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Price competition under linear demand and finite inventories: Contraction and approximate equilibria

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ABSTRACT

We consider a multi-period price competition among multiple firms with limited inventories of substitutable products, and study two types of equilibrium: with and without recourse. Under a linear demand model, we show that an equilibrium without recourse uniquely exists. In contrast, we show an equilibrium with recourse need not exist, nor be unique. In a low-influence regime, using the equilibrium without recourse, we construct an approximate equilibrium with recourse with the same equilibrium price trajectory.

1. Introduction

In many practical situations, multiple firms selling substitutable products set their prices competitively to sell limited inventories over a finite selling horizon, given that the demand of each firm jointly depends on the prices charged by all firms. For example, airlines competitively set the prices for their limited seat inventories in a particular market. Firms selling electronic products take the prices of their competitors into consideration when setting their prices. In this paper, we consider multiple firms with limited inventories of substitutable products. Each firm chooses the prices that it charges for its product over a finite selling horizon. The demand that each firm faces is a deterministic function of the prices charged by all of the firms, where the demand of a firm is linearly decreasing in its price and linearly increasing in the prices of the other firms. Each firm chooses its prices over a finite selling horizon to maximize its total revenue.

Main Contributions. We study two types of equilibrium for the competitive pricing setting described above. In an equilibrium without recourse, at the beginning of the selling horizon, each firm selects and commits to the prices it charges over the whole selling horizon, assuming that the other firms do the same. In an equilibrium with recourse, at each time period in the selling horizon, each firm observes the inventories of all of the firms and chooses its price at the current time period, again under the assumption that the other firms do the same. Essentially, an equilibrium without recourse corresponds to an open-loop equilibrium [4], whereas an equilibrium with recourse corresponds to a Markov perfect equilibrium (MPE) [5] in the dynamic game among the firms. Despite the fact that the demand of each firm is a deterministic function of the prices so that there is no uncertainty in the firms' responses, we show a clear contrast between the two equilibrium notions.

We consider the diagonal dominant regime, where the price charged by each firm affects its demand more than the prices charged by the other firms. In other words, if all of the competitors of a firm decrease their prices by a certain amount, then the firm can decrease its price by the same amount to ensure that its demand does not decrease. This regime is rather standard in the existing literature and it is used in, for example, [2] and [6]. Focusing on the equilibrium without recourse, we show in Section 2 that the best response of each firm to the price trajectories of the other firms is a contraction mapping, when viewed as a function of the prices of the other firms. In this case, it immediately follows that the equilibrium without recourse always exists and it is unique (see [17, Section 2.5]).

We give counterexamples in Section 3 to show that an equilibrium with recourse may not exist or may not be unique. Motivated by this observation, we look for an approximate equilibrium that is guaranteed to exist. We call a strategy profile for the firms an ϵ-equilibrium with recourse if no firm can improve its total revenue by more than ϵ by deviating from its strategy profile. We consider a low influence regime, where the effect of the price of a firm on the demand of another firm is diminishing, which naturally holds when the number of firms is large. We show in Section 4

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that the equilibrium without recourse can be used to construct an ϵ-equilibrium with recourse that has the same price trajectory as the equilibrium without recourse. So, intuitively speaking, an ϵ-equilibrium with recourse is expected to exist when the number of firms is large.

Our results fill a gap in a fundamental class of revenue management problems. Although there is no uncertainty in the firms' responses, the equilibria with and without recourse are not the same concept and can be qualitatively quite different. While the equilibrium without recourse uniquely exists, the same need not hold for the equilibrium with recourse. Also, our contraction argument for showing the existence and uniqueness of the equilibrium without recourse uses the Karush–Kuhn– Tucker (KKT) conditions for the firm's problem. Though contraction arguments are standard for showing existence and uniqueness of equilibrium [17], to the best of our knowledge, this duality-based contraction argument is new for price competition under limited inventories. This argument becomes surprisingly effective when dealing with linear demand functions, but it is an open question whether similar arguments hold for other demand functions. Lastly, our results indicate that in a low influence regime the equilibrium without recourse can be used to construct an ϵ-equilibrium with recourse with the same price trajectory as the equilibrium without recourse.

Literature Review. Similar to us, [6] considers price competition among multiple firms with limited inventories over a finite selling horizon. There are three key differences between their work and ours. First, they focus on a continuous-time setting, whereas we study a discrete-time formulation. Second, they consider a generalized Nash game [16] where each firm considers all firms' capacity constraints while setting their prices, whereas in our model, each firm only considers its own capacity constraints. Most importantly, they focus on open-loop and closed-loop equilibria, and show that in the diagonally dominant regime a unique open-loop equilibrium exists and coincides with a closed-loop equilibrium. Although an equilibrium without recourse in our setting is the same as an open-loop equilibrium, our equilibrium with recourse is more restrictive than their closed-loop equilibrium. In particular, their closed-loop equilibria need not be *perfect*, whereas our equilibrium with recourse is a Markov perfect equilibrium. Thus, we show that the equilibrium with recourse can be different from the equilibrium without recourse. More precisely, although the former equilibrium need not exist or be unique, the latter is an approximate equilibrium with recourse in the low influence regime.

