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KdV equation beyond standard assumptions on initial data

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HIGHLIGHTS

- We present a new approach to inverse scattering transform based upon Hankel operators.
- For KdV only decay of initial data at plus infinity is required.
- We establish well-posedness for data with no decay at minus infinity.
- We derive explicit formula for such solution in terms of the I+ Hankel determinant.

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We dedicate this paper to the memory of Ludwig Faddeev, one of the founders of soliton theory

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ABSTRACT

We show that the Cauchy problem for the KdV equation can be solved by the inverse scattering transform (IST) for any initial data bounded from below, decaying sufficiently rapidly at $+\infty$, but unrestricted otherwise. Thus our approach does not require any boundary condition at $-\infty$.

1. Introduction

This note is motivated in part by the open question posed by Vladimir Zakharov in his plenary talk in July of 2016 at the XXXV Workshop on Geometric Methods in Physics in Bialowieza, Poland. Zakharov stated the problem of understanding the *KdV equation*

$$\begin{cases} \partial_t u - 6u\partial_x u + \partial_x^3 u = 0 \\ u(x,0) = q(x) \end{cases} \tag{1.1}$$

with generic bounded but not decaying initial data q. He specifically pointed out that (1.1) no longer has (finite) conservation laws while existence of infinitely many such laws is one of the main features of any completely integrable system. A similar question was stated in his 2016 paper [1]: 'Suppose that the initial condition of (1.1) is a bounded function, which is neither rapidly vanishing nor periodic. What is its behavior under time evolution?'

To show how nontrivial this problem is let us put it in the historic context. For smooth rapidly decaying q's (1.1) was solved in closed form in the short 1967 paper [2] by Gardner–Greene–Kruskal–Miura (GGKM) . As is well-known, the paper [2] introduces what we now call the *inverse scattering transform* (IST), one of the major discoveries in the twentieth century mathematics. This paper was immediately followed by [3] where the famous *Lax pair* first appeared. More specifically, associate with $u(x,t)$ in (1.1) two linear operators, called the Lax pair,

$$L(t) = -\partial_x^2 + u(x,t) \text{ (the } \textit{Schrödinger operator}\text{)}, \tag{1.2}$$

$$P(t) = -4\partial_x^3 + 6u(x,t)\,\partial_x + 3\,(\partial_x u(x,t))\,.$$

The main observation made in [3] is that the KdV equation can be represented as

$$\partial_t L(t) = [P(t), L(t)]\,, \text{ (the } \textit{Lax representation}\text{)} \tag{1.3}$$

which immediately implies that if u solves (1.1) then $L(t)$ is unitary equivalent to $L(0) =: \mathbb{L}_q = -\partial_x^2 + q(x)$. The latter means that the spectrum of $L(t)$ is preserved under the KdV flow. This in turn implies that ψ and $\partial_t\psi - P(t)\,\psi$ are both eigenfunctions (of discrete or continuous spectrum) associate with the same point

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of spectrum. While the Lax representation (1.3) is a manifestation of a very specific structure of the KdV equation, (1.3) alone is not of much help to solving the Cauchy problem (1.1) as finding ψ is not any easier than solving (1.1). The main reason why the Lax representation works is that in some important cases direct computation of ψ can be effectively circumvented. It is the case in the original GGKM setting. More specifically, if we assume that (1.1) has a smooth rapidly decaying solution $u(x,t)$ for all $t \geq 0$ then the Jost solutions $\psi_\pm$ remain Jost under the KdV flow. This readily implies that the scattering data for the pair $(L(t), \mathbb{L}_0)$ evolves in time by a very simple law. The solution $u(x,t)$ to (1.1) for each $t > 0$ is now obtained by the formula

$$u(x,t) = -2\partial_x^2 \log \tau(x,t), \tag{1.4}$$

where τ is the so-called *Hirota tau-function* introduced in [4] and it admits an explicit representation in terms of the scattering data.[1] The solution has a relatively simple and by now well understood wave structure of running (finitely many) solitons accompanied by radiation of decaying waves (see e.g. [5,6]).

Another equally important case is when q is periodic. Like in the previous case the problem (1.1) is a priori well-posed and a unique periodic solution to (1.1) exists. While it was quite clear from the beginning that the GGKM approach should work but it was not until 1974 when the actual IST was found in the periodic context by a considerable effort of such top experts as Dubrovin, Flascka, Its, Marchenko, Matveev, McKean, Novikov, Trubowitz, to name just few. We refer to the historically very first survey [7] by Dubrovin–Matveev–Novikov and the 2003 Gesztesy–Holden book [8] where a complete history is given. The periodic IST is quite different from the GGKM one and is actually the *inverse spectral transform* (also abbreviated as IST) since it relies on the Floquet theory for $\mathbb{L}_q$ and analysis of Riemann surfaces and hence is much more complex than the rapidly decaying case. The time evolution of the spectral data is nevertheless simple (but not simple to derive) and the solution $u(x,t)$ is given essentially by the same formula (1.4), frequently referred to as the *Its–Matveev formula* [9], but τ is a multidimensional[2] theta-function of real hyperelliptic algebraic curves explicitly computed in terms of spectral data of the associated Dirichlet problem for $\mathbb{L}_q$. It is therefore very different from the rapidly decaying case. The main feature of a periodic solution is its quasi-periodicity in time t.

We have outlined two main classes of initial data q in (1.1) for which a suitable form of the IST was found during the initial boom followed by [2]. We emphasize again that existence of the Lax pair merely means only that the KdV flow is isospectral but it does not in general offer an algorithm to find the solution. It is the simple law of time evolution of the scattering/spectral data that makes the IST work in these two cases. That is why Krichever and Novikov claim in [10] that (1.1) is completely integrable essentially only in these two cases. In fact, the question if (1.1)[3] could be solved by a suitable IST outside of these two classes, has been raised in one form or another by (in chronological order) McLeod–Olver [11], Ablowitz–Clarkson [12], Marchenko [13], Krichever–Novikov [10], Deift [14], Matveev [15], and Zakharov [1] to name just a few. These authors also expand on many challenges and complications that arise and some regard it as a major unsolved problem.

