Dynamics of Coupled Microresonator-Based Degenerate Optical Parametric Oscillators

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Abstract: We theoretically study a network of microresonator-based $\chi^{(3)}$ degenerate optical parametric oscillators (DOPO's). We investigate the influence of coupling on the global oscillation condition and show that the system can emulate the Ising model.
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Combinatorial optimization problems are ubiquitous in modern science and are of substantial interest in subfields such as computer science and biology [1]. Many such problems and the corresponding decision problems are classified as NP-hard and NP-complete, respectively, and efficient classical algorithm for solving them are not known to exist. A well-known instance of NP-hard problems is the classical Ising problem [1,2], in which the objective is to find the optimal spin configuration $\{\sigma_j\}$ (j=1,2,…,N) of a spin glass that minimizes the corresponding Ising Hamiltonian $H = -\sum_{j\neq k} J_{jk}\sigma_j\sigma_k$, where σ_j is the magnetic spin of the jth element of the spin glass (with one of two possible equiprobable values, +1 or -1), and J_{jk} is the coupling coefficient between the jth and kth spin elements. Recently, a network of coupled degenerate optical parametric oscillators (DOPO's) has been conceived as a coherent Ising machine, that is, a physical system that could efficiently "calculate" the ground state of the Ising model [2-5]. These demonstrations utilized a time-domain multiplexing scheme based on a $\chi^{(2)}$ crystal contained within a long fiber cavity in which the binary phase state of the degenerate signal represents the "spin". Alternatively, realizing a $\chi^{(3)}$ DOPO system in a chip-based platform with microresonators [6,7] offers a platform where such a network could be created in a more compact, scalable device [see Fig. 1(a) for 2 coupled-DOPO configuration]. However, to date, the dynamics of such coupled microresonator-based DOPO's have not been explored.

Here, we theoretically investigate a system of coupled microresonator-based DOPO's. We show that the N-microresonator Lugiato-Lefever model [8] can be reduced to a simpler and computationally efficient set of equations. Via a straightforward analysis, we show that coupling reduces the oscillation threshold of DOPO's, in comparison to a single isolated DOPO. We also numerically solve the NP-hard MAX-CUT problem [3-5] for two instances of undirected graphs with 100 nodes and reveal the dependence of the success probability on several system parameters.

The starting point of our study is a set of N coupled normalized Lugiato Lefever equations [8],

$$\frac{\partial E_j}{\partial t} = \left(-1 - i\Delta + i\left|E_j\right|^2 - id_2\frac{\partial^2}{\partial \tau^2}\right)E_j + S\left(e^{-i\Omega\tau} + e^{i\Omega\tau}\right) + \sum_{k\neq j}\kappa_{jk}E_k, \quad (1)$$

where t governs the slow-time evolution while τ is a fast-time variable, Δ and S are the detuning parameter and pump strength, respectively, d_2 is the sign of GVD (+1, normal in this case), and Ω is the spectral detuning of each pump from the degeneracy point [see Fig. 1(a)]. The last summation term in Eq. (1) describes linear coupling between the DOPO's, where κ_{jk} is a complex coupling coefficient from kth to jth DOPO, with $|\kappa_{jk}|$ and its argument describing the coupling strength and phase, respectively. The final phase of the signal component of each DOPO encodes the solution. For this system the detuning Δ is the natural control parameter, unlike the previously studied systems where the pump power was ramped up directly [2-5]. The cavity enhancement is a function of the detuning and, as shown later, both the pump power and the detuning determine the system dynamics. Direct computation of the full coupled system [Eq. (1)] is challenging, especially as the system size scales to a larger number of DOPO's. We can reduce the computational complexity by assuming for this application that for each DOPO only Fourier components of interest are the two pump fields and the degenerate signal field, which is spectrally centered between the pumps. Thus, we introduce the total field $E_j(t,\tau) = E_{S,j}(t) + E_{+P,j}(t)e^{-i\Omega\tau} + E_{-P,j}(t)e^{i\Omega\tau}$, with $E_{S,j}$, $E_{+P,j}$ and $E_{-P,j}$ describing the amplitudes of the signal, pump 1 and pump 2, respectively. After substituting this field into Eq. (1), we obtain

$$\frac{dE_{+P,j}}{dt} = \left[-1 - i\Delta + id_2\Omega^2 + i\left(\left|E_{+P,j}\right|^2 + 2\left|E_{-P,j}\right|^2 + 2\left|E_{S,j}\right|^2\right)\right]E_{+P,j} + iE_{S,j}^{\;2}E_{-P,j}^{\;*} + S \quad (2.1)$$

$$\frac{dE_{-P,j}}{dt} = \left[-1 - i\Delta + id_2\Omega^2 + i\left(2\left|E_{+P,j}\right|^2 + \left|E_{-P,j}\right|^2 + 2\left|E_{S,j}\right|^2\right)\right]E_{-P,j} + iE_{S,j}^{\;2}E_{+P,j}^{\;*} + S \quad (2.2)$$

$$\frac{dE_{S,j}}{dt} = \left[-1 - i\Delta + i\left(2\left|E_{+P,j}\right|^2 + 2\left|E_{-P,j}\right|^2 + \left|E_{S,j}\right|^2\right)\right]E_{S,j} + 2iE_{+P,j}E_{-P,j}E_{S,j}^{\;*} + \sum_{k\neq j}\kappa_{jk}E_{S,k}. \quad (2.3)$$

where we assume that the pumps are not coupled as pump coupling was shown to deteriorate the system performance.