There are a number of papers that study price competition over a single period. [12] shows that pure Nash equilibrium (NE) exists for a wide class of supermodular demand models. [7] provides sufficient conditions for uniqueness of equilibrium in the Bertrand game when the demands of the firms are nonlinear functions of the prices, there is a non-linear cost associated with satisfying a certain volume of demand and each firm seeks to maximize its expected profit. [14] identifies the conditions for existence and uniqueness of pure NE when the demands are characterized by a mixture of multinomial logit models and the cost of satisfying a certain volume of demand is linear in the demand volume. [8] considers price competition among multiple firms when the relationship between demand and price is characterized by the nested logit model and provides conditions to ensure the existence and uniqueness of the equilibrium. [13] proves the existence of pure strategy equilibrium in a price competition between two suppliers when capacity is private information.

Considering the papers on price competition over multiple time periods, [9] studies a stochastic game when there are strategic consumers choosing the time to purchase. [10] studies a competitive pricing problem when the relationship between demand and price is captured by the multinomial logit model and inventory levels are public information. [1] studies the pricing game between two firms with limited inventories facing stochastic demand. The authors characterize the unique subgame perfect Nash equilibrium. [11] shows the existence of a unique pure MPE in a pricing game between two firms offering vertically differentiated products.

2. Equilibrium without recourse

There are n firms indexed by $N = \{1, \dots, n\}$. Firm i has c_i units of initial inventory, which cannot be replenished over the selling horizon. There are τ time periods in the selling horizon indexed by $T = \{1, \dots, \tau\}$. We use p_i^t to denote the price charged by firm i at time period t. Using $\boldsymbol{p}^t = (p_1^t, \dots, p_n^t)$ to denote the prices charged by all of the firms at time period t, the demand faced by firm i at time period t is given by $D_i^t(\boldsymbol{p}^t) = \alpha_i^t - \beta_i^t p_i^t + \sum_{j\neq i}\gamma_{i,j}^t p_j^t$, where $\alpha_i^t > 0$, $\beta_i^t > 0$ and $\gamma_{i,j}^t > 0$. We assume that the price charged by each firm affects its demand more than the prices charged by the other firms, in the sense that $\sum_{j\neq i}\gamma_{i,j}^t < \beta_i^t$ for all $i \in N$, $t \in T$. Also, using $\boldsymbol{p}_{-i}^t = (p_1^t, \dots, p_{i-1}^t, p_{i+1}^t, \dots, p_n^t)$ to denote the prices charged by firms other than firm i at time period t, to avoid negative demand quantities, we restrict the strategy space of the firms such that each firm i charges the price p_i^t at time period t that satisfies $\alpha_i^t - \beta_i^t p_i^t + \sum_{j\neq i}\gamma_{i,j}^t p_j^t \geq 0$, given the prices $\boldsymbol{p}_{-i}^t$ charged by the other firms. If the firms other than firm i commit to the price trajectories $\boldsymbol{p}_{-i} = \{\boldsymbol{p}_{-i}^t : t \in T\}$, then we can obtain the best response of firm i by solving the problem

$$\max\Big\{ \sum_{t\in T}\Big(\alpha_i^t - \beta_i^t p_i^t + \sum_{j\neq i}\gamma_{i,j}^t p_j^t\Big) p_i^t : \sum_{t\in T}\Big(\alpha_i^t - \beta_i^t p_i^t + \sum_{j\neq i}\gamma_{i,j}^t p_j^t\Big) \leq c_i, \quad \alpha_i^t - \beta_i^t p_i^t + \sum_{j\neq i}\gamma_{i,j}^t p_j^t \geq 0 \;\; \forall\, t \in T, \;\; p_i^t \geq 0 \;\; \forall\, t\in T \Big\}. \tag{1}$$

Since $\beta_i^t > 0$, problem (1) has a strictly concave objective function and linear constraints, which implies that the best response of firm i is unique.

Using the non-negative dual multipliers v_i and $\{u_i^t : t \in T\}$ for the first and the second constraint in problem (1), the KKT conditions for this problem are

$$\begin{aligned}
&\Big(\sum_{t\in T}\Big(\alpha_i^t - \beta_i^t p_i^t + \sum_{j\neq i}\gamma_{i,j}^t p_j^t\Big) - c_i\Big) v_i = 0,\\
&\Big(\alpha_i^t - \beta_i^t p_i^t + \sum_{j\neq i}\gamma_{i,j}^t p_j^t\Big) u_i^t = 0 \;\; \forall\, t \in T,\\
&\alpha_i^t - 2\,\beta_i^t p_i^t + \sum_{j\neq i}\gamma_{i,j}^t p_j^t + \beta_i^t (v_i - u_i^t) = 0 \;\; \forall\, t \in T.
\end{aligned} \tag{2}$$

