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Dilations of operator-valued measures with bounded p-variations and framings on Banach spaces ☆

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The dilations for operator-valued measures (OVMs) and bounded linear maps indicate that the dilation theory is in general heavily dependent on the Banach space nature of the dilation spaces. This naturally led to many questions concerning special type of dilations. In particular it is not known whether ultraweakly continuous (normal) maps can be dilated to ultraweakly continuous homomorphisms. We answer this question affirmatively for the case when the domain algebra is an abelian von Neumann algebra. It is well known that completely bounded Hilbert space operator valued measures correspond to the existence of orthogonal projection-valued dilations in the sense of Naimark and Stinespring, and OVMs with bounded total variations are completely bounded but not the vice-versa. With the aim of classifying OVMs from the dilation point of view, we introduce the concept of total p-variations for OVMs. We prove that any completely bounded OVM has finite 2-variation, and any OVM with finite p-variation can be dilated to a (but usually non-Hilbertian) projection-valued measure of the same type. With the help of framing induced OVMs, we prove that conventional minimal

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dilation space of a non-trivial framing contains c_0, then does not have bounded p-variation.

Contents

1. Introduction

While many known operator-valued measures (resp. bounded linear maps) acting on Hilbert spaces admit projection-valued measure (resp. bounded homomorphism) dilations to Hilbert spaces, i.e., Hilbertian dilation (cf. [1,2,12,20,27,28]), there are many examples that do not admit Hilbertian dilations [16]. In fact, there exists an example of a framing induced operator-valued measure on $\mathbb{N}$ that does not admit a Hilbert space dilation. Motivated by various known dilation theory for frames, framings, operator valued measures or linear maps, we recently established a general dilation theory for operator-valued measures and bounded linear maps that are not necessarily completely bounded [16–18]. These results can also be viewed as generalizations of the known result of Casazza, Han and Larson in [4] that arbitrary framings have Banach space dilations (for frames in Banach spaces, please see [3,7,9,22,31,32]), and also motivated by the dilation results for discrete structured frames (cf. [10,11,13–15,21]). Since Banach space techniques played an essential role in the dilations of OVMs and bounded linear maps, the theory applies to OVMs and bounded linear maps on arbitrary Banach algebras.

Let E be an operator-valued measure on a Banach space X. The projection-valued measure F (resp. bounded homomorphism π) on a Banach space Z in the above theorem is called a *dilation* of E (resp. ϕ), and the dilation is called *Hilbertian* if the dilation space Z can be taken as a Hilbert space. It is important to point out the Banach space nature of the above dilation theorem, i.e., unless E (resp. ϕ) belongs to a special class, the dilation space Z is usually not a Hilbert space even if X is a Hilbert space. A number of consequences and applications of this dilation theory have been studied, among which include its connection with Kadison's similarity problem for bounded homomorphisms on C^*-algebras, the connection with Mackey's theory of systems of imprimitivities, and with the completely bounded approximation property for operator spaces and reduced group

C^*-algebras [16–19,23,24]. However, some basic problems concerning the dilations of operator-valued measures and bounded linear maps remain to be addressed (for example, see Problems A, B, C, D, E in [16]). In this paper we settle two problems involving the dilations of ultraweakly continuous maps and operator-valued measures with bounded variations.

While the general dilation theory for bounded linear maps guarantees that the dilation homomorphism to be norm continuous, it is not known whether other continuity properties can be also preserved by the dilation (see Problem E in [16]). Our first result of this paper concerns the dilation of bounded linear maps that also preserve the ultraweakly continuity. Why do we care about this? In the dilation theory of operator-valued measures quite often we associate an operator-valued measure with a linear map on some abelian von Neumann algebra (or more general non-commutative von Neuamnn algebra in the case of quantum measures), and then dilate the associated map to a homomorphism. In order to get a projection-valued measure from the associated homomorphism, it will require some kind of continuity. The ultraweakly continuity ensures the induced projection-valued measure to be countably additive and so it will be a dilation of the original operator-valued measure. In [16] we obtained some partial results for arbitrary von Neumann algebras with continuity restrictions to bounded subsets, and a positive answer for purely atomic abelian von Neumann algebras. However, the general question remains open. Our first main result of this paper is to settle this problem affirmatively for arbitrary abelian von Neumann algebras.

Theorem 1.1. *Let (Ω, Σ, μ) be a finite measure space. If X is a Banach space, then for every ultraweak-wot continuous linear map $\phi : L^\infty(\mu) \to B(X)$, there exist a Banach space Z, a unital ultraweak-wot continuous linear homomorphism $\pi : L^\infty(\mu) \to B(Z)$, and bounded linear maps $T : X \to Z$ and $S : Z \to X$ with*

$$\|\pi\| = \|S\| = 1 \quad \text{and} \quad \|T\| = \|\phi\|$$

such that

$$\phi(f) = S\pi(f)T$$

for all $f \in L^\infty(\mu)$. Moreover, if $L^1(\mu)$ and X are separable, then the dilated space Z can be separable.

A special class of operator-valued measures is the class of OVMs with finite total variations. It is known that any operator-valued measure with finite total variation is completely bounded in the sense that its associated map defined by Paulsen ([29], page 105) is completely bounded. Don Hadwin and Vern Paulsen constructed examples that are completely bounded but do not have finite total variations [12,29]. This motivates us to examine the dilation theory of operator-valued measures that have various type of finite total variations. We will introduce the concept of p-variations for operator-valued measures, and show that any completely bounded operator-valued measure has bounded 2-variation. Our second main result of this paper deals with the question whether every operator-valued measure with bounded p-variation can be dilated to a projection-valued measure that also has p-bounded variation. At first glance this might seem too much to ask since even for Hilbert space operator-valued measures, the dilation space is usually a Banach space that is far from Hilbertian. However, we will prove that such a dilation indeed is always possible.

Theorem 1.2. *Every operator-valued measure with bounded p-variation has a dilation to a projection-valued measure with bounded p-variation.*

Along with some other related results, we present the proofs of Theorem 1.1 and Theorem 1.2 in sections 3 and 4. Sections 5 and 6 will be devoted to several important classes of examples and related problems. In particular we show that completely bounded operator-valued measure have bounded 2-variations and examine the dilations of framing induced operator-valued measures. There is a natural dilation space introduced in [16] that is called the minimal dilation space, and a naturally induced projection-valued measure that is a dilation of the original operator-valued measure. The above dilation p-norm used in the proof of Theorem 1.2 is completely new, and it is different from the ones constructed in [4,16]. We explain this in section 5 by showing that the conventional minimal framing model of a non-trivial redundant framing constructed in [16] always contains an isomorphic copy of c_0, and hence the dilatation does not have the bounded p-variation property for any $p \geq 1$ (please see Definition 5.4).

Theorem 1.3. *Let (x_i, f_i) be a framing of a Banach space X satisfying property (u). Let $(E, (e_i))$ be the above minimal framing model for $(X, (x_i, f_i))$. If (x_i, f_i) is not a near-unconditional basis, then (e_i) contains a block (unconditional-basic) sequence equivalent to the unit vector basis of c_0. Thus, E contains an isomorphic copy of c_0 as a subspace, and the projection-valued measure (minimal dilation) on E induced by (e_i) does not have bounded p-variation for any $p \geq 1$.*

2. Preliminaries

In this section we introduce some necessary terminologies and recall several basic properties that will be needed for the rest of the paper.

Let $F : \Sigma \to B(X)$ be an operator-valued measure (OVM for short), where (Ω, Σ) is a measurable space, and X is a Banach space. For $x \in X$ and $x^* \in X^*$, define the vector measure $F_x : \Sigma \to X$ and the complex measure $F_{x,x^*} : \Sigma \to \mathbb{C}$ respectively by

$$F_x(B) = F(B)x, \text{ and } F_{x,x^*}(B) = x^*(F(B)x)$$

for all $B \in \Sigma$.

