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Joint similarities and parameterizations for Naimark complementary frames ☆

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ABSTRACT

It is known that the Naimark complementary frames for a given frame are not necessarily unique up to the similarity. In this paper we introduce the concept of joint complementary frame pairs for a given dual frame pair, and prove that they are unique up to the joint similarity. As an application, we give a necessary and sufficient condition under which two Naimark complementary frames are similar. For different pairs of dual frames, we present an operator parameterization for their joint complementary frame pairs.

1. Introduction

The concept of frames first appeared in the late 40's and early 50's (see [1,11,12]) for the purpose of nonharmonic expansions, and the development of wavelet theory during the last couple of decades injected new ideas and led to extensive research to the theory of frames and related topics (cf. [4,5,8–10]). Recall that a sequence $(x_n)_{n\in\mathcal{N}}$ of elements in a Hilbert space H is called a *frame* for H if there are constants $A, B > 0$ so that

$$A\|f\|^2 \le \sum_{n\in\mathcal{N}} |\langle f, x_n\rangle|^2 \le B\|f\|^2, \text{ for all } f \in H.$$

The constants A and B are called the *lower* and *upper* frame bounds, respectively. A frame is called *tight* if $A = B$ and *normalized tight* or *Parseval* if $A = B = 1$.

Let (x_n) be a frame for H. Its *frame operator* $S : H \longrightarrow H$ is the bounded invertible linear operator defined by $Sf = \sum_{n\in\mathcal{N}}\langle f, x_n\rangle x_n$. It is easy to verify that $S = \theta_X^*\theta_X$, where $\theta_X : H \longrightarrow l^2(\mathcal{N})$, $\theta_X(f) =$

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$\sum_{n\in\mathcal{N}}\langle f, x_n\rangle e_n$, is the *analysis operator* of (x_n) and its adjoint θ_X^* is the *synthesis operator* of (x_n) (here (e_n) is the standard orthonormal basis of $l^2(\mathcal{N})$). Let (x_n, y_n) be a pair of frames for H. If for any $f \in H$ we have that

$$f = \sum_{n\in\mathcal{N}}\langle f, y_n\rangle x_n,$$

then we call (x_n, y_n) a *pair of dual frames or dual frame pair* for H. It is well-known that the positions of (x_n) and (y_n) are interchangeable. Dual frame pairs provide signal decomposition and reconstruction schemes and so they are important tools for applications. Usually, for a given frame (x_n), there are infinite many frames (y_n) such that (x_n, y_n) form a pair of dual frames, among which there is a special one called the *canonical or standard* dual frame (x_n^*) of (x_n), where $x_n^* = S^{-1}x_n$. Clearly if T is a bounded invertible operator and $w_n = Tx_n$, then $\theta_W = \theta_X T^*$ and $w_n^* = (T^*)^{-1}x_n^*$.

Frames are generalizations of Riesz bases, and it is known that a frame is a Riesz basis if and only if its analysis operator is surjective. Moreover, there is a nice geometric interpretation of frames in terms of dilations: Normalized tight frames are precisely the orthogonal compressions of orthonormal bases, and frames are the orthogonal compressions of Riesz bases for larger Hilbert spaces. This geometric interpretation has many natural generalization to various setups (cf. [2,3,6,7]). Casazza, Han and Larson studied in [3] the dilations for pairs of dual frames and proved that (x_n, y_n) is a pair of dual frames for a Hilbert space H if and only if there exists a Hilbert space $K \supset H$ and a pair of dual Riesz bases (z_n, z_n^*) such that $x_n = P_H z_n$ and $y_n = P_H z_n^*$, where P_H is the orthogonal projection from K onto H, and (z_n, z_n^*) is called *a dual Riesz basis pair dilation* for the dual frame pair (x_n, y_n).

The study of dilations of frames involves the concepts of (strongly) complementary frames and joint complementary frames, whose definitions will be given in the next section. In terms of (strongly) complementary frames, the dilation of a single frame can be rephrased as the existence of a (strongly) complementary frame, and the dilation of a pair of dual frames can be rephrased as the existence of a pair of joint complementary frames. It was pointed out in [8] that for a given frame, its strongly complementary frame is unique up to similarity. However, the complementary frames of a given frame are not necessarily similar anymore (see Example B in [8]). In this paper, we show that all the pairs of joint complementary frames for a given pair of dual frames are actually unique up to the joint similarity. In other words, we do have the uniqueness if take the dual frame into the consideration of the dilation. As an application of this result, we obtain a necessary and sufficient condition for two complementary frames of a given frame to be similar. By our uniqueness result of the joint complementary frame pairs for a given dual frame pair, we obtain that the joint similarity can be realized by a diagonal operator $I_H \oplus T$. This leads to the question of characterizing the operators which induce the joint similarity for different pairs of dual frame dilations. We will answer this question in Section 3 by presenting an operator parameterization of the Riesz basis dilation pairs for different pairs of dual frames (Theorem 3.7 and Theorem 3.6).

