into the M-theory action leads to a Chern-Simons mass term for the $\boldsymbol{V}^I$ proportional to

$$
\Delta\mathcal{L}_3 = \boldsymbol{V}^I\boldsymbol{F}^J\int_{M_8} G\Omega_I\Omega_J\ . \tag{3.28}
$$

To explore the kernel of this mass term we write out

$$
V^I\Omega_I = V_+^a j_a + V_-^\alpha\omega_\alpha + \widetilde{V}_+^{\dot a}\widetilde{\jmath}_{\dot a} + \widetilde{V}_-^{\dot\alpha}\widetilde{\omega}_{\dot\alpha} \tag{3.29}
$$

and combine these components into a 44–dimensional vector

$$
\mathbb{V}^T = (V_+^T\ \widetilde{V}_+^T\ V_-^T\ \widetilde{V}_-^T)\ , \tag{3.30}
$$

and similarly for the field-strengths, which are packaged in a vector $\mathbb{F}$. With a little bit of algebra we find

$$
\Delta\mathcal{L}_3 = 4v\widetilde{v}\mathbb{V}^T\mathbb{M}\mathbb{F}\ , \tag{3.31}
$$

where

$$
\mathbb{M} = \begin{pmatrix} \mathbb{M}_+ & 0 \\ 0 & \mathbb{M}_- \end{pmatrix}\ , \tag{3.32}
$$

and

$$\mathbb{M}_+ = \begin{pmatrix} (2A-C)\mathbb{1}_3 & M \\ M^T & (2A-C)\mathbb{1}_3 \end{pmatrix} , \qquad \mathbb{M}_- = \begin{pmatrix} (C-2A)\mathbb{1}_{19} & f \\ f^T & (C-2A)\mathbb{1}_{19} \end{pmatrix} . \tag{3.33}$$

So, the number of massless vectors is $\dim\ker \mathbb{M}_+ + \dim\ker \mathbb{M}_-$. The latter depend on the value of $(2A-C)$:

$$\begin{aligned} \dim\ker \mathbb{M}_+ &= \begin{cases} 2\dim\ker M & 2A = C \\ \dim\ker(M^TM - (2A-C)^2\mathbb{1}_3) & 2A \neq C \end{cases} , \\ \dim\ker \mathbb{M}_- &= \begin{cases} 2\dim\ker f & 2A = C \\ \dim\ker(f^Tf - (2A-C)^2\mathbb{1}_{19}) & 2A \neq C \end{cases} , \end{aligned} \tag{3.34}$$

Summary for $\mathcal{N} = 2, 3, 4$

We now combine the previous results with the constraints on M and $\mathcal{N}$ found in the previous section. In each case we will find a result consistent with the three-dimensional multiplet structure for the particular $\mathcal{N}$.

1. $\mathcal{N} = 4$. This requires $M = 0$ and therefore leads to

$$\begin{aligned} \#(\text{metric moduli}) &= 2 + 6\dim\ker f \ , \\ \#(\text{massless vectors}) &= 6 + 2\dim\ker f \ . \end{aligned} \tag{3.35}$$

   Recall that the massless vector and hyper multiplets of $\mathcal{N} = 4$ each contain 4 scalar degrees of freedom; this is consistent with the total number of massless scalars obtained here (which is in fact divisible by 8). We do not expect quantum corrections to lift any of these massless degrees of freedom.

2. $\mathcal{N} = 3$. In this case $\mu_a = |2A - C| \neq 0$ for $a = 1, 2, 3$, and therefore

$$\begin{aligned} \#(\text{metric moduli}) &= 1 + 3\dim\ker\{f^Tf - (2A-C)^2\mathbb{1}_{19}\} \ , \\ \#(\text{massless vectors}) &= 3 + \dim\ker\{f^Tf - (2A-C)^2\mathbb{1}_{19}\} \ . \end{aligned} \tag{3.36}$$

   Since the massless supermultiplets for $\mathcal{N} = 3$ have exactly the same structure as the more familiar $\mathcal{N} = 4$ multiplets [27], we expect that the total number of scalars is divisible by 4, and indeed it is. The moduli space of $\mathcal{N} = 3$ theories is quaternionic [27], and we suspect but have not checked that, as in the $\mathcal{N} = 4$ case, supersymmetry is sufficient to rule out quantum corrections that might lift these degrees of freedom.

3. $\mathcal{N} = 2$. In this case we expect quantum corrections to lift some of the classically massless fields, so our results are merely upper bounds on the massless spectrum. The

content of $\mathcal{N} = 2$ chiral and vector multiplets easily follows by reduction from $d = 4$ $\mathcal{N} = 1$ multiplets, and each massless multiplet contains two scalar degrees of freedom. Based on the analysis above, we find the following massless spectrum; in each case we do find the expected even number of scalars.

(a) The generic case is when $\mu_1 = |2A - C| \neq 0$ and $0 < \mu_2 = \mu_3 \neq \mu_1$.

$$\begin{aligned} \#(\text{metric moduli}) &= 1 + \dim\ker\{f^T f - \mu_1^2 \mathbb{1}\} + 2\dim\ker\{f^T f - \mu_2^2 \mathbb{1}\} \ , \\ \#(\text{massless vectors}) &= 1 + \dim\ker\{f^T f - \mu_1^2 \mathbb{1}\} \ . \end{aligned} \tag{3.37}$$

(b) A less generic possibility $\mu_1 = |2A - C| \neq 0$ and $\mu_2 = \mu_3 = 0$ leads to

$$\begin{aligned} \#(\text{metric moduli}) &= 1 + 4\dim\ker f + \dim\ker\{f^T f - \mu_1^2 \mathbb{1}\} \ , \\ \#(\text{massless vectors}) &= 1 + \dim\ker\{f^T f - \mu_1^2 \mathbb{1}\} \ . \end{aligned} \tag{3.38}$$

(c) The final possibility, $\mu_1 = |2A - C| = 0$ and $0 < \mu_2 = \mu_3$, leads to

$$\begin{aligned} \#(\text{metric moduli}) &= 2 + 2\dim\ker f + 2\dim\ker\{f^T f - \mu_2^2 \mathbb{1}\} \ , \\ \#(\text{massless vectors}) &= 2 + 2\dim\ker f \ . \end{aligned} \tag{3.39}$$

4 Heterotic 3d compactifications

The preceding sections identified and studied a large class of M-theory vacua based on the relatively simple geometry of K3×K3. In this section we will consider potential dual heterotic descriptions of these vacua in three dimensions. There are many examples of dual pairs based on the 7–dimensional duality between a heterotic string on T^3 and M-theory on K3. For instance, we expect to be able to find M-theory descriptions of heterotic backgrounds satisfying the following two conditions:

1. the three–dimensional gauge group is abelian;
2. the compactification manifold X_7 is a principal T^3 fibration over K3, with the bundle obtained by a combination of Wilson lines and a pull-back of a bundle from the base K3 geometry.