We give a complete answer to the following question: Assuming rapid decay of $q(x)$ at $+\infty$, what conditions do we have to impose at $-\infty$ for (1.1) to be well-posed in a certain sense and solvable by a suitable IST? We show that the only condition to be imposed is that q is bounded from below. More specifically, we assume the following condition.

Hypothesis 1.1. $q(x)$ is a real, locally bounded function such that

(1) For some $0 \leq h < \infty$, $q(x) \geq -h^2$ (boundedness from below);
(2) For some $\alpha > 4$, $q(x) = O\left(x^{-\alpha}\right)$, $x \to +\infty$ (decay at $+\infty$).

We call such q *step-type*. Thus any q subject to Hypothesis 1.1 is bounded from below, decays sufficiently fast at $+\infty$ but is arbitrary otherwise resulting in a much more complicated spectrum. The general spectral theory of second-order ordinary differential operators says that the negative spectrum of $\mathbb{L}_q$ has multiplicity one but could be of any type (including absolutely continuous (a.c.)) and the positive spectrum has a.c. component filling $(0,\infty)$ but need not be uniform (however no embedded bound states.[4])

Note that under our conditions neither well-posedness nor IST are a priori available and we have to deal with both. Our approach is based upon the theory of Hankel operators (see Section 2.2). Our Hankel operator is unitary equivalent to the well-known Marchenko integral operator but particularly convenient for limiting procedures we crucially rely on. Following our [16], we first introduce the one-sided scattering theory from the right. The scattering data can be conveniently represented in terms of the reflection coefficient R from the right and certain positive measure ρ (see Section 3.2) via

$$\varphi_{x,t}(k) = \xi_{x,t}(k)R(k) + \int_0^h \frac{\xi_{x,t}(is)\,d\rho(s)}{s+ik}, \tag{1.5}$$

where

$$\xi_{x,t}(k) := \exp\{i(8k^3t + 2kx)\} \quad \text{(cubic exponential)}. \tag{1.6}$$

This function $\varphi_{x,t}$ appears as the symbol of our Hankel operator $\mathbb{H}(\varphi_{x,t})$ which solely carries over the scattering data and the variables (x,t) in the KdV equation. In particular, if q is rapidly decaying also at $-\infty$ then ρ becomes discrete and the integral in (1.5) becomes a finite sum.

Our main result is the following theorem.

Theorem 1.2 (*Main Theorem*). *Suppose that the initial data q in (1.1) satisfy Hypothesis 1.1. Let*

$$q_b(x) = \begin{cases} 0, & x < b \\ q(x), & x \geq b \end{cases}$$

and denote by $u_b(x,t)$ the (necessarily unique) classical[5] *solution of (1.1) with data q_b. Then for every x and $t > 0$ the solutions $u_b(x,t)$ converge*[6] *to some $u(x,t)$ as $b \to -\infty$ which is also a classical solution to the KdV equation. Moreover,*

$$u(x,t) = -2\partial_x^2 \log\det\left(1 + \mathbb{H}(\varphi_{x,t})\right), \tag{1.7}$$

with $\varphi_{x,t}$ defined by (1.5), where the infinite determinant is understood in the classical Fredholm sense.

Note that (1.1) with data $q_b = q|_{(b,\infty)}$ is well-posed [17,18] and therefore Theorem 1.2 also says that (1.1) with data q subject to Hypothesis 1.1 is globally well-posed in the following sense: classical solutions $q_n(x,t)$ with compactly supported initial data $q_n(x)$ converge to a classical solution $u(x,t)$ uniformly on any compact x-domain for any $t > 0$ and independently of the choice of $q_n(x)$ approximating $q(x)$. This definition is consistent with [19], where it is also emphasized that existence implies uniqueness

1 Will be given later.
2 Infinite dimensional in general.
3 Or any other integrable system.
4 This is due to fast decay at $+\infty$ which rules out solutions square integrable at $+\infty$.
5 I.e., at least three times continuously differentiable in x and once in t.
6 In fact, uniform convergence on compact sets of (x,t) takes place but we do not need it here. We hope to present much more specific statements about the convergence elsewhere.

and certain continuous dependence on the data. For general background reading on well-posedness we refer the interested reader to the book [20] and literature cited therein. For results on well-posedness of the KdV equation in Sobolev spaces obtained by IST see the book [21] and in weighted Sobolev spaces see the recent [22]. We are unaware of any well-posedness results on the KdV equation with unrestricted behavior at $-\infty$.

The main reason why our Hankel operator approach works is that it allows us to show that classical solutions for restricted data q_b given by (1.7) suitably converge as $b \to -\infty$ to the classical solution of the KdV given by the same formula (1.7).

Our result includes as particular cases, all q's approaching a constant at $-\infty$ (considered first in physical literature and rigorously in 1976 by Hruslov[7] [23]) and a periodic function (considered in 1994 by Kotlyarov–Hruslov [24]). But it was not until very recently when a compete rigorous investigation of (1.1) with such initial profiles and their generalizations was done by Teschl with his collaborators (see e.g. [25–29]). We discuss some of their results in Section 5 where we also give a brief review of some other results on nonclassical situations.

The paper is organized as follows. In Section 2.1 we discuss the classical IST and give the solution to (1.1) in terms of the Hankel operator. In Section 3 we discuss the scattering theory for potentials satisfying to Hypothesis 1.1. In Section 4 we prove Theorem 1.2. Section 5 is devoted to some relevant discussions and connections of our results to those of others.

2. Classical IST and Hankel operators

2.1. The classical IST [12,30]

For the *Cauchy problem for the KdV equation* (1.1) with real rapidly decaying q's the IST method consists, as the standard Fourier transform method, of the three steps:

Step 1. *Direct transform*: $q(x) \longrightarrow S_q$, where S_q is a new set of variables turning (1.1) into a simple order 1 linear ODE for $S_q(t)$ with the initial condition $S_q(0) = S_q$.

