Acta Appl Math
https://doi.org/10.1007/s10440-018-0194-x

Functional Matrix Multipliers for Parseval Gabor Multi-frame Generators

Zhongyan Li[1] · **Deguang Han**[2]

Received: 12 October 2017 / Accepted: 2 June 2018

Abstract We consider the problem of characterizing the bounded linear operator multipliers on $L^2(\mathbb{R})$ that map Gabor frame generators to Gabor frame generators. We prove that a functional matrix $M(t) = [f_{ij}(t)]_{m\times m}$ (where $f_{ij} \in L^\infty(\mathbb{R})$) is a multiplier for Parseval Gabor multi-frame generators with parameters $a, b > 0$ if and only if $M(t)$ is unitary and $M^*(t)M(t + \frac{1}{b}) = \lambda(t)I$ for some unimodular a-periodic function $\lambda(t)$. As a special case ($m = 1$) this recovers the characterization of functional multipliers for Parseval Gabor frames with single function generators.

Keywords Gabor family · Gabor multi-frame generators · Functional matrix multipliers

Mathematics Subject Classification (2010) 42C15 · 46C05 · 47B10

1 Introduction

Motivated by the characterization of functional wavelet multipliers in the Wutam Consortium paper [11], Gu and Han [8] investigated the functional Gabor frame multipliers and obtained the following characterization: a L^∞-function h on $\mathbb{R}$ is a functional Gabor frame multiplier for the time-frequency lattice $a\mathbb{Z} \times b\mathbb{Z}$ if and only if it is unimodular and $h(t)/h(t + 1/b)$ is a-periodic, where by a *functional Gabor frame multiplier h* we mean that

Zhongyan Li acknowledges the support from the National Natural Science Foundation of China (Grant No. 11571107); Deguang Han acknowledges the support from NSF under the grants DMS-1403400 and DMS-1712602.

✉ Z. Li
lzhongy@ncepu.edu.cn

D. Han
deguang.han@ucf.edu

[1] Department of Mathematics and Physics, North China Electric Power University, Beijing, 102206, China

[2] Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA

Published online: 28 June 2018

hg is a Parseval Gabor frame generator for $L^2(\mathbb{R})$ whenever g is a Parseval Gabor frame generator for $L^2(\mathbb{R})$. It is natural to ask if there is a similar characterization for Gabor frames with multi-generators in view of many studies of Gabor multi-frames (cf. [3, 4, 6, 7, 9, 10] and the references therein). The purpose of this note is to present such a generalization.

Given $a, b > 0$ and $g \in L^2(\mathbb{R})$. The Gabor (or Weyl-Heisenberg) family (g, a, b) is defined to be the collection of functions:

$$\left\{e^{2\pi ikbt} g(t - na) : k, n \in \mathbb{Z}\right\}.$$

Definition 1.1 Let $g_1, \dots, g_L \in L^2(\mathbb{R})$. We say that $G(t) = (g_1(t), \dots, g_L(t))^\tau$ is a Gabor multi-frame generator of length L if $\{(g_j, a, b), j = 1, 2, \dots, L\}$ is a frame for $L^2(\mathbb{R})$, i.e., there exist $A, B > 0$, such that

$$A\|f\|^2 \le \sum_{i=1}^{L} \sum_{k,n\in\mathbb{Z}} \left|\left\langle f, e^{2\pi ikbt} g_j(t - na)\right\rangle\right|^2 \le B\|f\|^2$$

holds for all $f \in L^2(\mathbb{R})$. It is called a Parseval Gabor multi-frame generator if $A = B = 1$.

It is well-known that a single function Gabor frame generator exists if and only if $ab \le 1$ (cf. [2]). For the case $L - 1 < ab \le L$, we need at least L functions to generate a Gabor frame for $L^2(\mathbb{R})$ (cf. [5]). Therefore if there exists a Gabor multi-frame generator $(g_1, \dots, g_m)$ for $L^2(\mathbb{R})$ and $L - 1 < ab \le L$, then we have $m \ge L$. Our goal is to characterize all the $m \times m$ function matrices $M(t) = (h_{ij}(t))_{m\times m}$ that send every Gabor multi-frame generator $(g_1, \dots, g_m)$ to a Gabor multi-frame generator.

Definition 1.2 Let $f_{ij} \in L^\infty(\mathbb{R})$. Then the matrix $M(t) = (f_{ij}(t))_{m\times m}$ is called a functional matrix Gabor multi-frame multiplier if $E(t) = (\eta_1, \eta_2, \dots, \eta_m)^\tau$ is a Parseval Gabor multi-frame generator whenever $G = (g_1, g_2, \dots, g_m)^\tau$ is a Parseval Gabor multi-frame generator for $L^2(\mathbb{R})$, where $E(t) = M(t)G(t)$.

We will prove the following theorem:

Theorem 1.1 *Let $a, b > 0$, and $M(t) = (f_{ij}(t))_{m\times m}$ with $f_{ij} \in L^\infty(\mathbb{R})$, $i, j = 1, \dots, m$. Then $M(t)$ is a functional matrix Gabor multi-frame multiplier if and only if the following three conditions are satisfied*:

(1) *$M(t)$ is unitary for a.e. $t \in \mathbb{R}$,*
(2) *$M^*(t)M(t + \frac{1}{b})$ is a-periodic,*
(3) *$M^*(t)M(t + \frac{1}{b}) = \lambda(t)I$ (a.e. $t \in \mathbb{R}$) for some unimodular function $\lambda(t)$,*

where I is the identity matrix and M^* denotes the conjugate transpose of M.

The proof of this result will be presented in Sect. 2. The following lemmas will be needed for the rest of the paper.