We turn to the variation $|F_{x,x^*}|$ of the complex measure F_{x,x^*}. For each $B \in \Sigma$, let $|F_{x,x^*}|(B)$ be the supremum of the numbers $\sum_{j=1}^{n} |F_{x,x^*}(B_j)|$, where $\{B_j\}_{j=1}^{n}$ ranges over all finite partitions of B into Σ-measurable sets. The semivariation of F_x is the nonnegative function $\|F_x\|$ whose value on a set $B \in \Sigma$ is given by

$$\|F_x\|(B) = \sup\{|F_{x,x^*}|(B) : x^* \in X^*, \|x^*\| \le 1\},$$

which is finite by the Uniform Boundedness Principle. Then we have for every $B \in \Sigma$ that

$$\|F_x\|(B) = \sup \Big\| \sum_n \varepsilon_n F_x(B_n) \Big\| = \sup \Big\| \sum_n \varepsilon_n F(B_n) x \Big\|,$$

where the supremum is taken over all partitions B_n's of B into finitely many disjoint members of Σ, and all finite collections ε_n's satisfying $|\varepsilon_n| \le 1$.

The semivariation of F is the nonnegative function $\|F\|$ whose value on a set $B \in \Sigma$ is given by

$$\|F\|(B) = \sup\{\|F_x\|(B) : \|x\| \le 1\} = \sup\{|F_{x,x^*}|(B) : \|x\| \le 1, \|x^*\| \le 1\},$$

which again is finite by the Uniform Boundedness Principle. This implies that

$$\|F\|(B) = \sup \Big\| \sum_n \varepsilon_n F(B_n) \Big\|,$$

where the supremum is taken over all partitions B_n's of B into finitely many disjoint members of Σ, and all finite collections ε_n's satisfying $|\varepsilon_n| \le 1$.

The following shows that any operator-valued measure ϕ associates with a positive scalar valued measure μ. This allows us to identify the operator-valued measure with a countably additive map on $L^\infty(\mu)$.

Proposition 2.1. *Let X be a separable Banach space, and $\varphi : \Sigma \to B(X)$ be an OVM. Then there exists a nonnegative real-valued measure μ on Σ such that φ vanishes on sets of μ-measure zero. Moreover, we have $0 \le \mu(E) \le \|\varphi\|(E)$ for all $E \in \Sigma$.*

Proof. Since the unit sphere $S_1(X)$ is separable, there is a dense sequence $\{x_i\}$ in $S_1(X)$. For each x_i, $\varphi_{x_i} : \Sigma \to X$ is a vector measure. By Corollary 6 in [8, page 14], there exists a nonnegative real-valued measure μ_i on Σ such that $\mu_i(E) \to 0$ if and only if $\|\varphi_{x_i}\|(E) \to 0$, and that $0 \le \mu_i(E) \le \|\varphi_{x_i}\|(E)$ for all $E \in \Sigma$. Then define $\mu : \Sigma \to [0, \infty)$ by

$$\mu(E) = \sum_{i=1}^{\infty} \frac{1}{2^i} \mu_i(E)$$

for all $E \in \Sigma$. Since for each $i \in \mathbb{N}$

$$0 \leq \mu_i(E) \leq \|\varphi_{x_i}\|(E) \leq \|\varphi\|(E)\|x_i\| = \|\varphi\|(E) \leq \|\varphi\|(\Omega),$$

we get that μ is a nonnegative measure on Σ with $0 \leq \mu(E) \leq \|\varphi\|(E)$ for all $E \in \Sigma$. Moreover, we have the following implications:

$$\begin{aligned}\mu(E) = 0 &\Rightarrow \mu_i(E) = 0 \text{ for each } i \in \mathbb{N}\\ &\Rightarrow \|\varphi_{x_i}\|(E) = 0 \text{ for each } i \in \mathbb{N}\\ &\Rightarrow \varphi_{x_i}(E) = 0 \text{ for each } i \in \mathbb{N}\\ &\Rightarrow \varphi(E) = 0 \text{ by the density of } \{x_i\}.\end{aligned}$$

This completes the proof. □

The following results will be needed in the proof of Theorem 1.1.

Lemma 2.2. *[8] Let $F : \Sigma \to X$ be a countably additive vector measure and μ be a nonnegative real-valued measure on Σ. Then F is μ-continuous, i.e.,*

$$\lim_{\mu(E)\to 0} F(E) = 0$$

if and only if F vanishes on sets of μ-measure zero.

Lemma 2.3. *[16] Let $F : \Sigma \to B(X)$ be a countably additive operator-valued measure and μ be a nonnegative real-valued measure on Σ. Then the followings are equivalent:*

(a) *F is μ-continuous, that is, F vanishes on sets of μ-measure zero.*
(b) *sot-*$\lim_{\mu(E)\to 0} F(E) = 0$.
(c) *wot-*$\lim_{\mu(E)\to 0} F(E) = 0$.

While the ultraweak topology on $B(\mathcal{H})$ for a Hilbert space $\mathcal{H}$ is well-understood, we define the ultraweak topology on $B(X)$ for a Banach space X through the natural embedding $B(X) \hookrightarrow B(X, X^{**})$ and tensor products: Let $X \otimes Y$ be the tensor product of the Banach space X and Y. The projective norm on $X \otimes Y$ is defined by:

$$\|u\|_\wedge = \inf\left\{\sum_{i=1}^n \|x_i\|\|y_i\| : u = \sum_{i=1}^n x_i \otimes y_i\right\}.$$

We will use $X \otimes_\wedge Y$ to denote the tensor product $X \otimes Y$ endowed with the projective norm $\|\cdot\|_\wedge$. Its completion will be denoted by $X\widehat{\otimes}Y$. From [30] Section 2.2, for any Banach spaces X and Y, we have the identification:

$$(X\widehat{\otimes}Y)^* = B(X, Y^*).$$

Thus $B(X, X^{**}) = (X\widehat{\otimes}X^*)^*$. Viewing $X \subseteq X^{**}$, we define the *ultraweak topology* on $B(X)$ to be the weak* topology induced by the predual $X\widehat{\otimes}X^*$. We will usually use the term *normal* to denote an ultraweakly continuous linear map.

Notation: J is the canonical embedding from $B(X)$ into $B(X, X^{**}) = (X\widehat{\otimes}X^*)^*$.

Proposition 2.4. *Let X be a Banach space and (Ω, Σ, μ) be a σ-finite measure space. If $T : L^\infty(\mu) \to B(X)$ is a linear map, then T is ultraweak-wot continuous if and only if JT is normal.*

Proof. ($\Leftarrow$) For any $x \in X$, $x^* \in X^*$, and $f_\gamma \xrightarrow{w^*} 0$ in $L^\infty(\mu)$, we have $x \otimes x^* \in X\widehat{\otimes}X^*$ and

$$x^*(T(f_\gamma)x) = (JT(f_\gamma)x)x^* = JT(f_\gamma)(x \otimes x^*) \underset{\gamma}{\to} 0.$$

Then T is weak*-wot continuous.

($\Rightarrow$) Since T is weak*-wot continuous, for each $x \in X$ and $x^* \in X^*$, the linear functional $x^*(T(\cdot)x) : L^\infty(\mu) \to \mathbb{C}$ is weak* continuous on $L^\infty(\mu)$. Then there is a unique $g \in L^1(\mu)$ such that

$$x^*(T(f)x) = \int f\bar{g}\, d\mu$$

for all $f \in L^\infty(\mu)$.