Step 2. *Time evolution*: $S_q \longrightarrow S_q(t)$.

Step 3. *Inverse transform*: $S_q(t) \longrightarrow q(x, t)$.

The set S_q is formed as follows. Associate with q the full line *Schrödinger operator* $\mathbb{L}_q = -\partial_x^2 + q(x)$. As well-known, $\mathbb{L}_q$ is self-adjoint on $L^2 := L^2(\mathbb{R})$ and its spectrum consists of finitely many simple (negative) *bound states* $\{-\kappa_n^2\}$, and a twofold *absolutely continuous (a.c.) spectrum* filling $\mathbb{R}_+ := (0,\infty)$. The *Schrödinger equation* $\mathbb{L}_q\psi = k^2\psi$ has two *(Jost) solutions*: $\psi_\pm(x,k) = e^{\pm ikx} + o(1)$, $x \to \pm\infty$. The Jost solutions are analytic for $\operatorname{Im} k > 0$ and continuous for $\operatorname{Im} k \geq 0$. Moreover,

$$\psi_\pm(x,k) = e^{\pm ikx}\left(1 \pm \frac{i}{2k}\int_x^{\pm\infty} q + O\left(\frac{1}{k^2}\right)\right),\quad k \to \infty,\ \operatorname{Im} k \geq 0, \tag{2.1}$$

and

$$\psi_\pm(x,-k) = \overline{\psi_\pm(x,k)},\ k \in \mathbb{R}. \tag{2.2}$$

The pair $\{\psi_+, \overline{\psi_+}\}$ forms a fundamental set and hence[8]

$$T(k)\psi_-(x,k) = \overline{\psi_+(x,k)} + R(k)\psi_+(x,k),\ k \in \mathbb{R}, \quad (\textit{basic scattering identity}) \tag{2.3}$$

with some T and R called the *transmission* and (right) *reflection coefficients* respectively. $R(k)$ has important properties [31]:

$$R(-k) = \overline{R(k)}\ (\text{symmetry}) \tag{2.4}$$

$$|R(k)| < 1,\ k \neq 0,\ (\text{contraction}) \tag{2.5}$$

$$R(k) = o(1/k),\ |k| \to \infty\ (\text{decay}) \tag{2.6}$$

$$R \in C(\mathbb{R})\ (\text{continuity}). \tag{2.7}$$

Associate with q the *scattering data*

$$S_q := \{R(k),\ k \geq 0,\ (\kappa_n, c_n),\ 1 \leq n \leq N\}, \tag{2.8}$$

where c_n's are positive numbers called *norming constants* of bound states $-\kappa_n^2$. In terms of Jost solutions $\psi_\pm$ one has

$$R = \frac{W(\overline{\psi_+}, \psi_-)}{W(\psi_-, \psi_+)}\ \ (W(f,g) := fg' - f'g),$$
$$c_n = \left(\int |\psi_+(x, i\kappa_n)|^2 dx\right)^{-1} \tag{2.9}$$

and Step 1 is solved. As is well-known, S_q determines q uniquely. It is the fundamental classical fact that under the KdV flow the scattering data evolves in time as follows

$$S_q(t) = \left\{R(k)\exp 8ik^3t,\ k \geq 0, \left(\kappa_n, c_n \exp 8\kappa_n^3 t\right), 1 \leq n \leq N\right\} \tag{2.10}$$

which solves Step 2. We emphasize that the Lax pair considerations do not imply an explicit time evolution $\psi_\pm(x,t,k)$ for Jost solutions but do imply that so for quantities (2.9).

Step 3 amounts to solving the *inverse scattering problem* of recovering the potential $u(x,t)$ from $S_q(t)$ and can be done in many ways. Historically, the first one is due to Gelfand–Levitan–Marchenko and it requires solving an integral (Marchenko) equation. The most powerful one is based on the *Riemann–Hilbert problem* which is solved by means of singular integral equations (cf. Deift–Zhou [32] or recent Grunert–Teschl [33] for a streamlined exposition of [32]). Our approach also starts out from a Riemann–Hilbert problem (the basic scattering relation (2.3)) which we solve in terms of Hankel operators.

2.2. Hankel operators and the IST

A Hankel operator is an infinite-dimensional analog of a Hankel matrix, a matrix whose (j,k) entry depends only on $j+k$, i.e. a matrix Γ of the form

$$\Gamma = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \cdots \\ \gamma_2 & \gamma_3 & \cdots & \\ \gamma_3 & \cdots & & \\ \cdots & & & \gamma_n \end{pmatrix}.$$

The immediate Hilbert space generalization of a Hankel matrix is an integral operator on $L^2(\mathbb{R}_+)$ whose kernel depends on the sum of the arguments

$$(\mathbb{H}f)(x) = \int_0^\infty h(x+y)f(y)dy,\ f \in L^2(0,\infty),\ x \geq 0, \tag{2.11}$$

and it is in this form that Hankel operators typically appear in the inverse scattering formalism and are referred to as Marchenko's operator. The form (2.11) however does not prove to be convenient for our purposes and in fact it is not used much in the Hankel operator community either. Instead, we consider Hankel operators on Hardy spaces. (see e.g. excellent books [34,35] for more information and numerous references). We recall that a function f analytic in $\mathbb{C}^\pm$ is in the Hardy space $H^2_\pm$ if

$$\sup_{y>0}\int_{-\infty}^{\infty} |f(x \pm iy)|^2\, dx < \infty.$$

[7] Also transcripted as Khruslov.

[8] We call (2.3) the (right) *basic scattering relation*. Similarly, we define the left one by using $\{\psi_-, \overline{\psi_-}\}$.

