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Stable recovery of signals from frame coefficients with erasures at unknown locations

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Abstract In an earlier work, we proposed a frame-based kernel analysis approach to the problem of recovering erasures from unknown locations. The new approach led to the stability question on recovering a signal from noisy partial frame coefficients with erasures occurring at unknown locations. In this continuing work, we settle this problem by obtaining a complete characterization of frames that provide stable reconstructions. We show that an encoding frame provides a stable signal recovery from noisy partial frame coefficients at unknown locations if and only if it is totally robust with respect to erasures. We present several characterizations for either totally robust frames or almost robust frames. Based on these characterizations several explicit construction algorithms for totally robust and almost robust frames are proposed. As a consequence of the construction methods, we obtain that the probability for a randomly generated frame to be totally robust with respect to a fixed number of erasures is one.

Keywords frames, erasures, signal recovery, almost robust frames, totally robust frames

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1 Introduction

In applications frames are often used for analyzing and reconstructing signals, and it is well known that frames are generally robust to erasures, distortions and noises. Moreover, the redundancy property of frames also makes it possible in many cases to have perfect reconstruction from erasure-corrupted frame coefficients (see [1–8]). This has led to many approaches dealing with reconstructions of signals from noisy data (see [9–11, 13, 15–18]) and generated lots of research on characterizations and constructions of optimal or near-optimal frames for different considerations (see [14, 19–22, 24–26]). In addition, we refer to [27, 28] for more constructions on optimal frames.

If the encoding frame is properly selected, then a perfect reconstruction from erasure-corrupted frame coefficients can be achieved by using the so called "partial frame operators" (see [12, 26]). However, this approach often causes some stability problem in the process of inverting the partial frame operators. We proposed in [12] a frame-based kernel analysis approach to the problems of recovering the erasures that

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occurred at either known or unknown locations. This method recovers the lost data by solving, in most cases, a simple system of linear equations. In addition, the new method is particularly efficient when the number of erasures is relatively small. It is known that a frame allows perfect reconstruction with respect to any m-erasures occurring at known locations if and only if any subsequence obtained by removing any m-vectors from the frame sequence remains to be a frame. We proved in [12] that a frame allows erasure location recovery for almost all input signals if and only if it is almost robust. Unfortunately, while the reconstruction is stable from noise-free partial frame coefficients (at unknown locations), it is generally unstable when the received data (again from unknown locations) also carries noise. This suggests that we need to impose additional restrictions on almost robust frames in order to have a stable reconstruction. The main purpose of this paper is to address this problem.

In [11], Han et al. considered the stability of the reconstruction with more general settings, where the known partial frame coefficients are supposed to be unordered. Although the frames considered in [11] also solve the problem considered in this paper, it is still interesting to consider the stability problem when we assume only that the erasure locations are unknown, i.e., the partial frame coefficients are in correct order. In fact, the characterization and construction of frames in this paper are more simple than that in [11], which is expected since the problem we considered in [11] are more complicated.

We first recall and introduce a few definitions and terminologies that are needed throughout this paper. A frame for a finite-dimensional Hilbert space $\mathcal{N}$ is a sequence $\{\varphi_i\}_{1\leqslant i\leqslant N}$ such that there exist positive constants α and β such that

$$\alpha\|f\|^2 \leqslant \sum_{i=1}^{N} |\langle f, \varphi_i\rangle|^2 \leqslant \beta\|f\|^2, \quad \forall f \in \mathcal{N},$$

where α and β are called the lower and upper frame bounds, respectively. A frame $\{\varphi_i\}_{1\leqslant i\leqslant N}$ is said to be tight if $\alpha=\beta$, and Parseval if $\alpha=\beta=1$.

Let $\{\varphi_i\}_{1\leqslant i\leqslant N}$ be a frame for $\mathcal{N}$. Its analysis operator T is defined by

$$Tf = \{\langle f, \varphi_i\rangle\}_{1\leqslant i\leqslant N}, \quad \forall f \in \mathcal{N}.$$

It is easy to see that T^*T is invertible on $\mathcal{N}$ and $\{\tilde{\varphi}_i = (T^*T)^{-1}\varphi_i\}_{1\leqslant i\leqslant N}$ is also a frame for $\mathcal{N}$, which is called the *canonical or standard dual frame*. The canonical dual provides us the following reconstruction formula:

$$f = \sum_{i=1}^{N} \langle f, \varphi_i\rangle \tilde{\varphi}_i, \quad f \in \mathcal{N}. \tag{1.1}$$

Note that whenever $\{\varphi_i\}_{1\leqslant i\leqslant N}$ is a frame but not a basis, there are many (actually, infinitely many) other choices of $\tilde{\varphi}_i$ for which (1.1) holds. In general, if two frames $\{\varphi_i\}_{1\leqslant i\leqslant N}$ and $\{\tilde{\varphi}_i\}_{1\leqslant i\leqslant N}$ satisfy (1.1), we call them a *pair of dual frames*.

In this paper, we only consider frames $\{\varphi_i\}_{1\leqslant i\leqslant N}$ with pairwise different elements. In addition similarly for a matrix A, we always assume that its column vectors are pairwise different. We often identify a frame with the matrix consisting of frame vectors as its column vectors.

In applications the frame coefficients $c_i = \langle f, \varphi_i\rangle$ of a signal f (encoding f) are transmitted to a receiver and then the receiver reconstructs (decodes) the original signal f from the received data set. Erasures occur often in data transmission due to various reasons. How to recover signals from frame coefficients with erasures is an interesting problem both in theory and in practice. In modern communication networks, this problem is usually solved with coding theory. Here, we consider an alternate solution to this problem and show that the redundancy of a frame can mitigate the effect of losses in packet-based communication systems.

Assume that m frame coefficients are erased during the data transmission. In the case that the receiver knows the location of erasures, then the erasures can be easily recovered by solving a simple system of linear equations as long as the encoding frame $\{\varphi_i\}_{i=1}^{N}$ has the property that it remains to be a frame whenever any of its m elements are removed (see [12]). In the case that erasures occurred at unknown locations, then we face the following natural problems: suppose that N coefficients $c_1, \ldots, c_N$ were sent and we only received $c_{i_1}, \ldots, c_{i_{N-m}}$, where we only know that $i_1 < i_2 < \cdots < i_{N-m}$ but we have no information about the exact values of these indices. Is it still possible to recover the original signal with these coefficients? The answer to this question depends on both the encoding frames and the input signals. One of the main results in [12] provided a complete characterization of the encoding frames that ensure erasure location recovery for almost all input signals.

Definition 1.1 (See [12]). A frame is called almost robust with respect to m-erasures if we can recover any $f \in \mathcal{N} \setminus \mathcal{N}_0$ from its frame coefficients with m-erasures at unknown locations, where $\mathcal{N}_0$ is the union of finitely many proper subspaces of $\mathcal{N}$ and therefore is of measure zero.

Assume that $\{\varphi_i\}_{1\leqslant i\leqslant N}$ is an almost robust frame with respect to m-erasures. For $i_1 < \cdots < i_{N-m}$, write $\tilde{\mathcal{N}} := \mathrm{span}\{\varphi_{i_l}\}_{1\leqslant l\leqslant N-m}$. Then we have $\tilde{\mathcal{N}} = \mathcal{N}$. In fact, if $\tilde{\mathcal{N}} \neq \mathcal{N}$, then $\tilde{\mathcal{N}}$ is a proper subspace of $\mathcal{N}$ and its n-dimensional measure is 0. For any $f \in \mathcal{N} \setminus \tilde{\mathcal{N}}$, we can only recover the projection of f on $\tilde{\mathcal{N}}$ from the coefficients $\{\langle f, \varphi_{i_l}\rangle\}_{1\leqslant l\leqslant N-m}$, while f itself cannot be recovered, which contradicts with the almost robustness of $\{\varphi_i\}_{1\leqslant i\leqslant N}$.

The following characterizes almost robust frames.

Proposition 1.2 (See [12, Theorem 4.2]). *A frame $\{\varphi_i\}_{1\leqslant i\leqslant N}$ is almost robust with respect to m-erasures if and only if $\{T^{i_1,\ldots,i_{N-m}}\mathcal{N} : 1 \leqslant i_1 < \cdots < i_{N-m} \leqslant N\}$ consists of pairwise different n-dimensional subspaces, where $T^{i_1,\ldots,i_{N-m}}$ is the analysis operator corresponding to $\{\varphi_{i_\ell}\}_{1\leqslant \ell\leqslant N-m}$,*

$$T^{i_1,\ldots,i_{N-m}} f = \{\langle f, \varphi_{i_\ell}\rangle\}_{\ell=1}^{N-m}.$$

Now let us briefly discuss how to recover the erasure locations with an almost m-erasure robust frame $\{\varphi_i\}_{1\leqslant i\leqslant N}$. For each $1 \leqslant i_1 < \cdots < i_{N-m}$, we see from Proposition 1.2 that $\dim T^{i_1,\ldots,i_{N-m}}\mathcal{N} = n < N-m$. Hence, there exists some $(N-m-n)\times(N-m)$ matrix $M(i_1,\ldots,i_{N-m})$ such that

$$T^{i_1,\ldots,i_{N-m}} f = \ker(M(i_1,\ldots,i_{N-m})). \tag{1.2}$$

Assume that m erasures at unknown locations occur during data transformation. In addition, the received coefficient sequence is $x := (x_1, \ldots, x_{N-m})'$. Then we can recover the erasures with the following steps:

(i) Let

$$(i_1^0, \ldots, i_{N-m}^0) = \mathop{\arg\min}_{i_1<\cdots<i_{N-m}} \|M(i_1,\ldots,i_{N-m})x\|. \tag{1.3}$$

(ii) Set $c = \{c_i : 1 \leqslant i \leqslant N\}$ with $c_{i_l^0} = x_l$, $1 \leqslant l \leqslant N-m$. By solving the equation

$$M(1,\ldots,N)c = 0, \tag{1.4}$$

we get erased coefficients, where $M(1,\ldots,N)$ is a matrix satisfying $T\mathcal{N} = \ker(M(1,\ldots,N))$.

In the case that the received data is noise free, then both theory and numerical experiments tell us that we can get perfect reconstruction of f with very high probability (see [12]). However, in real applications the received data almost always carries noise. Denote the received coefficients by $T^{i_1,\ldots,i_{N-m}} f + \varepsilon$. Let

$$(i_1,\ldots,i_{N-m}) = \mathop{\arg\min}_{j_1<\cdots<j_{N-m}} \|M(j_1,\ldots,j_{N-m})(T^{i_1,\ldots,i_{N-m}} f + \varepsilon)\|.$$

Here, we remark that due to the noise interference it might occur that $(i_1,\ldots,i_{N-m}) \neq (i_1,\ldots,i_{N-m})$. Since $\{\varphi_i\}_{i=1}^N$ is almost robust, there is some $N\times(N-m)$ matrix $H(i_1,\ldots,i_{N-m})$ such that

$$T = H(i_1,\ldots,i_{N-m}) T^{i'_1,\ldots,i'_{N-m}}, \tag{1.5}$$

where $T := T^{1,\ldots,N}$. Then the reconstructed signal in this way is given by

$$Rf = (T'T)^{-1}T' H(i_1,\ldots,i_{N-m})(T^{i_1,\ldots,i_{N-m}} f + \varepsilon).$$

We say that the reconstruction algorithm is *stable* if there exists some constant C, which is independent of choices of $(i_1, \ldots, i_{N\ \ m})$, such that

$$\backslash Rf \quad f\backslash_2 \leqslant C\backslash\varepsilon\backslash_2. \tag{1.6}$$

Note that the stability defined in this way is only theoretical, i.e., the reconstruction error tends to zero as the input error does. In practice, if the constant C is large enough, then a small noise level ε might result in a big reconstruction error. In this paper, we focus on the theoretical part, which ensures exact reconstruction from noise free frame coefficients.