For every $u \in X\widehat{\otimes}X^*$, write $u = \sum_j x_j \otimes x_j^*$ with $\sum_j \|x_j\|\|x_j^*\| < \infty$. For each j, there is $g_j \in L^1(\mu)$ such that $x_j^*(T(\cdot)x_j) = \int(\cdot)\bar{g}_j\, d\mu$. For any $f \in L^\infty(\mu)$, we have

$$\begin{aligned}\int f \sum_j |\bar{g}_j|\, d\mu &= \sum_j \int f|\bar{g}_j|\, d\mu = \sum_j \int \frac{|\bar{g}_j|}{\bar{g}_j} f\bar{g}_j\, d\mu \\ &= \sum_j x_j^*\Big(T\Big(\frac{|\bar{g}_j|}{\bar{g}_j} f\Big)x_j\Big) \\ &\le \sum_j \|T\|\|f\|\|x_j\|\|x_j^*\| < \infty.\end{aligned}$$

This implies that $\sum_j |\bar{g}_j| \in L^1(\mu)$. If $f_\gamma \xrightarrow{w^*} 0$ in $L^\infty(\mu)$, then

$$\begin{aligned}JT(f_\gamma)(u) &= \sum_j (JT(f_\gamma)x_j)(x_j^*) = \sum_j x_j^*(T(f_\gamma)x_j) \\ &= \sum_j \int f_\gamma \bar{g}_j\, d\mu\end{aligned}$$

$$= \int f_\gamma \sum_j \bar{g_j}\, d\mu \underset{\gamma}{\longrightarrow} 0.$$

Thus, JT is normal. □

3. Normal dilations

In order to prove Theorem 1.1 we need some preparations. Let (Ω, Σ) be a measurable space and $F : \Sigma \to B(X)$ be an μ continuous OVM (see Proposition 2.1 for the existence of μ). If f is a scalar-valued simple function on Ω, say $f = \sum_{i=1}^n \alpha_i \chi_{E_i}$ where α_i are nonzero scalars and E_i are pairwise disjoint members of Σ, define

$$T_F(f) = \sum_{i=1}^n \alpha_i F(E_i).$$

Then T_F is a linear map from the space of simple functions of the above form into $B(X)$. Moreover, we have

$$\|T_F(f)\| \le \|F\|(\Omega)\|f\|.$$

Since the simple functions are dense in $L^\infty(\mu)$, T_F can be uniquely extended to the entire space $L^\infty(\mu)$, still denoted by T_F. It is easy to show that $\|T_F\| = \|F\|(\Omega)$. For each $f \in L^\infty(\mu)$, we use the notation $\int f dF$ for $T_F(f)$.

If $Q \in B(X, Y)$ and $R \in B(Y, X)$, then $QF(\cdot)R$ is a μ-continuous OVM from Σ to $B(Y)$, denoted by E. Then we have

$$\int f dE = T_E(f) = QT_F(f)R = Q\Big(\int f dF\Big)R$$

for all $f \in L^\infty(\mu)$. In particular, if $x \in X$ and $x^* \in X^*$, then the following useful equality holds

$$x^*\Big(\Big(\int f dF\Big)x\Big) = \int f dF_{x,x^*}.$$

Indeed, for simple functions f the above equalities are trivial and density of simple functions in $L^\infty(\mu)$ proves the identity for all $f \in L^\infty(\mu)$.

Lemma 3.1 *([8]). Let μ be a nonnegative measure on Σ, and X be a Banach space. If $T : L^\infty(\mu) \to X$ is a bounded linear map, then the followings are equivalent:*

(a) *T is ultraweak-weakly continuous.*
(b) *The representing measure of T is countably additive.*
(c) *The representing measure of T is μ-continuous.*

Thus we have

Theorem 3.2. *Let X be a Banach space and (Ω, Σ, μ) be a finite nonnegative measure. Then there is a one-to-one linear correspondence between the space of all ultraweak-wot continuous linear map T from $L^\infty(\mu)$ to $B(X)$ and the space of all μ-continuous OVM F from Σ to $B(X)$ defined by*

$$F \leftrightarrow T_F \quad \textit{if} \quad T_F f = \int f dF$$

for all $f \in L^\infty(\mu)$. Moreover, $\|T_F\| = \|F\|(\Omega)$.

We will also need the following corollary in the proof of Theorem 1.1.

Corollary 3.3. *Let X be a Banach space and (Ω, Σ, μ) be a finite nonnegative measure. Then there is a one-to-one linear correspondence between the space of all ultraweak-wot continuous linear homomorphism T from $L^\infty(\mu)$ to $B(X)$ and the space of all μ-continuous spectral OVM F from Σ to $B(X)$ defined by*

$$F \leftrightarrow T_F \quad \textit{if} \quad T_F f = \int f dF$$

for all $f \in L^\infty(\mu)$. Moreover, $\|T_F\| = \|F\|(\Omega)$.

Proof. By Theorem 3.2, let T be a ultraweak-wot continuous linear map from $L^\infty(\mu)$ to $B(X)$ and F be the uniquely corresponding representing OVM from Σ to $B(X)$.

If T is a homomorphism, then for each $B_1, B_2 \in \Sigma$,

$$\begin{aligned} F(B_1 \cap B_2) &= T(\chi_{B_1 \cap B_2}) = T(\chi_{B_1} \cdot \chi_{B_2}) \\ &= T(\chi_{B_1}) T(\chi_{B_2}) = F(B_1) F(B_2). \end{aligned}$$

That is, F is a spectral OVM. If F is spectral, since T is a bounded linear map, then it suffices to prove it for simple functions by density, which we leave for interested readers. □

Proof of Theorem 1.1. Since ϕ is ultraweakly-wot continuous, by Theorem 3.2, the representing measure $\varphi : \Sigma \to B(X)$ is a μ-continuous OVM. Thus, by Theorem 2.29 of [16], φ has a minimal dilation system (ρ, Z, Q, T). For every $z \in Z$, we first prove that the vector measure ρ_z is μ-continuous. From Lemma 2.2 we know that it suffices to prove that ρ_z vanishes on sets of μ-measure zero.

Now assume that $\mu(B) = 0$. Then we have $\mu(B \cap C \cap D) = 0$ for any $C, D \in \Sigma$. Thus for every $x \in X$ and $D \in \Sigma$ we have

$$\rho(B) \varphi_{x,C}(D) = \varphi_{x,B\cap C}(D) = \varphi(B \cap C \cap D)x = 0$$

which implies that $\rho_{\varphi_{x,C}}(B) = \rho(B)\varphi_{x,C} = 0$. Since $\|\rho(B)\| \leq 1$ and the linear span of $\varphi_{x,C}$ is dense in Z, we obtain that $\rho_z(B) = 0$ for all $z \in Z$. Therefore ρ_z is μ-continuous for all $z \in Z$ as claimed.

By Theorem 3.2, there is a unique ultraweak-wot continuous linear (unital) homomorphism $\pi : L^\infty(\mu) \to B(Z)$ induced by ρ. Then it is easy to prove that

$$\begin{aligned} Q\,\pi(f)T &= Q\Big(\int f d\rho\Big)T = \int fd(Q\rho(\cdot)T) \\ &= \int fd\varphi = \phi(f) \end{aligned}$$

for all $f \in L^\infty(\mu)$. Thus π is ultraweak-wot continuous linear (unital) homomorphism that dilates the ultraweakly-wot continuous bounded linear map ϕ.

Now assume that both $L^1(\mu)$ and X are separable. Since $L^1(\mu)$ is separable, then the closed unit ball $B_1(L^\infty(\mu))$ is a compact metrizable space with respect to the ultraweak-topology, and hence is separable. There is a sequence $\{f_i\}$ which is ultraweakly dense in $B_1(L^\infty(\mu))$. Since X is separable, there is a sequence $\{x_j\}$ that is dense in $B_1(X)$. Define

$$Z_0 = \text{span}\{\pi(f_i)T(x_j)\}_{i,j\in\mathbb{N}}.$$

For each $f \in B_1(L^\infty(\mu))$ and $x \in B_1(X)$, there are subsequences $\{f_{i_k}\}$ and $\{x_{j_k}\}$ such that $f_{i_k} \to f$ in the ultraweak topology and $x_{j_k} \to x$ in norm, then $T(x_{j_k}) \to T(x)$. Since π is ultraweak-wot continuous, $\pi(f_{i_k}) \to \pi(f)$ in the wot-topology, we get that $\pi(f_{i_k})T(x_{j_k}) \to \pi(f)T(x)$ in the weak topology. Then we have

$$\overline{Z_0}^{\|\cdot\|} = \overline{Z_0}^{weak} = \overline{\text{span}}^{\|\cdot\|}\{\pi(f)T(x) : f \in L^\infty(\mu), x \in X\} = Z.$$

Therefore Z is separable. $\square$

4. OVM with *p*-bounded variations

Let Σ be a σ-algebra of subsets of a set Ω and let $\mathcal{M}(\Sigma)$ denote the Banach space of scalar valued measures on Σ with the variation norm, $\|\mu\| = |\mu|(\Omega)$. We denote by $\mathcal{M}(\Sigma, X)$ the Banach space of vector measures on Σ with values in X. The semivariation norm is defined by $\|\mu\|_\infty = \|\mu\|(\Omega)$. Since $\varphi\mu$ is a scalar measure for every $\varphi \in X^*$ and so we may define a linear mapping $M_\mu : X^* \to \mathcal{M}(\Sigma)$ by $M_\mu\varphi = \varphi\mu$ with $\|\mu\|_\infty = \|\mu\|(\Omega) = \|M_\mu\|$.