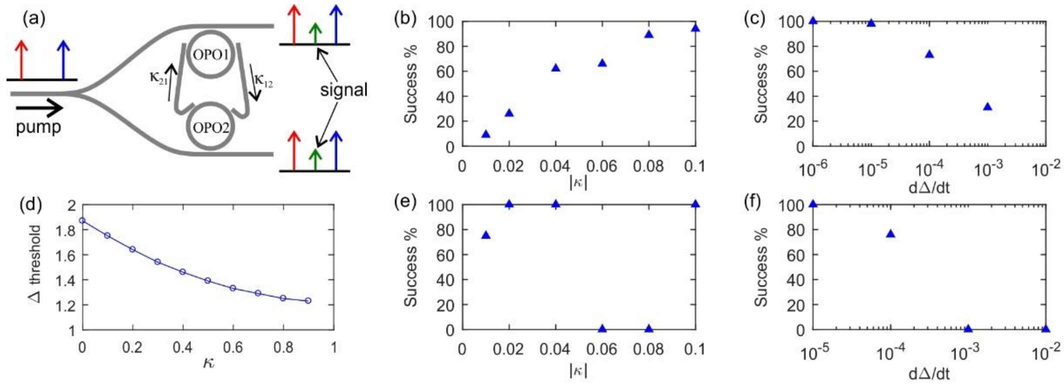

Fig. 1. (a) shows a schematic of two coupled DOPO system while (d) shows dependence of the oscillation threshold detuning on the coupling strength for this system. Circles denote simulation and solid curves analytical results. (b) and (c) show the dependence of the ground state probability on the coupling strength (for fixed detuning ramp rate at 0.0001 per unit *t*) and the detuning ramp rate (for fixed coupling strength at 0.05), respectively, for the Mobius-ladder configuration. (e) and (f) show similar plots for the random graph problem, except for the fact that the fixed coupling strength of 0.01 is used for simulation of (f).

First, we analyze the influence of coupling on the DOPO oscillation condition by considering two coupled DOPO's, as shown in Fig. 1(a). We set $|S|^2 = 5$ and focus on the case of symmetric out-of-phase coupling ($\kappa_{12} = \kappa_{21} = -|\kappa|$). Assuming the two DOPO's are identical and the signal components are initially negligible, the steady-state fields of the two pumps can be written as $E_{+P,1} = E_{-P,1} = S/[1 + i(\Delta - 3P - d_2\Omega^2)]$, where $P = |E_{+P,1}|^2 = |E_{-P,1}|^2$ is the solution of the cubic polynomial $9P^3 - 6(\Delta - d_2\Omega^2)P^2 + [\Delta^2 + (d_2\Omega^2)^2 - 2d_2\Delta\Omega^2 + 1]P - |S|^2 = 0$. Substituting these pump fields into Eq. (2.3) and solving the resulting evolution equation, we obtain the eigenvalues

$$\mu = -1 \pm \frac{1}{1-|\kappa|}\sqrt{8\Delta P - \Delta^2 - 12P^2}\,, \tag{3}$$

where the oscillation threshold condition is $\mu = 0$. The eigenvalues in the limit $\kappa \to 0$ correspond to the uncoupled case. Equation (3) predicts that the upper limit for the allowed coupling strength is $|\kappa| = 1$. This can be interpreted as the case where the coupling strength equals the cavity roundtrip loss. Simulations of Eq. (2) confirm that in this limit, the system oscillates and never stabilizes. The threshold dependence on the coupling strength is plotted in Fig. 1(d).

We investigate the ability of this system to solve MAX-CUT problem of two unweighted undirected graphs with 100 nodes (N = 100). The first instance is a highly symmetric cubic graph known as Mobius-ladder configuration [4,5], while the second is a random graph with 495 edges (10% edge density), generated using Python NetworkX module. In all runs, the detuning is initially set below the threshold level and tuned at a steady rate above the system oscillation. We investigate the dependence of the system on the coupling strength $|\kappa|$ for a given detuning ramp rate, and the detuning ramp rate dΔ/dt for a given coupling strength. We perform 100 numerical trials for each setting of parameters to estimate the success probability of the optimum solution search. While the optimum solution of the former problem is easily deduced, the solution of the latter problem is not known and thus, we assume that the lowest Hamiltonian value found in our search is the ground state. The results are summarized in Fig. 1. In the case of highly-symmetric Mobius-ladder graph, as shown in Figs. 1(b) and 1(c), a larger coupling strength and a slower detuning ramp increase the probability of finding the ground state. Similar trends are observed for the random graph in Fig. 1(e) and (f), however Fig. 1(e) indicates that there are values of coupling strength that are unfavorable for solving this specific problem. This is likely due to high density of coupling among the DOPO's, where one DOPO is coupled to as many as 18 other DOPO's. We speculate that the individual coupling strength must be reduced as the problem size and, more importantly, the coupling density increase. This could be a fundamental property of such a coupled DOPO system and will be a subject of future investigation.

In conclusion, our study extends the dynamical model of DOPO network by introducing the cavity detuning parameter and is expected to be generally applicable to other similar systems.

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