Lemma 1.2 (Cf. [1]) *Let $a, b > 0$, and $g_1, g_2, \dots, g_m \in L^2(\mathbb{R})$. Then $G = (g_1, \dots, g_m)^\tau$ is a Parseval Gabor multi-frame generator for $L^2(\mathbb{R})$ if and only if the following equations are satisfied* (*for a.e. $t \in \mathbb{R}$*):

$$\sum_{i=1}^{m} \sum_{n\in\mathbb{Z}} \left|g_i(t - na)\right|^2 = b, \tag{1.1}$$

$$\sum_{i=1}^{m}\sum_{n\in\mathbb{Z}} g_i(t-na)\overline{g_i\left(t-na-\frac{k}{b}\right)}=0,\quad \textit{for any } k\in\mathbb{Z}\setminus\{0\}. \tag{1.2}$$

Lemma 1.3 (Cf. [5]) *Let $a,b>0$. The following are equivalent*:

(i) $L^2(\mathbb{R})$ *admits a Parseval Gabor frame generator of length at least m*,
(ii) $m-1<ab\le m$.

Moreover, in this case for each $x\in\mathbb{R}$, there exist continuous disjoint intervals $E_1,\dots,E_m$, such that $x\in\bigcup_{i=1}^m E_i$, $g_i=\sqrt{b}\chi_{E_i}$, $i=1,\dots,m$ with $(g_1,\dots g_m)$ is a Parseval Gabor multi-frame generator.

Suppose that $M(t)=(f_{ij}(t))_{m\times m}$ is unitary for a.e. $t\in\mathbb{R}$. Then for every $k\in\mathbb{Z}$ we have

$$M^*(t)M\left(t+\frac{k}{b}\right)=M^*(t)M\left(t+\frac{k-1}{b}\right)M^*\left(t+\frac{k-1}{b}\right)M\left(t+\frac{k}{b}\right).$$

Thus we immediately have the following:

Lemma 1.4 *Assume that $M(t)=(f_{ij}(t))_{m\times m}$ is unitary for a.e. $t\in\mathbb{R}$. Then $M^*(t)M(t+\frac{1}{b})=\lambda(t)I$ for a.e $t\in\mathbb{R}$ if and only if $M^*(t)M(t+\frac{k}{b})=\lambda_k(t)I$ for a.e. $t\in\mathbb{R}$ and every $k\in\mathbb{Z}$.*

We also need following partition lemma:

Lemma 1.5 *Assume that $m-1<ab\le m$ and $\forall c\in\mathbb{R}$. Then $[c,c+a)$ can be partitioned into $2m-1$ subintervals in the form of $(\bigcup_{j=1}^m E_j)\cup(\bigcup_{j=1}^{m-1} F_j)$, where $E_1=[c,c+a+\frac{1-m}{b})$, $F_1=[c+a+\frac{1-m}{b},c+\frac{1}{b})$, $E_j=E_1+\frac{j-1}{b}$ for $2\le j\le m$ and $F_j=F_1+\frac{j-1}{b}$ for $2\le j\le m-1$.*

2 Proof of the Main Theorem

While the proof of the sufficient part of Theorem 1.1 is straightforward, the proof of necessary part is subtle and we divide it into several propositions. In what follows we will use $\|\cdot\|$ to denote the Euclidean norm of $\mathbb{C}^m$.

In sequence, for any $x\in\mathbb{R}$, by the properties of Parseval Gabor multi-frame generators, we will show that the claims in the following propositions are valid for all t in a neighborhood of x. It then follows that the claims are valid for a.e. $x\in\mathbb{R}$.

Proposition 2.1 *If $M(t)=(f_{ij}(t))_{m\times m}$, with $f_{ij}\in L^\infty(\mathbb{R})$, is a functional matrix Gabor multi-frame multiplier, then $M(t)$ is unitary for a.e. $t\in\mathbb{R}$.*

Proof Let $G(t)=(g_1(t),\dots,g_m(t))^\tau$ and $E(t)=(\eta_1(t),\dots,\eta_m(t))^\tau=M(t)G(t)$. Suppose that $G(t)$ is a Parseval Gabor multi-frame generator for $L^2(\mathbb{R})$. Then $E(t)$ is also a Parseval Gabor multi-frame generator for $L^2(\mathbb{R})$. Thus, by Lemma 1.2, we have

$$b=\sum_{i=1}^{m}\sum_{n\in\mathbb{Z}}\left|\eta_i(t-na)\right|^2$$

$$
\begin{aligned}
&= \sum_{n\in\mathbb{Z}} \big\| E(t-na) \big\|^2 = \sum_{n\in\mathbb{Z}} \big\| M(t-na)G(t-na) \big\|^2 \\
&= \sum_{n\in\mathbb{Z}} \big\langle M(t-na)G(t-na), M(t-na)G(t-na) \big\rangle \\
&= \sum_{n\in\mathbb{Z}} \big\langle G(t-na), M(t-na)^* M(t-na)G(t-na) \big\rangle
\end{aligned}
$$

and

$$
b = \sum_{i=1}^{m} \sum_{n\in\mathbb{Z}} \big|g_i(t-na)\big|^2 = \sum_{n\in\mathbb{Z}} \big\langle G(t-na), G(t-na) \big\rangle.
$$

These two equations imply that

$$
\sum_{n\in\mathbb{Z}} \big\langle G(t-na), \big(I - M^*(t-na)M(t-na)\big)G(t-na) \big\rangle = 0. \tag{2.3}
$$

Our goal is to choose special Parseval Gabor multi-frame generators $G(t)$ to force $M(t)$ to be unitary.

For $m = L$, $L-1 < ab \le L$. By Lemma 1.3, we know that $ab \le m$ and so $a/m \le 1/b$. Now fix a fixed $x \in \mathbb{R}$, let $[c,d)$ be an interval of length a and $x \in (c,d)$. Let $\{E_i, F_j\}$ as the same as Lemma 1.5 for $i = 1, .., m$, $j = 1, \dots, m-1$.