2. Joint similarity for complementary frame pairs

In this section, we classify the dilations of a fixed dual frame pair and examine the uniqueness of dual frame pair dilations in terms of joint similarity for their joint complementary frames.

Definition 2.1. Suppose that (x_n) and (y_n) are frames for Hilbert spaces H and M, respectively.

(i) If $(x_n \oplus y_n)$ is a frame for $H \oplus M$, then we call (x_n) and (y_n) are *disjoint* or call (x_n, y_n) a *pair of disjoint frames*.
(ii) If $(x_n \oplus y_n)$ is a Riesz basis for $H \oplus M$, then we call (x_n, y_n) a *pair of complementary frames*, and (y_n) is called an *(Naimark) complementary frame* of (x_n).

(iii) If there exist invertible operators $A \in B(H), B \in B(M)$ such that $(Ax_n \oplus By_n)$ is a normalized tight frame for $H \oplus M$, then we call (x_n) and (y_n) are *strongly disjoint* or call (x_n, y_n) *a pair of strongly disjoint frames*. If $(Ax_n \oplus By_n)$ is an orthonormal basis for $H \oplus M$, then we call (x_n, y_n) a *pair of strongly complementary frames* and (y_n) is called a *strongly complementary frame* of (x_n).

(iv) If $span\{x_n \oplus y_n\}$ is dense in $H \oplus M$, then we call (x_n) and (y_n) are *weakly disjoint* or call (x_n, y_n) a *pair of weakly disjoint frames*.

The following lemma characterizes different kinds of disjoint frames in terms of the range space properties of their analysis operators [8]:

Lemma 2.2. *Let (x_n) and (y_n) be frames for Hilbert spaces H and M respectively. Let θ_X and θ_Y be the analysis operators of (x_n) and (y_n) respectively. Then*

(i) *(x_n) and (y_n) are strongly disjoint if and only if $\theta_X(H) \perp \theta_Y(M)$.*

(ii) *(x_n, y_n) is a pair of strongly complementary frames if and only if $\theta_X(H) \oplus \theta_Y(M) = l^2(\mathcal{N})$.*

(iii) *(x_n) and (y_n) are disjoint if and only if $\theta_X(H) \cap \theta_Y(M) = \{0\}$ and $\theta_X(H) + \theta_Y(M)$ is a closed set in $l^2(\mathcal{N})$.*

(iv) *(x_n) and (y_n) are weakly disjoint if and only if $\theta_X(H) \cap \theta_Y(M) = \{0\}$.*

(v) *(x_n, y_n) is a pair of complementary frames if and only if $\theta_X(H) \cap \theta_Y(M) = \{0\}$ and $\theta_X(H) + \theta_Y(M) = l^2(\mathcal{N})$.*

The well-known (Naimark) dilation theorem (cf. [8]) tells us that every frame can be dilated (lifted) to a Riesz basis, and consequently complementary frames exist for every frame. Casazza, Han and Larson proved in [3] the following much stronger version of the dilation theorem by considering the dilations of dual frame pairs. This is also true for Banach space frames (framings) and so Banach space techniques are heavily involved in the proofs in [3]. Here we include a more transparent proof only for the case of Hilbert space frames since some of the ideas used in the proof will be needed in the sequel (e.g. Lemma 3.2).

Theorem 2.3. *Suppose (x_n, y_n) is a pair of dual frames for a Hilbert space H. Then there exists a Hilbert space $K \supseteq H$ and a Riesz basis (z_n) for K such that $x_n = P_H z_n$ and $y_n = P_H z_n^*$, where P_H denotes the orthogonal projection from K onto H and (z_n^*) denotes the dual Riesz basis of (z_n).*

Proof. Let θ_Y be the analysis operator of (y_n), $Q : l^2(\mathcal{N}) \longrightarrow \theta_Y(H)$ be the orthogonal projection from $l^2(\mathcal{N})$ onto $\theta_Y(H)$. Let $K = H \oplus \theta_Y(H)^\perp = H \oplus M$, $z_n = x_n \oplus Q^\perp e_n$, where (e_n) denotes the standard orthonormal basis of $l^2(\mathcal{N})$. Now we verify that (z_n) satisfies the requirements.