These geometries have a lift to 7 dimensions, and fiberwise duality with M-theory on K3 should make sense.

On the other hand, as we already saw, M-theory solutions with G-flux that threads the volumes of the K3s do not have simple heterotic duals. We outlined some of the challenges of finding the duality in terms of the massive 7–dimensional theory in section 2. We will now consider the problem directly in 3 dimensions, and we will show that there are no heterotic geometries that lead to exactly $\mathcal{N} = 3$ supersymmetry in three dimensions: a solution with 6 supercharges actually preserves 8 or 16 supercharges.

4.1 Review of heterotic G_2 geometry

Consider a three–dimensional compactification of the heterotic string with $\mathcal{N} \geq 1$ on a seven–dimensional compact manifold X_7. In order to discuss spinors and their properties on X_7 let us first fix a basis for the Clifford algebra.[7]

Clifford algebra on X_7

We choose the Γ_i, $i = 1, \dots, 7$ to be a pure imaginary antisymmetric basis satisfying

$$\{\Gamma_i, \Gamma_j\} = 2g_{ij} \ .$$

The matrices $\{\mathbb{1}, i\Gamma_{ijk}\}$ are real symmetric, while $\{i\Gamma_i, \Gamma_{jk}\}$ are real anti-symmetric. Together they span the Clifford algebra: given a non-zero real spinor ϵ_0 a basis of spinors is $\{\epsilon_0, i\Gamma_i\epsilon_0\}$. That is, we have the completeness relation

$$\Gamma^i \epsilon_0 \epsilon_0^t \Gamma_i + \epsilon_0 \epsilon_0^t = \mathbb{1}_8 \ . \tag{4.1}$$

In the usual way we define $\Gamma^{i_1 \cdots i_k} = \frac{1}{k!}\Gamma^{[i_1}\Gamma^{i_2}\cdots\Gamma^{i_k]}$, and we lower and raise the (co)tangent space indices with the metric g_{ij} and its inverse g^{ij}.

Minimal supersymmetry requirements

Minimal supersymmetry requires that the geometry satisfies the following conditions.[8]

1. The gauge bundle $\mathcal{P} \to X_7$ has structure group in $\mathrm{Spin}(32)/\mathbb{Z}_2$ or $E_8 \times E_8$ and satisfies the heterotic Bianchi identity in integral cohomology.

2. The vanishing of the gravitino variation requires that X_7 admits a ∇^-–constant spinor ϵ_0. The ∇^- connection is the Levi-Civita connection twisted by the 3–form H:

$$\left(\Gamma^-\right)^l_{jk} = g^{li}\left(\tfrac{1}{2}\left[g_{ji,k} + g_{ki,j} - g_{jk,i}\right] - \tfrac{1}{2}H_{ijk}\right) = \Gamma^l_{jk} - \tfrac{1}{2}H^l{}_{jk} \ .$$

   This means X_7 has G_2 structure.

3. The vanishing of the dilatino variation requires

$$\left[\partial_i\varphi\Gamma^i - \tfrac{1}{12}H_{ijk}\Gamma^{ijk}\right]\epsilon_0 = 0 \ . \tag{4.2}$$

   Here φ is the dilaton field.

4. The gauge curvature $\mathcal{F}$ annihilates the spinor: $\mathcal{F}_{ij}\Gamma^{ij}\epsilon_0 = 0$.

5. The Bianchi identity has a solution in the formal α' expansion [7].

[7] A thorough and readable review of G_2–structure compactification is given in [28]; we follow it in a number of conventions, including that for the spinors.

[8] The general result goes back to [29]; applications to X_7 may be found in, for instance, [25, 30].

Conditions 2,3, and 4 will be sufficient for our purposes, but any putative solution must satisfy all of these necessary conditions.

The existence of a ∇^-–constant spinor ϵ_0 implies the existence of ∇^-–constant associative and co-associative forms

$$\Phi_{ijk} = i\epsilon_0^T \Gamma_{ijk}\epsilon_0 \ , \qquad \Psi_{ijkl} = \epsilon_0^T \Gamma_{ijkl}\epsilon_0 \ . \tag{4.3}$$

The metric g relates these two by $*_g\Phi = \Psi$, and $\Phi \wedge *_g\Phi = 7\,\mathrm{dVol}_g(X_7)$. Moreover, we have the helpful relations

$$\Gamma_{ij}\epsilon_0 = -i\Phi_{ijk}\Gamma^k\epsilon_0 \ , \qquad i\Gamma_{ijk}\epsilon_0 = \Phi_{ijk}\epsilon_0 - i\Psi_{ijkl}\Gamma^l\epsilon_0 \ . \tag{4.4}$$

The Φ and Ψ obey a number of useful relations summarized in appendix A of [28]. We will find use for two of these:

$$\Psi_{ijnm}\Phi^{klm} = 6\delta_{[i}^{[k}\Phi_{jn]}{}^{l]} \ , \qquad \Phi_{ijk}\Phi_{lm}{}^{k} = g_{il}g_{mj} - g_{im}g_{lj} - \Psi_{ijlm} \ . \tag{4.5}$$

Turning the construction around, suppose X has a G_2 structure, i.e. a non-degenerate 3-form Φ that in a local orthonormal frame $\{e^i\}_{i=1,\ldots,7}$ with respect to metric g has the canonical form

$$\begin{aligned} \Phi &= e^{246} - e^{235} - e^{145} - e^{136} + e^{127} + e^{347} + e^{567} \ , \\ *\Phi &= e^{1234} + e^{1256} + e^{3456} + e^{1357} - e^{1467} - e^{2367} - e^{2457} \ . \end{aligned} \tag{4.6}$$

We use a condensed notation, where we omit the wedge symbol when it is unlikely to cause confusion, and we collapse labels on products of 1-forms; thus, $e^{246} = e^2 \wedge e^4 \wedge e^6$, etc.