It is particularly important that $H^2_\pm$ is a Hilbert space with the inner product induced from L^2:

$$\langle f,g\rangle_{H^2_\pm}=\langle f,g\rangle_{L^2}=\langle f,g\rangle=\int_{-\infty}^{\infty}f(x)\,\bar g(x)\,dx.$$

It is well-known that $L^2=H^2_+\oplus H^2_-$, the orthogonal (Riesz) projection $\mathbb{P}_\pm$ onto $H^2_\pm$ being given by

$$(\mathbb{P}_\pm f)(x)=\pm\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(s)\,ds}{s-(x\pm i0)}. \tag{2.12}$$

Notice that for any $f\in H^2_+$ and $\lambda\in\mathbb{C}^+$

$$\mathbb{P}_-\frac{f(\cdot)}{\cdot-\lambda}=\mathbb{P}_-\frac{f(\cdot)-f(\lambda)}{\cdot-\lambda}+\mathbb{P}_-\frac{f(\lambda)}{\cdot-\lambda}=\frac{f(\lambda)}{\cdot-\lambda}. \tag{2.13}$$

We will also need $H^\infty_\pm$, the algebra of analytic functions uniformly bounded in $\mathbb{C}^\pm$.

Let $(\mathbb{J}f)(x)\overset{\text{def}}{=}f(-x)$ be the operator of reflection. It is clearly an isometry on L^2 intertwining the Riesz projections

$$\mathbb{J}\mathbb{P}_\mp=\mathbb{P}_\pm\mathbb{J}. \tag{2.14}$$

Given $\varphi\in L^\infty$ the operator $\mathbb{H}(\varphi):H^2_+\to H^2_+$ is called *Hankel* if

$$\mathbb{H}(\varphi)f\overset{\text{def}}{=}\mathbb{J}\mathbb{P}_-\varphi f,\quad f\in H^2_+, \tag{2.15}$$

and φ is called its *symbol*.

It immediately follows from the definition (2.15) that

$$\mathbb{H}(\varphi+h)=\mathbb{H}(\varphi)\text{ for any }h\in H^\infty_+. \tag{2.16}$$

meaning that only $\mathbb{P}_-\varphi$ (the so-called co-analytic) part of the symbol matters.

Observe that if $\mathbb{J}\varphi=\overline{\varphi}$ then $\mathbb{H}(\varphi)$ is obviously selfadjoint. Note that $\mathbb{H}(\varphi)$ is unitary equivalent to the operator $\mathbb{H}$ given by (2.11), φ being the Fourier transform of h in (2.11).

For a given bounded operator $\mathbb{A}$ on a Hilbert space, we recall that its nth singular value $s_n(\mathbb{A})$ is defined as the nth eigenvalue of the operator $(\mathbb{A}^*\mathbb{A})^{1/2}$. We say that $\mathbb{A}$ is compact if $s_n(\mathbb{A})\to0$ and we write $\mathbb{A}\in\mathfrak{S}_\infty$. If $\Sigma_n s_n(\mathbb{A})=:\|A\|_{\mathfrak{S}_1}<\infty$ then $\mathbb{A}$ is called a *trace class* operator and we write $\mathbb{A}\in\mathfrak{S}_1$.

The membership of a Hankel operator $\mathbb{H}(\varphi)$ in the trace class is determined by smoothness of its symbol φ. In particular, the following criterion holds (see e.g. Theorems 9.1 and 9.2 in [16]).

Proposition 2.1 (Adamyan–Arov–Krein)**.** *If a bounded function φ is twice differentiable on $\mathbb{R}$ then $\in\mathfrak{S}_1$. Moreover,*[9]

$$\|\mathbb{H}(\varphi)\|_{\mathfrak{S}_1}\le const\,\|\varphi''\|_\infty.$$

The proof of Proposition 2.1 is based upon the seminal Adamyan–Arov–Krein Theorem on singular numbers of Hankel operators (see, e.g. [34,35] and the original literature cited therein). A necessary and sufficient condition for $\mathbb{H}(\varphi)\in\mathfrak{S}_1$ is found by Peller [35]. We are not sure if the beautiful Adamyan–Arov–Krein theory has been used in soliton theory.

Proposition 2.2. *Let φ_1,φ_2 are bounded and suppose $\mathbb{H}(\varphi_1),\mathbb{H}(\varphi_2)\in\mathfrak{S}_1$, then $\mathbb{H}(\varphi_1\varphi_2)\in\mathfrak{S}_1$.*

This statement is of course well-known but for the reader's convenience we give its elementary proof.

Proof. Note first that $\mathbb{H}(\varphi)$ is different from[10]

$$\widetilde{\mathbb{H}}(\varphi)f=\mathbb{P}_-\varphi f,\quad f\in H^2_+$$

only by isometry $\mathbb{J}$ and hence it is sufficient to prove the statement for $\widetilde{\mathbb{H}}(\varphi)$. Since $\mathbb{P}_++\mathbb{P}_-=I$, we then have

$$\begin{aligned}\widetilde{\mathbb{H}}(\varphi_1\varphi_2)f&=\mathbb{P}_-\varphi_1\varphi_2f=\mathbb{P}_-\varphi_1\left(\mathbb{P}_++\mathbb{P}_-\right)\varphi_2f\\&=\mathbb{P}_-\varphi_1\mathbb{P}_+\varphi_2f+\mathbb{P}_-\varphi_1\mathbb{P}_-\varphi_2f\\&=\widetilde{\mathbb{H}}(\varphi_1)\mathbb{P}_+\varphi_2f+\mathbb{P}_-\varphi_1\widetilde{\mathbb{H}}(\varphi_2)f.\end{aligned}$$

But $\|AKB\|_{\mathfrak{S}_1}\le\|A\|\,\|K\|_{\mathfrak{S}_1}\|B\|$ and therefore

$$\left\|\widetilde{\mathbb{H}}(\varphi_1\varphi_2)\right\|_{\mathfrak{S}_1}\le\left\|\widetilde{\mathbb{H}}(\varphi_1)\right\|_{\mathfrak{S}_1}\|\varphi_2\|_\infty+\|\varphi_1\|_\infty\left\|\widetilde{\mathbb{H}}(\varphi_2)\right\|_{\mathfrak{S}_1}.\quad\square$$

As mentioned before the Hankel operator appears in the classical IST as Marchenko's integral operator. But the integral realization (2.11) of a Hankel operator has some serious technical disadvantages and for some serious reasons is less popular in the Hankel operator community. On the other hand, we have not seen Marchenko's operator written in the form (2.15).