Firstly, we show that (z_n) is a Riesz basis for K. Let $Q^\perp e_n = w_n$, let θ_X, θ_W be the analysis operators of (x_n) and (w_n) respectively. We want to show that $\theta_X(H) \cap \theta_W(M) = \{0\}$. Since for any $y \in M = \theta_Y(H)^\perp$, we have

$$\theta_W(y) = \sum_{n\in\mathcal{N}} \langle y, w_n\rangle e_n = \sum_{n\in\mathcal{N}} \langle y, Q^\perp e_n\rangle e_n$$
$$= \sum_{n\in\mathcal{N}} \langle Q^\perp y, e_n\rangle e_n = \sum_{n\in\mathcal{N}} \langle y, e_n\rangle e_n = y.$$

So $\theta_W(M) = M = \theta_Y(H)^\perp = Ker(\theta_Y^*)$. Since (x_n, y_n) is a dual frame pair, we have

$$x = \sum_{n\in\mathcal{N}} \langle x, x_n\rangle y_n = \theta_Y^* \theta_X(x).$$

If there exist $x_0 \in \mathcal{H}, y_0 \in M$ such that $\theta_X(x_0) = \theta_W(y_0)$, then

$$x_0 = \theta_Y^* \theta_X(x_0) = \theta_Y^* \theta_W(y_0) = 0.$$

Hence $\theta_X(x_0) = 0$. So $\theta_X(H) \cap \theta_W(M) = \{0\}$. This tells us that (x_n) and (w_n) are weakly disjoint by Lemma 2.2. Now we show that the range space of the analysis operator of $(x_n \oplus w_n)$ is $l^2(\mathcal{N})$, i.e.,

$$\theta_X(H) + \theta_W(M) = \theta_X(H) + M = l^2(\mathcal{N}).$$

Since for any $\xi \in l^2(\mathcal{N})$, we have $\xi = \xi - \theta_X \theta_Y^* \xi + \theta_X \theta_Y^* \xi$. Note that

$$\theta_Y^*(\xi - \theta_X \theta_Y^* \xi) = \theta_Y^* \xi - \theta_Y^* \theta_X \theta_Y^* \xi = \theta_Y^* \xi - \theta_Y^* \xi = 0.$$

Thus $\xi - \theta_X \theta_Y^* \xi \in Ker(\theta_Y^*) = M$ and so $\xi \in \theta_X(H) + M$. This implies that $\theta_X(H) + M \supseteqq l^2(\mathcal{N})$, and therefore we get $\theta_X(H) + M = l^2(\mathcal{N})$. By Lemma 2.2 we have that $(x_n \oplus Q^\perp e_n)$ is a Riesz basis for $H \oplus M$.

Secondly, we show that $z_n^* = y_n \oplus v_n$. Let $z_n^* = u_n \oplus v_n$. We verify that $u_n = y_n$. Since (z_n, z_n^*) is a dual Riesz basis pair for $K = H \oplus M$, for any $y \in H$ we have

$$\begin{aligned} y \oplus 0 &= \sum_{n \in \mathcal{N}} \langle y \oplus 0, u_n \oplus v_n \rangle x_n \oplus Q^\perp e_n = \sum_{n \in \mathcal{N}} \langle y, u_n \rangle x_n \oplus Q^\perp e_n \\ &= \sum_{n \in \mathcal{N}} \langle y, u_n \rangle x_n \oplus \sum_{n \in \mathcal{N}} \langle y, u_n \rangle Q^\perp e_n. \end{aligned}$$

So, for any $y \in H$, $\sum_{n \in \mathcal{N}} \langle y, u_n \rangle Q^\perp e_n = 0$ and $\sum_{n \in \mathcal{N}} \langle y, u_n \rangle x_n = y$. Hence, for any $y \in H$ we have

$$y \oplus 0 = \sum_{n \in \mathcal{N}} \langle y, u_n \rangle (x_n \oplus Q^\perp e_n).$$

On the other hand, $y = \sum_{n \in \mathcal{N}} \langle y, y_n \rangle x_n$ and

$$\sum_{n \in \mathcal{N}} \langle y, y_n \rangle Q^\perp e_n = Q^\perp (\sum_{n \in \mathcal{N}} \langle y, y_n \rangle e_n) = 0.$$

It follows that $y \oplus 0 = \sum_{n \in \mathcal{N}} \langle y, y_n \rangle (x_n \oplus Q^\perp e_n)$. Since $(x_n \oplus Q^\perp e_n)$ is a Riesz basis, the expansion of $y \oplus 0$ is unique. Thus $\langle y, u_n \rangle = \langle y, y_n \rangle$ for any $y \in H$ and so $u_n = y_n$ for any $n \in \mathcal{N}$. □

Recall that two frame (x_n) and (y_n) for Hilbert spaces H and M are called *similar or equivalent* if there exists an invertible operator $T \in B(H, M)$ such that $y_n = T x_n$.

Lemma 2.4. *[8] Let (x_n) and (y_n) be normalized tight frames for Hilbert spaces H and M respectively. Let θ_X and θ_Y be analysis operators for (x_n) and (y_n) respectively. Then (x_n) and (y_n) are unitary equivalent if and only if θ_X and θ_Y have the same range. Likewise, two frames are similar if and only if their analysis operators have the same range.*

For the purpose of classifying dual frame pair dilations we introduce the following definition:

Definition 2.5. Suppose that (x_n, y_n) and (u_n, v_n) are two pairs of frames. If there exist bounded invertible operators T_1 and T_2 such that $u_n = T_1 x_n$ and $v_n = T_2 y_n$, then we call (x_n, y_n) and (u_n, v_n) are *joint similar* by (T_1, T_2).