The necessary and sufficient conditions to satisfy conditions 2 and 3 are that, in addition to the algebraic conditions of (4.6), we also have the differential conditions

$$\Phi \wedge d\Phi = 0 \ , \qquad d\left[e^{-2\varphi} * \Phi\right] = 0 \ , \qquad *H = e^{2\varphi} d\left[e^{-2\varphi}\Phi\right] \ . \tag{4.7}$$

Note that the last one determines the torsion H, and the last two involve the dilaton φ.

4.2 Extended supersymmetry : conditions on X_7

In order to have extended supersymmetry in $d = 3$, X_7 must admit additional linearly independent ∇^-–constant spinors. Suppose there are $p+1$ such linearly independent spinors $\{\epsilon_0, \epsilon_1, \ldots, \epsilon_p\}$. Let $A = 1, \ldots, p$ index the "extra" spinors. Since $\{i\Gamma_i\epsilon_0, \epsilon_0\}$ are a complete basis, we can find vector fields V_A^i and functions u^A so that

$$\epsilon_A = iV_A^i\Gamma_i\epsilon_0 + u^A\epsilon_0 \tag{4.8}$$

for each A. Covariant constancy of ϵ_0 requires $\nabla^- V_A = 0$ and $\nabla^- u_A = 0$; the latter means that the u_A are just constants; we can set $u^A = 0$ without loss of generality [31].

We conclude that extended supersymmetry requires X_7 to admit of ∇^-–constant vector fields. Conversely, given p linearly-independent ∇^-–constant vector fields V_A, we can construct p additional spinors ϵ_A. We can take the V_A to be orthonormal.[9]

The reader may recall that any compact G_2 structure manifold admits 3 nowhere vanishing vector fields which reduce the structure further to SU(2) [32]. However, we stress that the supersymmetry conditions are stronger: the vectors must be annihilated by ∇^-.

Constraints on the number of vectors

Suppose that X_7 satisfies the minimal supersymmetry conditions and admits exactly p linearly independent ∇^-–constant vectors V_A. We will now show that the number of vectors V_A, $A = 1, \ldots, p$ can only take on specific values: $p \in \{0, 1, 3, 7\}$. Realizations of each of these cases are well known.

1. $p = 0$ corresponds to an irreducible X_7 — this is minimally supersymmetric and exemplified by, for example, one of Joyce's manifolds of G_2 holonomy [8] (standard embedding for the gauge bundle is a standard solution of the other supersymmetry constraints).

2. $p = 1$, which leads to $\mathcal{N} = 2$ in three dimensions, is also familiar: for instance we can take $X_7 = X_6 \times S^1$, where X_6 is a Calabi-Yau 3–fold; more generally, we can take X_7 to be a principal circle bundle over X_6.

3. $p = 3$, which leads to $\mathcal{N} = 4$ in three dimensions can be obtained from $X_7 = \mathrm{K3} \times T^3$; again, it is easy to make more general solutions by fibering the T^3 over K3.

4. $p = 7$, which leads to $\mathcal{N} = 8$ in three dimensions can be obtained by taking $X_7 = T^7$.

Two vectors imply a third

Suppose we have two vectors V_A, $A = 1, 2$. Given these, we can construct the dual 1-forms Θ^A, and we can also find a third 1-form

$$\Theta^3 = V_1 \llcorner V_2 \llcorner \Phi \ . \tag{4.9}$$

The $\llcorner$ denotes contraction of the vector field into the form: given a k–form $\omega = \frac{1}{k!}\omega_{i_1 \cdots i_k} dx^{i_1} \cdots dx^{i_k}$, the $k-1$–form $V \llcorner \omega$ is

$$V \llcorner \omega = \frac{1}{(k-1)!} V^{i_1} \omega_{i_1 i_2 \cdots i_k} dx^{i_2} \cdots dx^{i_k} \ .$$

By construction Θ^3 is ∇^-–constant and annihilated by V_1 and V_2. Hence, if $\Theta^3 \neq 0$, its dual V_3 will be a third ∇^-–constant vector linearly independent from V_1 and V_2.

[9] It is not hard to show that the V_A are Killing vectors; moreover their commutator is determined by a pairwise contraction with the torsion H.

To show that Θ^3 is non-zero, we compute its norm:

$$\|\Theta^3\|^2 = V_1^i V_2^j \Phi_{ijk} V_1^l V_2^m \Phi_{lmn} g^{km} = V_1^i V_1^l V_2^j V_2^m \Phi_{ijk} \Phi_{lm}{}^k \ . \tag{4.10}$$

Using (4.5) we find (recall that the V_A are orthonormal by assumption)

$$\|\Theta^3\|^2 = \|V_1\|^2 \|V_2\|^2 = 1 \ . \tag{4.11}$$

Thus, if X_7 has $p \geq 2$ ∇^-–constant vectors, then $p \geq 3$.

From 4 to 7 vectors

We will now show that if $p \geq 4$, then $p = 7$. Suppose that we have exactly p ∇^-–constant vectors V^A and their dual 1–forms Θ_A. We can choose all of these to be orthonormal and in any patch complete the basis with some 1-forms e^α, $\alpha = 1, \ldots, 7-p$. The Hodge star then decomposes as $*_7 = *_p *_{7-p}$, and the 3-form Φ is

$$\Phi = \Phi^{(3)} + \Phi^{(2)}_A \Theta^A + \tfrac{1}{2} \Phi^{(1)}_{AB} \Theta^{AB} + \tfrac{1}{3!} \Phi^{(0)}_{ABC} \Theta^{ABC} \ , \tag{4.12}$$

where the $\Phi^{(s)}$ are s–forms constructed from the e^α:

$$\begin{aligned} \Phi^{(0)}_{ABC} &= \Theta^{ABC} \llcorner \Phi \ , \\ \Phi^{(1)}_{AB} &= \Theta^{AB} \llcorner \Phi - \tfrac{1}{2} \Phi^{(0)}_{ABC} \Theta^C \ , \\ \Phi^{(2)}_{A} &= \Theta^{A} \llcorner \Phi - \Phi^{(1)}_{AB} \Theta^B - \tfrac{1}{2} \Phi^{(0)}_{ABC} \Theta^{BC} \ , \end{aligned} \tag{4.13}$$

and $\Phi^{(3)}$ is found by taking the difference of these terms with Φ. Clearly the $\Phi^{(s)}$ are ∇^-–constant. In particular, the dual $\Phi^{(1)}$, if non-zero, would yield an additional vector that is linearly independent from the Θ^A. So, we set $\Phi^{(1)} = 0$ and work with

$$\begin{aligned} \Phi &= \Phi^{(3)} + \Phi^{(2)}_A \Theta^A + \tfrac{1}{3!} \Phi^{(0)}_{ABC} \Theta^{ABC} \ , \\ *\Phi &= (*_{7-p} \Phi^{(3)})(*_p 1) + (*_{7-p} \Phi^{(2)}_A)(*_p \Theta^A) + \tfrac{1}{3!} (*_{7-p} \Phi^{(0)}_{ABC})(*_p \Theta^{ABC}) \\ \Psi &= \Psi^{(4-p)} (*_p 1) + \Psi^{(5-p)}_A (*_p \Theta^A) + \tfrac{1}{3!} \Psi^{(7-p)}_{ABC} (*_p \Theta^{ABC}) \ . \end{aligned} \tag{4.14}$$

Since $\Psi = *\Phi$, the last line is merely convenient notation for the contents of the second one. By the same arguments as above, the $\Psi^{(7-p-s)}$ are ∇^-–constant and linearly independent from the Θ^A.

Now consider the possibility $p = 4$. This requires $\Phi^{(2)}_A = 0$, since otherwise $\Psi^{(1)}_A$ yields an additional 1–form. On the other hand, since $\wedge^3 \mathbb{R}^4 = \mathbb{R}$ we can write $\Phi^{(0)}_{ABC} = \epsilon_{ABCD} Y^D$ for some constants Y^D, but this contradicts non-degeneracy of Φ because Φ is annihilated by $\sum_A Y^A V_A$.

Similarly, $p = 5$ is not compatible with a non-degenerate metric. To see this, recall that

Φ determines the metric as

$$g_{ij} = \frac{1}{144}\epsilon^{klmnpqr}\Phi_{ikl}\Phi_{jmn}\Phi_{pqr}\ . \tag{4.15}$$

If $p = 5$, then Φ is given by

$$\Phi = e^{12}k_A\Theta^A + \Phi^{(0)}_{ABC}\Theta^{ABC}\ , \tag{4.16}$$

and a moment's thought shows the contradiction: on one hand we assumed without loss of generality that $\{e^1, e^2, \Theta^1, \ldots, \Theta^5\}$ is an orthonormal basis, but on the other hand from the explicit formula for g we have $g_{11} = 0$.

Finally, $p \geq 6$ implies $p = 7$ because otherwise Φ is annihilated by the dual of e^1.

4.3 A no-go theorem

As we just argued, the geometry of X_7 admits exactly p ∇^-–constant vectors V_A only if $p \in \{0, 1, 3, 7\}$. Naively such X_7 lead to $\mathcal{N} = \{1, 2, 4, 8\}$ supersymmetry in three dimensions. Of course this requires that the remaining supersymmetry conditions are obeyed with ϵ_0 replaced by the corresponding ϵ_A, and it may be that this only holds for some $k < p$ spinors. This would lead to extended supersymmetry with $\mathcal{N} = k+1$. We will now prove the following no-go result: if $p > 1$ then $k > 2$, so that a solution with $\mathcal{N} \geq 3$ necessarily has $\mathcal{N} \geq 4$. Similarly, $\mathcal{N} \geq 5$ implies $\mathcal{N} = 8$.

To get started, we note that (4.2) holds if and only if

$$\Phi \llcorner H = 0\ , \qquad\qquad 2d\varphi = -H \llcorner \Psi\ . \tag{4.17}$$

To show this we apply the completeness relation (B.8) to (4.2), which shows the latter to be equivalent to

$$0 = \epsilon_0^T \left[\nabla^i\varphi\Gamma_i - \tfrac{1}{12}H^{ijk}\Gamma_{ijk}\right]\epsilon_0\ , \qquad 0 = \epsilon_0^T\Gamma_m \left[\nabla^i\varphi\Gamma_i - \tfrac{1}{12}H^{ijk}\Gamma_{ijk}\right]\epsilon_0\ .$$

Since Γ_i is antisymmetric, the first equation is the statement $\Phi \llcorner H = 0$; using (4.4) the second condition leads to $2d\varphi = -H \llcorner \Psi$.

Similarly, applying (4.4) to the gaugino variation, we learn that

$$\mathcal{F}^{ij}\Gamma_{ij}\epsilon_0 = 0 \qquad \Longleftrightarrow \qquad \mathcal{F} \llcorner \Phi = 0 \qquad \Longleftrightarrow \qquad \mathcal{F} = \mathcal{F} \llcorner \Psi\ . \tag{4.18}$$

The third relation follows from the second by contracting $\mathcal{F} \llcorner \Phi$ into (the non-degenerate) Ψ and using (4.5) . With these preparations in hand, we assume minimal supersymmetry, and we turn to extended supersymmetry.