We denote by $\mathcal{M}(\Sigma, X, Y)$ the vector space of operator-valued measures on Σ with values in $B(X, Y)$. The semivariation norm is defined on this space by

$$\|\mu\|_\infty = |\mu|_\infty(\Omega) = \|V_\mu\|.$$

Let μ be a vector measure on Σ with values in a Banach space X. We consider the definition of the p-variation for vector measures as follows

$$|\mu|_p(E) = \sup\left\{ \left(\sum_{i=1}^{n} \|\mu(A_i)\|^p \right)^{1/p} : \{A_1, \ldots, A_n\} \text{ a partition of } E \right\}.$$

The vector measure μ is said to have bounded p-variation (or bounded variation for $p = 1$) if $|\mu|_p(E)$ is finite for every $E \in \Sigma$, or equivalently, if $|\mu|_p(\Omega)$ is finite. Furthermore, it is not difficult to show that the set of vector measure with values in X that have bounded p-variation is a Banach space with respect to the p-variation norm:

$$\|\mu\|_p = |\mu|_p(\Omega).$$

We now consider the definition of the p-variation for OVM's. Let

$$|\varphi|_p(E) = \sup_{\|x\| \le 1} |\varphi_x|_p(E).$$

The OVM φ is said to have bounded p-variation (or bounded variation for $p = 1$) if $|\varphi|_p(E)$ is finite for every $E \in \Sigma$, or equivalently, if $|\varphi|_p(\Omega)$ is finite. By Uniform Boundedness Principle, we know that the operator-valued measure φ has bounded p-variation if and only if, for all $x \in X$, the vector measure φ_x has bounded p-variation.

Example 4.1. Let (Ω, Σ, μ) be a measure space with $1 \le p < \infty$, and $\rho : \Sigma \to B(L_p(\mu))$ be given by multiplication $\rho(E) = \chi_E$ for all $E \in \Sigma$. Then ρ is a projection-valued measure with bounded p-variation and $\|\rho\|_p = 1$.

Example 4.2. Recall that a sequence (x_n) in a Hilbert space $\mathcal{H}$ is Bessel if there exists a constant $C > 0$ (a Bessel bound) such that $\sum_n |\langle x, x_n\rangle|^2 \le C||x||^2$ holds for every $x \in \mathcal{H}$. So in this case the operator defined Θ_X by $\Theta_X(x) = (\langle x, x_n\rangle)_n$ is a bounded linear operator from $\mathcal{H}$ to $\ell^2(\mathbb{N})$. A pair of sequences (x_n, y_n) is a dual frame pair for a Hilbert space $\mathcal{H}$ if both of them are Bessel sequences and $I = \sum_n x_n \otimes y_n$, where the convergence is in norm and unconditional. A dual frame pair (x_n, y_n) naturally induces an operator-valued measure φ: $\varphi(A) = \sum_{n\in A} x_n \otimes y_n$ for any subset A of $\mathbb{N}$. We claim that every dual frame pair induces operator-valued measure in Hilbert spaces has bounded 2-variation.

Proof. For a dual frame pair (x_n) and (y_n) in a Hilbert space $\mathcal{H}$, let φ be the operator-valued measure induced by (x_n, y_n). Assume that B is the optimal Bessel bound of (y_n), and that $G = \Theta_X \Theta_X^*$ is the Gramian matrix of (x_n). Then, for all $x \in \mathcal{H}$ and any partition (A_k) of $\mathbb{N}$, we have

$$
\begin{aligned}
\sum_{A_k}\left\|\sum_{n\in A_k}\langle x,y_n\rangle x_n\right\|^2 &= \sum_{A_k}\left\langle \sum_{n\in A_k}\langle x,y_n\rangle x_n, \sum_{m\in A_k}\langle x,y_m\rangle x_m\right\rangle \\
&= \sum_{A_k}\sum_{n\in A_k}\sum_{m\in A_k}\langle x,y_n\rangle\langle x_n,x_m\rangle\overline{\langle x,y_m\rangle} \\
&= \sum_{A_k}(\langle x,y_n\rangle)_{n\in A_i}(\langle x_n,x_m\rangle)_{n,m\in A_i}(\langle x,y_m\rangle)^*_{n\in A_i} \\
&= \sum_{A_k}(\langle x,y_n\rangle)_{n\in A_i}G|_{A_i}(\langle x,y_m\rangle)^*_{n\in A_i} \\
&\le \sum_{A_k}\sqrt{\sum_{n\in A_i}|\langle x,y_n\rangle|^2}\|G\|\sqrt{\sum_{m\in A_i}|\langle x,y_m\rangle|^2} \\
&= \sum_{A_k}\sum_{n\in A_i}|\langle x,y_n\rangle|^2\|G\| \\
&= \|G\|\sum_{n\in\mathbb{N}}|\langle x,y_n\rangle|^2 \\
&\le B^2\|G\|\|x\|^2.
\end{aligned}
$$

Thus, φ has $B\sqrt{\|G\|}$-bounded 2-variation. □

Example 4.3. Let H be a Hilbert space. Then for any $p \ge 1$, there is a $B(H)$-valued measure with bounded q-variation for any $q > p$ but not bounded p-variation. Actually, let $\{e_n\}$ be the unit vector basis of c_0, and define the operator-valued measure on the subsets of $\mathbb{N}$ as follows

$$\varphi(E) = \sum_{n\in E}\frac{1}{n^{1/p}}e_n\otimes e_n$$

for all $E \subset \mathbb{N}$.

Proposition 4.4. *Let $\rho : \Sigma \to B(Z)$ is an operator-valued measure with bounded p-variation. If $S : Z \to X$ and $T : X \to Z$ are bounded linear maps, then the compressed operator-valued measure $\varphi = S\rho(\cdot)T : \Sigma \to B(X)$ has bounded p-variation with*

$$|\varphi|_p(E) \le \|S\||\rho|_p(E)\|T\|$$

for any $E \in \Sigma$.

Proof. For all $E \in \Sigma$, we have

$$|\varphi|_p(E) = \sup_{\|x\|\le 1}|\varphi_x|_p(E)$$

$$
\begin{aligned}
&= \sup_{\|x\|\le 1} \sup_{\{A_n\} \text{ a partition of } E} \left(\sum \|\varphi(A_n)x\|^p\right)^{1/p} \\
&= \sup_{\|x\|\le 1} \sup_{\{A_n\} \text{ a partition of } E} \left(\sum \|S\rho(A_n)Tx\|\right)^{1/p} \\
&\le \sup_{\|x\|\le 1} \sup_{\{A_n\} \text{ a partition of } E} \|S\| \left(\sum \|\rho_{Tx}(A_n)\|\right)^{1/p} \\
&= \sup_{\|x\|\le 1} \|S\||\rho_{Tx}|_p(E) \\
&\le \sup_{\|x\|\le 1} \|S\||\rho|_p(E)\|Tx\| \\
&= \|S\||\rho|_p(E)\|T\|. \quad \square
\end{aligned}
$$

Now we are ready to prove our second main theorem.