Lemma 2.6. *(i) Suppose that* (x_n) *is a frame for a Hilbert space* H. *Suppose* M *is a Hilbert space and* $T \in B(H, M)$ *with* T *being bounded invertible. If* $y_n = Tx_n$, *then* (y_n) *is a frame for* M *and* $y_n^* = (T^*)^{-1}x_n^*$.

(ii) Suppose that (x_n) *and* (y_n) *are frames for* H. *If* (y_n) *is similar* (x_n) *by* T, *then* (y_n, y_n^*) *is joint similar to* (x_n, x_n^*) *by* $(T, (T^*)^{-1})$.

Proof. Clearly (ii) follows from (i). To prove (i), suppose θ_X and θ_Y are the analysis operators for (x_n) and (y_n), respectively, and S_X and S_Y are their frame operators. Then it is easy to check that $\theta_Y = \theta_X T^*$. Hence $S_Y = \theta_Y^*\theta_Y = T\theta_X^*\theta_Y T^* = TS_XT^*$. So

$$y_n^* = S_Y^{-1}y_n = (T^*)^{-1}S_X^{-1}T^{-1}Tx_n = (T^*)^{-1}S_X^{-1}x_n = (T^*)^{-1}x_n^*. \quad \square$$

Definition 2.7. Suppose (x_n, y_n) is a pair of dual frames for a Hilbert space H. If (u_n) and (v_n) are frames for a Hilbert spaces M such that (z_n, z_n^*) is a pair of dual Riesz bases for $H \oplus M$ with $z_n = x_n \oplus u_n$ and $z_n^* = y_n \oplus v_n$, then (u_n, v_n) is called a pair of *joint complementary frames* of (x_n, y_n).

Our main result of this section is to operator parameterize all the dilations of a given dual frame pair and show that joint complementary frame pairs are unique up to joint similarity. We divide the proof into two lemmas.

Lemma 2.8. *Assume* (x_n, y_n) *is a pair of dual frames for a Hilbert space* H *and* (u_n^1, v_n^1) *is a pair of joint complementary frames of* (x_n, y_n) *for a Hilbert space* M. *If* (u_n^2, v_n^2) *is a pair of frames for a Hilbert space* N *which is joint similar to* (u_n^1, v_n^1) *by* $(T, (T^*)^{-1})$ *with* T *being an invertible operator in* $B(M, N)$, *then* (u_n^2, v_n^2) *is a pair of joint complementary frames of* (x_n, y_n).

Proof. Let $z_n^1 = x_n \oplus u_n^1$ and $(z_n^1)^* = y_n \oplus v_n^1$. Since (u_n^1, v_n^1) is a pair of joint complementary frames of (x_n, y_n) for the Hilbert space M, by definition $(z_n^1, (z_n^1)^*)$ is a pair of dual Riesz bases for $H \oplus M$. Let $z_n^2 = x_n \oplus u_n^2$. Since (u_n^2, v_n^2) is joint similar to (u_n^1, v_n^1) by $(T, (T^*)^{-1})$, we have $u_n^2 = Tu_n^1$ and $v_n^2 = (T^*)^{-1}v_n^1$. So

$$z_n^2 = x_n \oplus u_n^2 = (I_H \oplus T)(x_n \oplus u_n^1) = (I_H \oplus T)z_n^1,$$

which implies that (z_n^2) is a Riesz basis for $H \oplus N$. Since $(z_n^2)^* = (I_H \oplus (T^*)^{-1})(z_n^1)^*$ by Lemma 2.6, hence

$$(z_n^2)^* = (I_H \oplus (T^*)^{-1})(y_n \oplus v_n^1) = y_n \oplus (T^*)^{-1}v_n^1 = y_n \oplus v_n^2.$$

Thus (u_n^2, v_n^2) is a pair of joint complementary frames of (x_n, y_n). $\square$

Lemma 2.9. *Assume* (x_n, y_n) *is a pair of dual frames for a Hilbert space* H. *If* (u_n^1, v_n^1) *and* (u_n^2, v_n^2) *are two pairs of joint complementary frames of* (x_n, y_n) *for a Hilbert space* M *and* N, *respectively, then there exists an invertible operator* $T \in B(M, N)$ *such that* (u_n^2, v_n^2) *is joint similar to* (u_n^1, v_n^1) *by* $(T, (T^*)^{-1})$.