Since problem (1) has a concave objective function and linear constraints, the KKT conditions above are necessary and sufficient at optimality; see [3]. In other words, for a feasible solution $\{p_i^t : t \in T\}$ to problem (1), there exist corresponding non-negative dual multipliers v_i and $\{u_i^t : t \in T\}$ that satisfy the KKT conditions in (2) if and only if $\{p_i^t : t \in T\}$ is the optimal solution to problem (1). Note that we do not associate dual multipliers with the constraints $p_i^t \geq 0$ for all $t \in T$ in problem (1) since it is never optimal for firm i to charge a negative price. Therefore, we can actually view the constraints $p_i^t \geq 0$ for all $t \in T$ as redundant constraints. We use the KKT conditions in (2) extensively to characterize the best response of firm i to the price trajectories $\boldsymbol{p}_{-i}$ of the other firms. In the rest of this section, we exclusively focus on the *strategies without recourse*, where each firm i commits to a price trajectory $\{p_i^t : t \in T\}$ at the beginning of the selling horizon and does not adjust these prices during the course of the selling horizon. If the price trajectory $\{p_i^t : t \in T\}$ chosen by each firm i is the best response to the price trajectories $\boldsymbol{p}_{-i}$ chosen by the other firms, then we say that the price trajectories $\{\boldsymbol{p}^t : t \in T\}$ chosen by the firms is an *equilibrium without recourse*. We show that there exists a unique equilibrium without recourse. Furthermore, if we start with any price trajectory $\{\boldsymbol{p}^t : t \in T\}$ for the firms and successively compute the best response of each firm to the price trajectories of the other firms, then the best response of each firm forms a contraction mapping when viewed as a function of the prices charged by the other firms. Using this result, we show that there exists a unique equilibrium without recourse. To capture the best response of firm i to the prices charged by the other firms, we define for each $\nu \geq 0$, the set of time periods

$$\mathcal{T}_i(\nu, \boldsymbol{p}_{-i}) = \left\{ t \in T : \frac{\alpha_i^t + \sum_{j \neq i} \gamma_{i,j}^t p_j^t}{\beta_i^t} > \nu \right\}.$$

In the next lemma, we use $\mathcal{T}_i(\cdot, \boldsymbol{p}_{-i})$ to give a succinct characterization of the solution $\{p_i^t : t \in T\}$ and the corresponding dual multipliers υ_i and $\{u_i^t : t \in T\}$ that satisfy the KKT conditions.

Lemma 1. *If a feasible solution $\{p_i^t : t \in T\}$ to problem (1) and the corresponding non-negative dual multipliers υ_i and $\{u_i^t : t \in T\}$ satisfy the KKT conditions in (2), then we have*

$$p_i^t = \begin{cases} \left(\alpha_i^t + \sum_{j \neq i} \gamma_{i,j}^t p_j^t\right) / (2\,\beta_i^t) + \upsilon_i/2 & \text{if } t \in \mathcal{T}_i(\upsilon_i, \boldsymbol{p}_{-i}) \\ \left(\alpha_i^t + \sum_{j \neq i} \gamma_{i,j}^t p_j^t\right) / \beta_i^t & \text{if } t \notin \mathcal{T}_i(\upsilon_i, \boldsymbol{p}_{-i}), \end{cases}$$

$$u_i^t = \begin{cases} 0 & \text{if } t \in \mathcal{T}_i(\upsilon_i, \boldsymbol{p}_{-i}) \\ \upsilon_i - \left(\alpha_i^t + \sum_{j \neq i} \gamma_{i,j}^t p_j^t\right) / \beta_i^t & \text{if } t \notin \mathcal{T}_i(\upsilon_i, \boldsymbol{p}_{-i}). \end{cases}$$

Proofs of all lemmas are in the e-companion. By Lemma 1, we can characterize the solution $\{p_i^t : t \in T\}$ and the dual multipliers υ_i and $\{u_i^t : t \in T\}$ that satisfy the KKT conditions in (2) only by using the value of υ_i. If we know the value of υ_i, then we can compute the set of time periods $\mathcal{T}_i(\upsilon_i, \boldsymbol{p}_{-i})$, in which case, we can choose the values of $\{p_i^t : t \in T\}$ and $\{u_i^t : t \in \mathcal{T}_i\}$ as given in Lemma 1. Throughout the rest of this section, we indeed choose the values of $\{p_i^t : t \in T\}$ and $\{u_i^t : t \in \mathcal{T}_i\}$ as given in Lemma 1, since we are interested in solutions that satisfy the KKT conditions. Naturally, we do not know the value of υ_i that allows us to obtain an optimal solution $\{p_i^t : t \in T\}$ to problem (1). In the next lemma, we give a characterization of the value of υ_i that corresponds to the solution $\{p_i^t : t \in T\}$ and the dual multipliers υ_i and $\{u_i^t : t \in T\}$ satisfying the KKT conditions in (2). In particular, we consider the function

$$G_i(\nu, \boldsymbol{p}_{-i}) = \begin{cases} \sum_{t \in \mathcal{T}_i(\nu, \boldsymbol{p}_{-i})} \left(\alpha_i^t - \beta_i^t \nu + \sum_{j \neq i} \gamma_{i,j}^t p_j^t\right) - 2c_i & \text{if } \nu > 0 \\ \left[\sum_{t \in \mathcal{T}_i(\nu, \boldsymbol{p}_{-i})} \left(\alpha_i^t + \sum_{j \neq i} \gamma_{i,j}^t p_j^t\right) - 2\,c_i\right]^+ & \text{if } \nu = 0. \end{cases}$$

Lemma 5 in the e-companion shows that $G_i(\cdot, \boldsymbol{p}_{-i})$ is strictly decreasing over some $[0, \nu^*]$ and has a unique root. In the next lemma, we use its root to characterize a solution to the KKT conditions.

