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# Recovery of signals from unordered partial frame coefficients ☆

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ABSTRACT

In this paper, we study the feasibility and stability of recovering signals in finite-dimensional spaces from unordered partial frame coefficients. We prove that with an almost self-located robust frame, any signal except from a Lebesgue measure zero subset can be recovered from its unordered partial frame coefficients. However, the recovery is not necessarily stable with almost self-located robust frames. We propose a new class of frames, namely self-located robust frames, that ensures stable recovery for any input signal with unordered partial frame coefficients. In particular, the recovery is exact whenever the received unordered partial frame coefficients are noise-free. We also present some characterizations and constructions for (almost) self-located robust frames. Based on these characterizations and construction algorithms, we prove that any randomly generated frame is almost surely self-located robust. Moreover, frames generated with cube roots of different prime numbers are also self-located robust.



## 1. Introduction

Frames have been widely used in many applications, in particular, in signal and data analysis. Different from orthogonal bases, one of the main features of frames is that they provide redundant representations of signals which makes it possible to resolve many practical problems. For example, when the frame coefficients of a signal are transferred in channels, it occurs often that some coefficients are erased. A lot of research has been done in recent years dealing with the various problems of recovering the original signal from erasure-corrupted frame coefficients. Before stating further results, we introduce the definition of frames.

Recall that $\{\varphi_i\}_{1\le i\le N}$ is said to be a frame for some finite-dimensional Hilbert space $\mathcal{H}$ if there exist constants $C_1$ and $C_2$ such that

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$$C_1\|f\|^2 \leq \sum_{i=1}^{N} |\langle f, \varphi_i\rangle|^2 \leq C_2\|f\|^2, \quad \forall f \in \mathcal{H},$$

where $C_1$ and $C_2$ are called the lower and upper frame bounds, respectively. A frame $\{\varphi_i\}_{1\leq i\leq N}$ is said to be of full spark if every $n$ elements in the frame form a Riesz basis for $\mathcal{H}$, where $n = \dim \mathcal{H}$.

Given a frame $\{\varphi_i\}_{1\leq i\leq N}$ for $\mathcal{H}$, one can find another frame $\{\tilde{\varphi}_i\}_{1\leq i\leq N}$, called the dual of $\{\varphi_i\}_{1\leq i\leq N}$, such that

$$f = \sum_{i=1}^{N} \langle f, \varphi_i\rangle \tilde{\varphi}_i, \quad \forall f \in \mathcal{H}.$$

A Parseval frame is a frame such that it is a dual of itself, or equivalently $C_1 = C_2 = 1$. For more details on frame theory, we refer to [7–9].

In this paper, we only consider frames with different real vectors. And $\mathcal{H}$ stands for the $n$-dimensional real or complex space.

Suppose that some frame coefficients $\{\langle f, \varphi_i\rangle\}_{i\in I}$ are erased in data transmission. If $\{\varphi_i\}_{i\in I^c}$ is also a frame for $\mathcal{H}$, then we can recover $f$ with a dual frame of $\{\varphi_i\}_{i\in I^c}$. However, since the index set for the erased coefficients varies, it is time consuming to compute a new dual frame every time, and therefore this recovery approach may not be suitable for real-time data processing applications. Another approach is to treat the erased frame coefficients as zeros and then reconstruct the input signal with the original reconstruction formula based on the canonical dual. Obviously, this will bring some reconstruction error. This led to some recent research in the literature about characterizing optimal frames that minimize the maximal reconstruction error for this approach (cf. [1–6,12–14,16–20,23] and references therein). For the case of $m$-erasures with $m \leq 2$, the optimal frames or dual frames that minimize the maximal reconstruction error can be characterized. However, not much is known if the number of erasures is greater than 2.

In [10], the first and the third named authors of this paper proposed a third approach to this problem. With well designed full spark frames, we can recover the erased frame coefficients by solving a very simple system of linear equations with erasures as unknowns, and consequently it overcomes the shortcomings of computing dual frames again and again and has no systematic error when comparing with the second approach.

In data transmission, the locations of erased coefficients might be unknown, or the received data might be disordered. To solve these problem, *almost robust frames* and *almost self-located frames* are introduced in [10].

**Definition 1.1.** (See [10].) A frame for $\mathcal{H}$ is said to be almost robust with respect to $m$-erasures if we can recover any $f \in \mathcal{H} \setminus \mathcal{H}_0$ from its frame coefficients with $m$-erasures at unknown locations, where $\mathcal{H}_0$ is the union of finitely many proper subspaces of $\mathcal{H}$.

A frame is said to be almost self-located if we can recover the sequence of frame coefficients (and therefore the corresponding signal) from any of its rearrangements for any signal in $\mathcal{H} \setminus \mathcal{H}_0$.

By the definitions, with almost robust frames, we can recover missing frame coefficients at unknown locations for almost all signals. And with almost self-located frames, we can recover the correct order of frame coefficients from any of their rearrangements for almost all signals. Characterizations and constructions for both almost robust frames and almost self-located frames were obtained in [10,11]. See also [15] for some explicitly given almost robust or almost self-located frames.

It is natural to ask if it is possible to recover the input signal if we only know unordered partial frame coefficients? In this paper, we show that the answer is yes with well-chosen frames. For our purpose, we introduce the following definition:

**Definition 1.2.** A frame for $\mathcal{H}$ is called *almost self-located robust* with respect to $m$-erasures, if we can recover any $f \in \mathcal{H} \setminus \mathcal{H}_0$ from any of its $N - m$ unordered frame coefficients, where $\mathcal{H}_0$ is the union of finitely many proper subspaces of $\mathcal{H}$.

We see from the definition that almost self-located robust frames are automatically both almost self-located and almost robust. It was shown in [10] that by choosing suitable rescaling constants, every full spark frame can be modified to be an almost self-located or almost robust frame. One of our main results in this paper is to show that the same is true for almost self-located robust frames. Moreover, we prove that for $N - m > n$, where $n$ is the dimension of the signal space, it is almost sure that the column vectors of any randomly generated $n \times N$ matrix form an $m$-erasure almost self-located robust frame.

In practice, the measured signals are usually noisy. If a signal $f$ is close to $\mathcal{H}_0$, the set consisting of unrecoverable signals, then the existence of noises makes it possible to get wrong erasure locations, and consequently we may not be able to recover $f$ from its unordered partial frame coefficients. So almost self-located robustness does not guarantee the stability for the signal recovering. Moreover, they only apply to the recovery for almost all the input signals even in the noise-free case. One of the goals of this paper is to find a new class of frames that not only guarantee the exact recovery for all signals in the noise-free case but also provide stable recovery when the set of unordered partial frame coefficients carries noise.

For $N \geq k$, denote by $I_{N,k}$ the set consisting of all rearrangements of $k$ elements in $\{1, \ldots, N\}$. We propose the following definition:

**Definition 1.3.** A frame $\{\varphi_i\}_{1 \leq i \leq N}$ for $\mathcal{H}$ is called *self-located robust* with respect to $m$-erasures, if for all $(i_1, \ldots, i_{N-m})$, $(j_1, \ldots, j_{N-m}) \in I_{N,N-m}$ and any signals $f$, $g \in \mathcal{H}$ satisfying

$$\langle f, \varphi_{i_l} \rangle = \langle g, \varphi_{j_l} \rangle, \quad 1 \leq l \leq N - m,$$

we have $f = g$.

We prove that with a self-located robust frame, we can reconstruct any signal from its unordered partial frame coefficients. Moreover, the reconstruction algorithm is stable in the sense that small changes of input signals only result in small changes of the output.

The rest of the paper is organized as follows. In Section 2, we obtain a characterization for almost self-located robust frames. As a consequence of this characterization we obtain that the column vectors of randomly generated matrices form almost self-located robust frames. Section 3 is devoted to the characterization/construction of self-located robust frames and the stability of our reconstruction algorithm. We give two types of construction, one is based on random matrices, and the other is based on prime numbers.

## 2. Almost self-located robust frames

In this section, we give a characterization for almost self-located robust frames, with which we provide construction method for such frames. We show that by choosing suitable parameters, every frame with full spark can be rescaled to an almost self-located robust frame. Moreover, typical randomly generated $n \times N$ matrix with $n < N$ corresponds to an almost self-located robust frame.

Let $\{\varphi_i\}_{1 \leq i \leq N}$ be a frame but not a basis for $\mathcal{H}$. Define the analysis operator $T$ as

$$T : \mathcal{H} \mapsto \mathbb{C}^N : \ Tf = (\langle f, \varphi_1 \rangle, \ldots, \langle f, \varphi_N \rangle)^T.$$

Then there is some matrix $M$ such that the null space of $M$ is exact the range of the analysis operator $T$.

For $(i_1, \ldots, i_k) \in I_{N,k}$, define

$$T_{i_1,\ldots,i_k} f = (\langle f, \varphi_{i_1}\rangle, \ldots, \langle f, \varphi_{i_k}\rangle)^T.$$

And we denote $T_{1,\cdots,N}$ simply by $T$.

The following is a characterization of almost self-located robust frames.

**Theorem 2.1.** *Suppose that* $\dim \mathcal{H} = n$. *Let* $\{\varphi_i\}_{1\le i\le N}$ *be a frame for* $\mathcal{H}$. *Then* $\{\varphi_i\}_{1\le i\le N}$ *is almost self-located robust with respect to* $m$*-erasures if and only if* $\{T_{i_1,\ldots,i_{N-m}}\mathcal{H} : (i_1, \ldots, i_{N-m}) \in I_{N,N-m}\}$ *consists of pairwise distinct* $n$*-dimensional subspaces.*

**Proof.** Necessity. If there exists some $(i_1, \ldots, i_{N-m}) \in I_{N,N-m}$ such that $\dim(T_{i_1,\ldots,i_{N-m}}\mathcal{H}) < n$, then $\{\varphi_{i_s}\}_{1\le s\le N-m}$ is not a frame for $\mathcal{H}$. Hence for almost all signals $f \in \mathcal{H}$, we can not recover $f$ from $T_{i_1,\ldots,i_{N-m}} f$.

On the other hand, assume that there exist $(i_1, \ldots, i_{N-m}), (l_1, \ldots, l_{N-m}) \in I_{N,N-m}$ such that $(i_1, \ldots, i_{N-m}) \ne (l_1, \ldots, l_{N-m})$ and

$$T_{i_1,\ldots,i_{N-m}}\mathcal{H} = T_{l_1,\ldots,l_{N-m}}\mathcal{H}. \tag{2.1}$$

Then there is some $s \in \{1, \ldots, N - m\}$ such that $i_s \ne l_s$. Let

$$\mathcal{H}_s = \{f \in \mathcal{H} : \langle f, \varphi_{i_s}\rangle = \langle f, \varphi_{l_s}\rangle\}.$$

Since $\varphi_{i_s} \ne \varphi_{l_s}$, $\mathcal{H}_s$ is a proper subspace of $\mathcal{H}$. For any $f \in \mathcal{H} \setminus \mathcal{H}_s$, we see from (2.1) that there exists some $g \in \mathcal{H}$ such that $T_{i_1,\ldots,i_{N-m}} f = T_{l_1,\ldots,l_{N-m}} g$. Since $f \notin \mathcal{H}_s$, we have $f \ne g$. Hence for almost all $f \in \mathcal{H}$, we can not recover $f$ from $T_{i_1,\ldots,i_{N-m}} f$.