Proof. Suppose $z_n^1 = x_n \oplus u_n^1$, $(z_n^1)^* = y_n \oplus v_n^1$ and $z_n^2 = x_n \oplus u_n^2$, $(z_n^2)^* = y_n \oplus v_n^2$ such that $(z_n^1, (z_n^1)^*)$ and $(z_n^2, (z_n^2)^*)$ are pairs of Riesz basis dilations of (x_n, y_n) on $H \oplus M$ and $H \oplus N$ respectively. Let $\theta_X, \theta_Y, \theta_{U^1}, \theta_{U^2}, \theta_{V^1}, \theta_{V^2}, \theta_{Z^1}, \theta_{Z^2}$ be the analysis operators of (x_n), (y_n), (u_n^1), (u_n^2), (v_n^1), (v_n^2), (z_n^1) and (z_n^2) respectively. Since $(z_n^1, (z_n^1)^*)$ is a dual Riesz basis pair on $H \oplus M$, for any $y \in H$ we have

$$\begin{aligned} y \oplus 0 &= \sum_{n\in\mathcal{N}} \langle y \oplus 0, (z_n^1)^*\rangle z_n^1 = \sum_{n\in\mathcal{N}} \langle y \oplus 0, y_n \oplus v_n^1\rangle x_n \oplus u_n^1 \\ &= \sum_{n\in\mathcal{N}} \langle y, y_n\rangle x_n \oplus u_n^1 = \sum_{n\in\mathcal{N}} \langle y, y_n\rangle x_n \oplus \sum_{n\in\mathcal{N}} \langle y, y_n\rangle u_n^1. \end{aligned}$$

So $\sum_{n\in\mathcal{N}}\langle y, y_n\rangle u_n^1 = 0$. Hence (y_n, u_n^1) are strongly disjoint and so $\theta_Y(H)\perp\theta_{U^1}(M)$ by Lemma 2.2. This implies that $\theta_{U^1}(M) \subseteqq \theta_Y(H)^\perp$.

Next we show that $\theta_Y(H)^\perp \subseteqq \theta_{U^1}(M)$. If there exists $\xi \in \theta_Y(H)^\perp = Ker\theta_Y^*$, but $\xi \notin \theta_{U^1}(M)$, we claim that $\xi \notin \theta_X(H) + \theta_{U^1}(M)$. In fact, if there exists $x_0 \in H$, $y_0 \in M$ such that $\xi = \theta_X(x_0) + \theta_{U^1}(y_0)$, then

$$0 = \theta_Y^*(\xi) = \theta_Y^*\theta_X(x_0) + \theta_Y^*\theta_{U^1}(y_0) = x_0$$

So $\xi = \theta_X(x_0) + \theta_{U^1}(y_0) = \theta_{U^1}(y_0) \in \theta_{U^1}(M)$, which contradicts with the assumption that $\xi \notin \theta_U^1(M)$. Thus $\xi \notin \theta_X(H) + \theta_{U^1}(M)$. On the other hand, since (z_n^1) is a Riesz basis for $H \oplus M$, we get

$$\theta_{Z^1}(H \oplus M) = \theta_X(H) + \theta_{U^1}(M_1) = l^2(\mathcal{N}).$$

This shows that $\xi \in \theta_X(H) + \theta_{U^1}(M)$, which leads to the contradiction. Thus we have $\theta_Y(H)^\perp \subseteqq \theta_{U^1}(M)$, and so $\theta_Y(H)^\perp = \theta_{U^1}(M)$. Similarly, we have $\theta_Y(H)^\perp = \theta_{U^2}(N)$. Therefore, $\theta_{U^1}(M) = \theta_{U^2}(N)$. Hence (u_n^1) is similar to (u_n^2) by Lemma 2.4. Let $T \in B(M, N)$ be the bounded invertible operator such that $u_n^2 = Tu_n^1$. Since

$$z_n^2 = x_n \oplus u_n^2 = (I_H \oplus T)(x_n \oplus u_n^1) = (I_H \oplus T)z_n^1,$$

we have

$$(z_n^2)^* = (I_H \oplus (T^*)^{-1})(z_n^1)^* = (I_H \oplus (T^*)^{-1})(y_n \oplus v_n^1) = y_n \oplus (T^*)^{-1}v_n^1,$$

which implies that $v_n^2 = (T^*)^{-1}v_n^1$. So (u_n^2, v_n^2) is joint similar to (u_n^1, v_n^1) by $(T, (T^*)^{-1})$. □

Remark 2.10. (1). By Lemma 2.8 and Lemma 2.9 we get that all the dilations of a given dual frame pair can be parameterized by diagonal operators $I_H \oplus T$ with T be invertible operators from some Hilbert space to another.

(2). Lemma 2.9 also show that joint complementary frame pairs are unique up to joint similarity. For more clarity we rephrase it as the following theorem.