The gravitino variation for ϵ_A

The existence of the spinors ϵ_A yields extra G_2 structures:

$$\Phi^A_{ijk} = i\epsilon^T_A \Gamma_{ijk}\epsilon_A \ , \qquad\qquad \Psi^A_{ijkl} = \epsilon^T_A \Gamma_{ijkl}\epsilon_A \ . \tag{4.19}$$

Using $\epsilon_A = iV^n_A \Gamma_n \epsilon_0$ we can also write this as

$$\Phi^A_{ijk} = iV^m_A V^n_A \epsilon^T_0 \Gamma_m \Gamma_i \Gamma_j \Gamma_k \Gamma_n \epsilon_0 \ . \tag{4.20}$$

By commuting the Γ_m through $\Gamma_i\Gamma_j\Gamma_k$, we obtain an elegant form for Φ^A:

$$\Phi^A = 2\Theta^A \wedge (V_A \llcorner \Phi) - \Phi \ . \tag{4.21}$$

In other words, to obtain Φ^A from Φ we write out Φ in a Θ expansion, and we flip the sign of every term that does contain Θ^A. The Φ^A will be ∇^-–constant since Φ and Θ^A are ∇^-–constant. Note that

$$\nabla^- \Theta^A = 0 \qquad \Longrightarrow \qquad d\Theta^A = V^A \llcorner H \ . \tag{4.22}$$

The dilatino variation for ϵ_A

The dilatino variation will vanish for ϵ_A provided that

$$V^m_A \left(\nabla^i \varphi \Gamma_i - \frac{1}{12} H^{ijk} \Gamma_{ijk} \right) \Gamma_m \epsilon_0 = 0 \ . \tag{4.23}$$

Since it vanishes for ϵ_0, we can replace this with the anti-commutator

$$V^m_A \Big\{ \left(\nabla^i \varphi \Gamma_i - \frac{1}{12} H^{ijk} \Gamma_{ijk} \right), \Gamma_m \Big\} \epsilon_0 = 0 \ , \tag{4.24}$$

and some Clifford algebra manipulations, together with (4.4), reduce this to

$$V_A \llcorner d\varphi = 0 \ , \qquad\qquad (V_A \llcorner H) \llcorner \Phi = 0 \ . \tag{4.25}$$

The gaugino variation for ϵ_A

Finally, we have

$$\mathcal{F}^{ij} \Gamma_{ij} \epsilon_A = iV^m_A \mathcal{F}^{ij} [\Gamma_i \Gamma_j, \Gamma_m] \epsilon_0 = 4iV^m_A \mathcal{F}^{ij} g_{jm} \epsilon_0 \ . \tag{4.26}$$

So, the vanishing of the gaugino variation for ϵ_A reduces to

$$V_A \llcorner \mathcal{F} = 0 \ . \tag{4.27}$$

$\mathcal{N} \geq 3$ **implies** $\mathcal{N} \geq 4$

Suppose that all of the supersymmetry conditions are satisfied by ϵ_0 and ϵ_A for $A = 1, 2$. We will call these the $\mathcal{N} = 3$ supersymmetry conditions. From the results above we know that there exists a third ∇^-–constant spinor

$$\epsilon_3 = iV_2^m \Gamma_m \epsilon_1 = iV_3^m \Gamma_m \epsilon_0 \ , \qquad\qquad V_3^m = V_2^i V_1^j \Phi_{ij}{}^m \ . \tag{4.28}$$

We will now show that ϵ_3 also yields a supersymmetry.

Let us start with the gaugino variation. Minimal supersymmetry requires $\mathcal{F} = \mathcal{F} \llcorner \Psi$, so

$$V_3^m \mathcal{F}_{mn} = \tfrac{1}{2} V_2^k V_1^l \Phi_{kl}{}^m \mathcal{F}^{ij} \Psi_{ijmn} = -\tfrac{1}{2} \Theta_k^2 \Theta_l^1 \mathcal{F}^{ij} \Psi_{ijnm} \Phi^{klm} \ . \tag{4.29}$$

Using (4.5) we then obtain

$$V_3^m \mathcal{F}_{mn} = -\Theta_k^2 \Theta_l^1 \mathcal{F}^{ij} \left(\delta_{[i}^{[k} \Phi_{jn]}{}^{l]} + \delta_{[j}^{[k} \Phi_{ni]}{}^{l]} + \delta_{[n}^{[k} \Phi_{ij]}{}^{l]} \right) = 0 \ . \tag{4.30}$$

The last equality follows because every term in the sum is proportional to either $V_1 \llcorner \mathcal{F}$, $V_2 \llcorner \mathcal{F}$, or to $\mathcal{F} \llcorner \Phi$, and all of these are zero by the $\mathcal{N} = 3$ conditions.

Next, we consider the term $V_3 \llcorner d\varphi$ that arises from the dilatino variation. Using minimal supersymmetry we have

$$-2V_3 \llcorner d\varphi = V_3 \llcorner (H \llcorner \Psi) \ . \tag{4.31}$$

In components we have

$$-2V_3 \llcorner d\varphi = \tfrac{1}{2} \Theta_p^2 \Theta_q^1 \tfrac{1}{3!} H^{ijk} \Psi_{ijkm} \Phi^{pqm} \ , \tag{4.32}$$

and (4.5) allows us to rewrite this as

$$-2V_3 \llcorner d\varphi = \tfrac{1}{2} V_1 \llcorner [(V_2 \llcorner H) \llcorner \Phi] - \tfrac{1}{2} V_2 \llcorner [(V_1 \llcorner H) \llcorner \Phi] = 0 \ . \tag{4.33}$$

The last equality follows because each square bracket is zero by $\mathcal{N} = 3$ conditions.

Finally, we need to show that $(V_3 \llcorner H) \llcorner \Phi = 0$. This requires more details on the structure of Φ and H.[10] The first ingredient is the form of Φ with $p = 3$ vectors. As we show in the appendix, we have

$$\Phi = \omega_1 \Theta^1 + \omega_2 \Theta^2 + \omega_3 \Theta^3 + \Theta^{123} \ , \tag{4.34}$$

where $\omega_A = \frac{1}{2} M_{Aij} e^i \wedge e^j$ are three self-dual 2–forms that satisfy the SU(2) structure relations

[10] At the level of representation theory the comparative difficulty can be traced to the fact that H has components in both **27** and **7** of $\wedge^3 T_X^*$ under the G_2 structure decomposition.