Now let $\alpha_1 = (\alpha_{11}, \dots, \alpha_{1m})$ be any vector in $\mathbb{C}^m$ with $\sum_{i=1}^m |\alpha_{1i}|^2 = 1$. Then there exist vectors $\beta_1 = (\beta_{11}, \dots, \beta_{1m}), \beta_2 = (\beta_{21}, \dots, \beta_{2m}), \dots, \beta_{m-1} = (\beta_{m-1,1}, \dots, \beta_{m-1,m}) \in \mathbb{C}^m$ with $\sum_{i=1}^m |\beta_{ji}|^2 = 1$, $j = 1, 2, \dots, m-1$ and $\alpha_1 \perp \beta_j$ and $\beta_i \perp \beta_j, i \ne j$. Define $g_i(t)$ by

$$
g_1(t) = \begin{cases}
\alpha_{11}\sqrt{b}, & t \in E_1 \cup F_1, \\
\beta_{11}\sqrt{b}, & t \in E_2 \cup F_2, \\
\beta_{21}\sqrt{b}, & t \in E_3 \cup F_3, \\
\quad\vdots & \\
\beta_{m-2,1}\sqrt{b}, & t \in E_{m-1} \cup F_{m-1}, \\
\beta_{m-1,1}\sqrt{b}, & t \in E_m, \\
0, & \text{otherwise},
\end{cases}
\qquad
g_2(t) = \begin{cases}
\alpha_{12}\sqrt{b}, & t \in E_1 \cup F_1, \\
\beta_{12}\sqrt{b}, & t \in E_2 \cup F_2, \\
\beta_{22}\sqrt{b}, & t \in E_3 \cup F_3, \\
\quad\vdots & \\
\beta_{m-2,2}\sqrt{b}, & t \in E_{m-1} \cup F_{m-1}, \\
\beta_{m-1,2}\sqrt{b}, & t \in E_m, \\
0, & \text{otherwise},
\end{cases}
\quad \dots,
$$

$$
g_m(t) = \begin{cases}
\alpha_{1m}\sqrt{b}, & t \in E_1 \cup F_1, \\
\beta_{1m}\sqrt{b}, & t \in E_2 \cup F_2, \\
\beta_{2m}\sqrt{b}, & t \in E_3 \cup F_3, \\
\quad\vdots & \\
\beta_{m-2,m}\sqrt{b}, & t \in E_{m-1} \cup F_{m-1}, \\
\beta_{m-1,m}\sqrt{b}, & t \in E_m, \\
0, & \text{otherwise},
\end{cases}
$$

then it can be verified that $(g_1, g_2, \dots, g_m)$ satisfies (1.1) and (1.2). Thus $(g_1, g_2, \dots, g_m)$ is a Parseval Gabor multi-frame by Lemma 1.2, and hence $M(s)(g_1, \dots, g_m)^\tau$ is also a Parseval Gabor multi-frame since $M(t)$ is a functional matrix Gabor multi-frame multiplier. By the definition of g_i, for any $n \neq 0$, $g_i(t-na)=0$. So, for any $t \in E_1 \cup F_1$, (2.3) becomes

$$\left\langle \begin{pmatrix} \alpha_{11} \\ \vdots \\ \alpha_{1m} \end{pmatrix}, \big(I - M^*(t)M(t)\big) \begin{pmatrix} \alpha_{11} \\ \vdots \\ \alpha_{1m} \end{pmatrix} \right\rangle = 0. \tag{2.4}$$

Because α_1 is an arbitrary unit vector in $\mathbb{C}^m$, we get that $M(t)$ is unitary on $E_1 \cup F_1$.

Now we revise the definition of $(g_1, g_2, \dots, g_m)$ as follows,

$$g_1(t) = \begin{cases} \beta_{11}\sqrt{b},\ t \in E_1 \cup F_1, \\ \alpha_{11}\sqrt{b},\ t \in E_2 \cup F_2, \\ \beta_{21}\sqrt{b},\ t \in E_3 \cup F_3, \\ \quad \vdots \\ \beta_{m-2,1}\sqrt{b},\ t \in E_{m-1} \cup F_{m-1}, \\ \beta_{m-1,1}\sqrt{b},\ t \in E_m, \\ 0,\ \text{otherwise}, \end{cases}$$

$$g_2(t) = \begin{cases} \beta_{12}\sqrt{b},\ t \in E_1 \cup F_1, \\ \alpha_{12}\sqrt{b},\ t \in E_2 \cup F_2, \\ \beta_{22}\sqrt{b},\ t \in E_3 \cup F_3, \\ \quad \vdots \\ \beta_{m-2,2}\sqrt{b},\ t \in E_{m-1} \cup F_{m-1}, \\ \beta_{m-1,2}\sqrt{b},\ t \in E_m, \\ 0,\ \text{otherwise}, \end{cases} \qquad \cdots, \qquad g_m(t) = \begin{cases} \beta_{1m}\sqrt{b},\ t \in E_1 \cup F_1, \\ \alpha_{1m}\sqrt{b},\ t \in E_2 \cup F_2, \\ \beta_{2m}\sqrt{b},\ t \in E_3 \cup F_3, \\ \quad \vdots \\ \beta_{m-2,m}\sqrt{b},\ t \in E_{m-1} \cup F_{m-1}, \\ \beta_{m-1,m}\sqrt{b},\ t \in E_m, \\ 0,\ \text{otherwise}. \end{cases}$$

Similarly, $(g_1, g_2, \dots, g_m)$ is also a Parseval Gabor multi-frame, and (2.3) becomes (2.4) for any $t \in E_2 \cup F_2$. This implies that $M(t)$ is unitary on $E_2 \cup F_2$. Repeating the above process, we can get that $M(t)$ is unitary on $E_3 \cup F_3, \dots, E_{m-1} \cup F_m$ and E_m. Thus $M(t)$ is unitary on $[c, c+a)$, so $M(x)$ is unitary. Since x is arbitrary, we get that $M(t)$ is unitary a.e. t in $\mathbb{R}$.