Theorem 2.11. *Suppose that (x_n, y_n) is a dual frame pair for a Hilbert space H. Then the joint complementary frame pairs of (x_n, y_n) are unique up to joint similarity.*

As we pointed out in the introduction the complementary frames for a given frame is not unique up to the similarity. However, Theorem 2.11 tells us that when taking its dual frame into the consideration, then we do have the uniqueness for the dilations. Moreover, from Theorem 2.11 we obtain the following conditions under which two complementary frames are similar.

Corollary 2.12. *Suppose (x_n) is a frame for a Hilbert space H and (u_n), (u_n') are two complementary frames for (x_n) on a Hilbert space M and N respectively, i.e., $(z_n) = (x_n \oplus u_n)$ and $(z_n') = (x_n \oplus u_n')$ are Riesz bases for $H \oplus M$ and $H \oplus N$ respectively. Then (u_n) and (u_n') are similar if and only if $P_H z_n^* = P_H(z_n')^*$.*

3. Operator parameterizations of dilations of dual frame pairs

While Section 2 is focused on the classifications of dilations when a dual frame pair is fixed, in this section we work on the case with different dual frame pairs. By building the connections of joint complementary frame pairs for a dual frame pair (x_n, y_n) with the joint complementary frame pairs for the canonical dual frame pair (x_n, x_n^*), we establish an operator parameterization for all the complementary frame pairs.

Lemma 3.1. *[3] Suppose (x_n, y_n) is a dual frame pair for a Hilbert space H and (z_n, w_n) is a dual frame pair for a Hilbert space M. If (x_n, w_n) and (y_n, z_n) are pairs of strongly disjoint frames, then $(x_n \oplus z_n, y_n \oplus w_n)$ is a dual frame pair for $H \oplus M$. Hence, if $(x_n \oplus z_n)$ is a Riesz basis, then $(y_n \oplus w_n)$ is a Riesz basis.*

Lemma 3.2. *Suppose that (x_n, x_n^*) is a canonical dual frame pair for a Hilbert space H. Let Q be the orthogonal projection from $l^2(\mathcal{N})$ onto $\theta_X(H)$ and $z_n = x_n \oplus Q^\perp e_n$, $z_n^* = x_n^* \oplus Q^\perp e_n$. Then (z_n, z_n^*) is a dual Riesz basis pair for $K = H \oplus \theta_X(H)^\perp$, hence (z_n, z_n^*) is a dual Riesz basis dilation for (x_n, x_n^*).*

Proof. By the proof of Theorem 2.3, we know that both (z_n) and (z_n^*) are Riesz bases for K. It is clear that $(x_n, Q^\perp e_n)$ and $(x_n^*, Q^\perp e_n)$ are pairs of strongly disjoint frames. Thus, by Lemma 3.1, we have that (z_n, z_n^*) is a pair of dual Riesz basis dilation of (x_n, x_n^*). □

For our convenience, the dual Riesz basis pair constructed in Lemma 3.2 will be refereed as the *natural pair of dual Riesz basis dilation* for (x_n, x_n^*).

Lemma 3.3. *Suppose that (x_n, x_n^*) is a canonical dual frame pair for a Hilbert space H and (z_n, z_n^*) is the natural pair of dual Riesz basis dilation of (x_n, x_n^*) on the Hilbert space $K = H \oplus M$ which is defined in Lemma 3.2. Suppose (x_n, y_n) is another dual frame pair for H. Then, there exists $A \in B(H, M)$ such that $(z_n^1, (z_n^1)^*)$ is a pair of dual Riesz basis dilation of (x_n, y_n) on K, where $z_n^1 = Tz_n$, for $T = \begin{pmatrix} I_H & 0 \\ A & I_M \end{pmatrix} \in B(K)$.*

Proof. Let $\xi_n = x_n^* - y_n$. Then (ξ_n) is a Bessel sequence (i.e., a sequence satisfies the right-hand side inequality in the definition of frames). It is easy to check that for any $x \in H$, $\theta_\xi(x) \in M$. Let $A = \theta_\xi \in B(H, M)$. Since $z_n = x_n \oplus Q^\perp e_n$, we have

$$Tz_n = \begin{pmatrix} I_H & 0 \\ A & I_M \end{pmatrix} \begin{pmatrix} x_n \\ Q^\perp e_n \end{pmatrix} = \begin{pmatrix} x_n \\ Ax_n + Q^\perp e_n \end{pmatrix}.$$

So $P_H(Tz_n) = x_n$. Since

$$(T^*)^{-1} = \begin{pmatrix} I_H & 0 \\ 0 & I_M \end{pmatrix} - \begin{pmatrix} 0 & A^* \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_H & -A^* \\ 0 & I_M \end{pmatrix},$$

we have

$$(T^*)^{-1} z_n^* = \begin{pmatrix} I_H & -A^* \\ 0 & I_M \end{pmatrix} \begin{pmatrix} x_n^* \\ Q^\perp e_n \end{pmatrix} = \begin{pmatrix} x_n^* - A^* Q^\perp e_n \\ Q^\perp e_n \end{pmatrix}$$