Sufficiency. Since $T_{i_1,\ldots,i_{N-m}}\mathcal{H}$ consists of different $n$-dimensional subspaces, we have $N - m > n$. Hence for any $(i_1, \ldots, i_{N-m}) \in I_{N,N-m}$, $\{\varphi_{i_s}\}_{1\le s\le N-m}$ is not a basis for $\mathcal{H}$. Therefore, there exists some $(N - m - n) \times (N - m)$ matrix $M_{i_1,\ldots,i_{N-m}}$ such that

$$\mathcal{N}(M_{i_1,\ldots,i_{N-m}}) = T_{i_1,\ldots,i_{N-m}}\mathcal{H}.$$

Take some $(i_1, \ldots, i_{N-m}) \ne (l_1, \ldots, l_{N-m})$. Then we have

$$T_{i_1,\ldots,i_{N-m}}\mathcal{H} \ne T_{l_1,\ldots,l_{N-m}}\mathcal{H}.$$

Since $\dim(T_{i_1,\ldots,i_{N-m}}\mathcal{H}) = \dim(T_{l_1,\ldots,l_{N-m}}\mathcal{H}) = n$, $T_{i_1,\ldots,i_{N-m}}\mathcal{H} \cap T_{l_1,\ldots,l_{N-m}}\mathcal{H}$ is a proper subspace of $T_{i_1,\ldots,i_{N-m}}\mathcal{H}$. Hence

$$T^{-1}_{i_1,\ldots,i_{N-m}}(T_{i_1,\ldots,i_{N-m}}\mathcal{H} \cap T_{l_1,\ldots,l_{N-m}}\mathcal{H})$$

is a proper subspace of $\mathcal{H}$. Let

$$\mathcal{H}_0 = \bigcup_{\substack{(i_1,\ldots,i_{N-m}),(l_1,\ldots,l_{N-m})\in I_{N,N-m}\\ (i_1,\ldots,i_{N-m})\ne(l_1,\ldots,l_{N-m})}} T^{-1}_{i_1,\ldots,i_{N-m}}(T_{i_1,\ldots,i_{N-m}}\mathcal{H} \cap T_{l_1,\ldots,l_{N-m}}\mathcal{H}).$$

For any $f \in \mathcal{H}\setminus\mathcal{H}_0$, let $\tilde{c}$ be the sequence of any unordered $N-m$ frame coefficients of $f$, i.e., $\tilde{c} = T_{i_1,\ldots,i_{N-m}} f$ for some unknown index sequence $(i_1, \ldots, i_{N-m})$. Since $f \notin \mathcal{H}_0$, the only index sequence $(j_1, \ldots, j_{N-m})$ satisfying

$$M_{j_1,\ldots,j_{N-m}}\tilde{c} = 0$$

is $(j_1, \ldots, j_{N-m}) = (i_1, \ldots, i_{N-m})$. Consequently, we can get the correct index sequence by the above test condition. Since $\{\varphi_{i_s}\}_{1\le s\le N-m}$ is a frame for $\mathcal{H}$, we can recover $f$ from its unordered partial frame coefficient sequence $\tilde{c}$. $\square$

Next we show that every full spark frame can be rescaled to an almost self-located robust frame.

**Theorem 2.2.** *Let $\{\varphi_i\}_{1\le i\le N}$ be a full spark frame for $\mathcal{H}$, where $N \ge n+2$. Then there exist constants $s_i, 1 \le i \le N$, such that $\{s_i\varphi_i\}_{1\le i\le N}$ is an almost self-located robust frame with respect to $(N-n-1)$-erasures.*

**Proof.** Pick any $n+1$ elements $\{\varphi_{i_s}\}_{1\le s\le n+1}$ from $\{\varphi_i\}_{1\le i\le N}$. Then there exists a unique sequence $\{a_s\}_{1\le s\le n}$ such that

$$\sum_{i=1}^{n} a_s\varphi_{i_s} = \varphi_{i_{n+1}}.$$

Since $\{\varphi_i\}_{1\le i\le N}$ is of full spark, we conclude that $a_s \ne 0, 1 \le s \le n$. Otherwise, there are $n$ elements in $\{\varphi_{i_s}\}_{1\le s\le n+1}$ which are linearly dependent, which is impossible.

Denote by $T_{i_1,\ldots,i_{n+1}}$ the analysis operator of $\{\varphi_{i_s}\}_{1\le s\le n+1}$. Define the vector

$$\alpha_{i_1,\ldots,i_{n+1}} = (a_1, a_2, \ldots, a_n, -1)^T.$$

Then for any $f \in \mathcal{H}$, we have

$$\langle T_{i_1,\ldots,i_{n+1}} f, \alpha_{i_1,\ldots,i_{n+1}}\rangle = \Big\langle f, \Big(\sum_{i=1}^{n} a_s\varphi_{i_s}\Big) - \varphi_{i_{n+1}}\Big\rangle = 0.$$

Hence $\alpha_{i_1,\ldots,i_{n+1}} \in (T_{i_1,\ldots,i_{n+1}}\mathcal{H})^\perp$. On the other hand, since $\{\varphi_{i_s}\}_{1\le s\le n+1}$ is a frame for $\mathcal{H}$, we have $\dim T_{i_1,\ldots,i_{n+1}}\mathcal{H} = n$. Hence $\dim(T_{i_1,\ldots,i_{n+1}}\mathcal{H})^\perp = n+1-n = 1$. Consequently,

$$(T_{i_1,\ldots,i_{n+1}}\mathcal{H})^\perp = \operatorname{span}\{\alpha_{i_1,\ldots,i_{n+1}}\}.$$

Let $X_{i_1,\ldots,i_{n+1}}$ be the set consisting of entries in $\alpha_{i_1,\ldots,i_{n+1}}$. Denote by $Y$ the set of all $X_{i_1,\ldots,i_{n+1}}$ corresponding to all choices of indices. Let

$$L = \max\Big\{\frac{|a'|}{|a|} : a, a' \in X_{i_1,\ldots,i_{n+1}}, X_{i_1,\ldots,i_{n+1}} \in Y\Big\}.$$

Then $L \ge 1$, and there exists some $M > 0$ such that $2^{2^M} > L$.

Now we let $s_i = 2^{-2^{M+i}}, 1 \le i \le N$. Denote by $T^s$ the analysis operator of $\{s_i\varphi_i\}_{1\le i\le N}$. Then

$$(T^s_{i_1,\ldots,i_{n+1}}\mathcal{H})^\perp = \operatorname{span}\{(2^{2^{M+i_1}}a_1, 2^{2^{M+i_2}}a_2, \ldots, 2^{2^{M+i_{n+1}}}a_{n+1})^T\},$$

where $(a_1, a_2, \ldots, a_{n+1})^T$ is chosen such that

$$(T_{i_1,\ldots,i_{n+1}}\mathcal{H})^\perp = \operatorname{span}\{(a_1, a_2, \ldots, a_{n+1})^T\}.$$

By Theorem 2.1, to finish the proof, it suffices to show that for $T^s_{i_1,\ldots,i_{n+1}}\mathcal{H} = T^s_{l_1,\ldots,l_{n+1}}\mathcal{H}$, we have

$$(i_1, \ldots, i_{n+1}) = (l_1, \ldots, l_{n+1}).$$

Since $T^s_{i_1,\ldots,i_{n+1}}\mathcal{H} = T^s_{l_1,\ldots,l_{n+1}}\mathcal{H}$, we have $(T^s_{i_1,\ldots,i_{n+1}}\mathcal{H})^\perp = (T^s_{l_1,\ldots,l_{n+1}}\mathcal{H})^\perp$. Let $(a_1, a_2, \ldots, a_{n+1})^T$ and $(a'_1, a'_2, \ldots, a'_{n+1})^T$ be such that

$$(T_{i_1,\ldots,i_{n+1}}\mathcal{H})^\perp = \mathrm{span}\{(a_1, a_2, \ldots, a_{n+1})^T\}$$

and

$$(T_{l_1,\ldots,l_{n+1}}\mathcal{H})^\perp = \mathrm{span}\{(a'_1, a'_2, \ldots, a'_{n+1})^T\},$$

respectively. Then we get that $(2^{2^{M+i_1}}a_1, 2^{2^{M+i_2}}a_2, \ldots, 2^{2^{M+i_{n+1}}}a_{n+1})$ and $(2^{2^{M+l_1}}a'_1, 2^{2^{M+l_2}}a'_2, \ldots, 2^{2^{M+l_{n+1}}}a'_{n+1})$ are linearly dependent. Picking the first two entries of both vectors, we have

$$\frac{2^{2^{M+i_1}}a_1}{2^{2^{M+i_2}}a_2} = \frac{2^{2^{M+l_1}}a'_1}{2^{2^{M+l_2}}a'_2}.$$

That is,

$$2^{2^{M+i_1}-2^{M+i_2}+2^{M+l_2}-2^{M+l_1}} = \frac{a'_1 a_2}{a'_2 a_1}.$$

By the definition of $L$ and $M$, we have

$$2^{2^{M+i_1}-2^{M+i_2}+2^{M+l_2}-2^{M+l_1}} \in (2^{-2^M\times 2}, 2^{2^M\times 2}).$$

Hence

$$2^{2^M(2^{i_1}-2^{i_2}+2^{l_2}-2^{l_1})} \in (2^{-2^M\times 2}, 2^{2^M\times 2}).$$

On the other hand, since $i_1$, $i_2$, $l_1$ and $l_2$ are positive integers, $2^{i_1} - 2^{i_2} + 2^{l_2} - 2^{l_1} \in 2\mathbb{Z}$. Hence $2^{i_1} - 2^{i_2} + 2^{l_2} - 2^{l_1} = 0$. That is, $2^{i_1} - 2^{i_2} = 2^{l_1} - 2^{l_2}$. Without loss of generality we suppose that $i_1 \geq i_2$. Then $l_1 \geq l_2$ and $(2^{i_1-i_2} - 1)2^{i_2} = (2^{l_1-l_2} - 1)2^{l_2}$. Hence $(i_1, i_2) = (l_1, l_2)$. Similarly we can prove that

$$(i_1, i_s) = (l_1, l_s), \quad 3 \leq s \leq n+1.$$

Consequently $(i_1, \ldots, i_{n+1}) = (l_1, \ldots, l_{n+1})$. This completes the proof. □

We conclude this section by showing a density result of almost self-located robust frames, i.e., every randomly generated $n \times N$ matrix with $n < N$ corresponds to an almost self-located robust frame.

**Theorem 2.3.** *Let $m$, $n$ and $N$ be positive integers such that $N - m > n$. Then we have*

(i). *The set of all vectors $(a_{1,1}, \ldots, a_{n,1}, \ldots, a_{1,N}, \ldots, a_{n,N})$ for which $\{\varphi_i := (a_{1,i}, \ldots, a_{n,i})^T\}_{1\leq i\leq N}$ is not an $m$-erasure almost self-located robust frame is Zariski-closed. Consequently, it is of Lebesgue measure zero in $\mathbb{R}^{nN}$ and its complement is open and dense in $\mathbb{R}^{nN}$.*

(ii). *Let $A$ be an $n \times N$ matrix whose entries are independent continuous random variables. Then it is almost sure that the column vectors of $A$ form an $m$-erasure almost self-located robust frame.*

The proof of Theorem 2.3 is mainly based on the following construction algorithm of almost self-located robust frames. And we leave the details to interested readers.