Finally, if the length m of a Parseval Gabor multi-frame G is bigger than L, then we can divide E_1 into $m-L$ subintervals and $(g_1, g_2, \dots, g_m)$ can be defined similarly to force $M(t)$ to be unitary. □

We will need the following special examples in the proofs of the following two propositions.

Example 2.2 (i) Suppose that $1 < ab \leq 2$. For $c \in \mathbb{R}$, and $x \in (c, c+a-\frac{1}{b})$, let

$$E_1^1 = \left[c,\ c+a-\frac{1}{b}\right), \qquad E_1^2 = \left[c+a-\frac{1}{b}, c+\frac{a}{2}\right),$$
$$E_2^1 = \left[c+\frac{a}{2}, c+\frac{1}{b}\right), \qquad E_2^2 = \left[c+\frac{1}{b}, c+a\right).$$

Then we have $E_1^1 + \frac{1}{b} = E_2^2$, $E_1^2 + \frac{1}{b} \cap (E_2^1 \cup E_2^2) = \emptyset$, and $E_2^1 - \frac{1}{b} \cap (E_1^1 \cup E_1^2) = \emptyset$. Define

$$g_1(t) = \sqrt{b}\chi_{E_1^1 \cup E_1^2}(t),$$

$$g_2(t)=\sqrt{b}\chi_{E_2^1\cup E_2^2}(t).$$

Then $G_2^0=(g_1(t),g_2(t))^\tau$ is a Parseval Gabor multi-frame generator by Lemma 1.2. Moreover for any $t\in E_1^1$, we have $g_1(t)=\sqrt{b}$, $g_2(t)=0$, $g_1(t+\frac{1}{b})=0$, $g_2(t+\frac{1}{b})=\sqrt{b}$, and for any $n(\neq 0)\in\mathbb{Z}$, $g_1(t-na)=0$, $g_2(t-na)=0$.

Similarly if we define,

$$h_1(t)=\begin{cases}\sqrt{\frac{b}{2}}, & t\in E_1^1\cup E_1^2,\\ \sqrt{\frac{b}{2}}, & t\in E_2^1\cup E_2^2,\\ 0, & \text{otherwise},\end{cases}\qquad h_2(t)=\begin{cases}\sqrt{\frac{b}{2}}, & t\in E_1^1\cup E_1^2,\\ -\sqrt{\frac{b}{2}}, & t\in E_2^1\cup E_2^2,\\ 0, & \text{otherwise},\end{cases}$$

then $G_2^1(t)=(h_1(t),h_2(t))^\tau$ is a Parseval Gabor multi-frame generator, and for any $t\in E_1^1$, we have $h_1(t)h_1(t+\frac{1}{b})=\frac{b}{2}$, $h_2(t)h_2(t+\frac{1}{b})=-\frac{b}{2}$. Moreover, $h_i(t-na)=0$, $i=1,2$ and $n(\neq 0)\in\mathbb{Z}$.

We will also need another Parseval Gabor multi-frame generator $G_2^2(t)=(h_1(t),T_{-a}h_2(t))^\tau=(h_1(t),h_2^1(t))^\tau$ which is derived from G_2^1:

$$h_1(t)=\begin{cases}\sqrt{\frac{b}{2}}, & t\in E_1^1\cup E_1^2,\\ \sqrt{\frac{b}{2}}, & t\in E_2^1\cup E_2^2,\\ 0, & \text{otherwise},\end{cases}\qquad T_{-a}h_2(t)=h_2(t+a)=\begin{cases}\sqrt{\frac{b}{2}}, & t\in E_1^1\cup E_1^2+a,\\ -\sqrt{\frac{b}{2}}, & t\in E_2^1\cup E_2^2+a,\\ 0, & \text{otherwise}.\end{cases}$$

In this case for $t\in E_1$, we have $h_1(t)h_1(t+\frac{1}{b})=\frac{b}{2}$ and $h_2^1(t+a)h_2^1(t+a+\frac{1}{b})=-\frac{b}{2}$.

(ii) If $L-1<ab\le L(L\ge 3)$, then we have $\frac{2a}{L}-\frac{1}{b}>0$. For $c\in\mathbb{R}$ and $x\in(c,c+\frac{2a}{L}-\frac{1}{b})$, let

$$E_1^1=\left[c,c+\frac{2a}{L}-\frac{1}{b}\right),\qquad E_1^2=\left[c+\frac{2a}{L}-\frac{1}{b},c+\frac{a}{L}\right),$$
$$E_2^1=\left[c+\frac{a}{L},c+\frac{1}{b}\right),\qquad E_2^2=\left[c+\frac{1}{b},c+\frac{2a}{L}\right),$$
$$E_3=\left[c+\frac{2a}{L},c+\frac{3a}{L}\right),\qquad E_4=\left[c+\frac{3a}{L},c+\frac{4a}{L}\right),$$
$$\vdots$$
$$E_L=\left[c+\frac{L-1}{L}a,c+a\right).$$

Define

$$\begin{aligned}g_1(t)&=\sqrt{b}\chi_{E_1^1\cup E_1^2}(t),\\ g_2(t)&=\sqrt{b}\chi_{E_2^1\cup E_2^2}(t),\\ g_3(t)&=\sqrt{b}\chi_{E_3}(t),\\ g_4(t)&=\sqrt{b}\chi_{E_4}(t),\\ &\cdots\end{aligned}$$

$$g_L(t) = \sqrt{b}\chi_{E_L}(t).$$

Then clearly $G_L^0 = (g_1, g_2, \ldots, g_L)^\tau$ is a Parseval Gabor multi-frame generator. Moreover, for any $t \in E_1^1$, we have $t + \frac{1}{b} \in E_2^2$, and thus we get

$$\begin{aligned}
g_1(t) &= \sqrt{b}, & g_1\left(t + \frac{1}{b}\right) &= 0,\\
g_2(t) &= 0, & g_2\left(t + \frac{1}{b}\right) &= \sqrt{b},\\
g_3(t) &= 0, & g_3\left(t + \frac{1}{b}\right) &= 0,\\
&\vdots & &\\
g_L(t) &= 0, & g_L\left(t + \frac{1}{b}\right) &= 0.
\end{aligned}$$

Also $g_i(t - na) = 0$ for $n \neq 0$.