Lemma 2. *If a feasible solution $\{p_i^t : t \in T\}$ to problem (1) and the corresponding non-negative dual multipliers υ_i and $\{u_i^t : t \in T\}$ satisfy the KKT conditions in (2), then we have $G_i(\upsilon_i, \boldsymbol{p}_{-i}) = 0$.*

By Lemma 2, if a feasible solution $\{p_i^t : t \in T\}$ to problem (1) and the corresponding non-negative dual multipliers υ_i and $\{u_i^t : t \in T\}$ satisfy the KKT conditions in (2), then υ_i must be the unique root of $G_i(\cdot, \boldsymbol{p}_{-i})$. Also, by Lemma 1, the values of $\{p_i^t : t \in T\}$ and $\{u_i^t : t \in T\}$ must be given as in Lemma 1. In the next theorem, we use these results to show that the best response of firm i is a contraction mapping when viewed as a function of the prices of the other firms.

Theorem 1. *Let $\{p_i^t(\boldsymbol{p}_{-i}) : t \in T\}$ be the optimal solution to problem (1) as a function of the prices charged by the firms other than firm i. For any two price trajectories $\hat{\boldsymbol{p}}_{-i} = \{\hat{\boldsymbol{p}}_{-i}^t : t \in T\}$ and $\tilde{\boldsymbol{p}}_{-i} = \{\tilde{\boldsymbol{p}}_{-i}^t : t \in T\}$ adopted by the firms other than firm i, we have*

$$|p_i^t(\hat{\boldsymbol{p}}_{-i}) - p_i^t(\tilde{\boldsymbol{p}}_{-i})| \leq \max_{t \in T} \left\{ \frac{\sum_{j \neq i} \gamma_{i,j}^t |\hat{p}_j^t - \tilde{p}_j^t|}{\beta_i^t} \right\}.$$

Proof outline. Use M_i to denote the right hand side of the inequality in the theorem. Let $\hat{p}_i^t = p_i^t(\hat{\boldsymbol{p}}_{-i})$ and $\tilde{p}_i^t = p_i^t(\tilde{\boldsymbol{p}}_{-i})$. Let $\hat{\upsilon}_i$ and $\tilde{\upsilon}_i$ be such that $G_i(\hat{\upsilon}_i, \hat{\boldsymbol{p}}_{-i}) = 0$ and $G_i(\tilde{\upsilon}_i, \tilde{\boldsymbol{p}}_{-i}) = 0$. Without loss of generality we assume $\hat{\upsilon}_i \geq \tilde{\upsilon}_i$. Otherwise, we interchange their roles. In the proof, we show that $|\hat{p}_i^t - \tilde{p}_i^t| \leq \frac{1}{2} M_i + \frac{1}{2} \max\{M_i, \hat{\upsilon}_i - \tilde{\upsilon}_i\}$, by considering four cases on whether t is in $\mathcal{T}_i(\hat{\upsilon}_i, \hat{\boldsymbol{p}}_{-i})$ and $\mathcal{T}_i(\tilde{\upsilon}_i, \tilde{\boldsymbol{p}}_{-i})$. In each case, we use Lemma 1 to get expressions for $\hat{p}_i^t$ and $\tilde{p}_i^t$. Once we have $|\hat{p}_i^t - \tilde{p}_i^t| \leq \frac{1}{2} M_i + \frac{1}{2} \max\{M_i, \hat{\upsilon}_i - \tilde{\upsilon}_i\}$, we only need to show that $\hat{\upsilon}_i - \tilde{\upsilon}_i \leq M_i$. We use Lemma 2 and the definition of $\mathcal{T}_i(\cdot, \cdot)$ to show that $G_i(\hat{\upsilon}_i - M_i, \tilde{\boldsymbol{p}}_{-i}) \geq 0$. Then $\hat{\upsilon}_i - M_i \leq \tilde{\upsilon}_i$ follows from simple monotonicity properties of $G(\cdot, \tilde{\boldsymbol{p}}_{-i})$ given in Lemma 5. The details are in the e-companion. □

For the vector $\boldsymbol{y} = \{y^t : t \in T\}$, define the norm on $\Re^\tau$ as $\|\boldsymbol{y}\|_\infty = \max_{t \in T} |y^t|$. Since $\sum_{j \neq i} \gamma_{i,j}^t < \beta_i^t$ for all $i \in N$ and $t \in T$, Theorem 1 implies that firm i's best response is a contraction under $\|\cdot\|_\infty$, when viewed as a function of the other firms' prices. Therefore, it immediately follows that if the price charged by each firm affects its demand more than the prices charged by the other firms, then there always exists a unique equilibrium without recourse.