$\omega_A \wedge \omega_B = 2\delta_{AB} e^{1234}$. This implies that Ψ is given by

$$\Psi = \tfrac{1}{2}\omega_A \epsilon_{ABC} \Theta^{BC} + e^{1234} \ . \tag{4.35}$$

Next, we obtain constraints on H. The H flux has a general expansion

$$H = H^{(3)} + H^{(2)}_A \Theta^A + \tfrac{1}{2} H^{(1)}_{AB} \Theta^{AB} + H^{(0)} \Theta^{123} \ , \tag{4.36}$$

and minimal supersymmetry requires

$$0 = H \llcorner \Phi = H^{(2)}_A \llcorner \omega_A + H^{(0)} \ . \tag{4.37}$$

A short computation shows that the $\mathcal{N} = 3$ supersymmetry conditions imply $H^{(1)}_{AB} = 0$ for all A and B, while

$$H^{(2)}_A \llcorner \omega_B = -\delta_{AB} H^{(0)} \qquad A = 1, 2, \qquad\qquad B = 1, 2, 3 \ . \tag{4.38}$$

There are further constraints on H from the minimal supersymmetry conditions. First, since H determines $d\Theta^A$ via

$$d\Theta^A = V_A \llcorner H = H^{(2)}_A + \tfrac{1}{2} H^{(0)} \epsilon_{ADE} \Theta^{DE} \tag{4.39}$$

we see that

$$d\Phi = \omega_A H^{(2)}_A + \{\text{terms with at least one } \Theta\} \ . \tag{4.40}$$

Therefore, $\Phi \wedge d\Phi = 0$ implies[11]

$$H^{(2)}_A \omega_A = 0 \qquad\qquad \Longleftrightarrow \qquad\qquad H^{(2)}_A \llcorner \omega_A = 0 \ . \tag{4.41}$$

Combining this result with (4.37), we conclude that $H^{(0)} = 0$, so that $V_A \llcorner H = H^{(2)}_A$.

For our last machination we note that since $H \llcorner \Psi = -2d\varphi$, and $V^A \llcorner d\varphi = 0$, $H \llcorner \Psi$ cannot have any Θ components. On the other hand, we have

$$H \llcorner \Psi = H^{(3)} \llcorner e^{1234} - H^{(2)}_A \llcorner \omega_B \epsilon_{ABC} \Theta^C \ . \tag{4.42}$$

The latter terms vanish if and only if

$$H^{(2)}_A \llcorner \omega_B = H^{(2)}_B \llcorner \omega_A \tag{4.43}$$

[11] The second condition follows from the first because $\omega_A = *\omega_A$.

for all A, B. But, combining this with (4.38) and $H^{(0)} = 0$, we finally have

$$H_A^{(2)} \llcorner \omega_B = 0 \tag{4.44}$$

for all A and B. So, at last, $(V_A \llcorner H) \llcorner \Phi = 0$ for all A, and, as promised, $\mathcal{N} \geq 3$ implies $\mathcal{N} \geq 4$.

Incidentally, $H^{(0)} = 0$ also implies that all three vectors V_A commute, so the $\mathcal{N} = 4$ solutions are all of the form of a T^3 bundle over a hyper-Hermitian surface. Just as in the analogous case of $d = 4$ $\mathcal{N} = 2$ compactifications [6], we expect that the most general geometric solution of this form is indeed a T^3 bundle over a K3.

$\mathcal{N} \geq 5$ implies $\mathcal{N} = 8$

Finally, we show that a compactification with $\mathcal{N} \geq 5$ necessarily preserves maximal supersymmetry. By assumption of $\mathcal{N} \geq 5$, we have ϵ_0 and $\epsilon_A = iV_A^m \Gamma_m \epsilon_0$, with $A = 1, 2, 3$, as well as $\epsilon_4 = iV_4^m \Gamma_m \epsilon_0$ that solve the supersymmetry constraints. We also know that X_7 admits three more independent ∇^-–constant vectors $\widetilde{V}_a$, with $a = 5, 6, 7$. Without loss of generality we take the $\widetilde{V}_a$ orthonormal and orthogonal to the V_A; we define their dual forms $\widetilde{\Theta}^a$.

From above we know that for $\mathcal{N} \geq 4$ Φ takes the form

$$\Phi = \omega_A \Theta^A + \Theta^{123} \ ,$$

where the ω_A are self-dual and satisfy $\omega_A \wedge \omega_B = 2\delta_{AB} \Theta^4 \widetilde{\Theta}^{567}$. The conditions on ω_A imply that

$$\omega_A = U_{Aa} \left[\Theta^4 \widetilde{\Theta}^a + \tfrac{1}{2} \epsilon^{abc} \widetilde{\Theta}^b \widetilde{\Theta}^c \right] \ , \tag{4.45}$$

with $U_{Aa} U_{Ba} = \delta_{AB}$. Hence $\Theta'^A = \Theta^4 \llcorner \omega_A$ are three orthonormal ∇^-–constant 1-forms that are also orthogonal to $\Theta^1, \ldots, \Theta^4$. The dual vectors V'_A complete the V_A to a basis for T_X. Moreover, we have

$$\Theta'^A = V_A \llcorner (V_4 \llcorner \Phi) \ , \tag{4.46}$$

and therefore the arguments we gave in the previous section guarantee that the spinors ϵ'_A constructed using the vectors V'_A satisfy all of the supersymmetry conditions and generate three additional supersymmetries.

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A Three vectors on X_7 and constraints on G_2 form

We argued in section 4.2 that with p nowhere vanishing vectors V_A we must have

$$\Phi = \Phi^{(3)} + \Phi_A^{(2)}\Theta^A + \tfrac{1}{3!}\Phi^{(0)}_{ABC}\Theta^{ABC} \ . \tag{A.1}$$

When $p = 3$ we need $\Phi^{(3)} = 0$, since otherwise $*_4\Phi^{(3)}$ will yield an additional vector. So, we have

$$\Phi = \omega_A\Theta^A + k\Theta^{123} \ . \tag{A.2}$$

We can take $k \geq 0$, since the sign of k can be changed by redefining $\Theta^A \to -\Theta^A$ and $\omega_A \to -\omega_A$. This is just a convenient choice of orientation on X_7.