**Construction of almost self-located robust frames:**

First, we introduce a preliminary result.

**Lemma 2.4.** *Let* $\{\varphi_i\}_{1\le i\le N}$ *be a full spark frame for* $\mathbb{R}^n$*. Define*

$$E_1 = \bigcup_{\substack{(i_1,\dots,i_n)\in I_{N,n}\\(j_1,\dots,j_n)\in I_{N,n}}} \Big\{c\in\mathbb{R}^n:\ c_1\ne 0 \text{ and } \sum_{l=1}^n c_l\varphi_{i_l} = \frac{1}{c_1}\varphi_{j_1} + \sum_{l=2}^n \frac{c_l}{c_1}\varphi_{j_l}\Big\},$$

$$E_2 = \bigcup_{(i_1,\dots,i_n)\ne(j_1,\dots,j_n)} \Big\{c\in\mathbb{R}^n:\ \sum_{l=1}^n c_l\varphi_{i_l} = \sum_{l=1}^n c_l\varphi_{j_l}\Big\}.$$

*Then we have* $|E_1| = |E_2| = 0$*, where* $|\cdot|$ *denotes the Lebesgue measure of a set.*

**Proof.** (i). Fix some $c_1, \dots, c_{n-1}\in\mathbb{R}$ with $c_1\notin\{0,1\}$. We see from $\sum_{l=1}^n c_l\varphi_{i_l} = \frac{1}{c_1}\varphi_{j_1} + \sum_{l=2}^n \frac{c_l}{c_1}\varphi_{j_l}$ that

$$c_n\Big(\varphi_{i_n} - \frac{1}{c_1}\varphi_{j_n}\Big) = \frac{1}{c_1}\varphi_{j_1} + \sum_{l=2}^{n-1}\frac{c_l}{c_1}\varphi_{j_l} - \sum_{l=1}^{n-1} c_l\varphi_{i_l}.$$

Since $\varphi_{i_n} - \frac{1}{c_1}\varphi_{j_n}\ne 0$, there is at most one $c_n$ satisfies the above equation. Hence

$$|E_1\cap\{c\in\mathbb{R}^n:\ c_1\ne 1\}| = 0.$$

Observe that $|E_1\cap\{c\in\mathbb{R}^n:\ c_1 = 1\}| = 0$. We have $|E_1| = 0$.

(ii). Since $(i_1,\dots,i_n)\ne(j_1,\dots,j_n)$, there is some $l_0$ such that $i_{l_0}\ne j_{l_0}$. For fixed $c_1,\dots,c_{l_0-1},c_{l_0+1},\dots,c_n$, there is at most one $c_{l_0}$ such that

$$c_{l_0}(\varphi_{i_{l_0}} - \varphi_{j_{l_0}}) = \sum_{l\ne l_0} c_l(\varphi_{j_l} - \varphi_{i_l}).$$

Hence $|E_2| = 0$. □

With the above lemma, we can construct $\varphi_i$ inductively.

(i). Set $\varphi_1 = (a_{1,1},\dots,a_{n,1})^T\in\mathbb{R}^n$ such that $a_{1,1}\ne 0$.

(ii). Assume that $\varphi_1, \dots, \varphi_i$ are well defined for some $1\le i\le n-1$. Let $\varphi_{i+1} = (a_{1,i+1},\dots,a_{n,i+1})^T$ be such that

$$\begin{vmatrix} a_{1,1} & \cdots & a_{1,i+1}\\ \vdots & & \vdots\\ a_{i+1,1} & \cdots & a_{i+1,i+1}\end{vmatrix}\ne 0.$$

(iii). Assume that the full spark frame $\{\varphi_i\}_{1\le i\le k}$ is well defined for some $n\le k<N$. Let $E_1$ and $E_2$ be defined as in Lemma 2.4 with $N$ being replaced by $k$. For $k=n$, let $E_3=\emptyset$. And for $k>n$, fix some $(i_1,\dots,i_{n+1})\in I_{k,n+1}$. Then there is a unique $c\in\mathbb{R}^n$ such that $\varphi_{i_{n+1}} = \sum_{l=1}^n c_l\varphi_{i_l}$. Let $E_3$ be the set of all such $c$ when $(i_1,\dots,i_{n+1})$ is varying from $I_{k,n+1}$. It is easy to see that $|E_1\cup E_2\cup E_3| = 0$.

Denote

$$\Phi_i = \bigcup_{(i_1,\ldots,i_n)\in I_{k,n}} \Big\{\sum_{l=1}^{n} c_l\varphi_{i_l} : c \in E_i\Big\}, \quad 1 \le i \le 3.$$

$$\Phi_4 = \bigcup_{(i_1,\ldots,i_{n-1})\in I_{k,n-1}} \{\varphi \in \mathbb{R}^n : \det(\varphi_{i_1},\ldots,\varphi_{i_{n-1}},\varphi) = 0\}$$

Then we have $\left|\bigcup_{i=1}^4 \Phi_i\right| = 0$. Take any $\varphi_{k+1} \in \mathbb{R}^n \setminus (\bigcup_{i=1}^4 \Phi_i)$.

It is obvious that frames constructed above are full spark. Moreover, we conclude that they are almost self-located robust frames. By Theorem 2.2, it suffices to show that for $(i_1,\ldots,i_{n+1}), (j_1,\ldots,j_{n+1}) \in I_{N,n+1}$ with $(i_1,\ldots,i_{n+1}) \neq (j_1,\ldots,j_{n+1})$, $T_{i_1,\ldots,i_{n+1}}\mathbb{R}^n \neq T_{j_1,\ldots,j_{n+1}}\mathbb{R}^n$.

Assume on the contrary that $T_{i_1,\ldots,i_{n+1}}\mathbb{R}^n = T_{j_1,\ldots,j_{n+1}}\mathbb{R}^n$. Then there is some $c \in \mathbb{R}^{n+1} \setminus \{0\}$ such that

$$\sum_{l=1}^{n+1} c_l\varphi_{i_l} = \sum_{l=1}^{n+1} c_l\varphi_{j_l} = 0. \tag{2.2}$$

Since $\{\varphi_i\}_{1\le i\le N}$ is of full spark, we have $c_l \neq 0$ for all $l$. Denote $L_0 = \max_{1\le l\le n+1}\{i_l\}$ and $L_0' = \max_{1\le l\le n+1}\{j_l\}$. There are two cases.

(i). $L_0 \neq L_0'$. Without loss of generality, assume that $L_0 < L_0' = j_{n+1}$. By (2.2), we have

$$\varphi_{i_{n+1}} = -\sum_{l=1}^{n} \frac{c_l}{c_{n+1}}\varphi_{i_l} \text{ and } \varphi_{j_{n+1}} = -\sum_{l=1}^{n} \frac{c_l}{c_{n+1}}\varphi_{j_l},$$

which contradicts with the fact that $\varphi_{j_{n+1}} \notin \Phi_3$.

(ii). $L_0 = L_0'$. Then there are some $l_0$ and $l_0'$ such that $i_{l_0} = L_0$ and $j_{l_0'} = L_0'$.

For $l_0 = l_0'$, it contradicts with the fact that $\varphi_{L_0} \notin \Phi_2$.

For $l_0 \neq l_0'$, without loss of generality, we assume that $l_0 = 1$, $l_0' = 2$ and $c_1 = 1$. Then we have

$$\varphi_{i_1} = -\sum_{l=2}^{n+1} c_l\varphi_{i_l} = -\frac{1}{c_2}\varphi_{j_1} - \sum_{l=3}^{n+1} \frac{c_l}{c_2}\varphi_{j_l},$$

which contradicts with the fact that $\varphi_{i_1} \notin \Phi_1$.

## 3. Stable recovery of signals from unordered partial frame coefficients

Based on the proofs from Section 2 we propose the following recovery algorithm.

**Signal Recovery Algorithm:**

Let $\{\varphi_i\}_{1\le i\le N}$ be an $m$-erasure almost self-located robust frame. For every $(i_1,\ldots,i_{N-m}) \in I_{N,N-m}$, since $\{\varphi_{i_l}\}_{1\le l\le N-m}$ is a frame for $\mathcal{H}$, there exists some matrix $M_{i_1,\ldots,i_{N-m}}$ such that

$$T_{i_1,\ldots,i_{N-m}} f = \mathcal{N}(M_{i_1,\ldots,i_{N-m}}).$$

In the following, we set

$$M_{i_1,\ldots,i_{N-m}} = I - T_{i_1,\ldots,i_{N-m}}(T^*_{i_1,\ldots,i_{N-m}}T_{i_1,\ldots,i_{N-m}})^{-1}T^*_{i_1,\ldots,i_{N-m}}. \tag{3.1}$$

Let $\tilde{c} := (\tilde{c}_1,\ldots,\tilde{c}_{N-m})^T$ be the $N-m$ unordered frame coefficients of some signal $f$. Then we can recover $f$ with the following steps.

(i). Let

$$(i_1^0, \ldots, i_{N-m}^0) = \underset{(i_1,\ldots,i_{N-m})\in I_{N,N-m}}{\arg\min} \|M_{i_1,\ldots,i_{N-m}}\tilde{c}\|. \tag{3.2}$$

(ii). Set $c = \{c_i : 1 \le i \le N\}$ with $c_{i_l^0} = \tilde{c}_l$, $1 \le l \le N-m$. By solving the equation

$$M_{1,\ldots,N}c = 0, \tag{3.3}$$

where $c_i$ with $i \notin \{i_l^0 : 1 \le l \le N-m\}$ are considered as unknowns, we get the erased coefficients with correct order.

(iii). Recover $f$ with the formula

$$Rf := \sum_{i=1}^{N} c_i\tilde{\varphi}_i, \tag{3.4}$$

where $\{\tilde{\varphi}_i\}_{1\le i\le N}$ is a dual frame of $\{\varphi_i\}_{1\le i\le N}$.

It is obvious that the first step is an NP hard problem [24]. In this paper, we will not be focusing on how to get an efficient and fast recovery algorithm, which we plan to investigate in some future work. Instead, we only focus on the theoretical side of the investigation on the feasibility and stability of recovery of signals from unordered partial frame coefficients.

As implied by the definition of almost self-located robustness, for signals coming from $\mathcal{H}_0$, a subset of $\mathcal{H}$ with Lebesgue measure zero, it might fail to be recovered properly. The reason is that for frame coefficients corresponding to signals in $\mathcal{H}_0$, the solution of (3.2) is not unique. As a result, signals recovered in this way might not be the original ones. However, this problem can be resolved by using self-located robust frames. That is, with such frames, even if we get a wrong index set of the unordered partial frame coefficients, the recovered signal is still the correct one. Moreover, the reconstruction error can be controlled by the input error. In other words, such frames provide a stable recovery.

To state the main result in this section, we introduce the following definitions.