Numerical experiments indicate that for some choices of almost robust frames, the above algorithm might be not stable. Our main result of this paper is to give a complete characterization of almost robust frames that provide stable reconstructions. For this purpose we introduce the following definition.

Definition 1.3. Let A be an $n \subset N$ matrix with column vectors $\}\varphi_i\langle_{1\leqslant i\leqslant N}$. We say that A (or $\}\varphi_i\langle_{1\leqslant i\leqslant N}$) is totally robust with respect to m-erasures if for any $1 \leqslant i_1 < \times\times\times< i_{N\ \ m} \leqslant N$, $1 \leqslant i_1 < \times\times\times< i_{N\ \ m} \leqslant N$ and $x, x \ / \mathcal{N}$ satisfying $\rangle x, \varphi_{i_l}| = \rangle x\ , \varphi_{i'_l}|$, $1 \leqslant l \leqslant N \quad m$, we have $x = x$.

We show later that totally robust frames are automatically almost robust (see Lemma 4.2). The following is one of the main results of this paper.

Theorem 1.4. *The above reconstruction algorithm is stable, i.e.,* (1.6) *holds, if and only if* $\}\varphi_i\langle_{1\leqslant i\leqslant N}$ *is totally robust.*

Totally robust frames can be explicitly constructed (see Sections 5 and 6). Moreover, we prove that any randomly generated frame has probability one of being totally robust.

Theorem 1.5. *Let* $A = (a_{i,j})_{1\leqslant i\leqslant n, 1\leqslant j\leqslant N}$ *be an* $n \subset N$ *matrix and* $N \quad m \geqslant 2n$*, where* $a_{i,j}$ *are independent continuous random variables. Then the probability for* A *being totally robust with respect to* m*-erasures is one.*

The rest of the paper is organized as follows. In Section 2, we give a proof of Theorem 1.4. In Section 3, we provide a concrete method for constructing almost robust frames. Sections 4 and 5 are devoted to totally robust frames. We first present some characterizations for totally robust frames in Section 4 and use them to show that totally robust frames are automatically almost robust. On the basis of characterizations of totally robust frames we provide in Section 5, an explicit construction method for such frames, and show that Theorem 1.5 follows naturally from the construction method. In Section 6, we use a set of different prime numbers to construct some explicit examples of frames that are either almost robust or totally robust with respect to different number of erasures.

Notation. For a given matrix A, we adopt the following notation for convenience: $A_{j_1,\ldots,j_k}$ stands for the submatrix of A consisting of the j_1-th, $\ldots$, j_k-th columns, $A^{i_1,\ldots,i_k}_{j_1,\ldots,j_k}$ is the submatrix of A consisting of the i_1-th, $\ldots$, i_k-th rows and the j_1-th, $\ldots$, j_k-th columns, and $(A^{i_1,\ldots,i_k}_{j_1,\ldots,j_k})^c$ stands for the submatrix of A that results from removing the i_1-th, $\ldots$, i_k-th rows and the j_1-th, $\ldots$, j_k-th columns.

2 Stability of the reconstruction algorithm

In order to prove Theorem 1.4, we need the following lemma.

Lemma 2.1. *Let* $\}\varphi_i\langle_{i=1}^N$ *be an almost robust frame for* $\mathcal{N}$ *with respect to* m*-erasures. For any* $i_1 < \times\times\times< i_{N\ \ m}$ *and* $i_1 < \times\times\times< i_{N\ \ m}$ *with* $(i_1, \ldots, i_{N\ \ m}) \forall (i_1, \ldots, i_{N\ \ m})$*, there exists some* $(n \quad s) \subset (N \quad m \quad n)$ *matrix* Q *such that for any* $f \ / \mathcal{N}$*,*

$$\backslash T^{i_1,\ldots,i_{N-m}}(f \quad f_0)\backslash_2 = \backslash QM(i_1, \ldots, i_{N\ \ m})T^{i_1,\ldots,i_{N-m}}(f \quad f_0)\backslash_2,$$

where $M(i_1, \ldots, i_{N\ \ m})$ *is defined by* (1.2)*,* $f_0 \ / \mathcal{N}$ *satisfies the condition that* $T^{i_1,\ldots,i_{N-m}}f_0$ *is the orthogonal projection of* $T^{i_1,\ldots,i_{N-m}}f$ *on* $T^{i_1,\ldots,i_{N-m}}\mathcal{N}\{\ T^{i'_1,\ldots,i'_{N-m}}\mathcal{N}$*, and* $s := \dim(T^{i_1,\ldots,i_{N-m}}\mathcal{N}\{\ T^{i'_1,\ldots,i'_{N-m}}\mathcal{N})$*.*

Proof. First, we consider the case of $s \geqslant 1$. Take an orthonormal basis $\{\eta_1, \dots, \eta_s\}$ of $T^{i_1,\dots,i_{N-m}}\mathcal{N} \cap T^{i'_1,\dots,i'_{N-m}}\mathcal{N}$. Select some vectors $\eta_{s+1}, \dots, \eta_n \in T^{i_1,\dots,i_{N-m}}\mathcal{N}$ and $\eta_{s+1}, \dots, \eta_n \in T^{i'_1,\dots,i'_{N-m}}\mathcal{N}$ such that $\{\eta_1, \dots, \eta_s, \eta_{s+1}, \dots, \eta_n\}$ and $\{\eta_1, \dots, \eta_s, \eta_{s+1}, \dots, \eta_n\}$ are orthonormal bases for $T^{i_1,\dots,i_{N-m}}\mathcal{N}$ and $T^{i'_1,\dots,i'_{N-m}}\mathcal{N}$, respectively.

Let W be an $(N-m-n) \times (N-m-n)$ matrix such that the row vectors of $W^{-1}M(i_1, \dots, i_{N-m})$ are orthonormal. Denote

$$(W^{-1}M(i_1, \dots, i_{N-m}))' = (\xi_1, \dots, \xi_{N-m-n}). \tag{2.1}$$

Then $\eta_1, \dots, \eta_s, \eta_{s+1}, \dots, \eta_n, \xi_1, \dots, \xi_{N-m-n}$ constitute an orthonormal basis for $\mathbb{C}^{N-m}$. Consequently, for $s+1 \leqslant i \leqslant n$, η_i can be linearly represented by $\eta_{s+1}, \dots, \eta_n, \xi_1, \dots, \xi_{N-m-n}$. Observe that $\eta_{s+1}, \dots, \eta_n, \eta_{s+1}, \dots, \eta_n$ are linearly independent. We conclude that the matrix

$$\begin{aligned} V &= \begin{pmatrix} \langle \eta_{s+1}, \xi_1 \rangle & \cdots & \langle \eta_n, \xi_1 \rangle \\ \vdots & \ddots & \vdots \\ \langle \eta_{s+1}, \xi_{N-m-n} \rangle & \cdots & \langle \eta_n, \xi_{N-m-n} \rangle \end{pmatrix} \\ &= \begin{pmatrix} \xi'_1 \\ \vdots \\ \xi'_{N-m-n} \end{pmatrix} (\, \eta_{s+1} \ \cdots \ \eta_n \,) = W^{-1}M(i_1, \dots, i_{N-m})(\, \eta_{s+1} \ \cdots \ \eta_n \,) \end{aligned}$$

is of full column rank. Otherwise, there are constants $c_1, \dots, c_{n-s}$, not all of which are zeros, such that

$$V \begin{pmatrix} c_1 \\ \vdots \\ c_{n-s} \end{pmatrix} = 0,$$

i.e.,

$$\begin{pmatrix} \xi'_1 \\ \vdots \\ \xi'_{N-m-n} \end{pmatrix} \sum_{t=1}^{n-s} c_t \eta_{t+s} = 0.$$

Consequently,

$$\begin{aligned} \sum_{t=1}^{n-s} c_t \eta_{t+s} &= \sum_{u=1}^{N-m-n} \left\langle \sum_{t=1}^{n-s} c_t \eta_{t+s}, \xi_u \right\rangle \xi_u + \sum_{v=1}^{n-s} \left\langle \sum_{t=1}^{n-s} c_t \eta_{t+s}, \eta_{v+s} \right\rangle \eta_{v+s} \\ &= \sum_{v=1}^{n-s} \left\langle \sum_{t=1}^{n-s} c_t \eta_{t+s}, \eta_{v+s} \right\rangle \eta_{v+s}, \end{aligned}$$

which contradicts with the fact that $\eta_{s+1}, \dots, \eta_n, \eta_{s+1}, \dots, \eta_n$ are linearly independent.

Therefore, we can find some $(n-s) \times (N-m-n)$ matrix U such that

$$UV = I. \tag{2.2}$$

Fix some vector $f \in \mathcal{N}$. Then there exists some vector $f_0 \in \mathcal{N}$ such that $T^{i_1,\dots,i_{N-m}} f_0$ is the orthogonal projection of $T^{i_1,\dots,i_{N-m}} f$ on $T^{i_1,\dots,i_{N-m}}\mathcal{N} \cap T^{i'_1,\dots,i'_{N-m}}\mathcal{N}$. It follows that

$$T^{i_1,\dots,i_{N-m}}(f - f_0) \in \operatorname{span}\{\eta_{s+1}, \dots, \eta_n\}.$$

Hence, we get

$$T^{i_1,\dots,i_{N-m}}(f - f_0) = (\, \eta_{s+1} \ \cdots \ \eta_n \,) \times \begin{pmatrix} \langle T^{i_1,\dots,i_{N-m}}(f - f_0), \eta_{s+1} \rangle \\ \vdots \\ \langle T^{i_1,\dots,i_{N-m}}(f - f_0), \eta_n \rangle \end{pmatrix}. \tag{2.3}$$

Therefore,

$$\|T^{i_1,\dots,i_{N-m}}(f - f_0)\|_2 = \|\{\langle T^{i_1,\dots,i_{N-m}}(f - f_0), \eta_t\rangle\}_{t=s+1}^{n}\|_2. \tag{2.4}$$

By (2.1) and (2.3), we obtain that

$$\begin{aligned} &M(i_1,\dots,i_{N-m})T^{i_1,\dots,i_{N-m}}(f - f_0) \\ &= W \begin{pmatrix} \xi_1' \\ \vdots \\ \xi_{N-m-n}' \end{pmatrix} (\eta_{s+1} \cdots \eta_n) \times \begin{pmatrix} \langle T^{i_1,\dots,i_{N-m}}(f - f_0), \eta_{s+1}\rangle \\ \vdots \\ \langle T^{i_1,\dots,i_{N-m}}(f - f_0), \eta_n\rangle \end{pmatrix} \\ &= WV \begin{pmatrix} \langle T^{i_1,\dots,i_{N-m}}(f - f_0), \eta_{s+1}\rangle \\ \vdots \\ \langle T^{i_1,\dots,i_{N-m}}(f - f_0), \eta_n\rangle \end{pmatrix}. \end{aligned}$$

It follows from (2.2) that

$$UW^{-1}M(i_1,\dots,i_{N-m})T^{i_1,\dots,i_{N-m}}(f - f_0) = \begin{pmatrix} \langle T^{i_1,\dots,i_{N-m}}(f - f_0), \eta_{s+1}\rangle \\ \vdots \\ \langle T^{i_1,\dots,i_{N-m}}(f - f_0), \eta_n\rangle \end{pmatrix}.$$

By setting $Q = UW^{-1}$, we get the desired conclusion.

For the case of $s = 0$, the same arguments work with $f_0 = 0$. This completes the proof. □

Proof of Theorem 1.4. (i) First, we prove the sufficiency.