We assume that $\{e^1, e^2, e^3, e^4, \Theta^1, \Theta^2, \Theta^3\}$ is an orthonormal basis for T^*_X and check the compatibility of this with the metric obtained from Φ via

$$g_{ij} = \frac{1}{144}\epsilon^{klmnpqr}\Phi_{ikl}\Phi_{jmn}\Phi_{pqr} \ . \tag{A.3}$$

A bit of algebra and (A.2) show that

$$\begin{aligned} 144 g_{ij} &= 3\epsilon^{ABC}\epsilon^{\alpha\beta\gamma\delta}(\Phi_{iAB}\Phi_{j\alpha\beta} + \Phi_{i\alpha\beta}\Phi_{jAB})\Phi_{C\gamma\delta} \\ &\quad - 12\epsilon^{ABC}\epsilon^{\alpha\beta\gamma\delta}\Phi_{iA\alpha}\Phi_{jB\beta}\Phi_{C\gamma\delta} + \epsilon^{ABC}\epsilon^{\alpha\beta\gamma\delta}\Phi_{i\alpha\beta}\Phi_{j\gamma\delta}\Phi_{ABC} \ . \end{aligned} \tag{A.4}$$

This can be unpacked into various components. Taking E_α to be the dual vectors to e^a, we have the following results:

$$\begin{aligned} g(E_\mu, V_A) &= 0 \ , \\ g(V_D, V_E) &= \frac{k}{8}\epsilon^{\alpha\beta\gamma\delta}\Phi_{D\alpha\beta}\Phi_{E\gamma\delta} \ , \\ g(E_\mu, E_\nu) &= -\frac{1}{12}\epsilon^{ABC}\epsilon^{\alpha\beta\gamma\delta}\Phi_{A\mu\alpha}\Phi_{B\nu\beta}\Phi_{C\gamma\delta} \ . \end{aligned} \tag{A.5}$$

Starting with the general form of Φ, we write $\omega_A = \frac{1}{2}M_{A\alpha\beta}e^\alpha \wedge e^\beta$, so that $\Phi_{A\alpha\beta} = M_{A\alpha\beta}$. Finally, setting

$$(*M)_{A\alpha\beta} = \tfrac{1}{2}\epsilon^{\alpha\beta\gamma\delta}M_{A\gamma\delta} \ , \tag{A.6}$$

we obtain a simple form for the metric components:

$$g(V_D, V_E) = -\tfrac{k}{4}\,\mathrm{Tr}(M_D(*M_E)) \ , \qquad g(E_\mu, E_\nu) = -\tfrac{1}{6}\epsilon^{ABC}(M_A(*M_B)M_C)_{\mu\nu} \ . \tag{A.7}$$

Since we already verified $g(E_\mu, V_A) = 0$, we now just need to check that

$$\delta_{DE} = -\tfrac{k}{4}\,\mathrm{Tr}(M_D(*M_E)) \,, \qquad \mathbb{1}_4 = -\tfrac{1}{6}\epsilon^{ABC}(M_A(*M_B)M_C) \,. \tag{A.8}$$

Reduction of parameters by SO(4) action

Since (A.8) is invariant under SO(4) rotations $M_A \to R^T M_A R$ we can bring the anti-symmetric matrices M_A to a canonical form. Without loss of generality we set

$$M_1 = \begin{pmatrix} x_1\rho & 0 \\ 0 & y_1\rho \end{pmatrix} \,, \tag{A.9}$$

where $\rho = i\sigma_2$ and $x_1 \geq 0$, $y_1 \geq 0$.[12] This is stabilized by an SO(2) × SO(2) action, which allows us to bring M_2 to the form

$$M_2 = \begin{pmatrix} x_2\rho & P_2 \\ -P_2^T & y_2\rho \end{pmatrix} \,, \qquad P_2 = \begin{pmatrix} 0 & b_2 \\ c_2 & 0 \end{pmatrix} \,, \tag{A.10}$$

with $c_2 \geq 0$. Finally, M_3 takes the general form

$$M_3 = \begin{pmatrix} x_3\rho & P_3 \\ -P_3^T & y_3\rho \end{pmatrix} \,, \qquad P_3 = \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} \,. \tag{A.11}$$

Solution of the constraints

We now have a system of 16 equations in (A.8) that depend on 13 parameters: 12 of these are in the reduced M_A, and k is the last one. The equations have a unique solution, with

$$M_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \,, \quad M_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix} \,, \quad M_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \,, \tag{A.12}$$

or in terms of Pauli matrices

$$M_1 = \mathbb{1}_2 \otimes i\sigma_2 \,, \qquad M_2 = i\sigma_2 \otimes \sigma_1 \,, \qquad M_3 = i\sigma_2 \otimes \sigma_3 \,. \tag{A.13}$$

These satisfy

$$*M_A = M_A \,, \qquad M_A M_B = -\delta_{AB}\mathbb{1}_4 + \epsilon_{ABC} M_C \,. \tag{A.14}$$

[12] The σ_i are the Pauli matrices.

Thus,

$$\Phi = \omega_A \Theta^A + \Theta^{123} \ , \tag{A.15}$$

and the ω_A are non-degenerate, self-dual, and satisfy $\omega_A \wedge \omega_B = 2\delta_{AB} e^{1234}$.

B An integral flux for $\mathcal{N} = 3$ supersymmetry

The case of the $\mathcal{N} = 3$ vacuum on $X \times \widetilde{X}$ is exotic enough that it is worthwhile to check that it can be obtained by some choice of integral flux and no space-filling M2-branes.[13]

Let $x = (2A - C)$, and write $f_{\alpha\dot{\alpha}} = x\phi_{\alpha\dot{\alpha}}$. To obtain exactly $\mathcal{N} = 3$ supersymmetry, we take $x \neq 0$ and the flux must be

$$G = -x j_a \widetilde{\jmath}_a + 2x[vE + \widetilde{v}\widetilde{E}] + x\omega_\alpha \phi_{\alpha\dot{\alpha}} \widetilde{\omega}_{\dot{\alpha}} \ . \tag{B.1}$$

This flux is integral if and only if

$$\frac{1}{2\pi}\left[-x j_a \widetilde{\jmath}_{\dot{a}} + \omega_\alpha f_{\alpha\dot{\alpha}} \widetilde{\omega}_{\dot{\alpha}}\right] \in H^4(X \times \widetilde{X}, \mathbb{Z}) \qquad \text{and} \qquad \frac{xv}{\pi} \in \mathbb{Z} \ , \qquad \frac{x\widetilde{v}}{\pi} \in \mathbb{Z} \ . \tag{B.2}$$

While the implications of the first of these are not immediately obvious, the last two are readily solved: there are non-zero integers m, $\widetilde{m}$ such that

$$v = \frac{m\pi}{x} \ , \qquad \qquad \widetilde{v} = \frac{\widetilde{m}\pi}{x} \ . \tag{B.3}$$

The integrated Bianchi identity now becomes

$$\frac{m\widetilde{m}}{2}\left[5 + \mathrm{tr}(\phi^T \phi)\right] = 24 - N(M_2) \ . \tag{B.4}$$

We will now demonstrate that we can choose $m, \widetilde{m}$ and ϕ so that the flux is integral and $N(M_2) = 0$.