Proof of Theorem 1.2. Let φ be an operator-valued measure that has the bounded p-variation. For any $x \in X$ and $E \in \Sigma$, we define $\varphi_{x,E} : \Sigma \to X$ by $\varphi_{x,E}(F) = \varphi(E \cap F)x$. We construct the dilation norm on the elementary dilation system introduced in [16] as follows. Let $\mathcal{V}_p$ be the completion of linear span of $\varphi_{x,E}$ for all $x \in X$ and $E \in \Sigma$ endowed with the norm $\|\cdot\|_{pV} = |\cdot|_p(\Omega)$. For each $F \in \Sigma$, it is easy to verify that

$$
\|\sum \varphi_{x_i,E_i\cap F}\|_{pV} \le \|\sum \varphi_{x_i,E_i}\|_{pV},
$$

which implies that $\|\rho(F)\| = 1$ or 0. Define S by

$$
S(\sum \varphi_{x_i,E_i}) = \sum \varphi(E_i)x_i
$$

for $x_i \in X$ and $E_i \in \Sigma$. Since

$$
\|S(\sum \varphi_{x_i,E_i})\| = \|\sum \varphi(E_i)x_i\| = \|(\sum \varphi_{x_i,E_i})(\Omega)\| \le \|\sum \varphi_{x_i,E_i}\|_{pV},
$$

S can be extended to the entire space $\mathcal{V}_p$ with $\|S\| \le 1$.

For the linear map $T : X \to \mathcal{V}_p$ defined by $T(x) = \varphi_{x,\Omega}$, we have

$$
\|T(x)\| = \|\varphi_{x,\Omega}\|_{pV} = |\varphi_x|_p \le \|\varphi\|_p\|x\|
$$

for all $x \in X$. It implies that $\|T\| \le \|\varphi\|_p$.

For each $F \in \Sigma$ define $\rho(F)\varphi_{x,E} = \varphi_{x,F\cap E}$. Then we know from [16] that $\rho(F)$ is a projection in $B(\mathcal{V}_p)$. Now we only need to prove the countable additivity of the projection-valued measure ρ. Let E_j's be a countable disjoint collection of members in Σ with union E. Then for any $x \in X$ and $F \in \Sigma$, we have

$$\left\|\rho(E)\varphi_{x,F} - \rho\left(\cup_{j=1}^{n} E_j\right)\varphi_{x,F}\right\|_p = \left\|\varphi_{x,F\cap E} - \varphi_{x,F\cap\left(\cup_{j=1}^{n}E_j\right)}\right\|_p$$
$$= \left\|\varphi_{x,F\cap\left(\cup_{j=n+1}^{\infty}E_j\right)}\right\|_p$$
$$= \left\|\varphi_{x,\cup_{j=n+1}^{\infty}F\cap E_j}\right\|_p .$$

Without loss of generality, assume that $F = \Omega$, then we need to prove that

$$\lim_{n\to\infty} \left\|\varphi_{x,\cup_{j=n+1}^{\infty}E_j}\right\|_p = 0. \tag{4.1}$$

By the definition of p-variation, we have $\|\varphi_{x,E}\|_p \leq \|\varphi_{x,F}\|_p$ for each $E \subset F$ in Σ. Hence the sequence $\|\varphi_{x,\cup_{j=n+1}^{\infty}E_j}\|_p$'s is decreasing. If (4.1) does not hold, then we assume that it converges to some $\delta > 0$. Please notice that, by the definition of p-variation, we have

$$\sum_j \|\varphi_{x,A_j}\|_p^p \leq \|\varphi_{x,A}\|_p^p,$$

where A_j's is a partition of A. For any $\epsilon_1 > 0$, there is n_1 such that for all $m \geq n_1$

$$\left\|\varphi_{x,\cup_{j=m+1}^{\infty}E_j}\right\|_p^p < \delta^p + \epsilon_1^p,$$

and for any $l > m$ we have

$$\left\|\varphi_{x,\cup_{j=m+1}^{l}E_j}\right\|_p^p \leq \left\|\varphi_{x,\cup_{j=m+1}^{\infty}E_j}\right\|_p^p - \left\|\varphi_{x,\cup_{j=l+1}^{\infty}E_j}\right\|_p^p < \epsilon_1^p.$$

Similarly, for any $\epsilon_2 > 0$, there is $n_2 > n_1$ such that for all $l > m \geq n_2$

$$\left\|\varphi_{x,\cup_{j=m+1}^{\infty}E_j}\right\|_p^p < \delta^p + \epsilon_2^p, \quad \left\|\varphi_{x,\cup_{j=m+1}^{l}E_j}\right\|_p < \epsilon_2.$$

Then, by induction, for any $\epsilon_{k+1} > 0$, there is $n_{k+1} > n_k$ such that for all $l > m \geq n_{k+1}$

$$\left\|\varphi_{x,\cup_{j=m+1}^{\infty}E_j}\right\|_p^p < \delta^p + \epsilon_{k+1}^p, \quad \left\|\varphi_{x,\cup_{j=m+1}^{l}E_j}\right\|_p < \epsilon_{k+1}.$$

Now let $\sum_{k=1}^{\infty}\epsilon_k < \delta$, and for all $k \in \mathbb{N}$ we put $F_k = \cup_{j=n_k+1}^{n_{k+1}}E_j$. Then, by the following norm inequality in $\ell_p^{(m)}$,

$$\left(\sum_{j=1}^{m}\left(\sum_{k=1}^{\infty}\alpha_{kj}\right)^p\right)^{1/p} \leq \sum_{k=1}^{\infty}\left(\sum_{j=1}^{m}\alpha_{kj}^p\right)^{1/p},$$

for arbitrary finite partition $\{B_j\}_{j=1}^{m}$ of $F = \cup_{k=1}^{\infty}F_k$, we have

$$
\begin{aligned}
\left(\sum_{j=1}^{m} \|\varphi_{x,F}(B_j)\|^p\right)^{1/p} &= \left(\sum_{j} \|\varphi(F\cap B_j)x\|^p\right)^{1/p} \\
&= \left(\sum_{j=1}^{m} \|\varphi\left(\cup_{k=1}^{\infty} F_k \cap B_j\right)x\|^p\right)^{1/p} \\
&= \left(\sum_{j=1}^{m} \left\|\sum_{k=1}^{\infty} \varphi(F_k\cap B_j)x\right\|^p\right)^{1/p} \\
&\le \left(\sum_{j=1}^{m} \left(\sum_{k=1}^{\infty} \|\varphi(F_k\cap B_j)x\|\right)^p\right)^{1/p} \\
&\le \sum_{k=1}^{\infty} \left(\sum_{j=1}^{m} \|\varphi(F_k\cap B_j)x\|^p\right)^{1/p} \\
&\le \sum_{k=1}^{\infty} \|\varphi_{x,F_k}\|_p \\
&\le \sum_{k=1}^{\infty} \epsilon_k \\
&< \delta.
\end{aligned}
$$

This implies that $\|\varphi_{x,F}\|_p < \delta$, which is a contradiction to

$$\|\varphi_{x,F}\|_p = \left\|\varphi_{x,\cup_{j=n_1+1}^{\infty} E_j}\right\|_p \ge \delta.$$

Then it is easy to see that ρ is countably additive on the dense linear subspace span$\{\varphi_{x,E} : x \in X, E \in \Sigma\}$ of $\mathcal{V}_p$. Since $||\rho(E)|| \le 1$ for all E, then by approximation we obtain that ρ is countably additive on the whole space $\mathcal{V}_p$.

Finally, we prove that ρ has bounded p-variation. In fact, for any $z \in \mathcal{V}_p$, we have a sequence of $\sum_i \varphi_{x_i,E_i}$ converges to z. Then for any partition A_j's of Ω, we have

$$
\begin{aligned}
\sum_{j} \|\rho_z(A_j)\|^p &= \sum_{j} \|\rho(A_j)z\|^p \\
&= \lim \sum_{j} \left\|\rho(A_j)\sum_{i} \varphi_{x_i,E_i}\right\|^p \\
&= \lim \sum_{j} \left\|\sum_{i} \varphi_{x_i,E_i\cap A_j}\right\|^p
\end{aligned}
$$

$$
\begin{aligned}
&= \lim \sum_j \sup_{B_k} \sum_k \left\| \sum_i \varphi(E_i \cap A_j \cap B_k) x_i \right\|^p \\
&= \lim \sup_{B_k} \sum_j \sum_k \left\| \sum_i \varphi(E_i \cap A_j \cap B_k) x_i \right\|^p \\
&\leq \lim \left\| \sum_i \varphi_{x_i, E_i} \right\|^p \\
&= \|z\|^p.
\end{aligned}
$$

This implies that

$$
|\rho|_p(\Omega) = \sup_{\|z\| \leq 1} |\rho_z|_p(\Omega) \leq 1,
$$

and hence ρ has bounded p-variation. □

Remark 4.5. Please notice that the above dilation p-norm is completely new, and it is different from the ones constructed in [4,16]. We explain this in more details in the next section by showing that the minimal framing model of a redundant framing constructed in [16] always contains an isomorphic copy of c_0, and hence the dilatation does not bounded p-variation for any $p \geq 1$.