Fix some vector $f \in \mathcal{N}$. Suppose that there are some $(i_1,\dots,i_{N-m}) \neq (i_1,\dots,i_{N-m})$ such that

$$\begin{aligned} &\|M(i_1,\dots,i_{N-m})(T^{i_1,\dots,i_{N-m}}f + \varepsilon)\|_2 \\ &\quad \leqslant \|M(i_1,\dots,i_{N-m})(T^{i_1,\dots,i_{N-m}}f + \varepsilon)\|_2 \\ &\quad = \|M(i_1,\dots,i_{N-m})\varepsilon\|_2. \end{aligned} \tag{2.5}$$

Let $f_0 \in \mathcal{N}$ be such that $T^{i_1,\dots,i_{N-m}}f_0$ is the projection of $T^{i_1,\dots,i_{N-m}}f$ on $T^{i_1,\dots,i_{N-m}}\mathcal{N} \cap T^{i_1',\dots,i_{N-m}'}\mathcal{N}$. By Lemma 2.1, there is a matrix Q such that

$$\begin{aligned} &\|T^{i_1,\dots,i_{N-m}}(f - f_0)\|_2 \\ &\quad = \|QM(i_1,\dots,i_{N-m})T^{i_1,\dots,i_{N-m}}(f - f_0)\|_2 \\ &\quad \leqslant \|Q\| \cdot \|M(i_1,\dots,i_{N-m})T^{i_1,\dots,i_{N-m}}(f - f_0)\|_2. \end{aligned} \tag{2.6}$$

Since $T^{i_1,\dots,i_{N-m}}f_0 \in T^{i_1',\dots,i_{N-m}'}\mathcal{N}$, we have

$$\begin{aligned} &\|M(i_1,\dots,i_{N-m})(T^{i_1,\dots,i_{N-m}}(f - f_0) + \varepsilon)\|_2 \\ &\quad = \|M(i_1,\dots,i_{N-m})(T^{i_1,\dots,i_{N-m}}f + \varepsilon)\|_2 \\ &\quad \leqslant \|M(i_1,\dots,i_{N-m})\varepsilon\|_2. \end{aligned}$$

Consequently,

$$\begin{aligned} &\|M(i_1,\dots,i_{N-m})T^{i_1,\dots,i_{N-m}}(f - f_0)\|_2 \\ &\quad \leqslant (\|M(i_1,\dots,i_{N-m})\| + \|M(i_1,\dots,i_{N-m})\|)\|\varepsilon\|_2. \end{aligned}$$

By (2.6), we have

$$\|T^{i_1,\dots,i_{N-m}}(f - f_0)\|_2 \leqslant \|Q\|(\|M(i_1,\dots,i_{N-m})\| + \|M(i_1,\dots,i_{N-m})\|)\|\varepsilon\|_2. \tag{2.7}$$

By the choice of f_0, we have $T^{i_1,\ldots,i_{N-m}} f_0 = T^{i'_1,\ldots,i'_{N-m}} f_0$. Hence,

$$(T'T)^{-1}T' H(i_1,\ldots,i_{N-m}) T^{i_1,\ldots,i_{N-m}} f_0 = f_0. \tag{2.8}$$

Recall that $H(i_1,\ldots,i_{N-m})$ is defined by (1.5). Since $\{\varphi_{i_l}\}_{l=1}^{N-m}$ is a frame for $\mathcal{N}$, there is some constant $C_1 > 0$ such that

$$\|f - f_0\|_2 \leqslant C_1 \|T^{i_1,\ldots,i_{N-m}}(f - f_0)\|_2. \tag{2.9}$$

Combining (2.7)–(2.9), we get

$$\begin{aligned}
&\|(T'T)^{-1}T' H(i_1,\ldots,i_{N-m})(T^{i_1,\ldots,i_{N-m}} f + \varepsilon) - f\|_2 \\
&\quad = \|(T'T)^{-1}T' H(i_1,\ldots,i_{N-m})(T^{i_1,\ldots,i_{N-m}}(f - f_0) + \varepsilon) - (f - f_0)\|_2 \\
&\quad \leqslant \|f - f_0\|_2 + \|(T'T)^{-1}T' H(i_1,\ldots,i_{N-m})\| \times \|\varepsilon\|_2 \\
&\qquad + \|(T'T)^{-1}T' H(i_1,\ldots,i_{N-m})\| \times \|T^{i_1,\ldots,i_{N-m}}(f - f_0)\|_2 \\
&\quad \leqslant (C_1 + \|(T'T)^{-1}T' H(i_1,\ldots,i_{N-m})\|) \\
&\qquad \times \|T^{i_1,\ldots,i_{N-m}}(f - f_0)\|_2 + \|(T'T)^{-1}T' H(i_1,\ldots,i_{N-m})\| \times \|\varepsilon\|_2 \\
&\quad \leqslant C\|\varepsilon\|_2.
\end{aligned}$$

(ii) Next, we prove the necessity. Assume that there are $f_0 \neq f_0$ such that $T^{i_1,\ldots,i_{N-m}} f_0 = T^{i'_1,\ldots,i'_{N-m}} f_0$. Then we have $f_0 \neq 0$, $f_0 \neq 0$, $(i_1,\ldots,i_{N-m}) \neq (i_1,\ldots,i_{N-m})$ and $s := \dim(T^{i_1,\ldots,i_{N-m}}\mathcal{N} \cap T^{i'_1,\ldots,i'_{N-m}}\mathcal{N}) \geqslant 1$.

Let $\eta_1,\ldots,\eta_s,\eta_{s+1},\ldots,\eta_n$, and $\eta_{s+1},\ldots,\eta_n$ be defined as in the proof of Lemma 2.1. Since $\{\varphi_i\}_{i=1}^N$ is almost robust, we have $T^{i_1,\ldots,i_{N-m}}\mathcal{N} \neq T^{i'_1,\ldots,i'_{N-m}}\mathcal{N}$. Hence $n - s > 0$.

Let $f = f_0$ and $\varepsilon = \lambda\eta_{s+1}$, where $\lambda > 0$. Since $\eta_{s+1} / T^{i'_1,\ldots,i'_{N-m}}\mathcal{N} \surd T^{i_1,\ldots,i_{N-m}}\mathcal{N}$, we have $M(i_1,\ldots,i_{N-m})\varepsilon = 0$ and $M(i_1,\ldots,i_{N-m})\varepsilon \neq 0$. Consequently,

$$M(i_1,\ldots,i_{N-m})(T^{i_1,\ldots,i_{N-m}} f_0 + \varepsilon) = M(i_1,\ldots,i_{N-m}) T^{i'_1,\ldots,i'_{N-m}} f_0 = 0$$

and

$$M(i_1,\ldots,i_{N-m})(T^{i_1,\ldots,i_{N-m}} f_0 + \varepsilon) = M(i_1,\ldots,i_{N-m})\varepsilon \neq 0. \tag{2.10}$$

Hence, $(i_1,\ldots,i_{N-m})$ is a minimizer of $\arg\min_{j_1<\cdots<j_{N-m}} \|M(j_1,\ldots,j_{N-m})(T^{i_1,\ldots,i_{N-m}} f_0 + \varepsilon)\|$. By the stability, there is some constant C such that

$$\|(T'T)^{-1}T' H(i_1,\ldots,i_{N-m})(T^{i_1,\ldots,i_{N-m}} f_0 + \varepsilon) - f_0\|_2 \leqslant C\|\varepsilon\|,$$

i.e.,

$$\|(T'T)^{-1}T' H(i_1,\ldots,i_{N-m})(T^{i_1,\ldots,i_{N-m}} f_0 + \lambda\eta_{s+1}) - f_0\|_2 \leqslant C\lambda.$$

By letting $\lambda \to 0$, we get $f_0 = (T'T)^{-1}T' H(i_1,\ldots,i_{N-m}) T^{i_1,\ldots,i_{N-m}} f_0 = f_0$, which contradicts with the choice of f_0 and f_0. This completes the proof. □

3 Construction of almost robust frames

In this section, we present a new characterization for almost robust frames with which we provide an algorithm for constructing almost robust frames. Since frames and matrices are equivalent, for convenience, we introduce the following definition.

Definition 3.1. Suppose that n and N are positive integers such that $n + 2 \leqslant N$. We call an $n \times N$ matrix M almost robust if the frame consisting of its column vectors is almost robust with respect to $(N - n - 1)$-erasures.

The following is a characterization of almost robust matrices.

Theorem 3.2. *An $n \subset N$ matrix A is almost robust if and only if for any $1 \leqslant j_1 < \times\times\times < j_{n+1} \leqslant N$ and $1 \leqslant j_1 < \times\times\times < j_{n+1} \leqslant N$ with $(j_1, \ldots, j_{n+1}) \not\forall (j_1, \ldots, j_{n+1})$,*

$$\operatorname{rank}\begin{pmatrix} A_{j_1,\ldots,j_{n+1}} \\ A_{j'_1,\ldots,j'_{n+1}} \end{pmatrix} = n + 1.$$

Recall that $A_{j_1,\ldots,j_{n+1}}$ is the submatrix of A consisting of the j_1-th, $\ldots, j_{n+1}$-th columns.

Proof. The necessity is a consequence of Proposition 1.2. For the sufficiency, we only need to show that for any $1 \leqslant j_1 < \times\times\times < j_{n+1} \leqslant N$, $\operatorname{rank}(A_{j_1,\ldots,j_{n+1}}) = n$.

Assume that $\operatorname{rank}(A_{j_1,\ldots,j_{n+1}}) < n$ for some $1 \leqslant j_1 < \times\times\times < j_{n+1} \leqslant N$. Then we can find some $1 \leqslant j_1 < \times\times\times < j_{n+1} \leqslant N$ such that only one l satisfies $j_l \not\forall j_l$. Without loss of generality, we assume $j_1 \not\forall j_1$ and $j_l = j_l$ for $2 \leqslant l \leqslant n+1$. Then we have

$$\operatorname{rank}\begin{pmatrix} A_{j_2,\ldots,j_{n+1}} \\ A_{j'_2,\ldots,j'_{n+1}} \end{pmatrix} = \operatorname{rank}(A_{j_2,\ldots,j_{n+1}}) \leqslant \operatorname{rank}(A_{j_1,\ldots,j_{n+1}}) \leqslant n \quad 1.$$

Hence

$$\operatorname{rank}\begin{pmatrix} A_{j_1,\ldots,j_{n+1}} \\ A_{j'_1,\ldots,j'_{n+1}} \end{pmatrix} \leqslant n,$$

which contradicts with the hypothesis. This completes the proof. □

Construction of almost robust frames. (i) Take some $(a_{i,1})_{1\leqslant i\leqslant n} / \mathbb{R}^n$ such that $a_{1,1} \not\forall 0$.

(ii) Take some $(a_{i,2})_{1\leqslant i\leqslant n} / \mathbb{R}^n$ such that $a_{1,2} \not\forall \}0, a_{1,1}\langle$ and

$$\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} \not\forall 0.$$

(iii) Assume that for some $2 \leqslant k \leqslant N \quad 1$ the matrix $(a_{i,j})_{1\leqslant i\leqslant n, 1\leqslant j\leqslant k}$ is well-defined such that for any $1 \leqslant t \leqslant \min\}k, n\langle$ and $1 \leqslant j_1 < \times\times\times < j_t \leqslant k$,

$$\operatorname{rank}\begin{pmatrix} a_{1,j_1} & \times\times\times a_{1,j_t} \\ \vdots & \ddots & \vdots \\ a_{t,j_1} & \times\times\times a_{t,j_t} \end{pmatrix} = t, \tag{3.1}$$

and for any $1 \leqslant s \leqslant \min\}k \quad 1, n\langle$, $1 \leqslant j_1 < \times\times\times < j_s \leqslant k$ and $1 \leqslant j_1 < \times\times\times < j_s \leqslant k$ with $j_l \not\forall j_l$, $\emptyset\, l$,

$$\operatorname{rank}\begin{pmatrix} a_{1,j_1} & a_{1,j'_1} & \times\times\times a_{1,j_s} & a_{1,j'_s} \\ \vdots & & \ddots & \vdots \\ a_{s,j_1} & a_{s,j'_1} & \times\times\times a_{s,j_s} & a_{s,j'_s} \end{pmatrix} = s.$$

We define $a_{1,k+1}, \ldots, a_{n,k+1}$ by induction. There are two cases.

Case 1. $2 \leqslant k \leqslant n$.