The contraction mapping also presents an efficient computation scheme. Let $M = \max_{i \in N, t \in T} \sum_{j \neq i} \gamma_{i,j}^t / \beta_i^t$ and note that since $\sum_{j \neq i} \gamma_{i,j}^t < \beta_i^t$ for all $i \in N, t \in T$, we have $M < 1$. Performing best-response iterations converges linearly to the unique equilibrium at rate M (see [15, Theorem 6.3.3]). In each iteration, one must solve the problems (1) for each $i \in N$. Using Lemma 5, each of these n problems can be solved by bisection on υ_i, as we can show that $\upsilon_i \geq 0$ must lie in a bounded interval. To see this, recall that the firms' prices must satisfy $\alpha_i^t - \beta_i^t p_i^t + \sum_{j \neq i} \gamma_{i,j}^t p_j^t \geq 0$ for all $i \in N$. Rearranging, we get $p_i^t \leq \alpha_i^t / \beta_i^t + \sum_{j \neq i} \gamma_{i,j}^t p_j^t / \beta_i^t \leq \max_{i \in N} \{\alpha_i^t / \beta_i^t\} + M \max_{j \in N} \{p_j^t\}$, which implies that $\max_{i \in N} \{p_i^t\} \leq \max_{i \in N} \{\alpha_i^t / \beta_i^t\} / (1 - M) = P_{\max}$. Then using the definition of $\mathcal{T}(\nu, \boldsymbol{p}_{-i})$, we obtain that $\mathcal{T}(\nu, \boldsymbol{p}_{-i})$ is empty if $\nu > P_{\max}$, which implies that $\upsilon_i \leq P_{\max}$.

3. Equilibrium with recourse

In this section, we consider *strategies with recourse*, where each firm can change its price at each time period based on its inventory and the inventories of the other firms. In other words, the firms do not commit to a price trajectory at the beginning of the selling horizon. We let x_i^t be the inventory of firm i at the beginning of time period t. Focusing on Markovian strategies without loss of generality, as a function of the inventories $\boldsymbol{x}^t = (x_1^t, \ldots, x_n^t)$ of all of the firms, we use $P_i^t(\boldsymbol{x}^t)$ to denote the price charged by firm i at time period t. It is useful to view $P_i^t(\cdot)$ as a function that determines the strategy of firm i at time period t as a function of the inventories of all of the firms. We use $\boldsymbol{P}^t = (P_1^t(\cdot), \ldots, P_n^t(\cdot))$ to capture the strategies of all of the firms at time period t and $\boldsymbol{P}_{-i}^t = (P_1^t(\cdot), \ldots, P_{i-1}^t(\cdot), P_{i+1}^t(\cdot), \ldots, P_n^t(\cdot))$ to capture the strategies of the firms other than firm i at time period t. If the firms other than firm

i use the strategies $\boldsymbol{P}_{-i} = \{\boldsymbol{P}^t_{-i} : t \in T\}$, then we can find the best response strategy of firm i by solving the dynamic program

$$V_i^t(\boldsymbol{x}^t) = \max\Bigg\{ \Big(\alpha_i^t - \beta_i^t p_i^t + \sum_{j\neq i} \gamma_{i,j}^t P_j^t(\boldsymbol{x}^t)\Big) p_i^t + V_i^{t+1}(\boldsymbol{x}^{t+1}) :$$
$$\alpha_i^t - \beta_i^t p_i^t + \sum_{j\neq i} \gamma_{i,j}^t P_j^t(\boldsymbol{x}^t) \geq 0,$$
$$x_i^{t+1} = \Big[x_i^t - (\alpha_i^t - \beta_i^t p_i^t + \sum_{j\neq i} \gamma_{i,j}^t P_j^t(\boldsymbol{x}^t))\Big]^+,$$
$$x_\ell^{t+1} = \Big[x_\ell^t - \big(\alpha_\ell^t - \beta_\ell^t P_\ell^t(\boldsymbol{x}^t) + \gamma_{\ell,i}^t p_i^t + \sum_{j\notin\{i,\ell\}} \gamma_{\ell,j}^t P_j^t(\boldsymbol{x}^t)\big)\Big]^+ \quad \forall \ell \in N \setminus \{i\},$$
$$p_i^t \geq 0,\ x_\ell^{t+1} \geq 0\ \forall\, \ell \in N \Bigg\},$$

with the boundary condition that $V_i^{\tau+1}(\cdot) = 0$. An optimal solution to the problem above characterizes the best response strategy of firm i at time period t.