For a matrix $A$, denote by $\sigma_{\min}(A)$ and $\sigma_{\max}(A)$ the minimal and maximal non-zero singular values of $A$, respectively. Set

$$\begin{aligned}
\sigma &= \min_{(i_1,\ldots,i_{N-m})\neq(i'_1,\ldots,i'_{N-m})} \sigma_{\min}(M_{i_1,\ldots,i_{N-m}}T_{i'_1,\ldots,i'_{N-m}}),\\
\alpha_m &= \min_{(i_1,\ldots,i_{N-m})\in I_{N,N-m}} \sigma^2_{\min}(T_{i_1,\ldots,i_{N-m}}),\\
\beta_m &= \max_{(i_1,\ldots,i_{N-m})\in I_{N,N-m}} \sigma^2_{\max}(T_{i_1,\ldots,i_{N-m}}).
\end{aligned}$$

**Theorem 3.1.** *Let $\{\varphi_i\}_{1\le i\le N}$ be an $m$-erasure self-located robust frame for $\mathcal{H}$. Then for any $f \in \mathcal{H}$ and $\varepsilon \in \mathbb{R}^N$, the reconstructed signal $Rf$ with the above algorithm from any unordered $N-m$ elements of $Tf+\varepsilon$ satisfies that*

$$\|Rf - f\| \le \left(\frac{2}{\sigma}\left(\frac{\sqrt{\beta_m}}{\sqrt{\alpha_m}} + 1\right) + \frac{1}{\sqrt{\alpha_m}}\right)\|\varepsilon\|. \tag{3.5}$$

*Moreover, if $\{\varphi_i\}_{1\le i\le N}$ is $m$-erasure almost self-located robust and the reconstruction algorithm is stable, i.e., there is some constant $C$ such that*

$$\|Rf - f\| \le C\|\varepsilon\|,$$

*then it must be $m$-erasure self-located robust.*

**Remark 3.2.** (3.5) implies that we can recover exactly every signal from its unordered partial frame coefficients with a self-located robust frame provided the inputs are noise-free. In particular, the robustness implies the almost robustness.

Before proving Theorem 3.1, we introduce a simple lemma, which is easy to prove with the singular value decomposition of matrices.

**Lemma 3.3.** *Let $A$ be a $k \times n$ matrix. For any $y$ in the orthogonal complement of $\{x \in \mathbb{R}^n : Ax = 0\}$, we have $\|Ay\|_2 \geq \sigma_{\min}(A)\|y\|_2$.*

We are now ready to prove Theorem 3.1.

**Proof of Theorem 3.1.** Fix some $f \in \mathcal{H}$, $(i_1, \ldots, i_{N-m}) \in I_{N,N-m}$ and $\varepsilon \in \mathbb{R}^N$. We only consider the case that the solution $(i_1^0, \ldots, i_{N-m}^0)$ of (3.2) is not equal to $(i_1, \ldots, i_{N-m})$, since the trivial case $(i_1^0, \ldots, i_{N-m}^0) = (i_1, \ldots, i_{N-m})$ is easy to check.

Let $H_0 = \{h \in \mathcal{H} : T_{i_1,\ldots,i_{N-m}}h = T_{i_1^0,\ldots,i_{N-m}^0}g$ for some $g\}$. Since $\{\varphi_i\}_{1\leq i\leq N}$ is $m$-erasure self-located robust, we have

$$H_0 = \{h \in \mathcal{H} : T_{i_1,\ldots,i_{N-m}}h = T_{i_1^0,\ldots,i_{N-m}^0}h\}.$$

Consequently, $H_0 \subset \{h \in \mathcal{H} : M_{i_1^0,\ldots,i_{N-m}^0}T_{i_1,\ldots,i_{N-m}}h = 0\}$. On the other hand, if $M_{i_1^0,\ldots,i_{N-m}^0}T_{i_1,\ldots,i_{N-m}}h = 0$, then there is some $g \in \mathcal{H}$ such that $T_{i_1,\ldots,i_{N-m}}h = T_{i_1^0,\ldots,i_{N-m}^0}g$. Hence $h \in H_0$. Therefore,

$$H_0 = \{h \in \mathcal{H} : M_{i_1^0,\ldots,i_{N-m}^0}T_{i_1,\ldots,i_{N-m}}h = 0\}.$$

Let $P_0$ be the orthogonal projection from $\mathcal{H}$ onto $H_0$. We abuse the symbol $\varepsilon$ to stand also for the erased vector $(\varepsilon_{i_1}, \ldots, \varepsilon_{i_{N-m}})$. We see from Lemma 3.3 that

$$\begin{aligned}
\|(I - P_0)f\| &\leq \frac{1}{\sigma_{\min}(M_{i_1^0,\ldots,i_{N-m}^0}T_{i_1,\ldots,i_{N-m}})}\|M_{i_1^0,\ldots,i_{N-m}^0}T_{i_1,\ldots,i_{N-m}}(f - P_0f)\| \\
&\leq \frac{1}{\sigma}\|M_{i_1^0,\ldots,i_{N-m}^0}T_{i_1,\ldots,i_{N-m}}f\| \\
&\leq \frac{1}{\sigma}\|M_{i_1^0,\ldots,i_{N-m}^0}(T_{i_1,\ldots,i_{N-m}}f + \varepsilon)\| + \frac{1}{\sigma}\|M_{i_1^0,\ldots,i_{N-m}^0}\varepsilon\| \\
&\leq \frac{1}{\sigma}\|M_{i_1,\ldots,i_{N-m}}(T_{i_1,\ldots,i_{N-m}}f + \varepsilon)\| + \frac{1}{\sigma}\|M_{i_1^0,\ldots,i_{N-m}^0}\varepsilon\| \\
&= \frac{1}{\sigma}\|M_{i_1,\ldots,i_{N-m}}\varepsilon\| + \frac{1}{\sigma}\|M_{i_1^0,\ldots,i_{N-m}^0}\varepsilon\|.
\end{aligned} \tag{3.6}$$

Let

$$Q_{i_1^0,\ldots,i_{N-m}^0} = TS_{i_1^0,\ldots,i_{N-m}^0},$$

where $S_{i_1^0,\ldots,i_{N-m}^0} := (T^*_{i_1^0,\ldots,i_{N-m}^0}T_{i_1^0,\ldots,i_{N-m}^0})^{-1}T^*_{i_1^0,\ldots,i_{N-m}^0}$ is the synthesis operator for the canonical dual of $\{\varphi_{i_l^0}\}_{1\leq l\leq N-m}$. Then we have

$$T = Q_{i_1^0,\ldots,i_{N-m}^0}T_{i_1^0,\ldots,i_{N-m}^0}.$$

Let $\tilde{T}$ be the analysis operator for $\{\tilde{\varphi}_i\}_{1\leq i\leq N}$. We see from the recovery algorithm that

$$Rf = \tilde{T}^*Q_{i_1^0,\ldots,i_{N-m}^0}(T_{i_1,\ldots,i_{N-m}}f + \varepsilon).$$

It follows that

$$\begin{aligned}\|Rf - f\| &= \|\tilde{T}^* Q_{i_1^0,\dots,i_{N-m}^0}(T_{i_1,\dots,i_{N-m}} f + \varepsilon) - f\| \\ &= \|\tilde{T}^* Q_{i_1^0,\dots,i_{N-m}^0} T_{i_1,\dots,i_{N-m}}(f - P_0 f) + \tilde{T}^* Q_{i_1^0,\dots,i_{N-m}^0}\varepsilon \\ &\qquad - (f - P_0 f)\| \\ &= \left\| S_{i_1^0,\dots,i_{N-m}^0} T_{i_1,\dots,i_{N-m}}(f - P_0 f) + S_{i_1^0,\dots,i_{N-m}^0}\varepsilon - (f - P_0 f)\right\| \\ &\le \left\| S_{i_1^0,\dots,i_{N-m}^0} T_{i_1,\dots,i_{N-m}}(f - P_0 f)\right\| + \left\| S_{i_1^0,\dots,i_{N-m}^0}\varepsilon\right\| + \|f - P_0 f\| \\ &\le (\|S_{i_1^0,\dots,i_{N-m}^0} T_{i_1,\dots,i_{N-m}}\| + 1)\|f - P_0 f\| + \|S_{i_1^0,\dots,i_{N-m}^0}\| \cdot \|\varepsilon\| \\ &\le \Bigg(\frac{1}{\sigma}(\|S_{i_1^0,\dots,i_{N-m}^0} T_{i_1,\dots,i_{N-m}}\| + 1)(\|M_{i_1,\dots,i_{N-m}}\| + \|M_{i_1^0,\dots,i_{N-m}^0}\|) \\ &\qquad + \|S_{i_1^0,\dots,i_{N-m}^0}\|\Bigg)\|\varepsilon\|. \end{aligned} \tag{3.7}$$

We see from (3.1) that $M_{i_1,\dots,i_{N-m}}$ is self-adjoint and its eigenvalues are either 0 or 1. Hence $\|M_{i_1,\dots,i_{N-m}}\| \le 1$. Similarly, $\|M_{i_1^0,\dots,i_{N-m}^0}\| \le 1$. On the other hand, since $S_{i_1^0,\dots,i_{N-m}^0}$ is the synthesis operator for the canonical dual of $\{\varphi_{i_l^0}\}_{1\le l\le N-m}$, we have

$$\|S_{i_1^0,\dots,i_{N-m}^0}\| \le \frac{1}{\sqrt{\alpha_m}}$$

and

$$\|S_{i_1^0,\dots,i_{N-m}^0} T_{i_1,\dots,i_{N-m}}\| \le \frac{\sqrt{\beta_m}}{\sqrt{\alpha_m}}.$$

Now we get (3.5) as desired.

Next we prove the second part. Assume that there exist some $f, h \in \mathcal{H}$ with $f \ne g$ such that

$$T_{i_1,\dots,i_{N-m}} f = T_{j_1,\dots,j_{N-m}} g.$$

Then we have $f \ne 0$, $g \ne 0$, $\dim T_{i_1,\dots,i_{N-m}}\mathcal{H} \cap T_{j_1,\dots,j_{N-m}}\mathcal{H} > 0$ and $(i_1,\dots,i_{N-m}) \ne (j_1,\dots,j_{N-m})$. Moreover, it follows from Theorem 2.1 that $T_{j_1,\dots,j_{N-m}}\mathcal{H} \ne T_{i_1,\dots,i_{N-m}}\mathcal{H}$.

Fix some $c \in T_{j_1,\dots,j_{N-m}}\mathcal{H} \setminus T_{i_1,\dots,i_{N-m}}\mathcal{H}$. Then for any $\alpha > 0$, we have $M_{j_1,\dots,j_{N-m}}\alpha c = 0$. Hence

$$M_{j_1,\dots,j_{N-m}}(T_{i_1,\dots,i_{N-m}} f + \alpha c) = M_{j_1,\dots,j_{N-m}} T_{j_1,\dots,j_{N-m}} g = 0.$$

Therefore, $(i_1^0,\dots,i_{N-m}^0) := (j_1,\dots,j_{N-m})$ is a minimizer of

$$\underset{(i'_1,\dots,i'_{N-m})\in I_{N,N-m}}{\arg\min} \|M_{i'_1,\dots,i'_{N-m}}(T_{i_1,\dots,i_{N-m}} f + \alpha c)\|.$$

By the assumption, there is some constant $C$ such that

$$\|\tilde{T}^* Q_{j_1,\dots,j_{N-m}}(T_{i_1,\dots,i_{N-m}} f + \alpha c) - f\| \le C\|\alpha c\|.$$

By letting $\alpha \to 0$, we get

$$f = \tilde{T}^* Q_{j_1,\dots,j_{N-m}} T_{i_1,\dots,i_{N-m}} f = \tilde{T}^* Q_{j_1,\dots,j_{N-m}} T_{j_1,\dots,j_{N-m}} g = g,$$

which contradicts with the choice of $f$ and $g$. This completes the proof. □

In the rest of this section, we focus on the construction of self-located robust frames. We first give some necessary and sufficient conditions for a sequence of vectors to be a self-located robust frame. Then we give two types of construction, one is based on random matrices [22], and the other is based on prime numbers.