Similarly, if we define

$$h_1(t) = \begin{cases} \sqrt{\frac{b}{2}}, & t \in E_1^1 \cup E_1^2,\\ \sqrt{\frac{b}{2}}, & t \in E_2^1 \cup E_2^2,\\ 0, & \text{otherwise,} \end{cases}$$

$$h_2(t) = \begin{cases} \sqrt{\frac{b}{2}}, & t \in E_1^1 \cup E_1^2,\\ -\sqrt{\frac{b}{2}}, & t \in E_2^1 \cup E_2^2,\\ 0, & \text{otherwise,} \end{cases}$$

$$\begin{aligned}
h_3(t) &= \sqrt{b}\chi_{E_3},\\
h_4(t) &= \sqrt{b}\chi_{E_4},\\
&\vdots\\
h_L(t) &= \sqrt{b}\chi_{E_L},
\end{aligned}$$

then we also have that $G_L^1 = (h_1, h_2, \ldots, h_L)^\tau$ is a Parseval Gabor multi-frame generator.

We will use another Parseval Gabor multi-frame generator $G_L^2 = (h_1, h_2^1, \ldots, h_L^1)^\tau$ which is derived from G_L^1:

$$h_1(t) = \begin{cases} \sqrt{\frac{b}{2}}, & t \in E_1^1 \cup E_1^2,\\ \sqrt{\frac{b}{2}}, & t \in E_2^1 \cup E_2^2,\\ 0, & \text{otherwise,} \end{cases}$$

$$h_2^1(t) = \begin{cases} \sqrt{\frac{b}{2}}, & t \in E_1^1 \cup E_1^2 + a,\\ -\sqrt{\frac{b}{2}}, & t \in E_2^1 \cup E_2^2 + a,\\ 0,\ \text{otherwise,} & \end{cases}$$

$$h_3^1(t) = \sqrt{b}\chi_{E_3},$$
$$h_4^1(t) = \sqrt{b}\chi_{E_4},$$
$$\vdots$$
$$h_L^1(t) = \sqrt{b}\chi_{E_L}.$$

Moreover for any $t \in E_1^1$, we have $h_1(t)h_1(t+\frac{1}{b}) = \frac{b}{2}$ and $h_2^1(t+a)h_2^1(t+a+\frac{1}{b}) = -\frac{b}{2}$.

Now we are ready to prove another two needed propositions.

Proposition 2.3 *Assume that $M(t) = (f_{ij}(t))_{m\times m}$, with $f_{ij} \in L^\infty(\mathbb{R})$, is a functional matrix Gabor multi-frame multiplier. Then $M^*(t-na)M(t-na+\frac{1}{b})$ is a diagonal matrix for a.e. $t \in \mathbb{R}$, and for each $n \in \mathbb{Z}$.*

Proof Write

$$M^*(t-na)M\left(t-na+\frac{1}{b}\right) = \begin{pmatrix} B_{11}^n(t) & B_{12}^n(t) & \cdots & B_{1m}^n(t) \\ \vdots & \vdots & \vdots & \vdots \\ B_{m1}^n(t) & B_{m2}^n(t) & \dots & B_{mm}^n(t) \end{pmatrix}.$$

Case (i): $1 < ab \leq 2$, and $m = 2$.

In this case we choose the Parseval Gabor multi-frame generator G_2^0 from Example 2.2. Then $M_{2\times 2}(t)(g_1(t), g_2(t))^\tau$ is also a Parseval Gabor multi-frame generator. Thus, $\forall t \in E_1$, we have

$$\begin{aligned} 0 &= \sum_{n\in\mathbb{Z}} \left\langle \begin{pmatrix} g_1(t-na) \\ g_2(t-na) \end{pmatrix}, \begin{pmatrix} B_{11}^n(t) & B_{12}^n(t) \\ B_{21}^n(t) & B_{22}^n(t) \end{pmatrix} \begin{pmatrix} g_1(t-na+\frac{1}{b}) \\ g_2(t-na+\frac{1}{b}) \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}, \begin{pmatrix} B_{11}^0(t) & B_{12}^0(t) \\ B_{21}^0(t) & B_{22}^0(t) \end{pmatrix} \begin{pmatrix} g_1(t+\frac{1}{b}) \\ g_2(t+\frac{1}{b}) \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} \sqrt{b} \\ 0 \end{pmatrix}, \begin{pmatrix} B_{11}^0(t) & B_{12}^0(t) \\ B_{21}^0(t) & B_{22}^0(t) \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{b} \end{pmatrix} \right\rangle \\ &= b\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} B_{12}^0(t) \\ B_{22}^0(t) \end{pmatrix} \right\rangle. \end{aligned} \tag{2.5}$$

Thus we get $B_{12}^0(t) = 0$. Similarly, by using the Parseval Gabor multi-frame generator $(g_2(t), g_1(t))^\tau$, we also get $B_{21}^0(t) = 0$. Therefore $M^*(t)M(t+\frac{1}{b})$ is a diagonal matrix for a.e. $t \in \mathbb{R}$.