For the strategies $\{\boldsymbol{P}^t : t \in T\}$ to form an *equilibrium with recourse*, we require that for each $t \in T$, all inventories $\boldsymbol{x}^t$, and each i, the strategy $\{P_i^s(\cdot) : s = t, \ldots, \tau\}$ chosen by firm i in the periods subsequent to time t is a best response against other firms' strategies $\{P_{-i}^s(\cdot) : s = t, \ldots, \tau\}$ in the subsequent time periods. In other words, we require the strategies $\{\boldsymbol{P}^t : t \in T\}$ to form a Markov perfect equilibrium [5]. In the previous section, we show that there always exists a unique equilibrium when we focus on strategies without recourse. We give two numerical examples to show that if we focus on strategies with recourse, then there may not exist an equilibrium or there may be multiple equilibria. Consider the case where there are two firms and the selling horizon has two time periods. For given inventories of the two firms at the second time period, the problem of computing the equilibrium strategy at the second time period is identical to finding an equilibrium without recourse. So, there exists a unique equilibrium strategy for the firms at the second time period for given inventories. Note that the prices charged by the firms in an equilibrium without recourse at the second time period depend on the inventories of the firms at the second time period, which, in turn, depend on the prices charged by the firms at the first time period. To obtain an equilibrium with recourse, we compute the best response strategy of each firm at the first time period as a function of the price of the other firm at the first time period. Recall that if we fix the prices of the firms at the first time period, then we fix the inventories at the second time period, in which case, we can compute the equilibrium strategies at the second time period. We plot the best response of each firm at the first time period as a function of the price of the other firm. An equilibrium with recourse corresponds to the intersection of the two best response curves.

Consider the parameters $\alpha_i^t = 4$, $\beta_i^1 = 4$, $\beta_i^2 = 2$, $\gamma_{i,j}^1 = 16/5$, $\gamma_{i,j}^2 = 1$, $c_i = 3$ for all $i \in \{1, 2\}$, $j \neq i$ and $t \in \{1, 2\}$, which satisfy $\sum_{j\neq i} \gamma_{i,t}^t < \beta_i^t$ for all $i, t \in \{1, 2\}$, so that we know that there exists a unique equilibrium without recourse. In Fig. 1, the solid line plots the best response of second firm at the first time period on the vertical axis, as a function of the price of the first firm on the horizontal axis, whereas the dashed line plots the best response of the first firm at the first time period on the horizontal axis as a function of the price of the second firm on the vertical axis. The two best response functions do not intersect. Therefore, an equilibrium with recourse does not exist. The main driver of the lack of equilibrium is the discontinuity in the best response function, which arises because the revenue of each firm is a multi-modal function of its first time period price. In Fig. 2, we show first firm's revenue as a function of its price at the first time period, when the second firm's price is fixed at 2.2. So, firm 1 can jump from one mode to another based on the price of the second firm. Considering the parameters $\alpha_i^t = 4$, $\beta_i^1 = 5$, $\beta_i^2 = 2$, $\gamma_{i,j}^1 = 0.1$, $\gamma_{i,j}^2 = 1$ and $c_i = 5$ for all $j \neq i$ and $i, t \in \{1, 2\}$, Fig. 3 shows the best response of each firm at the first time period as a function of the other firm's price. The best response functions intersect at two points, indicating multiple equilibria with recourse.

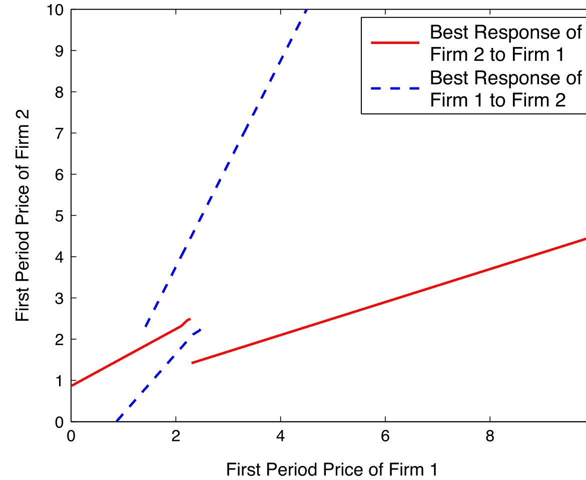

Fig. 1. Best responses when equilibrium does not exist.

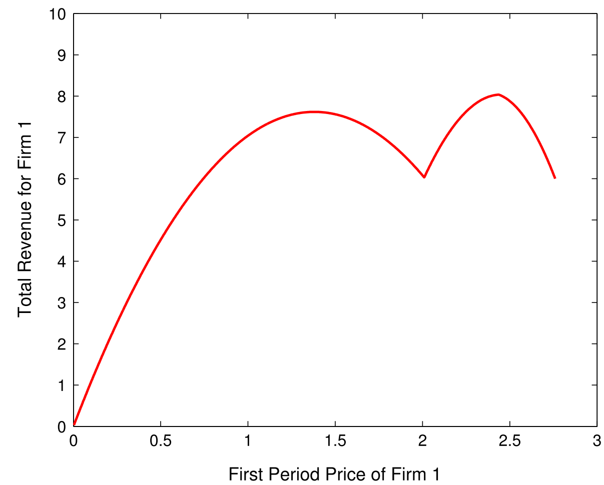

Fig. 2. Revenue of the first firm as a function of its first period price.