We now demonstrate how convenient the definition (2.15) of the Hankel operator is for solving (1.1) for rapidly decaying initial data in closed form.

Introduce

$$y_\pm(x,k):=e^{\mp ikx}\psi_\pm(x,k)\,\text{(Faddeev functions)}$$

and rewrite the basic scattering identity (2.3) in the form

$$Ty_-=\bar y_++R\xi_xy_+, \tag{2.17}$$

where $\xi_x(k):=e^{2ikx}$. Let us regard (2.17) as a Riemann–Hilbert problem of determining $y_\pm$ by given T,R which we will solve by Hankel operator techniques. For simplicity assume that there is only one bound state $-\kappa_0^2$ with the norming constant c_0.

The function Ty_- in (2.17) is meromorphic in $\mathbb{C}^+$ as a function of k for each x with a simple pole at $i\kappa_0$ and the residue

$$\operatorname*{Res}_{k=i\kappa_0}T(k)\,y_-(x,k)=ic_0\xi_x(i\kappa_0)\,y_+(x,i\kappa_0).$$

Therefore [31] for each fixed x

$$T(k)\,y_-(x,k)-1-\frac{ic_0\xi_x(i\kappa_0)}{k-i\kappa_0}y_+(x,i\kappa_0)\in H^2_+.$$

Rearrange (2.17) to read

$$\begin{aligned}&T(k)\,y_-(x,k)-1-\frac{ic_0\xi_x(i\kappa_0)}{k-i\kappa_0}y_+(x,i\kappa_0)\\&=\overline{(y_+(x,k)-1)}+(R\xi_x)(k)\,(y_+(x,k)-1)\\&+(R\xi_x)(k)-\frac{ic_0\xi_x(i\kappa_0)}{k-i\kappa_0}y_+(x,i\kappa_0).\end{aligned} \tag{2.18}$$

Noticing that the last term in (2.18) is in H^2_-, we can apply the Riesz projection $\mathbb{P}_-$ to (2.18). Introducing $Y:=y_+-1$, we have

$$\begin{aligned}&\mathbb{P}_-(\overline{Y}+R\xi_xY)+\mathbb{P}_-R\xi_x-ic_0\xi_x(i\kappa_0)\frac{Y(x,i\kappa_0)}{\cdot-i\kappa_0}\\&-\frac{ic_0\xi_x(i\kappa_0)}{\cdot-i\kappa_0}=0.\end{aligned} \tag{2.19}$$

It follows from (2.1) that $Y\in H^2_+$ for any $x\in\mathbb{R}$. Due to the symmetry (2.2), $\overline{Y}=\mathbb{J}Y$ and by (2.14) we have

$$\mathbb{P}_-\overline{Y}=\mathbb{P}_-\mathbb{J}Y=\mathbb{J}\mathbb{P}_+Y=\mathbb{J}Y. \tag{2.20}$$

Note that by (2.13)

$$\frac{Y(i\kappa_0,x)}{\cdot-i\kappa_0}=\mathbb{P}_-\frac{Y(\cdot,x)}{\cdot-i\kappa_0}. \tag{2.21}$$

Inserting (2.20) and (2.21) into (2.19), we obtain

$$\mathbb{J}Y+\mathbb{P}_-\left(R\xi_x-\frac{ic_0\xi_x(i\kappa_0)}{\cdot-i\kappa_0}\right)Y=-\mathbb{P}_-\left(R\xi_x-\frac{ic_0\xi_x(i\kappa_0)}{\cdot-i\kappa_0}\right).$$

[9] Recall $\|f\|_\infty=\sup|f(x)|\,,x\in\mathbb{R}$

[10] In fact, in the literature it is how Hankel operators are typically defined.

Applying $\mathbb{J}$ to both sides of this equation yields

$$(\mathbb{I}+\mathbb{H}(\varphi_x))Y=-\mathbb{H}(\varphi_x)1, \quad (2.22)$$

where $\mathbb{H}(\varphi_x)$ is the Hankel operator with symbol

$$\varphi_x(k)=R(k)\xi_x(k)+\frac{c_0\xi_x(i\kappa_0)}{\kappa_0+ik}.$$

Similarly, for N bound states one has

$$\varphi_x(k)=R(k)\xi_x(k)+\sum_{n=1}^{N}\frac{c_n\xi_x(i\kappa_n)}{\kappa_n+ik}.$$

By (2.4), $\mathbb{J}\varphi_x=\overline{\varphi_x}$ and hence $\mathbb{H}(\varphi_x)$ is selfadjoint. It follows from (2.7) that $\mathbb{H}(\varphi_x)$ is compact for any x. Note that $\mathbb{H}(\varphi_x)1$ on the right hand side of (2.22) should be interpreted as $\mathbb{H}(\varphi_x)1=\mathbb{P}_+\overline{\varphi_x}\in H^2_+$.

It is now clear that if we show that (2.22) is uniquely solvable and $Y(x,k)$ is its solution then the potential $q(x)$ can be found from (2.1) by

$$q(x)=\partial_x\lim 2ikY(x,k),\quad k\to\infty. \quad (2.23)$$

Alternatively,

$$q(x)=-2\partial_x^2\log\det\left(1+\mathbb{H}(\varphi_x)\right), \quad (2.24)$$

where the determinant is understood in the classical Fredholm sense. In a different (but equivalent) form (2.24) has been known since Bargmann and has been derived in a number of different ways (cf. e.g. [30,36,37]). Loosely speaking, it follows from solving (2.22) by Cramer's rule.