### 3.1. Characterization of self-located robust frames

First, we give a characterization of self-located robust frames.

**Theorem 3.4.** *Let $A$ be an $n \times N$ matrix with column vectors $\{\varphi_i\}_{1\le i\le N}$. Then $\{\varphi_i\}_{1\le i\le N}$ is a self-located robust frame for $\mathcal{H}$ with respect to $m$-erasures if and only if for any $(j_1,\dots,j_{N-m})$, $(j'_1,\dots,j'_{N-m}) \in I_{N,N-m}$,*

$$\operatorname{rank}\begin{pmatrix} A_{j_1,\dots,j_{N-m}} \\ A_{j_1,\dots,j_{N-m}} - A_{j'_1,\dots,j'_{N-m}} \end{pmatrix} = n + \operatorname{rank}(A_{j_1,\dots,j_{N-m}} - A_{j'_1,\dots,j'_{N-m}}), \tag{3.8}$$

*where $A_{j_1,\dots,j_{N-m}}$ stands for the submatrix of $A$ consisting of the $j_1$-th, ..., $j_{N-m}$-th columns.*

**Proof.** Necessity. Fix some $(j_1,\dots,j_{N-m})$, $(j'_1,\dots,j'_{N-m}) \in I_{N,N-m}$. Since $\{\varphi_i\}_{1\le i\le N}$ is a self-located robust frame for $\mathcal{H}$ with respect to $m$-erasures, $\{\varphi_{j_l}\}_{1\le l\le N-m}$ is complete in $\mathcal{H}$. Consequently, $\operatorname{rank}(A_{j_1,\dots,j_{N-m}}) = n$.

Let $S_1$ and $S_2$ be the linear spaces spanned by row vectors of $A_{j_1,\dots,j_{N-m}}$ and $A_{j_1,\dots,j_{N-m}} - A_{j'_1,\dots,j'_{N-m}}$, respectively. Suppose that $x = (x_1,\dots,x_{N-m}) \in S_1 \cap S_2$. Then there are constants $c_1,\dots,c_n$ and $\tilde c_1,\dots,\tilde c_n$ such that

$$x_l = \sum_{i=1}^n c_i a_{i,j_l} = \sum_{i=1}^n \tilde c_i (a_{i,j_l} - a_{i,j'_l}), \quad 1 \le l \le N-m.$$

Consequently,

$$\sum_{i=1}^n (\tilde c_i - c_i) a_{i,j_l} = \sum_{i=1}^n \tilde c_i a_{i,j'_l}, \quad 1 \le l \le N-m.$$

Let $c = (c_1,\dots,c_n)$ and $\tilde c = (\tilde c_1,\dots,\tilde c_n)$. Then we have

$$\langle \tilde c - c, \varphi_{j_l}\rangle = \langle \tilde c, \varphi_{j'_l}\rangle, \quad 1 \le l \le N-m.$$

Since $\{\varphi_i\}_{1\le i\le N}$ is $m$-erasure self-located robust, we have $\tilde c - c = \tilde c$. Hence $c = 0$. Therefore, $x = 0$. In other words, $S_1 \cap S_2 = \{0\}$. It follows that

$$\begin{aligned}
&\operatorname{rank}\begin{pmatrix} A_{j_1,\dots,j_{N-m}} \\ A_{j_1,\dots,j_{N-m}} - A_{j'_1,\dots,j'_{N-m}} \end{pmatrix} \\
&= \operatorname{rank}(A_{j_1,\dots,j_{N-m}}) + \operatorname{rank}(A_{j_1,\dots,j_{N-m}} - A_{j'_1,\dots,j'_{N-m}}) \\
&= n + \operatorname{rank}(A_{j_1,\dots,j_{N-m}} - A_{j'_1,\dots,j'_{N-m}}).
\end{aligned}$$

Sufficiency. Fix some $(j_1,\dots,j_{N-m})$, $(j'_1,\dots,j'_{N-m}) \in I_{N,N-m}$. Let $S_1$ and $S_2$ be defined as previous. Suppose that (3.8) is true. Since

$$\begin{aligned}
&\operatorname{rank}\begin{pmatrix} A_{j_1,\dots,j_{N-m}} \\ A_{j_1,\dots,j_{N-m}} - A_{j'_1,\dots,j'_{N-m}} \end{pmatrix} \\
&\le \operatorname{rank}(A_{j_1,\dots,j_{N-m}}) + \operatorname{rank}(A_{j_1,\dots,j_{N-m}} - A_{j'_1,\dots,j'_{N-m}}),
\end{aligned}$$

we see from (3.8) that

$$\mathrm{rank}(A_{j_1,\ldots,j_{N-m}}) \geq n.$$

But the inverse inequality is obvious. Hence

$$\mathrm{rank}(A_{j_1,\ldots,j_{N-m}}) = n \tag{3.9}$$

and $S_1 \cap S_2 = \{0\}$.

Assume that $f = (f_1, \ldots, f_n)$ and $g = (g_1, \ldots, g_n)$ satisfy

$$\langle f, \varphi_{j_l} \rangle = \langle g, \varphi_{j'_l} \rangle \quad 1 \leq l \leq N - m.$$

Then we have

$$\langle g - f, \varphi_{j_l} \rangle = \langle g, \varphi_{j_l} - \varphi_{j'_l} \rangle \quad 1 \leq l \leq N - m.$$

That is,

$$\sum_{i=1}^{n} (g_i - f_i) a_{i,j_l} = \sum_{i=1}^{n} g_i (a_{i,j_l} - a_{i,j'_l}) \quad 1 \leq l \leq N - m.$$

Since $S_1 \cap S_2 = \{0\}$, both sides must be zero. Hence

$$\langle g - f, \varphi_{j_l} \rangle = 0 \quad 1 \leq l \leq N - m.$$

By (3.9), we have $f = g$. □

By Theorem 3.4, we get the following necessary condition for a frame to be self-located robust.

**Corollary 3.5.** *Let $\{\varphi_i\}_{1 \leq i \leq N}$ be an $m$-erasure self-located robust frame for an $n$-dimensional Hilbert space $\mathcal{H}$. Then we have*

$$N - m \geq 2n - 1.$$

**Proof.** We see from (3.9) that $\mathrm{rank}(A_{1,\ldots,N-m}) = n$. Hence $N - m \geq n$ and there is some $(j_1, \ldots, j_n) \in I_{N-m,n}$ such that $\varphi_{j_1}, \ldots, \varphi_{j_n}$ are linearly independent. Consequently, $\varphi_{j_1} - \varphi_{j_2}, \ldots, \varphi_{j_{n-1}} - \varphi_{j_n}$ are linearly independent. Choose $N - m - n + 1$ elements $j_{n+1}, \ldots, j_{N-m+1}$ from $\{1, \ldots, N\} \setminus \{j_1, \ldots, j_n\}$. Then we have $\mathrm{rank}(A_{j_1,\ldots,j_{N-m}} - A_{j_2,\ldots,j_{N-m+1}}) \geq n - 1$. By (3.8), we have

$$\mathrm{rank}\begin{pmatrix} A_{j_1,\ldots,j_{N-m}} \\ A_{j_1,\ldots,j_{N-m}} - A_{j_2,\ldots,j_{N-m+1}} \end{pmatrix} \geq 2n - 1.$$

Hence $N - m \geq 2n - 1$. □

### *3.2. Randomly generated self-located robust frames*

Let $(\Omega, P)$ be a probability space, where $\Omega$ is the space of samples and $P$ is the probability measure. The second result in this section is the following.

**Theorem 3.6.** *Suppose that $N$, $m$ and $n$ are positive integers. Let $A$ be an $n \times N$ matrix whose entries are independent continuous random variables. Then it is almost sure that the column vectors of $A$ form an $m$-erasure self-located robust frame if any one of the following two conditions is satisfied,*

(i). $N - m \geq 2n$, *or*
(ii). $N - m \geq 2n - 1$ *and the first row of $A$ is replaced by* $(1, \ldots, 1)$.

Before proving this theorem, we present some preliminary results. The following simple lemma is useful in the proof of the main result.

**Lemma 3.7.** *Let $a_0, \ldots, a_n$ and $x$ be a sequence of continuous random variables. Suppose that $x$ and $(a_0, \ldots, a_n)$ are independent. If $a_n \neq 0$ almost surely, then we have*

$$\sum_{k=0}^{n} a_k x^k \neq 0, \quad a.s.$$

**Proof.** For fixed $(a_0, \ldots, a_n)$ with $a_n \neq 0$, the equation $\sum_{k=0}^{n} a_k x^k = 0$ has at most $n$ solutions. Since $x$ is a continuous random variable and $x$ is independent with $(a_0, \ldots, a_n)$, we have $\sum_{k=0}^{n} a_k x^k \neq 0$ almost surely. □

The following is an immediate consequence, which can be proved with induction over $n$.

**Lemma 3.8.** *Let $A$ be an $n \times N$ matrix whose entries are independent continuous random variables, where $N \geq n$. Then it is almost sure that every $n \times n$ submatrix of $A$ is nonsingular.*

Next we show that for a randomly generated matrix $A$, if the rows of $A_{j_1,\ldots,j_k} - A_{j'_1,\ldots,j'_k}$ are linearly dependent, then it remains this property whenever we add more rows to $A$.

**Lemma 3.9.** *Let $A$ be an $n \times N$ matrix whose entries are independent continuous random variables, where $N \geq n$. Suppose that $(j_1, \ldots, j_k)$, $(j'_1, \ldots, j'_k) \in I_{N,k}$, $k \geq n$. If $\operatorname{rank}(A_{j_1,\ldots,j_k} - A_{j'_1,\ldots,j'_k}) < n$ almost surely, then for any $m \geq 1$ and $m \times N$ matrix $B$ whose entries are independent continuous random variables, which are also independent with entries of $A$, we have*

$$\operatorname{rank}\begin{pmatrix} A_{j_1,\ldots,j_k} - A_{j'_1,\ldots,j'_k} \\ B_{j_1,\ldots,j_k} - B_{j'_1,\ldots,j'_k} \end{pmatrix} = \operatorname{rank}(A_{j_1,\ldots,j_k} - A_{j'_1,\ldots,j'_k}), \quad a.s.$$

**Proof.** Denote $A = (a_{i,j})_{1 \leq i \leq n, 1 \leq j \leq N}$ and $B = (b_{i,j})_{1 \leq i \leq m, 1 \leq j \leq N}$.

For $0 \leq r < n$, let

$$E_r = \{\omega \in \Omega : \operatorname{rank}(A_{j_1,\ldots,j_k} - A_{j'_1,\ldots,j'_k}) = r\}.$$

Then we have $P(\Omega \setminus \cup_{r=0}^{n-1} E_r) = 0$. For $\omega \in E_r$, we have

$$\begin{vmatrix} a_{1,j_1} - a_{1,j'_1} & \cdots & a_{1,j_{r+1}} - a_{1,j'_{r+1}} \\ \vdots & & \vdots \\ a_{r+1,j_1} - a_{r+1,j'_1} & \cdots & a_{r+1,j_{r+1}} - a_{r+1,j'_{r+1}} \end{vmatrix} = 0. \tag{3.10}$$

Expanding the determinant along the first row, we get

$$\sum_{l=1}^{r+1}(a_{1,j_l} - a_{1,j'_l})D_l = 0,$$

where $D_l$ is the $(1, l)$ minor.