Take some $a_{1,k+1} \not\forall \}0, a_{1,1}, \ldots, a_{1,k}\langle$. Suppose that $a_{1,k+1}, \ldots, a_{p,k+1}$ are well-defined for some $1 \leqslant p \leqslant n \quad 1$. For $t = \min\}k, p\langle$ and $1 \leqslant j_1 < \times\times\times < j_t \leqslant k$, there is a unique x such that

$$\begin{vmatrix} a_{1,j_1} & \times\times\times & a_{1,j_t} & a_{1,k+1} \\ \vdots & \ddots & \vdots & \vdots \\ a_{t,j_1} & \times\times\times & a_{t,j_t} & a_{t,k+1} \\ a_{t+1,j_1} & \times\times\times & a_{t+1,j_t} & x \end{vmatrix} = 0.$$

In addition, for $s=\min\{k-1,p\}$, $1\leqslant j_1<\cdots<j_{s+1}\leqslant k$ and $1\leqslant j'_1<\cdots<j'_{s+1}=k+1$ with $j_l\neq j'_l$, $\forall\, 1\leqslant l\leqslant s+1$, there is a unique y such that

$$\begin{vmatrix} a_{1,j_1} & a_{1,j'_1} & \cdots a_{1,j_{s+1}} & a_{1,j'_{s+1}} \\ \vdots & & \ddots & \vdots \\ a_{s,j_1} & a_{s,j'_1} & \cdots a_{s,j_{s+1}} & a_{s,j'_{s+1}} \\ a_{s+1,j_1} & a_{s+1,j'_1} & \cdots\ a_{s+1,j_{s+1}} & y \end{vmatrix}=0.$$

Let $B_{p+1,k}$ be the set consisting of all such x and y when i_l, j_l and j'_l vary from all possible choices. Take some $a_{p+1,k+1}\in B_{p+1,k}$.

Case 2. $n+1\leqslant k\leqslant N-1$.

For any $1\leqslant j_1\leqslant\cdots\leqslant j_{n+1}\leqslant k$, there exists a unique sequence $c=(c_1,\ldots,c_n)$ such that

$$\sum_{l=1}^{n}c_la_{i,j_l}=a_{i,j_{n+1}},\quad 1\leqslant i\leqslant n.$$

By (3.1), every n columns of $(a_{i,j})_{1\leqslant i\leqslant n,1\leqslant j\leqslant k}$ are linearly independent. Hence none of c_l is zero. Denote by S_k the set of all such sequences. Let

$$C_{i,k}=\{0,a_{i,1},\ldots,a_{i,k}\}\cap\left\{\sum_{l=1}^{n}c_la_{i,j_l}:c\in S_k,1\leqslant j_1\leqslant\cdots\leqslant j_n\leqslant k\right\}.$$

Set $a_{1,k+1}\in C_{1,k}$. Assume that $a_{1,k+1},\ldots,a_{p,k+1}$ are well-defined for some $1\leqslant p\leqslant n-1$. Let $B_{p+1,k}$ be defined as in Case 1. Take some $a_{p+1,k+1}\in B_{p+1,k}\cap C_{p+1,k}$.

Next, we show that frames consisting of column vectors of matrices constructed in this way are almost robust. To see this, fix such a matrix A. Denote its column vectors by $\varphi_1,\ldots,\varphi_N$, respectively. By Theorem 3.2, it suffices to show that for $1\leqslant j_1<\cdots<j_{n+1}<N$ and $1\leqslant j'_1<\cdots<j'_{n+1}<N$ with $(j_1,\ldots,j_{n+1})\neq(j'_1,\ldots,j'_{n+1})$,

$$\operatorname{rank}\begin{pmatrix}A_{j_1,\ldots,j_{n+1}}\\ A_{j'_1,\ldots,j'_{n+1}}\end{pmatrix}=n+1. \tag{3.2}$$

There are two cases.

Case 1. $j_l\neq j'_l$, $\forall\, 1\leqslant l\leqslant n+1$.

In this case, we see from the construction ($a_{i,k+1}\in C_{i,k}$) that the equations

$$A_{j_1,\ldots,j_{n+1}}c=0\quad\text{and}\quad A_{j'_1,\ldots,j'_{n+1}}c=0 \tag{3.3}$$

have no common solution but zero. Hence (3.2) is true.

Case 2. There is some l such that $j_l=j'_l$.

Without loss of generality, we assume that $j_l=j'_l$ for $1\leqslant l\leqslant s$ and $j_l\neq j'_l$ for $s+1\leqslant l\leqslant n+1$, where s is an integer satisfying $1\leqslant s\leqslant n$. Let c be a solution of (3.3). Then we have

$$\sum_{i=s+1}^{n+1}c_l(\varphi_{j_l}-\varphi_{j'_l})=A_{j_1,\ldots,j_{n+1}}c-A_{j'_1,\ldots,j'_{n+1}}c=0.$$

Since $\{\varphi_{j_l}-\varphi_{j'_l}\}_{s+1\leqslant l\leqslant n+1}$ is a sequence of independent vectors, we have $c_l=0$, $s+1\leqslant l\leqslant n+1$. Hence

$$\sum_{l=1}^{s}c_l\varphi_{j_l}=A_{j_1,\ldots,j_{n+1}}c=0.$$

Consequently, $c_1=\cdots=c_s=0$. Hence $c=0$. Therefore, (3.2) is true.

4 Characterizations of totally robust frames

In this section, we present a characterization of totally robust frames with which we show that every totally robust frame is automatically almost robust.

Theorem 4.1. *Let A be an $n \subset N$ matrix with column vectors $\}\varphi_i\langle_{1\leqslant i\leqslant N}$. Then $\}\varphi_i\langle_{1\leqslant i\leqslant N}$ is a totally robust frame for $\mathcal{N}$ with respect to m-erasures if and only if for any $1 \leqslant j_1 < \times\times\times < j_{N\ \ m} \leqslant N$ and $1 \leqslant j_1 < \times\times\times < j_{N\ \ m} \leqslant N$,*

$$n + \operatorname{rank}(A_{j_1,\ldots,j_{N-m}} \quad A_{j'_1,\ldots,j'_{N-m}}) = \operatorname{rank}\begin{pmatrix} A_{j_1,\ldots,j_{N-m}} \\ A_{j'_1,\ldots,j'_{N-m}} \end{pmatrix}. \tag{4.1}$$

Proof. (i) First, we prove the necessity. Fix some $1 \leqslant j_1 < \times\times\times < j_k \leqslant N$ and $1 \leqslant j_1 < \times\times\times < j_k \leqslant N$, where $k = N \quad m$. Let $M = (A'_{j_1,\ldots,j_k} \ A'_{j'_1,\ldots,j'_k})$ and $r = \operatorname{rank}(M)$. Then we have

$$\dim\}x \ / \ \mathbb{R}^{2n} : Mx = 0\langle = 2n \quad r. \tag{4.2}$$

Denote $x = (\ {}^{c}_{c'}\ )\ /\ \mathbb{R}^n \subset \mathbb{R}^n$. Then $Mx = 0$ is equivalent to $A'_{j_1,\ldots,j_k}c \quad A'_{j'_1,\ldots,j'_k}c \ = 0$, i.e.,

$$\rangle c, \varphi_{j_l}| = \rangle c\ , \varphi_{j'_l}|, \quad 1 \leqslant l \leqslant k.$$

Since A is totally robust, we have $c = c$. Hence,

$$(A'_{j_1,\ldots,j_k} \quad A'_{j'_1,\ldots,j'_k})c = 0. \tag{4.3}$$

On the other hand, if c is a solution of the above equation, then $x = (\ {}^{c}_{c}\ )$ is a solution of $Mx = 0$. Hence

$$2n \quad r = n \quad \operatorname{rank}(A_{j_1,\ldots,j_k} \quad A_{j'_1,\ldots,j'_k}).$$

This proves (4.1).

(ii) Next, we prove the sufficiency. Suppose that (4.1) is true. Use the previous symbols. Let c be a solution of (4.3). Then $x = (\ {}^{c}_{c}\ )$ is a solution of $Mx = 0$. By (4.1), every solution of $Mx = 0$ must be of this form, i.e., if

$$\rangle c, \varphi_{j_l}| = \rangle c\ , \varphi_{j'_l}|, \quad 1 \leqslant l \leqslant k,$$

then $c = c$. Hence $\}\varphi_i\langle_{1\leqslant i\leqslant N}$ is totally robust with respect to m-erasures. □

The following result shows that a totally robust frame is always an almost robust frame.

Lemma 4.2. *Let $\}\varphi_i\langle_{1\leqslant i\leqslant N}$ be a frame for $\mathcal{N}$, where $n = \dim\mathcal{N}$ and $N \geqslant n+2$. Suppose that $\}\varphi_i\langle_{1\leqslant i\leqslant N}$ is totally robust with respect to m-erasures. Then we have the following*:

(i) $\}\varphi_i\langle_{1\leqslant i\leqslant N}$ *is almost robust with respect to m-erasures.*

(ii) $N \quad m \geqslant 2n \quad 1$.

Proof. Put $k = N \quad m$ and $A = (\varphi_1, \ldots, \varphi_N)$.

(i) First, we show that $\dim T^{j_1,\ldots,j_k}\mathcal{N} = n$ for any $j_1 < \times\times\times < j_k$, which is equivalent to $\operatorname{rank}(A_{j_1,\ldots,j_k}) = n$. Assume on the contrary that it is not true. Then there is some $x \forall 0$ such that

$$\rangle x, \varphi_{j_l}| = 0 = \rangle 0, \varphi_l|, \quad 1 \leqslant l \leqslant k,$$

which contradicts with the definition of total robustness.

Next, we show that $T^{j_1,\ldots,j_k}\mathcal{N} \forall T^{j'_1,\ldots,j'_k}\mathcal{N}$ for $(j_1,\ldots,j_k) \forall (j_1,\ldots,j_k)$. Assume on the contrary that $T^{j_1,\ldots,j_k}\mathcal{N} = T^{j'_1,\ldots,j'_k}\mathcal{N}$. Then for any $x\ /\ \mathcal{N}$, there is some $x\ \ /\ \mathcal{N}$ such that $T^{j_1,\ldots,j_k}x = T^{j'_1,\ldots,j'_k}x$, i.e.,

$$\rangle x, \varphi_{j_l}| = \rangle x\ , \varphi_{j'_l}|, \quad 1 \leqslant l \leqslant k.$$

Since $\}\varphi_i\langle_{1\leqslant i\leqslant N}$ is totally robust, we have $x = x$. Hence

$$\rangle x, \varphi_{j_l}| = \rangle x, \varphi_{j'_l}|, \quad \emptyset\, x\ /\ \mathcal{N}, \quad 1 \leqslant l \leqslant k,$$

which implies that $j_l = j_l$ for $1 \leqslant l \leqslant k$ and therefore contradicts with the assumption.

(ii) Let $M = (A'_{1,\dots,k}\ A'_{2,\dots,k+1})$. Then rank $(M) \leqslant k$. By (4.1), we have

$$\text{rank}\,(A_{1,\dots,k} \quad A_{2,\dots,k+1}) = \text{rank}\,(M) \quad n \leqslant k \quad n. \tag{4.4}$$

Since rank $(A_{1,\dots,k}) = n$, there exist some $1 \leqslant j_1 < \times\times\times < j_n \leqslant k$ such that $\varphi_{j_1}, \dots, \varphi_{j_n}$ are linearly independent. Hence $\varphi_{j_2} \quad \varphi_{j_1}, \dots, \varphi_{j_n} \quad \varphi_{j_{n-1}}$ are linearly independent. Therefore,

$$\text{rank}\,(A_{1,\dots,k} \quad A_{2,\dots,k+1}) \geqslant n \quad 1. \tag{4.5}$$

By (4.4), we have $k \geqslant 2n \quad 1$. □

Lemma 4.3. *Suppose that* $N \geqslant 2n + m$. *Let* A *be an* $n \subset N$ *matrix with column vectors* $\}\varphi_i\langle_{1\leqslant i\leqslant N}$ *such that every* n *columns of* A *are linearly independent and for any* $j_1 < \times\times\times < j_{2n}$ *and* $j_1 < \times\times\times < j_{2n}$ *with at least* $n+1$ l*'s satisfying* $j_l \not\forall j_l$,

$$\text{rank}\begin{pmatrix} A_{j_1,\dots,j_{2n}} \\ A_{j'_1,\dots,j'_{2n}} \end{pmatrix} = 2n.$$

Then $\}\varphi_i\langle_{1\leqslant i\leqslant N}$ *is totally robust with respect to* m*-erasures.*

Proof. Fix some $1 \leqslant j_1 < \times\times\times < j_{N \quad m} \leqslant N$ and $1 \leqslant j_1 < \times\times\times < j_{N \quad m} \leqslant N$. Suppose that for some $c, c \ / \mathcal{N}$ and for all $1 \leqslant l \leqslant N \quad m$,

$$\rangle c, \varphi_{j_l}| = \rangle c\ , \varphi_{j'_l}|. \tag{4.6}$$

There are two cases.