For any $n_0 \neq 0$, we know that

$$T_{n_0}G_2^0(t) = \big(g_1(t-n_0a), g_2(t-n_0a)\big) = \big(\sqrt{b}\chi_{E_1^1\cup E_1^2-n_0a}, \sqrt{b}\chi_{E_2^1\cup E_2^2-n_0a}\big)$$

is also a Parseval Gabor multi-frame generator. Moreover for any $t \in E_1^1$, we have $g_1(t-n_0a) = \sqrt{b}$, $g_2(t-n_0a) = 0$, $g_1(t-n_0a+\frac{1}{b}) = 0$, $g_2(t-n_0a+\frac{1}{b}) = \sqrt{b}$. So for this

$(T_{n_0}g_1, T_{n_0}g_2)$, (2.5) becomes

$$
\begin{aligned}
0 &= \sum_{n\in\mathbb{Z}} \left\langle \begin{pmatrix} g_1(t-na) \\ g_2(t-na) \end{pmatrix}, \begin{pmatrix} B^n_{11}(t) & B^n_{12}(t) \\ B^n_{21}(t) & B^n_{22}(t) \end{pmatrix} \begin{pmatrix} g_1(t-na+\frac{1}{b}) \\ g_2(t-na+\frac{1}{b}) \end{pmatrix} \right\rangle \\
&= \left\langle \begin{pmatrix} g_1(t-n_0a) \\ g_2(t-n_0a) \end{pmatrix}, \begin{pmatrix} B^{n_0}_{11}(t) & B^{n_0}_{12}(t) \\ B^{n_0}_{21}(t) & B^{n_0}_{22}(t) \end{pmatrix} \begin{pmatrix} g_1(t-n_0a+\frac{1}{b}) \\ g_2(t-n_0a+\frac{1}{b}) \end{pmatrix} \right\rangle \\
&= \left\langle \begin{pmatrix} \sqrt{b} \\ 0 \end{pmatrix}, \begin{pmatrix} B^{n_0}_{11}(t) & B^{n_0}_{12}(t) \\ B^{n_0}_{21}(t) & B^{n_0}_{22}(t) \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{b} \end{pmatrix} \right\rangle \\
&= b \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} B^{n_0}_{12}(t) \\ B^{n_0}_{22}(t) \end{pmatrix} \right\rangle.
\end{aligned}
$$

Thus we have $B^{n_0}_{12}(t) = 0$. Similarly, by using the Parseval Gabor multi-frame generator $(T_{n_0}g_2, T_{n_0}g_1)$ we can get $B^{n_0}_{21}(t) = 0$. Hence $M^*(t-n_0a)M(t-n_0a+\frac{1}{b})$ is a diagonal matrix for a.e. $t \in \mathbb{R}$.

Case (ii): $L-1 < ab \le L$, $L \ge 3$ and $m = L$.

In this case we will use the Parseval Gabor multi-frame generator G^0_L from Example 2.2. Since $M(t)(g_1, \ldots, g_L)^\tau$ is a Parseval Gabor multi-frame generator, we obtain the following equation:

$$
\begin{aligned}
0 &= \sum_{n\in\mathbb{Z}} \left\langle \begin{pmatrix} g_1(t-na) \\ g_2(t-na) \\ \vdots \\ g_L(t-na) \end{pmatrix}, \begin{pmatrix} B^n_{11}(t) & \cdots & B^n_{1L}(t) \\ B^n_{21}(t) & \cdots & B^n_{2L}(t) \\ \vdots & & \\ B^n_{L1}(t) & \cdots & B^n_{LL}(t) \end{pmatrix} \begin{pmatrix} g_1(t-na+\frac{1}{b}) \\ g_2(t-na+\frac{1}{b}) \\ \vdots \\ g_L(t-na+\frac{1}{b}) \end{pmatrix} \right\rangle \\
&= b \left\langle \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} B^0_{11}(t) & \cdots & B^0_{1L}(t) \\ B^0_{21}(t) & \cdots & B^0_{2L}(t) \\ \vdots & & \\ B^0_{L1}(t) & \cdots & B^0_{LL}(t) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\rangle \\
&= b \left\langle \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} B^0_{12}(t) \\ B^0_{22}(t) \\ \vdots \\ B^0_{L2}(t) \end{pmatrix} \right\rangle = b\overline{B^0_{12}(t)}.
\end{aligned}
$$

Thus we get $B^0_{12}(t) = 0$.

Similarly, if we use the Parseval Gabor multi-frame generator $(g_2, g_1, g_3, \ldots, g_L)$ by interchanging g_1, g_2 in G^0_L, then we can get $B^0_{21}(t) = 0$. For $i \ge 3$, we will use the Parseval Gabor multi-frame generator $(g_1, g_3, \ldots, g_2, \ldots, g_L)$, where g_2 is the $i (i \ge 3)$ coordinate. Then we can get $B^0_{1i}(t) = 0$. By interchanging g_1 and g_2 in $(g_1, g_3, \ldots, g_2, \ldots, g_m)$, we also get $B^0_{i1}(t) = 0$.

Repeating this argument we get $B^0_{ij}(t) = 0$ for all $i \ne j$ (for example, by using $(g_3, g_1, g_2, \ldots, g_L)$, we can get $B^0_{23}(t) = 0$). Therefore $M^*(t)M(t+\frac{1}{b})$ is a diagonal matrix for a.e. $t \in \mathbb{R}$. With the same argument as in Case (i) we can also get that $B^{n_0}_{ij}(t) = 0$

for all $i \neq j$ and all $n_0 \neq 0$. Thus $M^*(t-n_0a)M(t-n_0a+\frac{1}{b})$ is a diagonal matrix for a.e. $t \in \mathbb{R}$.

Case (iii): $m > L$.