Observe that $a_{i,j}$ are independent continuous random variables. If $j_l \notin \{j'_s : 1 \leq s \leq r+1\}$, then we have $D_l = 0$ almost surely on $E_r$. And if $j_l = j'_s$ for some $l \neq s$, then we have $D_l - D_s = 0$ almost surely on $E_r$. Consequently, (3.10) remains true if we replace $a_{1,j_l}$ and $a_{1,j'_l}$ by $b_{1,j_l}$ and $b_{1,j'_l}$, respectively. That is,

$$\begin{vmatrix} b_{1,j_1} - b_{1,j'_1} & \dots & b_{1,j_{r+1}} - b_{1,j'_{r+1}} \\ a_{2,j_1} - a_{2,j'_1} & \dots & a_{2,j_{r+1}} - a_{2,j'_{r+1}} \\ \vdots & & \vdots \\ a_{r+1,j_1} - a_{r+1,j'_1} & \dots & a_{r+1,j_{r+1}} - a_{r+1,j'_{r+1}} \end{vmatrix} = 0, \quad \text{a.s. on } E_r.$$

Similarly we can prove that for any $1 \leq s_1 < \ldots < s_{r+1} \leq k$ and $1 \leq i_1 < \ldots < i_r \leq n$, we have

$$\begin{vmatrix} b_{1,j_{s_1}} - b_{1,j'_{s_1}} & \dots & b_{1,j_{s_{r+1}}} - b_{1,j'_{s_{r+1}}} \\ a_{i_1,j_{s_1}} - a_{i_1,j'_{s_1}} & \dots & a_{i_1,j_{s_{r+1}}} - a_{i_1,j'_{s_{r+1}}} \\ \vdots & & \vdots \\ a_{i_r,j_{s_1}} - a_{i_r,j'_{s_1}} & \dots & a_{i_r,j_{s_{r+1}}} - a_{i_r,j'_{s_{r+1}}} \end{vmatrix} = 0, \quad \text{a.s. on } E_r.$$

Hence the rank of $A_{j_1,\ldots,j_k} - A_{j'_1,\ldots,j'_k}$ remains unchanged if we add the first row of $B$ to $A$. By induction on $m$, we get the conclusion as desired. $\square$

In order to prove the main result (Theorem 3.6), we also need the following lemma, which itself is also interesting.

Denote by $A^{i_1,\ldots,i_n}_{j_1,\ldots,j_n}$ the submatrix of $A$ consisting of the $i_1$-th, $\ldots$, $i_n$-th rows and the $j_1$-th, $\ldots$, $j_n$-th columns.

**Lemma 3.10.** *Let $A$ be an $n \times N$ matrix whose entries are independent continuous random variables, where $N \geq 2n$. Then for any $(j_1, \ldots, j_{2n})$, $(j'_1, \ldots, j'_{2n}) \in I_{N,2n}$ with $j_l \neq j'_l$, $1 \leq l \leq 2n$, we have*

$$\text{rank}(A_{j_1,\ldots,j_{2n}} - A_{j'_1,\ldots,j'_{2n}}) = n, \quad \textit{a.s.}$$

**Proof.** We prove this lemma by induction on $n$. For $n = 1$, the conclusion is obvious.

Now we assume that it is true for $n = k$, where $k \geq 1$. Let us consider the case of $n = k + 1$.

By rearranging column vectors of $A_{j_1,\ldots,j_{2n}} - A_{j'_1,\ldots,j'_{2n}}$, we may assume that $j'_l \neq j_{2n}$ for all $1 \leq l \leq 2n-2$.

By the assumption, for almost all $\omega \in \Omega$, there is some $k \times k$ submatrix of $A^{1,\ldots,k}_{j_1,\ldots,j_{2k}} - A^{1,\ldots,k}_{j'_1,\ldots,j'_{2k}}$ which is of rank $k$. Hence there is a partition $\{E_i : 1 \leq i \leq m\}$ of $\Omega$ satisfying that $P(\cup_{i=1}^m E_i) = 1$ and for each $1 \leq i \leq m$, there is some sequence $(j_1, \ldots, j_k) \in I_{N,k}$ such that for almost all $\omega \in E_i$, $\text{rank}(A^{1,\ldots,k}_{j_1,\ldots,j_k} - A^{1,\ldots,k}_{j'_1,\ldots,j'_k}) = k$. Since $a_{j_{2n}}$ is an $n = k + 1$ dimensional continuous random variable and it is independent with $(a_{j_1}, \ldots, a_{j_{2k}})$, $(a_{j'_1}, \ldots, a_{j'_{2k}})$, and $a_{j'_{2n}}$, we see from Lemma 3.7 that

$$\det(A^{1,\ldots,k+1}_{j_1,\ldots,j_k,j_{2n}} - A^{1,\ldots,k+1}_{j'_1,\ldots,j'_k,j'_{2n}}) \neq 0, \quad \text{a.s. on } E_i.$$

Hence

$$\text{rank}(A_{j_1,\ldots,j_{2n}} - A_{j'_1,\ldots,j'_{2n}}) = k + 1, \quad \text{a.s.}$$

This completes the proof. $\square$

We are now ready to prove the density of self-located robust frames.

**Proof of Theorem 3.6.** We only prove (i). And (ii) can be proved similarly. By Theorem 3.4, it suffices to show that for any $(j_1, \ldots, j_{2n}), (j'_1, \ldots, j'_{2n}) \in I_{N,2n}$,

$$\operatorname{rank}\begin{pmatrix} A_{j_1,\ldots,j_{2n}} \\ A_{j_1,\ldots,j_{2n}} - A_{j'_1,\ldots,j'_{2n}} \end{pmatrix} = n + \operatorname{rank}(A_{j_1,\ldots,j_{2n}} - A_{j'_1,\ldots,j'_{2n}}), \quad \text{a.s.} \tag{3.11}$$

Denote $A = (a_{i,j})_{1\le i\le n, 1\le j\le N}$. By Lemma 3.8, we have

$$\begin{aligned}\operatorname{rank}&\begin{pmatrix} A_{j_1,\ldots,j_{2n}} \\ A_{j_1,\ldots,j_{2n}} - A_{j'_1,\ldots,j'_{2n}} \end{pmatrix} \\ &\le \operatorname{rank}(A_{j_1,\ldots,j_{2n}}) + \operatorname{rank}(A_{j_1,\ldots,j_{2n}} - A_{j'_1,\ldots,j'_{2n}}) \\ &= n + \operatorname{rank}(A_{j_1,\ldots,j_{2n}} - A_{j'_1,\ldots,j'_{2n}}), \quad \text{a.s.}\end{aligned}$$

Hence (3.11) is equivalent to

$$\operatorname{rank}\begin{pmatrix} A_{j_1,\ldots,j_{2n}} \\ A_{j_1,\ldots,j_{2n}} - A_{j'_1,\ldots,j'_{2n}} \end{pmatrix} \ge n + \operatorname{rank}(A_{j_1,\ldots,j_{2n}} - A_{j'_1,\ldots,j'_{2n}}), \quad \text{a.s.} \tag{3.12}$$

We prove (3.11) or (3.12) by induction over $n$.

To avoid complicated symbols, we use the same term "almost surely" or its abbreviation "a.s." for different probability measures, which in many cases are conditional ones. The exact meaning can be seen with the context.

First, we consider the case of $n = 1$. If $\operatorname{rank}(A_{j_1,j_2} - A_{j'_1,j'_2}) = 0$, then we have $(j_1, j_2) = (j'_1, j'_2)$. Consequently, $\operatorname{rank}\begin{pmatrix} A_{j_1,j_2} \\ A_{j_1,j_2} - A_{j'_1,j'_2} \end{pmatrix} = 1$. Hence (3.11) is true.

If $\operatorname{rank}(A_{j_1,j_2} - A_{j'_1,j'_2}) = 1$, then we have $(j_1, j_2) \ne (j'_1, j'_2)$. Since $a_{i,j}$ are independent continuous random variables. By Lemma 3.7, (3.11) is also true.

Now assume that (3.11) is true for $1 \le n \le k$. We consider the case of $n = k+1$. Since different $a_{i,j}$ are independent, we have

$$\begin{aligned}\operatorname{rank}\begin{pmatrix} A_{j_1,\ldots,j_{2n}} \\ A_{j_1,\ldots,j_{2n}} - A_{j'_1,\ldots,j'_{2n}} \end{pmatrix} &\ge \operatorname{rank}\begin{pmatrix} A_{j_1,\ldots,j_{2n}} \\ A^{2,\ldots n}_{j_1,\ldots,j_{2n}} - A^{2,\ldots n}_{j'_1,\ldots,j'_{2n}} \end{pmatrix} \\ &= 1 + \operatorname{rank}\begin{pmatrix} A^{2,\ldots n}_{j_1,\ldots,j_{2n}} \\ A^{2,\ldots n}_{j_1,\ldots,j_{2n}} - A^{2,\ldots n}_{j'_1,\ldots,j'_{2n}} \end{pmatrix} \\ &\ge 1 + \operatorname{rank}\begin{pmatrix} A^{2,\ldots n}_{j_3,\ldots,j_{2n}} \\ A^{2,\ldots n}_{j_3,\ldots,j_{2n}} - A^{2,\ldots n}_{j'_3,\ldots,j'_{2n}} \end{pmatrix} \\ &= 1 + \operatorname{rank}(A^{2,\ldots n}_{j_3,\ldots,j_{2n}}) + \operatorname{rank}(A^{2,\ldots n}_{j_3,\ldots,j_{2n}} - A^{2,\ldots n}_{j'_3,\ldots,j'_{2n}}) \\ &= n + \operatorname{rank}(A^{2,\ldots n}_{j_3,\ldots,j_{2n}} - A^{2,\ldots n}_{j'_3,\ldots,j'_{2n}}), \quad \text{a.s.,}\end{aligned} \tag{3.13}$$

where we use the inductive assumption in the second last step. If

$$\operatorname{rank}(A^{1,\ldots n}_{j_1,\ldots,j_{2n}} - A^{1,\ldots n}_{j'_1,\ldots,j'_{2n}}) = \operatorname{rank}(A^{2,\ldots n}_{j_3,\ldots,j_{2n}} - A^{2,\ldots n}_{j'_3,\ldots,j'_{2n}}),$$

we see from (3.13) that (3.12) is true. Next we assume that

$$\operatorname{rank}(A^{1,\ldots n}_{j_1,\ldots,j_{2n}} - A^{1,\ldots n}_{j'_1,\ldots,j'_{2n}}) > \operatorname{rank}(A^{2,\ldots n}_{j_3,\ldots,j_{2n}} - A^{2,\ldots n}_{j'_3,\ldots,j'_{2n}}). \tag{3.14}$$

There are two cases.

(i). There is some $l$ such that $j_l = j'_l$.

First, we assume that $\{j'_1, \ldots, j'_{2n}\}$ is a rearrangement of $\{j_1, \ldots, j_{2n}\}$. If $j_l = j'_l$ for all $1 \le l \le 2n$, then the conclusion is obvious. For other cases, by rearranging columns of $\begin{pmatrix} A_{j_1,\ldots,j_{2n}} \\ A_{j_1,\ldots,j_{2n}} - A_{j'_1,\ldots,j'_{2n}} \end{pmatrix}$, we may assume that $j_1 = j'_2$.