Case 1. There are at least n l's satisfying $j_l = j_l$. Since every n elements of $\}\varphi_i\langle$ form a basis for $\mathcal{N}$, we have $c = c$.

Case 2. There are at most $n \quad 1$ l's satisfying $j_l = j_l$. In this case, there are at least $n+1$ l's satisfying $j_l \not\forall j_l$. We see from the hypothesis that

$$\text{rank}\begin{pmatrix} A_{j_1,\dots,j_{N-m}} \\ A_{j'_1,\dots,j'_{N-m}} \end{pmatrix} = 2n.$$

By (4.6), we have

$$(A'_{j_1,\dots,j_{N-m}}\ A'_{j'_1,\dots,j'_{N-m}}) \begin{pmatrix} c \\ \ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence $c = c \ = 0$. This completes the proof. □

Lemma 4.4. *Suppose that* $N \geqslant 2n + m \quad 1$. *Let* A *be an* $n \subset N$ *matrix such that*

(i) $a_{1,j} = 1$ *for* $1 \leqslant j \leqslant N$,

(ii) *every* n *columns of* A *are linearly independent, and*

(iii) *for any* $j_1 < \times\times\times < j_{2n \quad 1}$ *and* $j_1 < \times\times\times < j_{2n \quad 1}$ *with at least* n l*'s satisfying* $j_l \not\forall j_l$,

$$\text{rank}\begin{pmatrix} A_{j_1,\dots,j_{2n-1}} \\ A_{j'_1,\dots,j'_{2n-1}} \end{pmatrix} = 2n \quad 1.$$

Then column vectors of A *form a totally robust frame with respect to* m*-erasures.*

Proof. As in the proof of Lemma 4.3, we fix some $1 \leqslant j_1 < \times\times\times < j_{N \quad m} \leqslant N$ and $1 \leqslant j_1 < \times\times\times < j_{N \quad m} \leqslant N$. Suppose that for some $c, c \ / \mathcal{N}$ and for all $1 \leqslant l \leqslant N \quad m$, (4.6) holds.

We consider only Case 2. In this case, there are at least n l's satisfying $j_l \not\forall j_l$. We see from the hypothesis that

$$\text{rank}\begin{pmatrix} A_{j_1,\dots,j_{N-m}} \\ A_{j'_1,\dots,j'_{N-m}} \end{pmatrix} = 2n \quad 1. \tag{4.7}$$

By (4.6), we have

$$(A'_{j_1,\dots,j_{N-m}}\ \ A'_{j'_1,\dots,j'_{N-m}})\begin{pmatrix} c \\ -c' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $a_{1,j}$ is equal to 1 for all j, $(c, -c') = (c_1, 0, \dots, 0, -c_1, 0, \dots, 0)$ is a solution of the above equation. Now we see from (4.7) that every solution of the above equation must be of this form. Hence $c = c'$. This completes the proof. □

At the end of this subsection, we show that for $N - m = 2n - 1$, totally robust frames have a special structure.

Theorem 4.5. *Suppose that $N = 2n - 1 + m$. Let A be an $n \times N$ matrix with column vectors $\{\varphi_i\}_{1\leqslant i\leqslant N}$. If $\{\varphi_i\}_{1\leqslant i\leqslant N}$ is totally robust with respect to m-erasures, then there is some invertible matrix U such that the first row of UA is $(1, \dots, 1)$.*

Proof. Since $\operatorname{rank}(A_{1,\dots,2n-1}) = n$, there exist some $1 \leqslant j_1 < \cdots < j_n \leqslant 2n - 1$ such that $\varphi_{j_1}, \dots, \varphi_{j_n}$ are linearly independent. Hence $\varphi_{j_2} - \varphi_{j_1}, \dots, \varphi_{j_n} - \varphi_{j_{n-1}}$ are linearly independent.

For $2n \leqslant j \leqslant N$, we see from Theorem 4.1 that

$$\operatorname{rank}(A_{1,\dots,2n-1} - A_{2,\dots,2n-1,j}) \leqslant n - 1.$$

On the other hand, since $\varphi_{j_2} - \varphi_{j_1}, \dots, \varphi_{j_n} - \varphi_{j_{n-1}}$ are linearly independent, we have

$$\operatorname{rank}(A_{1,\dots,2n-1} - A_{2,\dots,2n-1,j}) \geqslant n - 1.$$

Hence

$$\operatorname{rank}(A_{1,\dots,2n-1} - A_{2,\dots,2n-1,j}) = n - 1. \tag{4.8}$$

Denote $e_l = \varphi_{j_l} - \varphi_{j_{l-1}}$, $2 \leqslant l \leqslant n$. Then we have $e_l \in \operatorname{span}(\{\varphi_{i+1} - \varphi_i\}_{1\leqslant i\leqslant 2n-2})$. By (4.8), we have $\operatorname{span}(\{\varphi_{i+1} - \varphi_i\}_{1\leqslant i\leqslant 2n-2} \cup (\varphi_j - \varphi_{2n-1})) = \operatorname{span}(\{e_l\}_{2\leqslant l\leqslant n})$. Hence

$$\varphi_j - \varphi_1 \in \operatorname{span}(\{e_l\}_{2\leqslant l\leqslant n}), \quad 1 \leqslant j \leqslant N.$$

Let $V = (\varphi_1, e_2, \dots, e_n)$. Then we have $A = V\tilde{A}$, where $\tilde{A}$ is a matrix whose first row is $(1, \dots, 1)$. Since $\operatorname{rank}(A) = n$, V is nonsingular. Now we get the conclusion as desired by setting $U = V^{-1}$. □

5 Construction of totally robust frames

On the basis of characterizations of totally robust frames in Section 4, we provide an explicit construction method for such frames. This construction method also yields the density property of totally robust frames. We see from Lemma 4.2 that if an $n \times N$ matrix is totally robust with respect to m-erasures, then we have $N - m \geqslant 2n - 1$. In this section, we give two methods to construct totally robust frames, one for the case $N - m \geqslant 2n$ and the other for the case $N - m \geqslant 2n - 1$.

A: Construction of totally robust frames with $N - m \geqslant 2n$. Suppose that $N > 2n$. Let $A = (\varphi_1, \dots, \varphi_N)$. We define φ_i by induction.

First, set $k = n + 1$. As in the construction of almost robust frames, we can get an $n \times k$ matrix $A = (a_{i,j})_{1\leqslant i\leqslant n, 1\leqslant j\leqslant k}$ satisfying the following:

(P1) every n columns of A are linearly independent;

(P2) for any $1 \leqslant s \leqslant n$, $1 \leqslant j_1 < \cdots < j_s \leqslant k$ and $1 \leqslant j'_1 < \cdots < j'_s \leqslant k$ with $j_l \neq j'_l$, $\forall\, l$,

$$\operatorname{rank}\begin{pmatrix} a_{1,j_1} - a_{1,j'_1} & \cdots & a_{1,j_s} - a_{1,j'_s} \\ \vdots & \ddots & \vdots \\ a_{s,j_1} - a_{s,j'_1} & \cdots & a_{s,j_s} - a_{s,j'_s} \end{pmatrix} = s;$$

(P3) for any $n+1 \leqslant t \leqslant \min\{k, 2n\}$, $1 \leqslant j_1 < \cdots < j_t \leqslant k$ and $1 \leqslant j'_1 < \cdots < j'_t \leqslant k$ with at least $t-n$ l's satisfying $j_l \neq j'_l$,

$$\operatorname{rank}\begin{pmatrix} A_{j_1,\ldots,j_t} \\ A_{j'_1,\ldots,j'_t} \end{pmatrix} = t.$$

Note that (P3) is nothing for $k = n+1$.

Now we assume that the matrix $A = (a_{i,j})_{1\leqslant i\leqslant n, 1\leqslant j\leqslant k}$ is well-defined for some $k \geqslant n+1$ which satisfies (P1)–(P3). Next, we add the $(k+1)$-th column φ_{k+1} to A such that the new matrix also satisfies these properties.

Fix some $n+1 \leqslant t \leqslant \min\{k+1, 2n\}$, $1 \leqslant j_1 < \cdots < j_t \leqslant k+1$ and $1 \leqslant j'_1 < \cdots < j'_t = k+1$ such that at least $t-n$ l's satisfy $j_l \neq j'_l$. There are two cases.

Case 1. $j_t < j'_t$.

In this case, there are at least $t-n-1$ l's satisfying $j_l \neq j'_l$ and $1 \leqslant l \leqslant t-1$. By the assumption,

$$\operatorname{rank}\begin{pmatrix} A_{j_1,\ldots,j_{t-1}} \\ A_{j'_1,\ldots,j'_{t-1}} \end{pmatrix} = t-1$$

for $t \geqslant n+2$. In addition for $t = n+1$, the above equation is also true since every n columns of A are linearly independent.

If $\operatorname{rank}\left(\begin{smallmatrix} A_{j_1,\ldots,j_t} \\ A_{j'_1,\ldots,j'_t} \end{smallmatrix}\right) < t$, then there exists some $c \in \mathbb{R}^{t-1}$ such that

$$A_{j_1,\ldots,j_{t-1}} c = \varphi_{j_t}, \tag{5.1}$$

$$A_{j'_1,\ldots,j'_{t-1}} c = \varphi_{j'_t}. \tag{5.2}$$

Since the rank of $A_{j_1,\ldots,j_{t-1}}$ is no less than n, the dimension of the set consisting of all c satisfying (5.1) is no greater than $t-1-n \leqslant n-1$. Therefore, the dimension of the set $F_1 := \{A_{j'_1,\ldots,j'_{t-1}} c : A_{j_1,\ldots,j_{t-1}} c = \varphi_{j_t}\}$ is no greater than $n-1$. In addition for $\varphi_{j'_t} \in \mathbb{R}^n \setminus F_1$, $\operatorname{rank}\left(\begin{smallmatrix} A_{j_1,\ldots,j_t} \\ A_{j'_1,\ldots,j'_t} \end{smallmatrix}\right) = t$.

Case 2. $j_t = j'_t$.

In this case, there are at least $t-n$ l's satisfying $j_l \neq j'_l$ and $1 \leqslant l \leqslant t-1$. If $\operatorname{rank}\left(\begin{smallmatrix} A_{j_1,\ldots,j_t} \\ A_{j'_1,\ldots,j'_t} \end{smallmatrix}\right) < t$, then there exists again some $c \in \mathbb{R}^{t-1}$ such that

$$A_{j_1,\ldots,j_{t-1}} c = \varphi_{j_t}, \quad A_{j'_1,\ldots,j'_{t-1}} c = \varphi_{j_t}.$$

Consequently,

$$(A_{j_1,\ldots,j_{t-1}} - A_{j'_1,\ldots,j'_{t-1}}) c = 0. \tag{5.3}$$

By (P2), we have $\operatorname{rank}(A_{j_1,\ldots,j_{t-1}} - A_{j'_1,\ldots,j'_{t-1}}) \geqslant t-n$. Hence the dimension of the set consisting of solutions of (5.3) is no greater than $t-1-(t-n) = n-1$. Therefore, the dimension of the set $F_2 := \{A_{j'_1,\ldots,j'_{t-1}} c : (A_{j_1,\ldots,j_{t-1}} - A_{j'_1,\ldots,j'_{t-1}}) c = 0\}$ is no greater than $n-1$. And for $\varphi_{j'_t} \in \mathbb{R}^n \setminus F_2$, $\operatorname{rank}\left(\begin{smallmatrix} A_{j_1,\ldots,j_t} \\ A_{j'_1,\ldots,j'_t} \end{smallmatrix}\right) = t$.