Steps 2 and 3 of Section 2.1 now merely amount to replacing φ_x with

$$\varphi_{x,t}(k)=R(k)\xi_{x,t}(k)+\sum_{n=1}^{N}\frac{c_n\xi_{x,t}(i\kappa_n)}{\kappa_n+ik},$$

where $\xi_{x,t}(k)=\exp i(8k^3t+2kx)$ solely carries the dependence on (x,t). Thus Steps 1–3 can now be put together in a compact form

$$q(x)\longrightarrow\mathbb{H}(\varphi_{x,t})\longrightarrow q(x,t), \quad (2.25)$$

where $q(x,t)$ is explicitly given by

$$q(x,t)=-2\partial_x^2\log\det\left(1+\mathbb{H}(\varphi_{x,t})\right). \quad (2.26)$$

We will call $\mathbb{H}(\varphi_{x,t})$ the *IST Hankel operator*.

For rapidly decaying initial data our Hankel operator approach is shorter but completely equivalent to the classical treatment. However our edition (2.25)–(2.26) of the IST turns to be a very convenient starting point to extend IST far beyond standard assumptions on initial data.

3. Step-type potentials

In this section we discuss scattering theory for step-type potentials following our [16]. But first we need to review some facts from the classical Titchmarsh–Weyl theory.

3.1. Titchmarsh–Weyl Theory and m-function [38]

The main point of this theory is that the problem

$$\mathbb{L}_q u=\lambda u,\ x\in\mathbb{R}_\pm,\ u(\cdot,\lambda)\in L^2(0,\pm\infty),\lambda\in\mathbb{C}^+,$$

has a unique (up to a multiplicative constant) solution $\Psi_\pm(x,\lambda)$, called a *Weyl solution* for broad classes of q's (called *limit point case at* $\pm\infty$). Define then the *(Titchmarsh–Weyl) m-function* $m_\pm$ for $(0,\pm\infty)$ as follows:

$$m_\pm(\lambda)=\pm\partial_x\log\Psi_\pm(\pm0,\lambda),\quad \lambda\in\mathbb{C}^+. \quad (3.1)$$

One defines $m_\pm(\lambda,a)$ for $(a,\pm\infty)$ in a similar way.

By the *Borg–Marchenko uniqueness theorem* $\{m_+,m_-\}$ recovers q uniquely (see [39] and the original literature cited therein). While fundamentally important to spectral theory of OD operators,[11] its role in scattering theory is modest. Moreover, Steps 1–3 in the previous subsection with data $S_q=\{m_+,m_-\}$ do not work well [13]. For this reason the m-function is little known in the soliton community.

In our approach the m-function plays a supporting but nevertheless crucial role due to the following reasons: it is well-defined for any realistic q (with no decay assumptions) including q's subject to Hypothesis 1.1, and it is a *Herglotz function*, i.e. it is analytic and maps $\mathbb{C}^+$ analytically to $\mathbb{C}^+$. As is well-known, each such function f can be represented as

$$f(\lambda)=a+b\lambda+\int_{-\infty}^{\infty}\left(\frac{1}{s-\lambda}-\frac{s}{1+s^2}\right)d\mu(s) \quad (3.2)$$

with some

$$a\in\mathbb{C}^+,\quad b\geq 0,\quad d\mu(s)\geq 0,\quad \int_{-\infty}^{\infty}\frac{d\mu(s)}{1+s^2}<\infty.$$

If $m=m_+$ is given by (3.1) then μ is the spectral measure of the Schrödinger operator on $L^2(0,\infty)$ with a Dirichlet boundary condition at $x=0$. The latter implies that if the spectrum is bounded from below then so is the support of μ. It follows that m can be analytically extended into $\mathbb{C}^-$.

Theorem 3.1. *If q, q_n are limit point case at $\pm\infty$ and $q_n\to q$ in L^1_{loc} then $m_\pm(\lambda,q_n)\to m_\pm(\lambda,q)$ uniformly on compacts away from the spectra.*

A proof of this statement in the most general case is given in [40].

Note that by definition (3.1) the m-function is a 1D *Dirichlet-to-Neumann map*.

3.2. Scattering theory for step-type potentials

The main feature of such potentials is that we can do one-sided scattering theory replacing in (2.3) the Jost solution ψ_- with Weyl solution Ψ_-. This immediately yields

$$R=W(\overline{\psi_+},\Psi_-)/W(\Psi_-,\psi_+) \quad (3.3)$$

which is consistent with the classical reflection coefficient. While properties (2.4)–(2.7) all hold for rapidly decaying potentials, only (2.4) holds for our step-type potentials. The property (2.5) is replaced with $|R(k)|\leq 1$ but $|R(k)|=1$ may occur for almost all k. The properties (2.6)–(2.7) fail and this is a very serious circumstance even for the powerful Riemann–Hilbert problem approach. In [16] we found what makes up for the lost properties:

Theorem 3.2 (*Analytic Split Formula*)**.** *Under Hypothesis 1.1*

$$R(k)=R_a(k)+\xi_a^{-1}\{A_a(k)-T_a(k)/y_+(a,k)\}, \quad (3.4)$$

where R_a, T_a are respectively the right reflection, transmission coefficients from $q_a=q|_{(a,\infty)}$, and

$$A_a(k)=2ik\,y_+(a,k)^{-2}\left(m_-(k^2,a)+m_+(k^2,a)\right)^{-1} \quad (3.5)$$

[11] The dependence of m on q is very intricate and understood only in rather narrow classes.

is analytic in $\mathbb{C}^+$ *except for* $i\Delta := \left\{k : k^2 \in \operatorname{Spec} \mathbb{L}_q \cap (-\infty, 0)\right\}$. *Moreover,* (1)

$$R_a(k) = \frac{T_a(k)}{2ik} G_a(k), \quad G_a(k) := \int_a^\infty e^{-2iks} Q(s)\, ds, \tag{3.6}$$

with some function Q (independent of a) such that

$$|Q(s)| \le |q(s)| + const \int_s^\infty |q|; \tag{3.7}$$

(2) *for a large enough* $y_+(a,k)^{-1} \in H_+^\infty$*;*

$$T_a(k) = \frac{k + i\varkappa_a}{k - i\varkappa_a} g_a(k), \quad \varkappa_a \ge 0,$$

where $g_a \in H_+^\infty$ *has only one simple zero at* $k = 0$ *in* $\mathbb{C}^+ \cup \mathbb{R}$*;* (3) *for a large enough the jump* $A_a(is-0) - A_a(is+0)$ *across* $i\Delta$ *is independent of a and defines a non-negative finite measure*

$$\begin{aligned} d\rho(s) &:= i\{A_a(is-0) - A_a(is+0)\}\, ds/2\pi \\ &= \operatorname{Im} A_a(is+0) ds/\pi \end{aligned} \tag{3.8}$$

supported on $\Delta \subseteq [0, h]$.