4. An approximate equilibrium

If for each firm i, any deviation from the strategy $\{P_i^t(\cdot) : t \in T\}$ cannot increase the revenue of firm i by more than ϵ given that the other firms use the strategies $\boldsymbol{P}_{-i}$, then we say that the price strategies $\{\boldsymbol{P}^t : t \in T\}$ chosen by the firms is an ϵ*- equilibrium with recourse*. Since there may not exist an equilibrium with recourse or there may be multiple equilibria with recourse, we focus on ϵ-equilibria with recourse. We consider a low influence regime, where, roughly speaking, the price charged by a firm affects its demand more than the prices charged by each of the other firms. In particular, we consider the regime where the price charged by

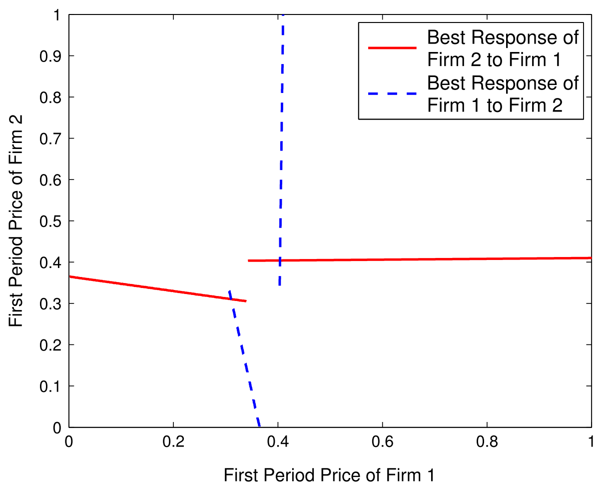

Fig. 3. Best responses when there are multiple equilibria.

a firm affects its demand so much more than the prices charged by each of the other firms such that we have $\gamma_{i,j}^t/\beta_i^t < (1/M) - 1$, where M is as defined at the end of Section 2. When $\sum_{j\neq i}\gamma_{i,j}^t < \beta_i^t$ and the number of firms is large, we expect this assumption to hold. For example, if we have a symmetric setting, where the parameters related to each firm are the same, then under the assumption that $\sum_{j\neq i}\gamma_{i,j}^t < \beta_i^t$, we have $\gamma_{i,j}^t/\beta_i^t < 1/(n-1)$, in which case, the low influence regime naturally holds as the number of firms gets large. In the low influence regime, we show that the equilibrium without recourse studied in the previous section can be used to construct an ϵ-equilibrium with recourse. Intuitively, this result uses the fact that if $\gamma_{i,j}^t/\beta_i^t$ is small, then any deviation of a firm from a given price trajectory has little influence on the prices of the other firms in the subsequent time periods. In the next lemma, we formalize this idea. Throughout the rest of this section, we use $\mu = \max_{i\in N,\, j\in N\setminus\{i\},\, t\in T}\gamma_{i,j}^t/\beta_i^t$ and $\bar\beta = \max_{i\in N,\, t\in T}\beta_i^t/\min_{i\in N,\, t\in T}\beta_i^t$. Note that the low influence regime is defined as the setting where $\mu < (1/M) - 1$.

Lemma 3. *Fixing the prices $\hat{\boldsymbol p}^1$ charged by the firms at the first time period, let the prices $\{\hat{\boldsymbol p}^t : t \in T\setminus\{1\}\}$ form the equilibrium without recourse in the remaining portion of the selling horizon. Define the prices $\tilde{\boldsymbol p}^1$ at the first time period as $\tilde p_i^1 = \hat p_i^1 + \delta$ and $\tilde p_j^1 = \hat p_j$ for all $j \in N\setminus\{i\}$ for some $\delta \ge 0$. Fixing the prices $\tilde{\boldsymbol p}^1$ charged by the firms at the first time period, let the prices $\{\tilde{\boldsymbol p}^t : t\in T\setminus\{1\}\}$ form the equilibrium without recourse in the remaining portion of the selling horizon. If we have $\mu < (1/M) - 1$, then $\max_{j\neq i,\, t\in T\setminus\{1\}}|\hat p_j^t - \tilde p_j^t| \le \frac{2\,\mu\,\bar\beta\,\delta}{1-M-M\mu}$.*