In both cases, we find a set F of dimension no greater than $n-1$ such that for $\varphi_{j'_t} \in \mathbb{R}^n \setminus F$, $\operatorname{rank}\left(\begin{smallmatrix} A_{j_1,\ldots,j_t} \\ A_{j'_1,\ldots,j'_t} \end{smallmatrix}\right) = t$. Let E_1 be the union of all such F when t, j_l and j'_l vary from all possible choices. Then E_1 is of measure 0 in $\mathbb{R}^n$.

Define $B_{i,k}$ and $C_{i,k}$ as in the construction of almost robust matrices. Then there are finitely many functions $f_{i,j}$ such that

$$B_{i,k} \cap C_{i,k} = \{f_{i,j}(a_{1,1}, \ldots, a_{n,k}, a_{1,k+1}, \ldots, a_{i-1,k+1}) : 1 \leqslant j \leqslant n_i\}.$$

Let

$$E_2 = \bigcup_{i=1}^{n} \{x \in \mathbb{R}^n : x_i = f_{i,j}(a_{1,1}, \ldots, a_{n,k}, x_1, \ldots, x_{i-1})\}.$$

Then E_2 is of measure 0 in $\mathbb{R}^n$.

Take some $\varphi_{k+1} \in \mathbb{R}^n \setminus (E_1 \cap E_2)$. Then (P1)–(P3) hold with k being replaced by $k+1$. By induction, we can construct an $n \times N$ matrix with these properties. It follows from Lemma 4.3 that $\{\varphi_i\}_{1\leqslant i\leqslant N}$ is a totally robust frame with respect to $N - 2n$ erasures.

Remark 5.1. We point out that frames constructed in this way are not totally robust with respect to $(N - 2n + 1)$-erasures.

To see this, set

$$(j_1, \dots, j_{2n-1}) = (1, \dots, n, n+2, \dots, 2n),$$
$$(j'_1, \dots, j'_{2n-1}) = (2, \dots, n+1, n+2, \dots, 2n).$$

Then there exists some $(x, x') \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(\vec{0}, \vec{0})\}$ such that

$$(A'_{j_1,\dots,j_{2n-1}} \; -A'_{j'_1,\dots,j'_{2n-1}}) \begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence

$$A'_{j_1,\dots,j_{2n-1}} x - A'_{j'_1,\dots,j'_{2n-1}} x' = 0. \tag{5.4}$$

Therefore,

$$\langle x, \varphi_{j_l} \rangle = \langle x', \varphi_{j'_l} \rangle, \quad 1 \leqslant l \leqslant 2n - 1.$$

By (P2), $\operatorname{rank}(A'_{j_1,\dots,j_{2n-1}} \; -A'_{j'_1,\dots,j'_{2n-1}}) = n$. It follows from (5.4) that $x \neq x'$. Otherwise, $x = x' = 0$, which contradicts with the choice of (x, x'). Hence A is not totally robust with respect to $(N - 2n + 1)$-erasures.

B: Construction of totally robust frames with $\boldsymbol{N - m \geqslant 2n - 1}$. Suppose that $N \geqslant 2n$. As in the case of $N - m \geqslant 2n$, we can construct an $n \times k$ matrix $A = (a_{i,j})_{1\leqslant i\leqslant n, 1\leqslant j\leqslant k}$ inductively such that

(Q1) $a_{1,j} = 1$ for $1 \leqslant j \leqslant k$,

(Q2) every n columns of A are linearly independent,

(Q3) for any $2 \leqslant s \leqslant n$, $1 \leqslant j_1 < \cdots < j_s \leqslant k$ and $1 \leqslant j'_1 < \cdots < j'_s \leqslant k$ with $j_l \neq j'_l$, $\forall\, l$,

$$\operatorname{rank} \begin{pmatrix} 1 - 1 & \cdots & 1 - 1 \\ a_{2,j_1} - a_{2,j'_1} & \cdots a_{2,j_s} & - a_{2,j'_s} \\ \vdots & \ddots & \vdots \\ a_{s,j_1} - a_{s,j'_1} & \cdots a_{s,j_s} & - a_{s,j'_s} \end{pmatrix} = s,$$

and

(Q4) for any $n \leqslant t \leqslant \min\{k, 2n - 1\}$, $1 \leqslant j_1 < \cdots < j_t \leqslant k$ and $1 \leqslant j'_1 < \cdots < j'_t \leqslant k$ with at least $t - n + 1$ l's satisfying $j_l \neq j'_l$,

$$\operatorname{rank} \begin{pmatrix} A_{j_1,\dots,j_t} \\ A_{j'_1,\dots,j'_t} \end{pmatrix} = t.$$

In addition, the proof is almost the same as that for the case of $N - m \geqslant 2n$ except that (5.3) is replaced by

$$\begin{cases} (A_{j_1,\dots,j_{t-1}} \; -A_{j'_1,\dots,j'_{t-1}})c = 0, \\ c_1 + \cdots + c_{t-1} = 1. \end{cases}$$

The following shows that totally robust frames are abundant.

Theorem 5.2. *Let $A = (a_{i,j})_{1\leqslant i\leqslant n, 1\leqslant j\leqslant N}$ be an $n \times N$ matrix, where $a_{i,j}$ are independent continuous random variables, $n \leqslant N$. Denote column vectors of A by $\{\varphi_i\}_{1\leqslant i\leqslant N}$. Then the following assertions are true with probability* 1:

(i) *$\{\varphi_i\}_{1\leqslant i\leqslant N}$ is almost robust with respect to m-erasures whenever $N - m \geqslant n + 1$.*

(ii) $\{\varphi_i\}_{1\leqslant i\leqslant N}$ *is totally robust with respect to* m*-erasures whenever* $N-m\geqslant 2n$.

(iii) $\{\varphi_i\}_{1\leqslant i\leqslant N}$ *is totally robust with respect to* m*-erasures whenever* $N-m\geqslant 2n-1$ *and the first row of* A *is replaced by* $(1,\ldots,1)$.

Proof. We prove only the first conclusion. The other two can be proved with the similar arguments.

We see from the construction that $\{\varphi_i\}_{1\leqslant i\leqslant N}$ is almost robust if φ_i is not a solution of finitely many systems of linear equations which depend on $\varphi_1,\ldots,\varphi_{i-1}$. Consequently, if $\{\varphi_i\}_{1\leqslant i\leqslant N}$ is not almost robust for some sample point, it must be contained in one of the following sets:

$$F_i=\{\varphi_i\in E_i\},$$

where $E_i\subset\mathbb{R}^n$ is of measure 0 and E_i depends on $\varphi_1,\ldots,\varphi_{i-1}$. For example, $E_1=\{x\in\mathbb{R}^n: x_1=0\}$ and $E_2=\{x\in\mathbb{R}^n: x_1=0 \text{ or } x_1=a_{1,1} \text{ or } a_{1,1}x_2-a_{2,1}x_1=0\}$.

Since entries of A are continuous random variables, we have $P(F_1)=0$.

Suppose that for some $s\geqslant 1$, $P(F_i)=0$, $\forall\, 1\leqslant i\leqslant s$. Then we have

$$P(F_{s+1})=P(F_{s+1}F_1^c\cdots F_s^c)=P(F_1^c\cdots F_s^c)P(F_{s+1}\,|\,F_1^c\cdots F_s^c)=0.$$

By induction, we see that $P(F_i)=0$ for any $1\leqslant i\leqslant N$. This completes the proof. □

The following is another statement on the density of robust frames, which can be proved similarly to Theorem 5.2.

Theorem 5.3. *For* $N-m>n$ (*resp.* $N-m>2n$), *the set of all vectors* $(a_{1,1},\ldots,a_{n,1},\ldots,a_{1,N},\ldots,a_{n,N})$ *for which* $\{\varphi_i:=(a_{1,i},\ldots,a_{n,i})'\}_{1\leqslant i\leqslant N}$ *is not an* m*-erasure almost robust* (*resp. totally almost*) *frame is of measure zero in* $\mathbb{R}^{nN}$.

6 Examples

In this section, we present some concrete examples. First, we give two simple examples of almost robust frames among which one is not totally robust and hence provides unstable reconstruction, and the other one is totally robust and hence provides stable reconstruction.

Example 6.1. Consider the frame consisting of column vectors of

$$A=\begin{pmatrix}1&0&1&1\\0&1&1&3\end{pmatrix}.$$

It is easy to check that

$$T^{2,3,4}=\begin{pmatrix}0&1\\1&1\\1&3\end{pmatrix},\quad T^{1,3,4}=\begin{pmatrix}1&0\\1&1\\1&3\end{pmatrix},$$
$$T^{1,2,4}=\begin{pmatrix}1&0\\0&1\\1&3\end{pmatrix},\quad T^{1,2,3}=\begin{pmatrix}1&0\\0&1\\1&1\end{pmatrix},$$

and the corresponding matrices $M(i_1,i_2,i_3)$ (see (1.2)) are

$$M(2,3,4)=(-2\ \ -1\ \ 1),\quad M(1,3,4)=(2\ \ -3\ \ 1),$$
$$M(1,2,4)=(-1\ \ -3\ \ 1),\quad M(1,2,3)=(-1\ \ -1\ \ 1).$$

Since $M(2,3,4)$, $M(1,3,4)$, $M(1,2,4)$ and $M(1,2,3)$ are pairwise linearly independent, A is almost robust with respect to 1-erasure. But

$$T^{2,3,4}\begin{pmatrix}1\\2\end{pmatrix}=T^{1,2,4}\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}2\\1\\5\end{pmatrix},$$

which implies that the frame consisting of column vectors of A is not totally robust and so the corresponding reconstruction algorithm is not stable.

Example 6.2. Consider the frame consisting of column vectors of

$$A=\begin{pmatrix}1&1&1&1\\1&2&3&6\end{pmatrix}.$$

In this case, we have

$$T^{2,3,4}=\begin{pmatrix}1&2\\1&3\\1&6\end{pmatrix},\quad T^{1,3,4}=\begin{pmatrix}1&1\\1&3\\1&6\end{pmatrix},$$

$$T^{1,2,4}=\begin{pmatrix}1&1\\1&2\\1&6\end{pmatrix},\quad T^{1,2,3}=\begin{pmatrix}1&1\\1&2\\1&3\end{pmatrix},$$

and

$$M(2,3,4)=(\,3\quad 4\ 1\,),\quad M(1,3,4)=(\,3/2\quad 5/2\ 1\,),$$
$$M(1,2,4)=(\,4\quad 5\ 1\,),\quad M(1,2,3)=(\,1\quad 2\ 1\,).$$

Again, $M(2,3,4)$, $M(1,3,4)$, $M(1,2,4)$ and $M(1,2,3)$ are pairwise linearly independent. Hence A is almost robust with respect to 1-erasure.

For any $1\leqslant i_1<i_2<i_3\leqslant 4$ and $1\leqslant j_1<j_2<j_3\leqslant 4$ with $(i_1,i_2,i_3)\not\equiv(j_1,j_2,j_3)$, we have

$$T^{i_1,i_2,i_3}\mathbb{R}^2\ \{\ T^{j_1,j_2,j_3}\mathbb{R}^2=\mathrm{span}\}(1,1,1)'\,\langle.$$

Observe that

$$T^{2,3,4}\begin{pmatrix}1\\0\end{pmatrix}=T^{1,3,4}\begin{pmatrix}1\\0\end{pmatrix}=T^{1,2,4}\begin{pmatrix}1\\0\end{pmatrix}=T^{1,2,3}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}.$$

This implies that the frame consisting of column vectors of A is totally robust and so the corresponding reconstruction algorithm is stable.