Thus for any $x, y \in H$ we have

$$\begin{aligned} \sum_{n \in \mathcal{N}} \langle x, \xi_n \rangle \langle x_n, y \rangle &= \sum_{n \in \mathcal{N}} \langle x, x_n^* - y_n \rangle \langle x_n, y \rangle \\ &= \langle (\sum_{n \in \mathcal{N}} \langle x, x_n^* \rangle x_n - \sum_{n \in \mathcal{N}} \langle x, y_n \rangle x_n), y \rangle \\ &= \langle x - x, y \rangle = 0. \end{aligned}$$

This shows that $\theta_\xi(H) \perp \theta_X(H)$ and so $Range(A) = Range(\theta_\xi) \subset \theta_X(H)^\perp$. Thus $Q^\perp A = A$, and so $A^* Q^\perp = A^* = \theta_\xi^*$. Therefore,

$$A^* Q^\perp e_n = \theta_\xi^* e_n = \xi_n = x_n^* - y_n.$$

So $x_n^* - A^*Q^\perp e_n = y_n$, which implies that $P_H((Tz_n)^*) = y_n$. Therefore $(z_n^1, (z_n^1)^*)$ is a pair of dual Riesz basis dilation of (x_n, y_n) on K. $\square$

We need the following slightly more general statement and its proof is almost identical to the proof of Lemma 3.3.

Lemma 3.4. *Suppose that (x_n, x_n^*) is a canonical dual frame pair for a Hilbert space H and (z_n, z_n^*) is the natural pair of dual Riesz basis dilation of (x_n, x_n^*) on the Hilbert space $K = H \oplus M$ which is defined in Lemma 3.2. Suppose (x_n, y_n) is another dual frame pair for H and N is a Hilbert space and B is invertible in $B(M,N)$. Then, there exists $A \in B(H,N)$ such that $(z_n^1, (z_n^1)^*)$ is a pair of dual Riesz basis dilation of (x_n, y_n) on $H \oplus N$, where $z_n^1 = Tz_n$, for $T = \begin{pmatrix} I_H & 0 \\ A & B \end{pmatrix} \in B(H \oplus M, H \oplus N)$.*

Proof. Let $\xi_n = x_n^* - y_n$ and $A = B\theta_\xi$. Then the rest of the proof is just a simple modification for the corresponding parts of proofs in Lemma 3.3. $\square$

The following lemma shows that a bounded linear operator that maps the natural dual Riesz basis dilation pair for the canonical dual frames pair to a Riesz basis dilation pair for another pair of dual frames has to be in the form in Lemma 3.4.

Lemma 3.5. *Suppose (x_n, x_n^*) is a canonical dual frame pair and (x_n, y_n) is another dual frame pair for a Hilbert space H. Suppose (z_n, z_n^*) is the natural pair of dual Riesz basis dilation of (x_n, x_n^*) on $K = H \oplus M$ which is defined in Lemma 3.2, $(z_n', (z_n')^*)$ is a dual Riesz basis dilations of (x_n, y_n) on $K' = H \oplus N$. If $z_n' = Tz_n$, where $T \in B(H \oplus M, H \oplus N)$ is invertible, then $T = \begin{pmatrix} I_H & 0 \\ A & B \end{pmatrix}$ for some $A \in B(H,N)$ and $B \in B(M,N)$.*

Proof. Let $T_0 = \begin{pmatrix} I_H & 0 \\ \theta_\xi & I_M \end{pmatrix}$, where θ_ξ is the analysis operator of Bessel sequence $(x_n^* - y_n)$ and let $z_n'' = T_0 z_n$. Then $(z_n'', (z_n'')^*)$ is a pair of Riesz basis dilation of (x_n, y_n) on $K = H \oplus M$ by Lemma 3.3. Suppose $z_n' = x_n \oplus u_n'$ and $z_n'' = x_n \oplus u_n''$. Since $(z_n', (z_n')^*)$ is a pair of Riesz basis dilation of (x_n, y_n) on $H \oplus N$, there exists an invertible operator $B \in B(M,N)$ such that $Bu_n'' = u_n'$ for any $n \in \mathcal{N}$ by Lemma 2.9. It follows that

$$\begin{pmatrix} I_H & 0 \\ 0 & B \end{pmatrix} z_n'' = \begin{pmatrix} I_H & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} x_n \\ u_n'' \end{pmatrix} = \begin{pmatrix} x_n \\ u_n' \end{pmatrix} = z_n' = Tz_n.$$

Since $z_n'' = T_0 z_n$, we have

$$\begin{pmatrix} I_H & 0 \\ 0 & B \end{pmatrix} T_0 z_n = \begin{pmatrix} I_H & 0 \\ 0 & B \end{pmatrix} z_n'' = Tz_n.$$

Since (z_n) is a Riesz basis for $H \oplus M$, it implies that

$$T = \begin{pmatrix} I_H & 0 \\ 0 & B \end{pmatrix} T_0 = \begin{pmatrix} I_H & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I_H & 0 \\ \theta_\xi & I_M \end{pmatrix} = \begin{pmatrix} I_H & 0 \\ B\theta_\xi & B \end{pmatrix}. \quad \square$$

Now we are ready to state and prove our main results of this section.