Our Ansatz for the flux is motivated by the counting of massless moduli for $\mathcal{N} = 3$ vacua: we see that at best, the number of geometric moduli preserved is that of a single K3 geometry, so that it is not unreasonable to tie the geometries of X and $\widetilde{X}$ together. In fact, we will take X and $\widetilde{X}$ to be identical.

Let us explain a little bit more what this means. We fix an integral basis $\{e^1, e^2, \ldots, e^{22}\}$ for $H^2(X, \mathbb{Z})$ such that

$$e^I e^J = D^{IJ} E \ , \tag{B.5}$$

[13] We thank Dave Morrison for stressing the importance of this point and for discussions regarding the solution presented here and its possible generalizations.

where $D = (-\mathrm{E}_8)^{\oplus 2} \oplus H^{\oplus 3}$ is the standard metric of signature $(3,19)$. Since $H^2(X,\mathbb{Z})$ is unimodular, we have the key fact that D^{-1}, with components denoted by D_{IJ} is also an integral matrix. There is a corresponding set of forms $\widetilde{e}^{\dot{I}}$ on $\widetilde{X}$ that have identical structure.

The j_a and ω_α can be written in terms of the integral basis:

$$j_a = \mathcal{E}_{aI} e^I \ , \qquad\qquad \omega_\alpha = \mathcal{E}_{\alpha I} e^I \ . \tag{B.6}$$

The coefficients obey

$$\mathcal{E}_{aI} D^{IJ} \mathcal{E}_{bJ} = 2\delta_{ab} v \ , \qquad \mathcal{E}_{aI} D^{IJ} \mathcal{E}_{\alpha J} = 0 \ , \qquad \mathcal{E}_{\alpha I} D^{IJ} \mathcal{E}_{\beta J} = -2\delta_{\alpha\beta} v \ . \tag{B.7}$$

There is also a useful completeness relation for the vielbeins $\mathcal{E}$:

$$\mathcal{E}_{aI}\mathcal{E}_{aJ} - \mathcal{E}_{\alpha I}\mathcal{E}_{\alpha J} = 2 v D_{IJ} \ . \tag{B.8}$$

We now describe our Ansatz for the flux.

1. We assume that X has an integral -4 class that is orthogonal to all of the j_a. That is, there exists $\xi \in H^2(X,\mathbb{Z})$ that is annihilated by the j_a and satisfies $\xi \wedge \xi = \xi \cdot \xi E = -4E$. It is easy to construct smooth K3 geometries with this property at low Picard number.

   Without loss of generality we can take ξ to be the direction of one of the anti-self-dual forms. More precisely, we set

$$\omega_1 = \sqrt{\tfrac{v}{2}}\xi \ . \tag{B.9}$$

2. Once we choose this data for X, we use the same $\mathcal{E}_{aI}$ and $\mathcal{E}_{\alpha I}$ to prescribe the $\widetilde{\jmath}_{\dot{a}}$ and $\widetilde{\omega}_{\dot{\alpha}}$, i.e. the geometry of $\widetilde{X}$:

$$\widetilde{\jmath}_a = \mathcal{E}_{a\dot{I}} \widetilde{e}^{\dot{I}} \ , \qquad\qquad \widetilde{\omega}_\alpha = \mathcal{E}_{\alpha\dot{I}} \widetilde{e}^{\dot{I}} \ . \tag{B.10}$$

   This implies that $v = \widetilde{v}$, and therefore $m = \widetilde{m}$ as well; we also have a form $\widetilde{\xi}$ as a special -4 class on $\widetilde{X}$.

3. We take the $\phi_{\alpha\dot{\alpha}}$ to be diagonal: $\phi_{\alpha\dot{\alpha}} = \phi_\alpha \delta_{\alpha\dot{\alpha}}$.

With these assumptions the flux takes the form

$$\begin{aligned} \frac{G}{2\pi} &= -\frac{x}{2\pi}\left[j_a \widetilde{\jmath}_a - \omega_\alpha \widetilde{\omega}_\alpha\right] + \frac{x}{2\pi}(\phi_\alpha - 1)\omega_\alpha \widetilde{\omega}_\alpha \\ &= -m D_{I\dot{I}} e^I \widetilde{e}^{\dot{I}} + \frac{x}{2\pi}(\phi_\alpha - 1)\omega_\alpha \widetilde{\omega}_\alpha \ , \end{aligned} \tag{B.11}$$

where in the second line we used the completeness relation (B.8).

The reason this works nicely is that the first term is automatically integral, and we just need to choose the ϕ_α appropriately so that the last term is integral as well. We accomplish this by setting $\phi_\alpha = 1$ for all $\alpha > 1$, so that now

$$\frac{G}{2\pi} = -mD_{I\dot{I}} e^I \widetilde{e}^{\dot{I}} + \frac{m(\phi_1 - 1)}{4} \xi\widetilde{\xi} \,. \tag{B.12}$$

Choosing $\phi_1 = 5$ leads to an integral flux.

For this choice of integral flux the Bianchi identity becomes

$$\frac{m^2}{2}\left[5 + 5^2 + 18\right] = 24 - N(M_2) \,, \tag{B.13}$$

and setting $m = 1$, we find the desired $N(M_2) = 0$.

We have shown that there is a choice of flux that leads to exactly $\mathcal{N} = 3$ supersymmetry without space-filling M2 branes. The choice leaves many moduli; indeed, the number of massless scalars is smaller than the maximum allowed by just one $\mathcal{N} = 3$ "hypermultiplet."

It is not so easy to generalize this solution. If one stays with the "completeness" relation trick above and simply modifies the $\phi_{\alpha\dot{\alpha}}$ it is quite likely there are no others with $N(M_2) = 0$.[14]

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