Next, we present a class of examples of totally robust frames constructed from a set of prime numbers. For $a_1,\ldots,a_k\ /\ \mathbb{R}$, denote by $\mathbb{Q}(a_1,\ldots,a_k)$ the minimal field containing all rational numbers and $a_1,\ldots,a_k$.

The following result may be well-known and the proof can be found in [23].

Proposition 6.3. *If $p_{i_1},\ldots,p_{i_m},p_{j_1},\ldots,p_{j_n}$ are $m+n$ different prime numbers, then*

$$\overline{p_{j_1}\cdots p_{j_n}}\ /\ \mathbb{Q}(\}\ \ \overline{p_{i_k}}\langle_{k=1}^{m}).$$

To present concrete examples of robust frames, we need the following lemma.

Lemma 6.4. *Let $A = (\sqrt{p_{i,j}})_{1\leqslant i,j\leqslant n}$ be an $n\times n$ matrix, where $p_{i,j}$ are different prime numbers. Then we have* $\det(A)\neq 0$.

Moreover, if the first row of A is replaced by $(1,\ldots,1)$, we also have $\det(A)\neq 0$.

Proof. By Laplace's formula, we have

$$\det(A)=\sum_{k=1}^{n}\sqrt{p_{n,k}}(-1)^{n+k}\det((A_k^n)^c).$$

Let P be the set consisting of all entries of A except $\sqrt{p_{n,n}}$. Observe that

$$\det((A_k^n)^c)\in\mathbb{Q}(P),\quad 1\leqslant k\leqslant n$$

and

$$\sqrt{p_{n,j}}\in\mathbb{Q}(P),\quad 1\leqslant j\leqslant n-1.$$

By Proposition 6.3, it suffices to prove that $\det((A_n^n)^c)\neq 0$. Using Proposition 6.3 again and again, we reduce it to show that $\sqrt{p_{1,1}}\neq 0$, which is obviously correct. Hence $\det(A)\neq 0$.

The second conclusion can be proved similarly. This completes the proof. □

We say that a frame for $\mathbb{R}^n$ is of uniform excess if any of its n elements form a basis for $\mathbb{R}^n$.

Theorem 6.5. *Let $\{p_i\}_{1\leqslant i\leqslant nN}$ be a sequence of different prime numbers. Set*

$$\varphi_i=(\sqrt{p_{(i-1)n+1}},\sqrt{p_{(i-1)n+2}},\ldots,\sqrt{p_{in}})',\quad 1\leqslant i\leqslant N.$$

Then $\{\varphi_i\}_{1\leqslant i\leqslant N}$ is an almost robust frame with respect to m-erasures whenever $m\leqslant N-n-1$.

Proof. Let $A=(\varphi_1,\ldots,\varphi_N)$. For any $1\leqslant i_1<\cdots<i_n\leqslant N$, we see from Lemma 6.4 that $\det(A_{i_1,\ldots,i_n})\neq 0$. Hence $\{\varphi_i\}_{1\leqslant i\leqslant N}$ is of uniform excess.

Next, we show that $\{\varphi_i\}_{1\leqslant i\leqslant N}$ is almost robust with respect to $(N-n-1)$ erasures. Fix some $1\leqslant i_1<\cdots<i_{n+1}\leqslant N$ and $1\leqslant i'_1<\cdots<i'_{n+1}\leqslant N$ with

$$(i_1,i_2,\ldots,i_{n+1})\neq(i'_1,i'_2,\ldots,i'_{n+1}).$$

Then there is some $1\leqslant s\leqslant n+1$ such that $i'_s\notin\{i_l\}_{1\leqslant l\leqslant n+1}$.

Consider the matrix $\tilde{A}$ consisting of rows of $A_{i_1,\ldots,i_{n+1}}$ and the first row of $A_{i'_1,\ldots,i'_{n+1}}$, i.e.,

$$\tilde{A}=\begin{pmatrix}\sqrt{p_{(i_1-1)n+1}} & \sqrt{p_{(i_2-1)n+1}} & \cdots & \sqrt{p_{(i_{n+1}-1)n+1}}\\ \sqrt{p_{(i_1-1)n+2}} & \sqrt{p_{(i_2-1)n+2}} & \cdots & \sqrt{p_{(i_{n+1}-1)n+2}}\\ \vdots & \vdots & \ddots & \vdots\\ \sqrt{p_{i_1 n}} & \sqrt{p_{i_2 n}} & \cdots & \sqrt{p_{i_{n+1}n}}\\ \sqrt{p_{(i'_1-1)n+1}} & \sqrt{p_{(i'_2-1)n+1}} & \cdots & \sqrt{p_{(i'_{n+1}-1)n+1}}\end{pmatrix}.$$

We conclude that $\det(\tilde{A})\neq 0$. To see this, expanding $\det(\tilde{A})$ along the last row, we get

$$\det(\tilde{A})=\sum_{k=1}^{n+1}\sqrt{p_{(i'_k-1)n+1}}(-1)^{n+1+k}\det((\tilde{A}_k^{n+1})^c).$$

Denote by P the set consisting of all entries of $A_{i_1,\ldots,i_{n+1}}$. Since $i'_s\notin\{i_1,\ldots,i_{n+1}\}$, we have $\sqrt{p_{(i'_s-1)n+1}}\notin P$. By Proposition 6.3, to show that $\det(\tilde{A})\neq 0$, it suffices to prove that the $(n+1,s)$ cofactor of $\tilde{A}$ is not equal to zero. Observe that

$$\det(\tilde{A}_s^{n+1})^c=\det(A_{i_1,\ldots,i_{s-1},i_{s+1},\ldots,i_{n+1}}).$$

We see from Lemma 6.4 that $\det(\tilde{A}_s^{n+1})^c\neq 0$. Hence $\det(\tilde{A})\neq 0$. Therefore,

$$\operatorname{rank}\begin{pmatrix}A_{i_1,\ldots,i_{n+1}}\\ A_{i'_1,\ldots,i'_{n+1}}\end{pmatrix}=n+1.$$

By Theorem 3.2, we get the conclusion as desired. □

Next, we show that the frame defined in Theorem 6.5 is totally robust whenever $N - m \geqslant 2n$.

Theorem 6.6. *Let $\{\varphi_i\}_{1\leqslant i\leqslant N}$ be defined as in Theorem 6.5. Then it is totally robust with respect to m-erasures whenever $N - m \geqslant 2n$.*

Proof. Fix some $1 \leqslant i_1 < \cdots < i_{N-m} \leqslant N$ and $1 \leqslant i'_1 < \cdots < i'_{N-m} \leqslant N$ with $(i_1,\ldots,i_{N-m}) \neq (i'_1,\ldots,i'_{N-m})$. It suffices to show that for any $x, y \in \mathcal{N}$, if

$$T^{i_1,\ldots,i_{N-m}}x = T^{i'_1,\ldots,i'_{N-m}}y, \tag{6.1}$$

then $x = y$. There are two cases.

Case 1. $\#\{l : i_l = i'_l\} \geqslant n$. In this case, there exist $1 \leqslant s_1 < \cdots < s_n \leqslant N - m$ such that $i_{s_l} = i'_{s_l}$ for $1 \leqslant l \leqslant n$. By (6.1), we have

$$T^{i_{s_1},\ldots,i_{s_n}}(x - y) = 0.$$

Since $\{\varphi_i\}_{1\leqslant i\leqslant N}$ is of uniform excess, $\{\varphi_{i_{s_l}}\}_{1\leqslant l\leqslant n}$ is a basis for $\mathcal{N}$. Hence $x = y$.

Case 2. $\#\{l : i_l = i'_l\} < n$. In this case, there exist some $0 \leqslant k \leqslant n - 1$, $1 \leqslant s_1 < \cdots < s_k \leqslant N - m$ and $1 \leqslant r_1 < \cdots < r_{2n-k} \leqslant N - m$ such that $\{r_1,\ldots,r_{2n-k}\} \cap \{s_1,\ldots,s_k\} = \emptyset$, $i_{s_l} = i'_{s_l}$ for $1 \leqslant l \leqslant k$ and $i_{r_l} \neq i'_{r_l}$ for $1 \leqslant l \leqslant 2n - k$. By (6.1), we have

$$\begin{pmatrix} T^{i_{s_1},\ldots,i_{s_k}} \\ T^{i_{r_1},\ldots,i_{r_{2n-k}}} \end{pmatrix} x = \begin{pmatrix} T^{i'_{s_1},\ldots,i'_{s_k}} \\ T^{i'_{r_1},\ldots,i'_{r_{2n-k}}} \end{pmatrix} y. \tag{6.2}$$

Let

$$L = \begin{pmatrix} T^{i_{s_1},\ldots,i_{s_k}} & T^{i_{s_1},\ldots,i_{s_k}} \\ T^{i_{r_1},\ldots,i_{r_{2n-k}}} & T^{i'_{r_1},\ldots,i'_{r_{2n-k}}} \end{pmatrix}.$$

Then (6.2) is equivalent to

$$L\begin{pmatrix} x \\ y \end{pmatrix} = 0.$$

To prove $x = y = 0$, it suffices to show that $\det(L) \neq 0$.

By definition, we have

$$L = (A \ \ B),$$

where

$$A = \begin{pmatrix} \sqrt{p_{(i_{s_1}-1)n+1}} & \cdots & \sqrt{p_{i_{s_1}n}} \\ \vdots & \ddots & \vdots \\ \sqrt{p_{(i_{s_k}-1)n+1}} & \cdots & \sqrt{p_{i_{s_k}n}} \\ \sqrt{p_{(i_{r_1}-1)n+1}} & \cdots & \sqrt{p_{i_{r_1}n}} \\ \vdots & \ddots & \vdots \\ \sqrt{p_{(i_{r_{2n-k}}-1)n+1}} & \cdots & \sqrt{p_{i_{r_{2n-k}}n}} \end{pmatrix},$$

$$B = \begin{pmatrix} \sqrt{p_{(i_{s_1}-1)n+1}} & \cdots & \sqrt{p_{i_{s_1}n}} \\ \vdots & \ddots & \vdots \\ \sqrt{p_{(i_{s_k}-1)n+1}} & \cdots & \sqrt{p_{i_{s_k}n}} \\ \sqrt{p_{(i'_{r_1}-1)n+1}} & \cdots & \sqrt{p_{i'_{r_1}n}} \\ \vdots & \ddots & \vdots \\ \sqrt{p_{(i'_{r_{2n-k}}-1)n+1}} & \cdots & \sqrt{p_{i'_{r_{2n-k}}n}} \end{pmatrix}.$$

Expanding $\det(L)$ along the last row, we get

$$\det(L) = \sum_{l=1}^{n} \sqrt{p_{(i_{r_{2n-k}}-1)n+l}}(-1)^{2n+l}\det((L_l^{2n})^c) + \sum_{l=1}^{n} \sqrt{p_{(i'_{r_{2n-k}}-1)n+l}}(-1)^{3n+l}\det((L_{n+l}^{2n})^c).$$

If $i_{r_{2n-k}} < i'_{r_{2n-k}}$, then $\sqrt{p_{i'_{r_{2n-k}}n}}$ appears only once in the entries of L. In addition for the case of $i_{r_{2n-k}} > i'_{r_{2n-k}}$, $\sqrt{p_{(i_{r_{2n-k}}-1)n+1}}$ appears only once in the entries of L. By Proposition 6.3, to prove $\det(L) \neq 0$, it suffices to show that $\det((L_{2n}^{2n})^c) \neq 0$ or $\det((L_1^{2n})^c) \neq 0$.