Since the sum of all the column vectors of $A_{j_1,\ldots,j_{2n}} - A_{j'_1,\ldots,j'_{2n}}$ is zero, we have

$$\operatorname{rank}(A_{j_1,\ldots,j_{2n}} - A_{j'_1,\ldots,j'_{2n}}) = \operatorname{rank}(A_{j_2,\ldots,j_{2n}} - A_{j'_2,\ldots,j'_{2n}}). \tag{3.15}$$

Let $r = \operatorname{rank}(A^{2,\ldots,n}_{j_3,\ldots,j_{2n}} - A^{2,\ldots,n}_{j'_3,\ldots,j'_{2n}})$.

If $r < n-1$, we see from Lemma 3.9 that

$$\operatorname{rank}(A^{1,\ldots,n}_{j_3,\ldots,j_{2n}} - A^{1,\ldots,n}_{j'_3,\ldots,j'_{2n}}) = r, \quad \text{a.s.}$$

Hence

$$\operatorname{rank}(A^{1,\ldots,n}_{j_2,\ldots,j_{2n}} - A^{1,\ldots,n}_{j'_2,\ldots,j'_{2n}}) \le 1 + r, \quad \text{a.s.}$$

By (3.15), we have

$$\operatorname{rank}(A^{1,\ldots,n}_{j_1,\ldots,j_{2n}} - A^{1,\ldots,n}_{j'_1,\ldots,j'_{2n}}) \le 1 + r, \quad \text{a.s.} \tag{3.16}$$

On the other hand, if $r = n-1$, (3.16) is obvious.

By the inductive assumption, there is some $(n-1+r)\times(n-1+r)$ submatrix of $\begin{pmatrix} A^{2,\ldots,n}_{j_3,\ldots,j_{2n}} \\ A^{2,\ldots,n}_{j_3,\ldots,j_{2n}} - A^{2,\ldots,n}_{j'_3,\ldots,j'_{2n}} \end{pmatrix}$, say $\begin{pmatrix} A^{2,\ldots,n}_{j_3,\ldots,j_{n+r+1}} \\ A^{2,\ldots,r+1}_{j_3,\ldots,j_{n+r+1}} - A^{2,\ldots,r+1}_{j'_3,\ldots,j'_{n+r+1}} \end{pmatrix}$, which is of rank $n-1+r$.

In the expansion of the determinant $\begin{vmatrix} A^{1,\ldots,n}_{j_1,\ldots,j_{n+r+1}} \\ A^{1,\ldots,r+1}_{j_1,\ldots,j_{n+r+1}} - A^{1,\ldots,r+1}_{j'_1,\ldots,j'_{n+r+1}} \end{vmatrix}$, the coefficient of $a^2_{1,j_1}$ is

$$(-1)^n \begin{vmatrix} A^{2,\ldots,n}_{j_3,\ldots,j_{n+r+1}} \\ A^{2,\ldots,r+1}_{j_3,\ldots,j_{n+r+1}} - A^{2,\ldots,r+1}_{j'_3,\ldots,j'_{n+r+1}} \end{vmatrix},$$

which is not zero almost surely. By Lemma 3.7, we have

$$\begin{vmatrix} A^{1,\ldots,n}_{j_1,\ldots,j_{n+r+1}} \\ A^{1,\ldots,r+1}_{j_1,\ldots,j_{n+r+1}} - A^{1,\ldots,r+1}_{j'_1,\ldots,j'_{n+r+1}} \end{vmatrix} \ne 0, \quad \text{a.s.}$$

Hence

$$\begin{aligned} \operatorname{rank} &\begin{pmatrix} A^{1,\ldots,n}_{j_1,\ldots,j_{2n}} \\ A^{1,\ldots,n}_{j_1,\ldots,j_{2n}} - A^{1,\ldots,n}_{j'_1,\ldots,j'_{2n}} \end{pmatrix} \\ &\ge \operatorname{rank} \begin{pmatrix} A^{1,\ldots,n}_{j_1,\ldots,j_{n+r+1}} \\ A^{1,\ldots,r+1}_{j_1,\ldots,j_{n+r+1}} - A^{1,\ldots,r+1}_{j'_1,\ldots,j'_{n+r+1}} \end{pmatrix} \\ &= n + r + 1 \\ &\ge n + \operatorname{rank}(A^{1,\ldots,n}_{j_1,\ldots,j_{2n}} - A^{1,\ldots,n}_{j'_1,\ldots,j'_{2n}}), \quad \text{a.s.}, \end{aligned}$$

where (3.16) is used in the last step. This proves that (3.12) is true.

Next we consider the case that $\{j_1', \ldots, j_{2n}'\}$ is not a rearrangement of $\{j_1, \ldots, j_{2n}\}$. Since there is some $l$ such that $j_l = j_l'$, by rearranging columns of $\begin{pmatrix} A_{j_1,\ldots,j_{2n}} \\ A_{j_1,\ldots,j_{2n}} - A_{j_1',\ldots,j_{2n}'} \end{pmatrix}$, we may assume that $j_1 = j_1'$ and $j_2' \notin \{j_2, \ldots, j_{2n}\}$. If

$$\mathrm{rank}(A^{2,\ldots,n}_{j_3,\ldots,j_{2n}} - A^{2,\ldots,n}_{j_3',\ldots,j_{2n}'}) < n-1, \quad \text{a.s. on } E$$

for some $E \subset \Omega$ with $P(E) > 0$, then we see from Lemma 3.9 that

$$\mathrm{rank}(A^{1,\ldots,n}_{j_3,\ldots,j_{2n}} - A^{1,\ldots,n}_{j_3',\ldots,j_{2n}'}) = \mathrm{rank}(A^{2,\ldots,n}_{j_3,\ldots,j_{2n}} - A^{2,\ldots,n}_{j_3',\ldots,j_{2n}'}), \quad \text{a.s. on } E.$$

Hence

$$\begin{aligned} \mathrm{rank}(A^{1,\ldots,n}_{j_1,\ldots,j_{2n}} - A^{1,\ldots,n}_{j_1',\ldots,j_{2n}'}) &= \mathrm{rank}(A^{1,\ldots,n}_{j_2,\ldots,j_{2n}} - A^{1,\ldots,n}_{j_2',\ldots,j_{2n}'}) \\ &\le 1 + \mathrm{rank}(A^{2,\ldots,n}_{j_3,\ldots,j_{2n}} - A^{2,\ldots,n}_{j_3',\ldots,j_{2n}'}), \quad \text{a.s. on } E. \end{aligned}$$

On the other hand, if

$$\mathrm{rank}(A^{2,\ldots,n}_{j_3,\ldots,j_{2n}} - A^{2,\ldots,n}_{j_3',\ldots,j_{2n}'}) = n-1,$$

we also have

$$\mathrm{rank}(A^{1,\ldots,n}_{j_1,\ldots,j_{2n}} - A^{1,\ldots,n}_{j_1',\ldots,j_{2n}'}) \le 1 + \mathrm{rank}(A^{2,\ldots,n}_{j_3,\ldots,j_{2n}} - A^{2,\ldots,n}_{j_3',\ldots,j_{2n}'}). \tag{3.17}$$

Hence (3.17) is true almost surely. By (3.14), we have

$$\mathrm{rank}(A^{1,\ldots,n}_{j_1,\ldots,j_{2n}} - A^{1,\ldots,n}_{j_1',\ldots,j_{2n}'}) = 1 + \mathrm{rank}(A^{2,\ldots,n}_{j_3,\ldots,j_{2n}} - A^{2,\ldots,n}_{j_3',\ldots,j_{2n}'}), \quad \text{a.s.} \tag{3.18}$$

Let $r = \mathrm{rank}(A^{2,\ldots,n}_{j_3,\ldots,j_{2n}} - A^{2,\ldots,n}_{j_3',\ldots,j_{2n}'})$. By the inductive assumption, there is some $(n-1+r)\times(n-1+r)$ submatrix of $\begin{pmatrix} A^{2,\ldots,n}_{j_3,\ldots,j_{2n}} \\ A^{2,\ldots,n}_{j_3,\ldots,j_{2n}} - A^{2,\ldots,n}_{j_3',\ldots,j_{2n}'} \end{pmatrix}$, say $\begin{pmatrix} A^{2,\ldots,n}_{j_3,\ldots,j_{n+r+1}} \\ A^{2,\ldots,r+1}_{j_3,\ldots,j_{n+r+1}} - A^{2,\ldots,r+1}_{j_3',\ldots,j_{n+r+1}'} \end{pmatrix}$, which is of rank $n-1+r$. Consequently, the coefficient of $a_{1,j_1}a_{1,j_2'}$ in the determinant $\begin{vmatrix} A^{1,\ldots,n}_{j_1,\ldots,j_{n+r+1}} \\ A^{1,\ldots,r+1}_{j_1,\ldots,j_{n+r+1}} - A^{1,\ldots,r+1}_{j_1',\ldots,j_{n+r+1}'} \end{vmatrix}$ is not zero almost surely. By Lemma 3.7, we get (3.11).

(ii). $j_l \ne j_l'$ for all $1 \le l \le 2n$.

If $\{j_1, \ldots, j_{2n}\} \cap \{j_1', \ldots, j_{2n}'\} = \emptyset$, we see from Lemma 3.8 that

$$\mathrm{rank}\begin{pmatrix} A_{j_1,\ldots,j_{2n}} \\ A_{j_1,\ldots,j_{2n}} - A_{j_1',\ldots,j_{2n}'} \end{pmatrix} = \mathrm{rank}\begin{pmatrix} A_{j_1,\ldots,j_{2n}} \\ A_{j_1',\ldots,j_{2n}'} \end{pmatrix} = 2n, \quad \text{a.s.}$$

Hence (3.12) is true.

Next we assume that there is some $l \ne l'$ such that $j_l = j_{l'}'$. Without loss of generality, we assume that $j_1 = j_2'$.

By Lemma 3.10, we have $\mathrm{rank}(A^{2,\ldots,n}_{j_3,\ldots,j_{2n}} - A^{2,\ldots,n}_{j_3',\ldots,j_{2n}'}) = n-1$ almost surely. It follows from the inductive assumption that

$$\mathrm{rank}\begin{pmatrix} A^{2,\ldots,n}_{j_3,\ldots,j_{2n}} \\ A^{2,\ldots,n}_{j_3,\ldots,j_{2n}} - A^{2,\ldots,n}_{j_3',\ldots,j_{2n}'} \end{pmatrix} = 2n-2, \quad \text{a.s.}$$

Observe that the coefficient of $a_{1,j_1}^2$ in the expansion of the determinant

$$\begin{vmatrix} A_{j_1,\ldots,j_{2n}}^{1,\ldots,n} \\ A_{j_1,\ldots,j_{2n}}^{1,\ldots,n} - A_{j'_1,\ldots,j'_{2n}}^{1,\ldots,n} \end{vmatrix}$$

is

$$(-1)^n \begin{vmatrix} A_{j_3,\ldots,j_{2n}}^{2,\ldots,n} \\ A_{j_3,\ldots,j_{2n}}^{2,\ldots,n} - A_{j'_3,\ldots,j'_{2n}}^{2,\ldots,n} \end{vmatrix},$$

which is not equal to zero almost surely. By Lemma 3.7, we have

$$\operatorname{rank} \begin{pmatrix} A_{j_1,\ldots,j_{2n}}^{1,\ldots,n} \\ A_{j_1,\ldots,j_{2n}}^{1,\ldots,n} - A_{j'_1,\ldots,j'_{2n}}^{1,\ldots,n} \end{pmatrix} = 2n.$$

Hence (3.12) is true. This completes the proof. □

### *3.3. Explicitly given self-located robust frames*

In this subsection, we give some explicit construction of self-located robust frames. The main result is the following.