In this case we insert zero functions into $(g_1, \ldots, g_L)$ to make it a Parseval Gabor multi-frame generator of length m. Then the same argument implies that for any $n_0 \in \mathbb{Z}$, $M^*(t-n_0a)M(t-n_0a+\frac{1}{b})$ is a diagonal matrix for a.e. $t \in \mathbb{R}$. □

Proposition 2.4 *Assume that $M(t) = (f_{ij}(t))_{m\times m}$, with $f_{ij} \in L^\infty(\mathbb{R})$, $i, j = 1, \ldots, m$ is a matrix functional Gabor multi-frame multiplier. Then $M^*(t)M(t+\frac{1}{b}) = \lambda(t)I$ for a.e. $t \in \mathbb{R}$, where $\lambda(t)$ is a unimodular (i.e. $|\lambda(t)| = 1$ for a.e. $t \in \mathbb{R}$) a-periodic function.*

Proof By Proposition 2.3, we have that

$$M^*(t-na)M\left(t-na+\frac{1}{b}\right) = \begin{pmatrix} B_{11}^n(t) & 0 & \cdots & 0 \\ 0 & B_{22}^n(t) & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & 0 & B_{mm}^n(t) \end{pmatrix},$$

and $|B_{ii}^n(t)| = 1$. Again we divide the proof into three cases.

Case (i): $1 < ab \leq 2$ and $m = 2$.

In this case we choose the frame G_2^1 in Example 2.2. Then $M(t)_{2\times 2}(G_2^1)^\tau$ is also Parseval Gabor multi-frame generator, and so it satisfies the following condition:

$$\begin{aligned} 0 &= \sum_{n\in\mathbb{Z}} \left\langle \begin{pmatrix} h_1(t-na) \\ h_2(t-na) \end{pmatrix}, \begin{pmatrix} B_{11}^n(t) & 0 \\ 0 & B_{22}^n(t) \end{pmatrix} \begin{pmatrix} h_1(t-na+\frac{1}{b}) \\ h_2(t-na+\frac{1}{b}) \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix}, \begin{pmatrix} B_{11}^0(t)h_1(t+\frac{1}{b}) \\ B_{22}^0(t)h_2(t+\frac{1}{b}) \end{pmatrix} \right\rangle \\ &= h_1(t)\overline{B_{11}^0(t)}\overline{h_1\left(t+\frac{1}{b}\right)} + h_2(t)\overline{B_{22}^0(t)}\overline{h_2\left(t+\frac{1}{b}\right)}. \end{aligned} \tag{2.6}$$

By the definition of G_2^1 we obtain that $B_{11}^0(t) = B_{22}^0(t)$.

To prove the a-periodic property, it is enough to prove that $B_{11}^0(t) = B_{22}^{-1}(t)$ and $B_{11}^{-1}(t) = B_{22}^0(t)$.

By using the Parseval Gabor multi-frame generator $G_2^2 = (h_1(t), h_2^1(t))^\tau$ from Example 2.2, we obtain that

$$\begin{aligned} 0 &= \sum_{n\in\mathbb{Z}} h_1(t-na)\overline{B_{11}^n(t)}\overline{h_1\left(t-na+\frac{1}{b}\right)} + \sum_{n\in\mathbb{Z}} h_2^1(t-na)\overline{B_{22}^n(t)}\overline{h_2^1\left(t-na+\frac{1}{b}\right)} \\ &= \overline{B_{11}^0(t)} \cdot \frac{b}{2} + \overline{B_{22}^{-1}(t)} \cdot \left(-\frac{b}{2}\right). \end{aligned}$$

Thus we get $B_{11}^0(t) = B_{22}^{-1}(t)$. Similarly, by using the Parseval Gabor multi-frame generator $(h_2^1(t), h_1(t))$ in above equation, we can get $B_{11}^{-1}(t) = B_{22}^0(t)$.

Case (ii): $L-1 < ab \le L$ for $L \ge 3$ and $m = L$.

In this case we use the Parseval Gabor multi-frame generator G_L^1 from Example 2.2. Then

$$M(t)\big(h_1(t), h_2(t), h_3(t), \dots, h_m(t)\big)^\tau$$

is a Parseval Gabor multi-frame generator, which implies that $B_{11}^0(t) = B_{22}^0(t)$. By using the Parseval Gabor multi-frame generator $(h_1(t), h_3(t), \dots, h_2(t), \dots, h_m(t))^\tau$, where $h_2(t)$ is the i-coordinate with $i = 3, \dots, m$, we get $B_{11}^0(t) = B_{ii}^0(t)$. Thus we obtain $B_{11}^0(t) = B_{22}^0(t) \cdots = B_{mm}^0(t)$.

To prove the a-periodic property, we use the Parseval Gabor multi-frame generator G_L^2 in Example 2.2. This implies that

$$\begin{aligned}
0 &= \sum_{n\in\mathbb{Z}} h_1(t-na)\overline{B_{11}^n(t)}\overline{h_1\left(t-na+\frac{1}{b}\right)} \\
&\quad + \sum_{n\in\mathbb{Z}} h_2^1(t-na)\overline{B_{22}^n(t)}\overline{h_2^1\left(t-na+\frac{1}{b}\right)} \\
&\quad \vdots \\
&\quad + \sum_{n\in\mathbb{Z}} h_m^1(t-na)\overline{B_{mm}^n(t)}\overline{h_m^1\left(t-na+\frac{1}{b}\right)} \\
&= \frac{b}{2}\overline{B_{11}^0(t)} + \left(-\frac{b}{2}\right)\overline{B_{22}^{-1}(t)}.
\end{aligned}$$

Thus $B_{11}^0(t) = B_{22}^{-1}(t)$.

By using the following Parseval Gabor multi-frame generators $(h_2^1, h_1, h_3^1, \dots, h_m^1)$, $(h_1, h_3^1, h_2^1, \dots, h_m^1)$, $(h_2^1, h_3^1, h_1, \dots, h_m^1)$ and so on, we can get $B_{11}^0(t) = B_{11}^{-1}(t) = B_{22}^0(t) = B_{22}^{-1}(t) = \cdots = B_{mm}^0(t) = B_{mm}^{-1}(t)$. Thus $M^*(t)M(t+\frac{1}{b})$ is a-periodic.