The set

$$S_q = \{R, \rho\}$$

plays the role of the classical scattering data (2.8) in our one sided scattering. The measure ρ carries over the information on the negative spectrum. In particular, if q is rapidly decaying at both $\pm\infty$ then $d\rho = \sum c_n^2 \delta(s - \kappa_n)\, ds$. Therefore we can call ρ a smeared norming constant measure. If $q(x)$ is a pure step function, i.e. $q(x) = -h^2,\ x < 0,\ q(x) = 0,\ x \ge 0$ then $\operatorname{Spec}(\mathbb{L}_q) = (-h^2, \infty)$ and purely a.c., $(-h^2, 0)$ and $(0, \infty)$ being components of the spectrum with respective multiplicities one and two. Moreover

$$R(k) = -\left(\frac{h}{\sqrt{k^2} + \sqrt{k^2 + h^2}}\right)^2,$$

$$\rho(s) = \frac{1}{3\pi h^2}\left(h^3 - \left(h^2 - s^2\right)^{3/2}\right).$$

The split (3.4) is surprisingly effective. Its main feature is that the analytic part A_a of R mimics the rough behavior of R and carries all the information about ρ in the set of scattering data $S_q = \{R, \rho\}$. The rest is at least continuous and its smoothness is determined by the decay of q at $+\infty$. This is crucially used in developing the IST for step-type initial data as all properties of R required for the IST are encoded in the analytic part A_a of A through the m-functions $m_\pm$. Thus the m-function works behind the scene but in a crucial way.

4. The IST for the KdV equation with step-type initial data

In this section we prove Theorem 1.2. Since the KdV equation is translation invariant by shifting q (if needed) we may assume in Theorem 3.2 that $a = 0$. Moreover, to avoid unnecessary technicalities we suppose that $\kappa_0 = 0$ (i.e. q_0 has no bound states). In this case $T_0 = g_0 \in H_+^\infty$ and (3.4) simplifies to read

$$R(k) = R_0(k) + A(k), \tag{4.1}$$

with

$$R_0(k) = f_0(k)\, G_0(k), \quad G_0(k) = \int_0^\infty e^{-2iks} Q(s)\, ds \tag{4.2}$$

and

$$\begin{aligned} A(k) &= 2ik\, y_+(0,k)^{-2}\left(m_-(k^2) + m_+(k^2)\right)^{-1} - T_0(k)/y_+(0,k) \\ &= f_1(k)\, \frac{2ik}{m_-(k^2) + m_+(k^2)} + f_2(k), \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} &f_0(k) = T_0(k)/2ik, \quad f_1(k) = y_+(0,k)^{-2}, \\ &f_2(k) = -T_0(k)/y_+(0,k) \quad \in H_+^\infty, \end{aligned} \tag{4.4}$$

i.e. are analytic functions all bounded in the upper half plane. Throughout this section we assume that q in (1.1) is subject to Hypothesis 1.1.

4.1. Fundamental properties of the IST Hankel operator

Theorem 4.1. *Let*

$$\varphi_{x,t}(k) = \xi_{x,t}(k) R(k) + \int_0^h \frac{\xi_{x,t}(is)\, d\rho(s)}{s + ik}.$$

Under Hypothesis 1.1 for any real x and positive t the operator $\mathbb{H}(\varphi_{x,t})$

(1) *is selfadjoint (also holds for* $t = 0$*),*
(2) *has no eigenvalue equal* -1*,*
(3) *its derivatives* $\partial_t \mathbb{H}(\varphi_{x,t})$, $\partial_x^m \mathbb{H}(\varphi_{x,t})$, $0 \le m \le 5$, *are compact Hankel operators,*
(4) *is trace class.*

Proof. Part (1) is trivial as clearly $\varphi_{x,t}(-k) = \overline{\varphi_{x,t}(-\overline{k})}$. Part (2) is the most difficult and it is proven in our [16].

Let us show (3). It follows from (4.1) that

$$\varphi_{x,t}(k) = \xi_{x,t}(k) R_0(k) + \left\{\xi_{x,t}(k) A(k) + \int_0^h \frac{\xi_{x,t}(is)\, d\rho(s)}{s + ik}\right\}.$$

It follows from (2.16) that

$$\mathbb{H}\left(\varphi_{x,t}\right) = \mathbb{H}\left(\widetilde{\varphi}_{x,t}\right), \tag{4.5}$$

with

$$\widetilde{\varphi}_{x,t}(k) = \xi_{x,t} R_0 + \Phi_{x,t},$$

where

$$\begin{aligned} \Phi_{x,t}(k) &= \mathbb{P}_-\left(\xi_{x,t} A\right)(k) + \int_0^h \frac{\xi_{x,t}(is)\, d\rho(s)}{s + ik} \\ &= \frac{i}{2\pi}\int_{\mathbb{R}} \frac{\xi_{x,t}(z) A(z)}{z - (k - i0)}\, dz + \int_0^h \frac{\xi_{x,t}(is)\, d\rho(s)}{s + ik} \ \text{(by (2.12))}. \end{aligned} \tag{4.6}$$