**Theorem 3.11.** *Let $N$, $m$ and $n$ be positive integers and $\{p_{i,j}\}_{1\le i\le n, 1\le j\le N}$ be a sequence of different prime numbers. Define*

$$\varphi_j = (p_{1,j}^{1/3}, \ldots, p_{n,j}^{1/3})^T, 1 \le i \le N.$$

*Then $\{\varphi_j\}_{1\le j\le N}$ is a self-located robust frame with respect to $m$-erasures whenever $N - m \ge 2n$.*

*Moreover, $\{\varphi_j\}_{1\le j\le N}$ remains a self-located robust frame with respect to $m$-erasures if $N - m \ge 2n - 1$ and the first entry of $\varphi_j$ is replaced by 1 for all $1 \le j \le N$.*

To prove Theorem 3.11, we need some preliminary results. First, we introduce a result on prime numbers [21].

**Proposition 3.12.** *(See [21, Theorem 1.1].) Let $K$ and $L$ be two fields such that $K \subseteq L \subseteq \mathbb{R}$. Let $A$ be a subset of $L$ satisfying the following conditions:*

(i). $|A| \ge 2$.
(ii). *For every $a \in A$ there is some $n_a \in \mathbb{N}$ with $a^{n_a} \in K$. In what follows we always assume $n_a$ is minimal.*
(iii). *$A$ is pairwise linearly independent over $K$.*
(iv). *If $\operatorname{char}(K) > 0$, then $(n_a, \operatorname{char}(K)) = 1$ for all $a \in A$.*

*Then $A$ is linearly independent over $K$.*

With Proposition 3.12, we have the following lemma, which can be considered as a generalization of [21, Proposition 4.1].

**Lemma 3.13.** *Let $\{p_i\}_{1\le i\le s}$ be a sequence of different prime numbers and $r_i = p_i^{1/3}, 1 \le i \le s$. Then the equation*

$$\sum_{i_1,\dots,i_s\in\{0,1,2\}} c_{i_1,i_2,\dots,i_s} r_1^{i_1} r_2^{i_2} \dots r_s^{i_s} = 0 \tag{3.19}$$

*with $c_{i_1,i_2,\dots,i_s}$ as unknowns has no nonzero solution in the field $\mathbb{Q}$.*

**Proof.** Let $K = \mathbb{Q}$, $L = \mathbb{R}$ and $A = \{r_1^{i_1} r_2^{i_2} \dots r_s^{i_s} : i_1, \dots, i_s \in \{0, 1, 2\}\}$. By Proposition 3.12, it suffices to show that $A$ is pairwise linearly independent over $\mathbb{Q}$. Suppose on the contrary that there exist $r_1^{i_1} r_2^{i_2} \dots r_s^{i_s} \neq r_1^{i'_1} r_2^{i'_2} \dots r_s^{i'_s}$ such that $r_1^{i_1} r_2^{i_2} \dots r_s^{i_s}$ and $r_1^{i'_1} r_2^{i'_2} \dots r_s^{i'_s}$ are linearly dependent over $\mathbb{Q}$. Then there exist two nonzero rational numbers $u_1$ and $u_2$ such that

$$u_1 r_1^{i_1} r_2^{i_2} \dots r_s^{i_s} = u_2 r_1^{i'_1} r_2^{i'_2} \dots r_s^{i'_s}.$$

That is,

$$r_1^{i_1-i'_1} r_2^{i_2-i'_2} \dots r_s^{i_s-i'_s} = \frac{u_2}{u_1}.$$

Hence

$$p_1^{i_1-i'_1} p_2^{i_2-i'_2} \dots p_s^{i_s-i'_s} = (\frac{u_2}{u_1})^3.$$

Since the right side is the cube of some rational number, for each $d \in \{1, 2, \dots, s\}$ we have $3|(i_d - i'_d)$.

On the other hand, since $r_1^{i_1} r_2^{i_2} \dots r_s^{i_s} \neq r_1^{i'_1} r_2^{i'_2} \dots r_s^{i'_s}$, there exists some $d_0 \in \{1, 2, \dots, s\}$ such that $i_{d_0} \neq i'_{d_0}$. Note that $i_{d_0}, i'_{d_0} \in \{0, 1, 2\}$. So $i_{d_0} - i'_{d_0} \in \{-2, -1, 1, 2\}$, which contradicts with $3|(i_{d_0} - i'_{d_0})$. This completes the proof. □

The following are two immediate consequences.

**Lemma 3.14.** *Let $\{p_i\}_{1\le i\le s}$ be a sequence of different prime numbers and $r_i = p_i^{1/3}$, $1 \le i \le s$. Let $f, g$ and $h$ be some multivariate polynomials of $r_2, r_3, \dots, r_s$ with rational coefficients. Then*

$$f r_1^2 + g r_1 + h = 0$$

*if and only if $f = g = h = 0$.*

**Lemma 3.15.** *Let $\{p_i\}_{1\le i\le s}$ be a sequence of different prime numbers and $r_i = p_i^{1/3}$, $1 \le i \le s$. Let $f$, $g$, $h$ and $u$ be multivariate polynomials of $r_3, r_4, \dots, r_s$ with rational coefficients. Then*

$$f r_1 r_2 + g r_1 + h r_2 + u = 0$$

*if and only if $f = g = h = u = 0$.*

And the following is a counterpart of Lemma 3.8, which is a consequence of Lemma 3.13.

**Lemma 3.16.** *Let $A = (p_{i,j}^{1/3})_{1\le i\le n, 1\le j\le N}$ be an $n \times N$ matrix, where $p_{i,j}$ are pairwise different prime numbers. Then every $n \times n$ submatrix of $A$ is nonsingular.*

Observe that Lemmas 3.9 and 3.10 also remain true if we replace independent continuous random variables by cube roots of different prime numbers, thanks to Lemma 3.13.

For example, let $A$ be an $n \times N$ matrix whose entries are cube roots of different prime numbers, where $N > n$. Suppose that $(j_1, \dots, j_k)$, $(j'_1, \dots, j'_k) \in I_{N,k}$, $k \ge n$. If $\text{rank}(A_{j_1,\dots,j_k} - A_{j'_1,\dots,j'_k}) < n$, then for any

$m \geq 1$ and $m \times N$ matrix $B$ whose entries are cube roots of different prime numbers, which are also different from entries of $A$, we have

$$\operatorname{rank}\begin{pmatrix} A_{j_1,\ldots,j_k} - A_{j'_1,\ldots,j'_k} \\ B_{j_1,\ldots,j_k} - B_{j'_1,\ldots,j'_k} \end{pmatrix} = \operatorname{rank}(A_{j_1,\ldots,j_k} - A_{j'_1,\ldots,j'_k}).$$

By Lemma 3.13, the above conclusion can be proved using almost the same arguments as that in the proof of Lemma 3.9 except that the probability is removed.

With similar arguments as in the proof of Theorem 3.6 we can prove Theorem 3.11. We leave the details to interested readers.

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## References

[1] B.G. Bodmann, Optimal linear transmission by loss-insensitive packet encoding, Appl. Comput. Harmon. Anal. 22 (2007) 274–285.
[2] B.G. Bodmann, V.I. Paulsen, Frames, graphs and erasures, Linear Algebra Appl. 404 (2005) 118–146.
[3] B.G. Bodmann, V.I. Paulsen, M. Tomforde, Equiangular tight frames from complex Seidel matrices containing cube roots of unity, Linear Algebra Appl. 430 (2009) 396–417.
[4] B.G. Bodmann, P.K. Singh, Burst erasures and the mean-square error for cyclic Parseval frames, IEEE Trans. Inform. Theory 57 (2011) 4622–4635.
[5] P.G. Casazza, A. Heinecke, F. Krahmer, G. Kutyniok, Optimally sparse frames, IEEE Trans. Inform. Theory 57 (2011) 7279–7287.
[6] P.G. Casazza, J. Kovačević, Equal-norm tight frames with erasures, Adv. Comput. Math. 18 (2003) 387–430.
[7] P.G. Casazza, G. Kutyniok, F. Philipp, Introduction to finite frame theory, in: Finite Frames, in: Appl. Numer. Harmon. Anal., Birkhäuser/Springer, New York, 2013, pp. 1–53.
[8] O. Christensen, An Introduction to Frames and Riesz Bases, Birkhäuser, Boston, 2003.
[9] D. Han, D. Larson, Frames, bases and group representations, Mem. Amer. Math. Soc. 147 (2000).
[10] D. Han, W. Sun, Reconstruction of signals from frame coefficients with erasures at unknown locations, IEEE Trans. Inform. Theory 60 (2014) 4013–4025.
[11] D. Han, F. Lv, W. Sun, Stable recovery of signals from frame coefficients with erasures at unknown locations, preprint, 2014.
[12] R.B. Holmes, V.I. Paulsen, Optimal frames for erasures, Linear Algebra Appl. 377 (2004) 31–51.
[13] J. Leng, D. Han, Optimal dual frames for erasures II, Linear Algebra Appl. 435 (2011) 1464–1472.
[14] J. Lopez, D. Han, Optimal dual frames for erasures, Linear Algebra Appl. 432 (2010) 471–482.
[15] F. Lv, W. Sun, Construction of robust frames in erasure recovery, Linear Algebra Appl. 479 (2015) 155–170.
[16] P.G. Massey, M.A. Ruiz, D. Stojanoff, Optimal dual frames and frame completions for majorization, Appl. Comput. Harmon. Anal. 34 (2013) 201–223.
[17] P.G. Massey, M.A. Ruiz, D. Stojanoff, Optimal frame completions, Adv. Comput. Math. 40 (2014) 1011–1042.
[18] H. Nguyen, V.K. Goyal, L.R. Varshney, Frame permutation quantization, Appl. Comput. Harmon. Anal. 31 (2011) 74–97.
[19] S. Pehlivan, D. Han, R. Mohapatra, Linearly connected sequences and spectrally optimal dual frames for erasures, J. Funct. Anal. 265 (2013) 2855–2876.
[20] S. Pehlivan, D. Han, R. Mohapatra, Spectrally two-uniform frames for erasures, Oper. Matrices 9 (2015) 383–399.
[21] C. Richard, O. Cormac, On the linear independence of roots, Int. J. Number Theory 5 (2009) 161–171.
[22] M. Rudelson, R. Vershynin, Smallest singular value of a random rectangular matrix, Comm. Pure Appl. Math. 62 (2009) 1707–1739.
[23] T. Strohmer, R.W. Heath Jr., Grassmannian frames with applications to coding and communication, Appl. Comput. Harmon. Anal. 14 (2003) 257–275.
[24] G.J. Woeginger, Exact algorithms for NP-hard problems: a survey, in: Combinatorial Optimization-Eureka, You Shrink!, in: Lecture Notes in Comput. Sci., vol. 2570, Springer, Berlin, 2003, pp. 185–207.