Case (iii): $m > L$.

This case can be dealt with by inserting $m - L$ zero functions into $(g_1, \dots, g_L)$ to make it a Parseval Gabor multi-frame generator of length m. □

Proof of Theorem 1.1 The necessity part follows from Proposition 2.1 and Proposition 2.4.

For the sufficiency part, by Lemma 1.2, it enough to show that for any Parseval Gabor multi-frame $(g_1, \dots, g_m)$, $M(s)(g_1, \dots, g_m)^\tau = (\eta_1, \dots, \eta_m)^\tau$ satisfies the equations in Lemma 1.2.

Since $M(t)$ is unitary, we obtain that

$$\begin{aligned}
\sum_{i=1}^m \sum_{n\in\mathbb{Z}} \left|\eta_i(t-na)\right|^2 &= \sum_{n\in\mathbb{Z}} \left\langle \begin{pmatrix} \eta_1(t-na) \\ \vdots \\ \eta_m(t-na) \end{pmatrix}, \begin{pmatrix} \eta_1(t-na) \\ \vdots \\ \eta_m(t-na) \end{pmatrix} \right\rangle \\
&= \sum_{n\in\mathbb{Z}} \left\| M(t-na) \begin{pmatrix} g_1(t-na) \\ \vdots \\ g_m(t-na) \end{pmatrix} \right\|^2
\end{aligned}$$

$$= \sum_{n\in\mathbb{Z}} \left\| \begin{pmatrix} g_1(t-na) \\ \vdots \\ g_m(t-na) \end{pmatrix} \right\|^2$$

$$= \sum_{i=1}^{m} \sum_{n\in\mathbb{Z}} \left|g_i(t-na)\right|^2 = b.$$

Thus (1.1) holds.

Now we verify (1.2).

$$\Delta(t,k) := \sum_{i=1}^{m} \sum_{n\in\mathbb{Z}} \big(\eta_i(t-na)\big)\overline{\eta_i\left(t-na-\frac{k}{b}\right)}$$

$$= \sum_{n\in\mathbb{Z}} \left\langle M(t-na) \begin{pmatrix} g_1(t-na) \\ \vdots \\ g_m(t-na) \end{pmatrix}, M\left(t-na-\frac{k}{b}\right) \begin{pmatrix} g_1(t-na-\frac{k}{b}) \\ \vdots \\ g_m(t-na-\frac{k}{b}) \end{pmatrix} \right\rangle$$

$$= \sum_{n\in\mathbb{Z}} \left\langle \begin{pmatrix} g_1(t-na) \\ \vdots \\ g_m(t-na) \end{pmatrix}, M^*(t-na)M\left(t-na-\frac{k}{b}\right) \begin{pmatrix} g_1(t-na-\frac{k}{b}) \\ \vdots \\ g_m(t-na-\frac{k}{b}) \end{pmatrix} \right\rangle.$$

By Proposition 1.4, and condition (iii), we obtain that

$$\Delta(t,k) = \overline{\lambda(t)} \sum_{n\in\mathbb{Z}} \left\langle \begin{pmatrix} g_1(t-na) \\ \vdots \\ g_m(t-na) \end{pmatrix}, \begin{pmatrix} g_1(t-na-\frac{k}{b}) \\ \vdots \\ g_m(t-na-\frac{k}{b}) \end{pmatrix} \right\rangle = 0$$

for a.e. $t \in \mathbb{R}$ and $k \neq 0$. Hence $M(t)$ is a Parseval Gabor multi-frame multiplier. □

Acknowledgement The authors would like to thank the referee for several helpful comments and suggestions that helped us improve the presentation of this paper.

References

1. Casazza, P.: Modern tools for Weyl-Heisenberg frame theory. Adv. Imaging Electron Phys. **115**, 1–127 (2001)
2. Han, D., Wang, Y.: Lattice tiling and the Weyl-Heisenberg frames. Geom. Funct. Anal. **11**, 742–758 (2001)
3. Feichtinger, H., Onchis, D.: Constructive reconstruction from irregular sampling in multi-window spline-type spaces. In: Progress in Analysis and Its Applications, pp. 257–265. World Scientific, Hackensack (2010)
4. Feichtinger, H., Onchis, D.: Constructive realization of dual systems for generators of multi-window spline-type spaces. J. Comput. Appl. Math. **234**, 3467–3479 (2010)
5. Gabardo, J., Han, D.: Frame representations for group-like unitary operator systems. J. Oper. Theory **49**, 223–244 (2003)
6. Jaillet, F., Torresani, B.: Time-frequency jigsaw puzzle: adaptive multiwindow and multilayered Gabor expansions. Int. J. Wavelets Multiresolut. Inf. Process. **5**, 293–315 (2007)
7. Li, S.: Discrete multi-Gabor expansions. IEEE Trans. Inf. Theory **45**, 1954–1967 (1999)
8. Gu, Q., Han, D.: Functional Gabor frame multipliers. J. Geom. Anal. **13**, 467–478 (2003)

9. Zibulski, M., Zeevi, Y.: Analysis of multiwindow Gabor-tpye schemes by frame methods. Appl. Comput. Harmon. Anal. **4**, 188–221 (1997)
10. Zeevi, Y., Zibulski, M., Porat, M.: Multi-window Gabor schemes in signal and image representations. In: Gabor Analysis and Algorithms. Appl. Numer. Harmon. Anal, pp. 381–407. Birkhäuser, Boston (1998).
11. The Wutam Consortium: Basic properties of wavelets. J. Fourier Anal. Appl. **4**, 575–594 (1998)