Theorem 3.6. *Let (x_n, y_n) and (x_n, v_n) be two dual frame pairs for a Hilbert space H, and N_1, N_2 are be two Hilbert spaces. Suppose (u_n, u_n^*) is a pair of Riesz basis dilation of (x_n, y_n) on $H \oplus N_1$ and B is an* *invertible operator in* $B(N_1, N_2)$*. Then, there exists* $A \in B(H, N_2)$ *such that* (w_n, w_n^*) *is a pair of Riesz basis dilation of* (x_n, v_n) *on* $H \oplus N_2$*, where,* $w_n = Tu_n$*, for* $T = \begin{pmatrix} I_H & 0 \\ A & B \end{pmatrix} \in B(H \oplus N_1, H \oplus N_2)$.

Proof. By Lemma 3.5, there exists $A_1 \in B(H, N_1)$ and $B \in B(M, N_1)$ such that $u_n = T_1 z_n$, where $T_1 = \begin{pmatrix} I_H & 0 \\ A_1 & B_1 \end{pmatrix}$ and $(z_n, (z_n)^*)$ is the natural pair of Riesz basis dilation of (x_n, x_n^*) on $H \oplus M$. Define $B_2 \in B(M, N_2)$ by $B_2 = BB_1$. Then, by Lemma 3.4, there exists $A_2 \in B(H, N_2)$ such that (w_n, w_n^*) is a Riesz basis dilation of (x_n, v_n), where $w_n = T_2 z_n$, and $T_2 = \begin{pmatrix} I_H & 0 \\ A_2 & B_2 \end{pmatrix}$. Finally, $w_n = T_2 z_n = T_2(T_1)^{-1} u_n = \begin{pmatrix} I_H & 0 \\ A_2 - BA_1 & B \end{pmatrix}$, so the invertible $A \in B(H, N_2)$ is $A = A_2 - BA_1$. □

Theorem 3.7. *Suppose that* (x_n, y_n) *and* (x_n, v_n) *are two dual frame pairs for a Hilbert space* H*. Let* (u_n, u_n^*) *be a pair of Riesz basis dilation of* (x_n, y_n) *on* $H \oplus N_1$ *and* (w_n, w_n^*) *be a pair of Riesz basis dilation of* (x_n, v_n) *on* $H \oplus N_2$*. If* T *is the invertible operator such that* $w_n = Tu_n$ *for all* $n \in \mathcal{N}$*, then* $T = \begin{pmatrix} I_H & 0 \\ A & B \end{pmatrix}$ *for some* $A \in B(H, N_2)$ *and* $B \in B(N_1, N_2)$.

Proof. Let $\xi_n^1 = x_n^* - y_n, \xi_n^2 = x_n^* - v_n$. Then (ξ_n^1) and (ξ_n^2) are Bessel sequences for H and $\theta_{\xi^1}, \theta_{\xi^2} \in B(H, M)$. Let (z_n, z_n^*) be the natural pair of dual Riesz basis dilation of (x_n, x_n^*) on $H \oplus Q^\perp l^2(\mathcal{N}) = H \oplus M$ which is defined in Lemma 3.2. By Lemma 3.5, there exists invertible operator A_1 in $B(M, N_1)$, invertible operator A_2 in $B(M, N_2)$ and

$$T_1 = \begin{pmatrix} I_H & 0 \\ A_1\theta_{\xi^1} & A_1 \end{pmatrix}, T_2 = \begin{pmatrix} I_H & 0 \\ A_2\theta_{\xi^2} & A_2 \end{pmatrix}$$

such that $u_n = T_1 z_n, w_n = T_2 z_n$. So for any $n \in \mathcal{N}$ we have

$$T_2 z_n = w_n = Tu_n = TT_1 z_n.$$

Thus $T_2 = TT_1$. So

$$T = T_2 T_1^{-1} = \begin{pmatrix} I_H & 0 \\ A_2\theta_{\xi^2} & A_2 \end{pmatrix} \begin{pmatrix} I_H & 0 \\ -\theta_{\xi^1} & A_1^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} I_H & 0 \\ A_2(\theta_{\xi^2} - \theta_{\xi^1}) & A_2 A_1^{-1} \end{pmatrix}. \quad \square$$

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