Consider the problem over the time periods $\kappa,\ldots,\tau$ when the inventories of the firms at time period κ are given by $\boldsymbol x = (x_1,\ldots,x_n)$. We use $p_i^{N,t}(\kappa,\boldsymbol x)$ to denote the price charged by firm i at time period t in the equilibrium without recourse. We consider the following strategy with recourse for firm i. If the inventories of the firms at time period t are given by $\boldsymbol x$, then firm i charges the price $p_i^{N,t}(t,\boldsymbol x)$. In other words, letting $P_i^{R,t}(\cdot)$ be the strategy function of firm i under this strategy with recourse, we have $P_i^{R,t}(\boldsymbol x) = p_i^{N,t}(t,\boldsymbol x)$. Using $\boldsymbol P^{R,t} = (P_1^{R,t}(\cdot),\ldots,P_n^{R,t}(\cdot))$ to capture the strategies of all of the firms at time period t and $\boldsymbol c = (c_1,\ldots,c_n)$ to denote the inventories of the firms at the first time period, note that if all firms use the strategies $\{\boldsymbol P^{R,t} : t\in T\}$ over the selling horizon, then the price charged by each firm i at each time period t is given by $p_i^{N,t}(1,\boldsymbol c)$, which is precisely the prices corresponding to the equilibrium without recourse when we consider the problem over the time periods T with the inventories of the firms at the first time period given by $\boldsymbol c$. However, if one of the firms deviates from the strategies $\{\boldsymbol P^{R,t} : t\in T\}$ at a time period, then the prices charged by the firms will be different from those in the equilibrium without recourse. Therefore, it is not generally true that the strategies $\{\boldsymbol P^{R,t} : t\in T\}$ correspond to an equilibrium with recourse. In the remainder of this section, we show that the strategies $\{\boldsymbol P^{R,t} : t\in T\}$ correspond to an ϵ-equilibrium with recourse in the low influence regime. In the next lemma, we show that if firm i unilaterally deviates from the strategy $\{P_i^{R,t}(\cdot) : t\in T\}$, but the other firms use the strategies $\{\boldsymbol P^{R,t} : t\in T\}$, then firm i does not increase its revenue by more than a simple function of μ.

Lemma 4. *Assume that the strategies of all of the firms are $\{\boldsymbol P^{R,t} : t\in T\}$. Let Π_i^N be the revenue of firm i under these strategies. Also, assume that the strategies of the firms other than firm i are $\{\boldsymbol P_{-i}^{R,t} : t\in T\}$, but firm i deviates to charge an arbitrary price at the first time period and uses the strategy $\{P_i^{R,t}(\cdot) : t\in T\setminus\{1\}\}$ at the other time periods. Let Π_i^D be the revenue of firm i under this strategy. Letting $\beta_{\max} = \max_{i\in N,t\in T}\beta_i^t$, we have for $\mu < (1/M) - 1$,*

$$\Pi_i^D - \Pi_i^N \le \frac{2\,\bar\beta\,M\beta_{\max}\,P_{\max}^2\,(\tau-1)\,\mu}{1-M-M\mu}.$$

In the next theorem, we show that the strategy $\{\boldsymbol P^{R,t} : t\in T\}$ is an ϵ-equilibrium with recourse, when the number of firms is large so that μ is small.

Theorem 2. *Assume that the strategies of all of the firms are $\{\boldsymbol P^{R,t} : t\in T\}$. Let Π_i^N be the revenue of firm i under these strategies. Also, assume that the strategy of the firms other than firm i are $\{\boldsymbol P_{-i}^{R,t} : t\in T\}$, but firm i uses an arbitrary strategy over the whole selling horizon. Let Π_i^A be the revenue of firm i under these strategies. Letting $\Gamma_\mu = \bar\beta\,M\beta_{\max}\,P_{\max}^2/(1-M-M\mu)$, we have, for $\mu < (1/M) - 1$,*

$$\Pi_i^A - \Pi_i^N \le \Gamma_\mu\,\tau\,(\tau-1)\,\mu.$$

Thus, $\{\boldsymbol P^{R,t} : t\in T\}$ is an ϵ-equilibrium with recourse, with $\epsilon = \Gamma_\mu\,\tau\,(\tau-1)\,\mu$.

Proof outline. We use induction to prove the result. The result trivially holds for $\tau = 1$, as there is no difference between equilibrium with and without recourse for $\tau = 1$. Assume the result is true for $\tau = k$. Let all firms other than i use the strategy $\{\boldsymbol P_{-i}^{R,t} : t\in T\}$. We use $\{Q_i^t : t\in T\}$ to denote the arbitrary strategy of firm i. Let Π_i^N and Π_i^A be firm i's revenue when it uses strategy $\{P_i^{R,t} : t\in T\}$ and $\{Q_i^t : t\in T\}$, respectively. Let Π_i^D be firm i's revenue when it uses strategy $\{P_i^{R,1}, Q_i^t : t\in T/\{1\}\}$. We use Lemma 4 to bound the difference between Π_i^D and Π_i^A. Similarly, we use the induction hypothesis at $\tau = k$ to bound the difference between Π_i^D and Π_i^N. Summing up the two bounds gives us the result for $\tau = k+1$. The details are in the e-companion. □

We observe that as μ approaches zero, $\Gamma_\mu\,\tau\,(\tau-1)\,\mu$ approaches zero as well. Therefore, by the theorem above, if we are in the low influence regime, then no firm can improve its revenue significantly by deviating from the policy $\{\boldsymbol P^{R,t} : t\in T\}$, which implies that $\{\boldsymbol P^{R,t} : t\in T\}$ is an ϵ-equilibrium with recourse. As discussed earlier, the price trajectory realized under the strategy $\{\boldsymbol P^{R,t} : t\in T\}$ is same as that in the unique equilibrium without recourse.

5. Future research

A natural research direction is to extend our contraction properties to more general demand models. Also, it would be useful to define an analogue of equilibrium without recourse under stochastic demand and check whether it uniquely exists.

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Appendix A. Supplementary data

Supplementary material related to this article can be found online at http://dx.doi.org/10.1016/j.orl.2017.05.005.

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