5. Framings for Banach spaces

Framings are generalizations of frames [4,16,21,22] and they present us a rich class of examples for operator valued measures that does not admit Hilbert space dilations. The main aim of this section is to show that if a framing is not trivial, i.e., it is not a near-unconditional basis, then its minimal framing model contains c_0 as a subspace. Thus, the dilation projection-valued measure induced by the unconditional basis can not have bounded p-variation for any $p \geq 1$.

Recall that a *frame* $\mathcal{F}$ for a Hilbert space H is a sequence of vectors $\{x_j\} \subset H$ indexed by a countable index set $\mathbb{J}$ for which there exist constants $0 < C_1 \leq C_2 < \infty$ such that, for every $x \in H$,

$$
C_1 \|x\|^2 \leq \sum_{j=1}^{\infty} |\langle x, x_j \rangle|^2 \leq C_2 \|x\|^2. \tag{5.1}
$$

For each frame $\{x_j\}$, there exist a dual frame $\{y_j\}$ (indeed, infinitely many dual frames if $\{x_j\}$ is not a Riesz basis) such that

$$
x = \sum_{j=1}^{\infty} \langle x, x_j \rangle y_j, \ \forall x \in H,
$$

where the convergence is unconditionally in norm. This concept has various generalizations for Banach spaces.

Definition 5.1 *([4])*. Let X be a separable Banach space. A sequence (x_j, f_j), with $(x_j) \subset X$ and $(f_j) \subset X^*$, is called a *framing of* X if for every $x \in X$

$$x = \sum_{j=1}^{\infty} f_j(x)x_j, \tag{5.2}$$

where the series in (5.2) converges unconditionally in norm.

Any framing (x_j, f_j) naturally induces an operator valued measure by

$$E(B)x = \sum_{j\in B} f_j(x)x_j$$

for any $B \subset \mathbb{N}$. We point out that framings are true generalization of frames even in the Hilbert spaces case. Those framings that are not frames in Hilbert spaces often provide us some examples with surprising properties. For example, every frame induced operator valued measure has a Hilbertian dilation. However, there are examples of framing induced operator valued measures on Hilbert spaces that do not allow any Hilbert space dilation [16]. The following concept of framing model introduced in [4] corresponds to the dilation projection-valued measure for the operator-valued measure induced by framing (x_j, f_j), which is defined by, for any $B \subset \mathbb{N}$

$$F(B)z = \sum_{j\in B} e_j^*(z)e_j.$$

It is direct by definition to see that (F, S, T) is a dilation system of E, which we leave to interested readers.

Definition 5.2 *([4])*. A framing model is a Banach space Z with a fixed unconditional basis $\{e_i\}$ for Z. A framing model on $(Z, \{e_i\}_{i\in\mathbb{N}})$ for a Banach space X is a pair of sequences $\{y_i\}$ in X^* and $\{x_i\}$ in X so that the analysis operator $T : X \to Z$ defined by

$$T(u) = \sum_{i\in\mathbb{N}} \langle u, y_i\rangle e_i,$$

is an into isomorphism and the reconstruction operator $S : Z \to X$ given by

$$S(\sum_{i\in\mathbb{N}} a_i e_i) = \sum_{i\in\mathbb{N}} a_i x_i$$

is bounded and $ST = \mathrm{Id}_X$.

Let (x_i, f_i) be a framing of a Banach space X. We denote the unit vector basis of c_{00} by (e_i) and define on c_{00} the following norm $\|\cdot\|_{\min}$:

$$\Big\|\sum a_i e_i\Big\|_{\min} = \sup_{\varepsilon_i=\pm1}\Big\|\sum \varepsilon_i a_i x_i\Big\|_X \quad \text{for all } (a_i) \in c_{00}. \tag{5.3}$$

It follows easily that (e_i) is a 1-unconditional basic sequence with respect to $\|\cdot\|_{\min}$, which we denote by $(\hat{e}_i)$, and thus, a 1-unconditional basis of the completion of c_{00} with respect to $\|\cdot\|_{\min}$, which we denote by $E_{\min}$ (see also [4, Theorem 2.6]). From the proof of [4, Theorem 2.6], we also know that (x_i, f_i) is a framing model for $(E_{\min}, (\hat{e}_i))$. Please notice that the definition (5.3) is 2-equivalent to the following one:

$$\Big\|\sum a_i e_i\Big\|_{\min} = \sup_{B\subset\mathbb{N}}\Big\|\sum_{i\in B} a_i x_i\Big\|_X \quad \text{for all } (a_i) \in c_{00}. \tag{5.4}$$

Then the projection-valued measure induced by $(\hat{e}_i, \hat{e}_i^*)$ is the dilation spectral measure of the OVM induced by (x_i, f_i) (ref. [16]).

Lemma 5.3. *$\sum_{i=1}^\infty a_i x_i$ converges unconditionally in X if and only if $\sum_{i=1}^\infty a_i \hat{e}_i$ converges (unconditionally) in $E_{\min}$.*

Proof. Sufficiency is trivial by using $S_{\min}$ the corresponding reconstruction operator. For necessity, if $\sum_{i=1}^\infty a_i x_i$ converges unconditionally, then for any $\epsilon > 0$, there is $N \in \mathbb{N}$ such that for any $N < m \le n$ and $\varepsilon_i = \pm1$, $\|\sum_{i=m}^n \varepsilon_i a_i x_i\| < \epsilon$, then

$$\Big\|\sum_{i=m}^n a_i \hat{e}_i\Big\| = \max_{\varepsilon_i}\Big\|\sum_{i=m}^n \varepsilon_i a_i x_i\Big\| < \epsilon.$$

It follows that $\sum_{i=1}^\infty a_i \hat{e}_i$ converges unconditionally. □

Definition 5.4. (1) Let X a Banach space with a framing $(x_i, f_i) \subset X \times X^*$. We call (x_i, f_i) a *near-unconditional basis of* X if there is a finite set $\sigma \subset \mathbb{N}$ such that $(x_i)_{i\notin\sigma}$ is an unconditional basis of X.

(2) A framing (x_i, f_i) is said to have *property (u)* if

$$\sum_{i=1}^\infty a_i x_i \text{ converges } \Leftrightarrow \sum_{i=1}^\infty a_i x_i \text{ converges unconditionally.}$$

Now we are ready to prove our third main theorem.

Proof of Theorem 1.3. First, we prove that there is $N \in \mathbb{N}$ such that $S|_{[e_i]_{i\ge N}} : [e_i]_{i\ge N} \to [x_i]_{i\ge N}$ is an isomorphic operator, that is, $\inf\{S(u) : u \in [e_i]_{i\ge N}, \|u\| = 1\} > 0$, where $S : E \to X$ is the corresponding operator on framing model with $S(\sum a_i e_i) = \sum a_i x_i$ for all $\sum a_i e_i \in E$.

Suppose not, then for every $n \in \mathbb{N}$, we can find a normalized block sequence (u_i) of (e_i) with $\|S(u_i)\| < 1/2^{i-1}$. By induction, it is easy to have an increasing sequences $\{n_i\}_{i=0}^{\infty}$ such that

$$\|u_{i+1}\| = \left\| \sum_{j=n_i+1}^{n_{i+1}} e_j^*(u_{i+1})e_j \right\| = 1 \quad \text{and} \quad \|S(u_{i+1})\| = \left\| \sum_{j=n_i+1}^{n_{i+1}} e_j^*(u_{i+1})x_j \right\| < \frac{1}{2^i}$$

for all $i \in \mathbb{N} \cup \{0\}$. Let K be the unconditional constant of (e_i) and

$$\tilde{u}_i = \sum_{j=n_{i-1}+1}^{n_i} e_j^*(u_i)e_j$$

for all $i \in \mathbb{N}$. Then

$$\|\tilde{u}_i\| = \| \sum_{j=n_{i-1}+1}^{n_i} e_j^*(u_i)e_j\| \leq 2K\|u_i\| = 2K.$$

Then $(\tilde{u}_i)$ is a semi-normalized block unconditional-basic sequence of (e_i). So whenever $\sum a_i\tilde{u}_i$ converges, it follows that $(a_i) \in c_0$. By the hypothesis that (e_i) contains no block sequence equivalent to the unit vector basis of c_0, there must exist $(c_i) \in c_0$ such that $\sum c_i\tilde{u}_i$ does not converges.