Assume that $i_{r_{2n-k}} < i'_{r_{2n-k}}$. Then we only need to show that $\det((L_{2n}^{2n})^c) \neq 0$. Expanding $\det((L_{2n}^{2n})^c)$ along the last row and repeat the previous procedure, we can reduce the problem to prove a smaller matrix to be nonsingular. Obviously, the procedure can be repeated many times.

Set $n_0 = \#\{u : i_u > i'_u, u = r_1, r_2, \ldots, r_{2n-k}\}$. There are three cases.

(i) $n_0 \leqslant n$ and $2n-k-n_0 \leqslant n$. In this case, the previous procedure can be executed $2n-k-1$ times and we reduce the problem to prove that

$$\det((L^{k+2,\ldots,2n}_{1,\ldots,n_0,n_0+k+2,\ldots,2n})^c) \neq 0, \quad \text{if} \quad i_{r_1} < i'_{r_1},$$

or

$$\det((L^{k+2,\ldots,2n}_{1,\ldots,n_0-1,n_0+k+1,\ldots,2n})^c) \neq 0, \quad \text{if} \quad i_{r_1} > i'_{r_1},$$

i.e.,

$$\det(L^{1,\ldots,k+1}_{n_0+1,\ldots,n_0+k+1}) \neq 0 \quad \text{or} \quad \det(L^{1,\ldots,k+1}_{n_0,\ldots,n_0+k}) \neq 0. \tag{6.3}$$

Since $k+1 \leqslant n$, it is easy to see that entries of $L^{1,\ldots,k+1}_{n_0+1,\ldots,n_0+k+1}$ and $L^{1,\ldots,k+1}_{n_0,\ldots,n_0+k}$ are pairwise different, respectively. By Lemma 6.4, both inequalities in (6.3) are true.

(ii) $n_0 > n$. In this case, there is an integer $n_1 \geqslant 2$ such that $i_{r_{n_1}} > i'_{r_{n_1}}$ and $\#\{u : i_u > i'_u, u = r_{n_1}, r_{n_1+1}, \ldots, r_{2n-k}\} = n$. By repeating the procedure $2n-k-n_1+1$ times, we reduce the problem to show that $\det((L^{n_1+k,\ldots,2n}_{1,\ldots,n,n_1+n+k,\ldots,2n})^c) \neq 0$, which is equivalent to $\det(L^{1,\ldots,n_1+k-1}_{n+1,\ldots,n+n_1+k-1}) \neq 0$. Since $n_1+k-1 \leqslant n$, entries of $L^{1,\ldots,n_1+k-1}_{n+1,\ldots,n+n_1+k-1}$ are pairwise different. Similar to Case (i), we can prove that the above inequality is true.

(iii) $2n-k-n_0 > n$. In this case, we can find some integer $n_2 \geqslant 2$ such that $i_{r_{n_2}} < i'_{r_{n_2}}$ and $\#\{u : i_u < i'_u, u = r_{n_2}, r_{n_2+1}, \ldots, r_{2n-k}\} = n$. By repeating the procedure $2n-k-n_2+1$ times, we reduce the problem to show that $\det((L^{n_2+k,\ldots,2n}_{1,\ldots,n-k-n_2+1,n+1,\ldots,2n})^c) \neq 0$, which can be proved similarly to Case (ii). This completes the proof. □

For the case of $N-m \geqslant 2n-1$, we get a similar result.

Theorem 6.7. *Let $\{p_i\}_{1\leqslant i\leqslant (n-1)N}$ be a sequence of different prime numbers, where $N \geqslant 2n$ and $n \geqslant 2$. For $1 \leqslant i \leqslant N$, let $\varphi_i = (1, \sqrt{p_{(i-1)(n-1)+1}}, \sqrt{p_{(i-1)(n-1)+2}}, \ldots, \sqrt{p_{i(n-1)}})'$. Then*

(i) *$\{\varphi_i\}_{1\leqslant i\leqslant N}$ is an almost robust frame with respect to m-erasures whenever $N-m \geqslant n+1$.*

(ii) *$\{\varphi_i\}_{1\leqslant i\leqslant N}$ is a totally robust frame with respect to m-erasures whenever $N-m \geqslant 2n-1$.*

Proof. (i) First, we show that $\{\varphi_i\}_{1\leqslant i\leqslant N}$ is of uniform excess. To see this, fix some $1 \leqslant i_1 < \cdots < i_n \leqslant N$. We see from Lemma 6.4 that the matrix $(\varphi_{i_1}, \ldots, \varphi_{i_n})$ is nonsingular. Hence $\{\varphi_{i_l}\}_{1\leqslant l\leqslant n}$ is a basis for $\mathcal{N}$. Therefore, $\{\varphi_i\}_{1\leqslant i\leqslant N}$ is of uniform excess.

Similar to Theorem 6.5 we can prove that $\{\varphi_i\}_{1\leqslant i\leqslant N}$ is almost robust with respect to m-erasures whenever $m \leqslant N-n-1$. We leave the details to interested readers.

(ii) Fix some $1 \leqslant i_1 < \cdots < i_{N-m} \leqslant N$ and $1 \leqslant i'_1 < \cdots < i'_{N-m} \leqslant N$ with

$$(i_1, \ldots, i_{N-m}) \neq (i'_1, \ldots, i'_{N-m}).$$

It suffices to show that for any $x, y \in \mathcal{N}$, if

$$T^{i_1,\ldots,i_{N-m}}x = T^{i'_1,\ldots,i'_{N-m}}y, \tag{6.4}$$

then $x = y$. There are two cases.

Case 1. $\#\{l: i_l = i'_l\} \geqslant n$. Similar to Theorem 6.6, we can prove that $x = y$.

Case 2. $\#\{l: i_l = i'_l\} < n$. In this case, there exist some $0 \leqslant k \leqslant n-1$, $1 \leqslant s_1 < \cdots < s_k \leqslant N-m$ and $1 \leqslant r_1 < \cdots < r_{2n-1-k} \leqslant N-m$ such that $\{r_1, \ldots, r_{2n-1-k}\} \cap \{s_1, \ldots, s_k\} = \emptyset$, $i_{s_l} = i'_{s_l}$ for $1 \leqslant l \leqslant k$ and $i_{r_l} \neq i'_{r_l}$ for $1 \leqslant l \leqslant 2n-1-k$. By (6.4), we have

$$\begin{pmatrix} T^{i_{s_1},\ldots,i_{s_k}} \\ T^{i_{r_1},\ldots,i_{r_{2n-1-k}}} \end{pmatrix} x = \begin{pmatrix} T^{i'_{s_1},\ldots,i'_{s_k}} \\ T^{i'_{r_1},\ldots,i'_{r_{2n-1-k}}} \end{pmatrix} y. \tag{6.5}$$

Let

$$L = \begin{pmatrix} T^{i_{s_1},\ldots,i_{s_k}} & T^{i_{s_1},\ldots,i_{s_k}} \\ T^{i_{r_1},\ldots,i_{r_{2n-1-k}}} & T^{i'_{r_1},\ldots,i'_{r_{2n-1-k}}} \end{pmatrix}.$$

Then (6.5) is equivalent to

$$L \begin{pmatrix} x \\ -y \end{pmatrix} = 0. \tag{6.6}$$

Denote the column vectors of L by ζ_i, $1 \leqslant i \leqslant 2n$. Then we have $\zeta_1 = \zeta_{n+1} = (1, \ldots, 1)'$. We rewrite (6.6) as

$$\zeta_1 x_1 + \zeta_2 x_2 + \cdots + \zeta_n x_n - (\zeta_{n+1} y_1 + \zeta_{n+2} y_2 + \cdots + \zeta_{2n} y_n) = 0,$$

i.e.,

$$\zeta_2 x_2 + \cdots + \zeta_n x_n + \zeta_{n+1}(x_1 - y_1) - \zeta_{n+2} y_2 - \cdots - \zeta_{2n} y_n = 0.$$

If $\zeta_2, \ldots, \zeta_{2n}$ are linearly independent, then the above equation implies that

$$x_2 = \cdots = x_n = x_1 - y_1 = y_2 = \cdots = y_n = 0.$$

Hence $x = y$.

Now it remains to prove that $\zeta_2, \ldots, \zeta_{2n}$ are linearly independent. Let $K = (\zeta_2, \zeta_3, \ldots, \zeta_{2n})$. Then K is a $(2n-1) \times (2n-1)$ matrix. It suffices to prove that $\det(K) \neq 0$. Observe that $K = (A\ e\ B)$, where

$$A = \begin{pmatrix} \sqrt{p_{(i_{s_1}-1)(n-1)+1}} & \cdots & \sqrt{p_{i_{s_1}(n-1)}} \\ \vdots & \ddots & \vdots \\ \sqrt{p_{(i_{s_k}-1)(n-1)+1}} & \cdots & \sqrt{p_{i_{s_k}(n-1)}} \\ \sqrt{p_{(i_{r_1}-1)(n-1)+1}} & \cdots & \sqrt{p_{i_{r_1}(n-1)}} \\ \vdots & \ddots & \vdots \\ \sqrt{p_{(i_{r_{2n-1-k}}-1)(n-1)+1}} & \cdots & \sqrt{p_{i_{r_{2n-1-k}}(n-1)}} \end{pmatrix},$$

$$e = (1, \ldots, 1)',$$

$$B = \begin{pmatrix} \sqrt{p_{(i_{s_1}-1)(n-1)+1}} & \cdots & \sqrt{p_{i_{s_1}(n-1)}} \\ \vdots & \ddots & \vdots \\ \sqrt{p_{(i_{s_k}-1)(n-1)+1}} & \cdots & \sqrt{p_{i_{s_k}(n-1)}} \\ \sqrt{p_{(i'_{r_1}-1)(n-1)+1}} & \cdots & \sqrt{p_{i'_{r_1}(n-1)}} \\ \vdots & \ddots & \vdots \\ \sqrt{p_{(i'_{r_{2n-1-k}}-1)(n-1)+1}} & \cdots & \sqrt{p_{i'_{r_{2n-1-k}}(n-1)}} \end{pmatrix}.$$

Let $n_0 = \#(\{u: i_u > i'_u, u = r_1, r_2, \ldots, r_{2n-1-k}\})$. There are three cases.

(i) $n_0 \leqslant n-1$ and $2n-1-k-n_0 \leqslant n-1$.

As Case (i) in the proof of Theorem 6.6, we reduce the problem to prove that

$$\det(K^{1,\ldots,k+1}_{n_0+1,\ldots,n_0+k+1}) \not\equiv 0 \quad \text{or} \quad \det(K^{1,\ldots,k+1}_{n_0,\ldots,n_0+k}) \not\equiv 0. \tag{6.7}$$

Since $k \leqslant n \quad 1$, $n_0 + 1 \leqslant n$ and $n_0 + k \geqslant n$, one column of $K^{1,\ldots,k+1}_{n_0+1,\ldots,n_0+k+1}$ or $K^{1,\ldots,k+1}_{n_0,\ldots,n_0+k}$ is $(1,\ldots,1)'$ and entries in other columns consist of different numbers. By Lemma 6.4, both inequalities in (6.7) are true.

(ii) $n_0 > n \quad 1$.

In this case, there is an integer $n_1 \geqslant 2$ such that

$$i_{r_{n_1}} > i_{r_{n_1}} \quad \text{and} \quad \#(\}u : i_u > i_u, u = r_{n_1}, r_{n_1+1}, \ldots, r_{2n \quad 1 \quad k}\langle) = n \quad 1.$$

As Case (ii) in the proof of Theorem 6.6, we reduce the problem to prove that $\det(L^{1,\ldots,n_1+k \quad 1}_{n,\ldots,n+n_1+k \quad 2}) \not\equiv 0$. Since $n_1 + k \quad 1 \leqslant n$, we see from Lemma 6.4 that the above inequality is true.

(iii) $2n \quad 1 \quad k \quad n_0 > n \quad 1$.

Again, the conclusion can be proved similarly to Case (iii) in the proof of Theorem 6.6. We omit the details. □

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