Now, we claim that

$$\sum_{j=1}^{\infty} b_j x_j = \sum_{i=0}^{\infty} \sum_{j=n_i+1}^{n_{i+1}} c_i e_j^*(u_{i+1})x_j$$

converges. Indeed, for any $\epsilon > 0$, choose N so big that

$$\sup_{i \geq N} |c_i| < \min\left\{\frac{\epsilon}{6K\|S\|}, \frac{\epsilon}{3}\right\}.$$

Then for any $n_N \leq l \leq m \in \mathbb{N}$, if there is $i_0 \geq 0$ such that $l, m \in [n_{i_0} + 1, n_{i_0+1}]$, then

$$\begin{aligned}
\left\| \sum_{j=l}^{m} b_j x_j \right\| &= \left\| \sum_{j=l}^{m} c_{i_0} e_j^*(u_{i_0+1})x_j \right\| = |c_{i_0}| \cdot \left\| \sum_{j=l}^{m} e_j^*(u_{i_0+1})S(e_j) \right\| \\
&\leq |c_{i_0}| \cdot \|S\| \cdot \left\| \sum_{j=l}^{m} e_j^*(u_{i_0+1})e_j \right\| \\
&\leq 2K \cdot |c_{i_0}| \cdot \|S\| \\
&\leq \frac{\epsilon}{6}.
\end{aligned}$$

If $\sum_{j=1}^{\infty} b_j x_j = \sum_{i=0}^{\infty} \sum_{j=n_i+1}^{n_{i+1}} c_i e_j^*(u_{i+1}) x_j$ does not converges, then we can find $0 \leq i_1 < i_2$ such that $l \in [n_{i_1}+1, n_{i_1+1}]$ and $m \in [n_{i_2}+1, n_{i_2+1}]$, then

$$\begin{aligned}
&\Big\| \sum_{j=l}^{m} b_j x_j \Big\| \\
&= \left\| \sum_{j=l}^{n_{i_1+1}} c_{i_1} e_j^*(u_{i_1+1}) x_j + \sum_{k=i_1+1}^{i_2} \sum_{j=n_k+1}^{n_{k+1}} c_k e_j^*(u_{k+1}) x_j + \sum_{j=n_{i_2}+1}^{m} c_{i_2} e_j^*(u_{i_2+1}) x_j \right\| \\
&\leq \left\| \sum_{j=l}^{n_{i_1+1}} c_{i_1} e_j^*(u_{i_1+1}) x_j \right\| + \sum_{k=i_1+1}^{i_2} |c_k| \cdot \left\| \sum_{j=n_k+1}^{n_{k+1}} e_j^*(u_{k+1}) x_j \right\| \\
&\quad + \left\| \sum_{j=n_{i_2}+1}^{m} c_{i_2} e_j^*(u_{i_2+1}) x_j \right\| \\
&\leq 2K \cdot |c_{i_1}| \cdot \|S\| + \sum_{k=i_1+1}^{i_2} \frac{1}{2^k} |c_k| + 2K \cdot |c_{i_2}| \cdot \|S\| \\
&\leq 2K \cdot |c_{i_1}| \cdot \|S\| + \sup_{k > i_1} |c_k| + 2K \cdot |c_{i_2}| \cdot \|S\| \\
&\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

Thus, $\sum_{j=1}^{\infty} b_j x_j = \sum_{i=0}^{\infty} \sum_{j=n_i+1}^{n_{i+1}} c_i e_j^*(u_{i+1}) x_j$ converges. Then it converges unconditionally by property (u). By that fact that $(x_i) \sim (e_i)$, we know that

$$\sum_{j=1}^{\infty} b_j e_j = \sum_{i=0}^{\infty} \sum_{j=n_i+1}^{n_{i+1}} c_i e_j^*(u_{i+1}) e_j$$

converges. It follows that $\sum c_i \tilde{u}_i = \sum_{i=0}^{\infty} c_i \sum_{j=n_i+1}^{n_{i+1}} e_j^*(u_{i+1}) e_j$ converges, which leads to a contradiction.

In this case, the projection-valued measure (minimal dilation) on E induced by the unconditional basis (e_i) does not have bounded p-variation for any $p \geq 1$. Since as in the above proof (e_i) contains a block (unconditional-basic) sequence $u_i = \sum_{j=n_{i-1}+1}^{n_i} e_j^*(u_i) e_j$ equivalent to the unit vector basis of c_0. Then

$$u = \sum_{i=1}^{\infty} \frac{1}{\ln(i)} u_i = \sum_{i=1}^{\infty} \frac{1}{\ln(i)} \sum_{j=n_{i-1}+1}^{n_i} e_j^*(u_i) e_j \in E.$$

Then for the projection-valued measure (minimal dilation) induced by (e_i) with $\rho(B) = \sum_{n \in B} e_n \otimes e_n^*$ for all $B \subset \mathbb{N}$, we have the vector-valued measure ρ_u with $\rho_u(B) = \sum_{n \in B} e_n^*(u) e_n$. Now we get

$$\sup\left\{\left(\sum_{i=1}^{n}\|\rho_u(A_i)\|^p\right)^{1/p} : \{A_1,\ldots,A_n\} \text{ a partition of } \mathbb{N}\right\}$$
$$\geq \left(\sum_{i=1}^{\infty}\left\|\frac{1}{\ln(i)}\sum_{j=n_{i-1}+1}^{n_i} e_j^*(u_i)e_j\right\|^p\right)^{1/p} \geq C\left(\sum_{i=1}^{\infty}\left(\frac{1}{\ln(i)}\right)^p\right)^{1/p} = \infty.$$

Then, ρ does not have bounded p-variation for any $p \geq 1$. □

Example 5.5. It is easy to know that every framing in Banach spaces which can be decomposed to finite union of unconditional-basic sequences satisfies property (u). It contains many important frames as examples, such as, Wavelets, Gabor frames, even all frames generated by unitary systems, etc. Actually, all semi-normalized frames in Hilbert spaces satisfies property (u), since the answer to Feichtinger conjecture, equivalent to Kadison–Singer problem ([5,6]), is positive [25,26].

6. Examples and questions

6.1. Bounded 2-variation and complete boundedness

Now assume that φ is a $B(H)$-valued measure on a measurable space (Ω, Σ, μ) with the property that φ is μ-continuous, where H is a Hilbert space and μ is a positive measure. We identify each $B \in \Sigma$ with the orthogonal projection $P_B \in L^\infty(\Omega, \mu)$ defined by $P_B f = \chi_B f$ for any $f \in L^2(\Omega, \mu)$. Then E extends (uniquely) to an ultra weakly continuous map $\Phi : L^\infty(\Omega, \mu) \to B(H)$. We say that E is completely bounded if Φ is completely bounded (cf. [16]).

Corollary 6.1. *Let φ be an $B(H)$-valued measure on a measurable space (Ω, Σ, μ). If it is completely bounded, then it has bounded 2-variation.*

Proof. Since the induced map Φ is completely bounded, by Stinespring's dilation theorem there exist a Hilbert space Z, bounded linear operators $S : Z \to H$ and $T : H \to Z$ and a *-homomorphism $\pi : L^\infty(\Omega, \mu) \to B(Z)$ such that $\Phi(\cdot) = S\pi(\cdot)T$. Let ρ be the induced operator-valued measure by π. Then it is an orthogonal projection valued measure which has bounded 2-variation. Thus, by Proposition 4.4, we have that φ is 2-variation bounded. □

Remark 6.2. Paulsen etc. ([12,29]) used the following definition for total variations: Let φ be Hilbert space operator valued measure. Then the total variation is defined to be

$$\sup\left\{\left\|\sum |\varphi(A_i)|\right\| : A_i \text{ disjoint partition of } \Omega\right\},$$

where $|T| = (T^*T)^{1/2}$. Let use $\tau(\varphi)$ to denote about quantity.

Note that for every $x \in H$ we have $||\varphi(A)x|| = || \, |\varphi(A)|(x)||$. Thus we have

$$|| \sum |\varphi(A_i)|x|| \leq \sum || \, |\varphi(A_i)|x|| = \sum ||\varphi(A_i)x||$$

By taking sup over $\{x : ||x|| \leq 1\}$, we get

$$\tau(\varphi) \leq |\varphi|_1(\Omega).$$

It is known that φ is completely bounded if $\tau(\varphi) < \infty$ (and hence if $|\varphi|_1(\Omega) < \infty$). But completely boundedness does not imply the finiteness of $\tau(\varphi)$, i.e. there exists an example such that $|\varphi|_1(\Omega) = \infty$ but is completely bounded.

Since the $|\varphi|_2(\Omega) \leq |\varphi|_1(\Omega)$, it naturally leads us to the following problem:

Question 6.3. Is ever Hilbert space operator-valued measure with bounded 2-variation completely bounded?

6.2. Bounded p-variation and p-summable property

Example 6.4. For any measure space (Ω, Σ, μ), there is a Banach space U such that for any $1 \leq p \leq \infty$ there is a projection-valued measure χ_p from Σ to $B(U)$ such that χ_p has bounded p-variation but not bounded q-varion for any $q < p$. Actually, if $\{p_n\}$ is a dense sequence of $[1, \infty)$, then we take the ℓ_1-direct sum as follows

$$U = \bigoplus_{n=1}^{\infty} L^{p_n}(\mu),$$

and we leave the proof to interested readers.

Now we provide some examples that illustrate the connections between the p-summable property and bounded p-variation property by examining the purely atomic operator valued measures. Throughout the rest of this section, we assume that $\|x_j\|\|f_j\| \neq 0$ for all $j \in \mathbb{N}$.

Definition 6.5. (1) A pair of sequences $(x_n, f_n) \subset X \times X^*$ is said to have bounded p-variation if the induced operator-valued measure has bounded p-variation, that is,

$$\sup_{\|x\| \leq 1} \left\{ \left(\sum_j \left\| \sum_{n \in N_j} \langle x, f_n \rangle x_n \right\|^p \right)^{1/p} : \; N_j\text{'s a partition of } \mathbb{N} \right\} < \infty.$$

(2) It is called p-summable if

$$\sum_n \|\langle x, f_n \rangle x_n\|^p = \sum_n (|\langle x, f_n \rangle| \|x_n\|)^p < \infty$$

for all $x \in X$, then by Uniform Boundedness Principle, we have

$$\sup_{\|x\|\le 1} \sum_n (|\langle x, f_n\rangle| \|x_n\|)^p < \infty.$$

It is obvious that bounded p-variation $\Rightarrow$ p-summable. However, the reverse direction does not hold. For example, for each nonzero $x \in X$ and $f \in X^*$, let the pair of sequences to be $x_n = \frac{1}{n}x$ and $f_n = f$. Then it is easy to check that (x_n, f_n) is 2-summable but not bounded 2-variation. The following is an example that has bounded 2-variation, but it is not even p-summable for any $p < 2$.

Example 6.6. Let (z_n) be a Riesz basis of a Hilbert space $\mathcal{H}$, and $0 < \sum_j |\alpha_j|^2 < \infty$. Then there exists an isomorphism $T : \ell_2 \to \mathcal{H}$ such that $T(e_n) = z_n$ for all n. Put $f_{j,n} = \alpha_j z_n$, for all $x \in \mathcal{H}$, we have

$$\begin{aligned}\sum_{j,n} |\langle x, f_{j,n}\rangle|^2 &= \sum_{j,n} |\langle x, \alpha_j z_n\rangle|^2 = \sum_{j,n} |\langle x, z_n\rangle|^2 |\alpha_j|^2 \\ &= \sum_j |\alpha_j|^2 \sum_n |\langle x, T(e_n)\rangle|^2 = \sum_j |\alpha_j|^2 \sum_n |\langle T^*(x), e_n\rangle|^2 \\ &= \sum_j |\alpha_j|^2 \|T^*(x)\|^2.\end{aligned}$$

It implies that $\{f_{j,n}\}_{j,n}$ is a frame of $\mathcal{H}$. For any scalars β_j with

$$0 < \sum_j |\beta_j|^2 < \infty \text{ and } \sum_j \overline{\alpha_j}\beta_j = 1.$$

Thus, $\{g_{j,n}\} = \{\beta_j (T^*)^{-1}(e_n)\}$ is a frame of $\mathcal{H}$. For all $x \in \mathcal{H}$,

$$\begin{aligned}\sum_{j,n} \langle x, f_{j,n}\rangle g_{j,n} &= \sum_{j,n} \langle x, \alpha_j T(e_n)\rangle \beta_j (T^*)^{-1}(e_n) = \sum_j \overline{\alpha_j}\beta_j \sum_n \langle x, T(e_n)\rangle (T^*)^{-1}(e_n) \\ &= (T^*)^{-1}\left(\sum_n \langle T^*(x), e_n\rangle e_n\right) = (T^*)^{-1}(T^*(x)) = x.\end{aligned}$$

Thus $\{g_{j,n}\}$ is a dual frame of $\{f_{j,n}\}$, and thus $\{g_{j,n}, f_{j,n}\}$ has bounded 2-variation.

For any $p < 2$ and $x \in \mathcal{H}$, we have

$$\begin{aligned}\sum_{j,n} \|\langle x, f_{j,n}\rangle g_{j,n}\|^p &= \sum_{j,n} |\langle x, \alpha_j T(e_n)\rangle|^p \|\beta_j (T^*)^{-1}(e_n)\|^p \\ &= \sum_{j,n} |\langle x, e_n\rangle|^p |\alpha_j \beta_j|^p \|(T^*)^{-1}(e_n)\|^p\end{aligned}$$

$$= \sum_j |\alpha_j \beta_j|^p \sum_n |\langle x, e_n\rangle|^p \|(T^*)^{-1}(e_n)\|^p.$$

It is easy to see that $\{g_{j,n}, f_{j,n}\}$ is not p-summable for any $p < 2$.

Next example shows that it is possible that an operator valued measure could have bounded variation and unbounded p-variation, but still p-summable for every $p < 2$.

Example 6.7. There exists a Parseval frame such that its induced framing is p-summable for any $1 < p < 2$, but still has unbounded q-variation for any $q < 2$. Let (e_n) be an orthonormal basis of a Hilbert space $\mathcal{H}$. Define

$$f_{j,n} = \frac{1}{2^n} e_n, \quad 1 \le j \le 4^n, \quad n \in \mathbb{N}.$$

Then $\{f_{j,n}\}_{j,n}$ is a Parseval frame of $\mathcal{H}$, and also a framing of $\mathcal{H}$ with bounded 2-variation. For any $x \in \mathcal{H}$ with $\|x\| \le 1$, we have

$$\sum_{j,n} |\langle x, f_{j,n}\rangle|^p \|f_{j,n}\|^p = \sum_n \sum_{1 \le j \le 4^n} \frac{1}{4^{pn}} |\langle x, e_n\rangle|^p = \sum_n \frac{1}{4^{(p-1)n}} |\langle x, e_n\rangle|^p$$
$$\le \max_n |\langle x, e_n\rangle|^p \sum_n \left(\frac{1}{4^{(p-1)}}\right)^n \le \frac{1}{4^{(p-1)} - 1}.$$

But it is still type 2, since for the partition as follows:

$$\sum_n \left\| \sum_{1 \le j \le 4^n} \langle x, f_{j,n}\rangle f_{j,n} \right\|^p = \sum_n \left\| \sum_{1 \le j \le 4^n} \left\langle x, \frac{1}{2^n} e_n \right\rangle \frac{1}{2^n} e_n \right\|^p = \sum_n |\langle x, e_n\rangle|^p.$$

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