By Theorem 3.2 A is analytic in $\mathbb{C}^+ \setminus i\Delta$. Since $m_\pm$ are Herglotz functions, so is $-(m_- + m_+)^{-1}$ and hence by (3.2) $iz\left\{m_-(z^2) + m_+(z^2)\right\}^{-1}$ does not grow faster that z^3 along the line $\mathbb{R} + ih_0$ for any $h_0 > h$.[12] Due to the rapid decay of $\xi_{x,t}(z)$ along $\mathbb{R} + ih_0$ we can deform the contour in the first integral of the right hand side of (4.6) to $\mathbb{R} + ih_0$. We have

$$\begin{aligned} &\frac{i}{2\pi}\int_{\mathbb{R}} \frac{\xi_{x,t}(z) A(z)}{z - (k - i0)}\, dz \quad \text{(by (3.8))} \\ &= \frac{i}{2\pi}\int_{\mathbb{R}+ih_0} \frac{\xi_{x,t}(z) A(z)}{z - k}\, dz - \int_0^h \frac{\xi_{x,t}(is)\, d\rho(s)}{s + ik}. \end{aligned}$$

Inserting this formula into (4.6) yields

$$\Phi_{x,t}(k) = \frac{i}{2\pi}\int_{\mathbb{R}+ih_0} \frac{\xi_{x,t}(z) A(z)}{z - k}\, dz.$$

We emphasize that the cancellation of the second integral of the right hand side of (4.6) is the main reason why our approach works.

[12] In fact, it is even bounded.

Thus we have obtained the crucially important split of our symbol:

$$\widetilde{\varphi}_{x,t} = \varphi^0_{x,t} + \Phi_{x,t}, \tag{4.7}$$

where

$$\varphi^{(0)}_{x,t} := \xi_{x,t} R_0,\ \ \Phi_{x,t}(k) = \int_{\mathbb{R}+ih_0} \frac{\phi_{x,t}(z)}{z-k} dz,\ \ \phi_{x,t} := \frac{i}{2\pi}\xi_{x,t} A.$$

Since we can take h_0 in (4.7) as large as we want, the function $\Phi_{x,t}$ is entire. Due to the rapid decay of $\xi_{x,t}(z)$ along each $\mathbb{R}+ih_0$ one easily sees that $\partial_t^n \partial_x^m \Phi_{x,t}$ is also entire for any nonnegative integers n, m and hence by Proposition 2.1

$$\partial_t^n \partial_x^m \mathbb{H}(\Phi_{x,t}) = \mathbb{H}(\partial_t^n \partial_x^m \Phi_{x,t}) \in \mathfrak{S}_1. \tag{4.8}$$

The Hankel operator $\mathbb{H}(\varphi^0_{x,t})$ is implicitly studied in [18]. In particular, it follows from the proof of Theorem 5.1 in [18] (the main result of this paper) that for $0 \le m \le 5$

$$\partial_x^m \mathbb{H}\left(\varphi^0_{x,t}\right), \partial_t \mathbb{H}\left(\varphi^0_{x,t}\right) \in \mathfrak{S}_\infty. \tag{4.9}$$

Indeed, as was discussed in Section 2.2, $\mathbb{H}\left(\varphi^0_{x,t}\right)$ is unitarily equivalent to the integral Hankel operator given by (2.11) with the kernel

$$H_{x,t}(s) = \frac{1}{\pi}\int_{-\infty}^{\infty} \varphi^0_{x,t}(k)\, e^{2iks} dk.$$

Under our conditions on q_0, by (b) on page 1012 of [18] we have that $\partial_x^m H_{x,t} \in L^1(0,\infty)$, $0 \le m \le 5$. By Corollary 8.11 of [35] the integral Hankel operator with kernel $\partial_x^m H_{x,t}$ is compact[13] and therefore, by unitary equivalence, so is $\partial_x^m \mathbb{H}(\varphi^0_{x,t})$. Since $\partial_t H_{x,t} = \partial_x^3 H_{x,t}$ one also concludes that $\partial_t \mathbb{H}(\varphi^0_{x,t}) \in \mathfrak{S}_\infty$. Thus (4.9) is proven and due to (4.8) so is (3).

It remains to show (4). Due to (4.8) one only needs to demonstrate that $\mathbb{H}(\varphi^0_{x,t})$ is trace class. It follows from (4.2) that

$$\varphi^0_{x,t} = \xi_{x,t} f_0 G_0.$$

By Proposition 2.2 $\mathbb{H}\left(\xi_{x,t} R_a\right)$ is trace class if each $\mathbb{H}\left(\xi_{x,t}\right), \mathbb{H}(f_0)$, $\mathbb{H}(G_0)$ is. For $\mathbb{H}\left(\xi_{x,t}\right)$ as before we write

$$\mathbb{H}\left(\xi_{x,t}\right) = \mathbb{H}\left(\Phi^{(0)}_{x,t}\right),\quad \Phi^{(0)}_{x,t}(k) := \frac{i}{2\pi}\int_{\mathbb{R}+ih_0} \frac{\xi_{x,t}(z)}{z-k} dz$$

and hence $\mathbb{H}(\xi_{x,t})$ is trace class for any real x and positive t.

Since $f_0 \in H^\infty_+$ (see (4.4)) we simply have $\mathbb{H}(f_0) = 0$. It remains to show that $\mathbb{H}(G_0) \in \mathfrak{S}_1$. It follows from condition 2 of Hypothesis 1.1 that $Q(s) = O\left(s^{-\alpha+1}\right), s \to \infty$, and hence

$$\left|\partial_k^2 G_0(k)\right| \le 4\int_0^\infty s^2 |Q(s)|\, ds < \infty.$$

By Proposition 2.1 $\mathbb{H}(G_0)$ is trace class which completes the proof. □

Theorem 4.1 says that $I + \mathbb{H}(\varphi_{x,t})$ is invertible globally in time which is the main reason for validity of the IST for any step-type data. It is also the most nontrivial part of Theorem 4.1. The proof is based on Theorems 3.1, 3.2, properties of the m-function discussed in Eq. (3.1), and subtle arguments and facts from the theory of Hankel/Toeplitz operators. This has been incrementally improved in the course of our [41–45,16].

Remark 4.2. Theorem 4.1 does not say that the Hankel operator corresponding to each piece in (1.5) is trace class. In fact, it is shown in [16] that the Hankel operator with symbol $\phi_{x,t}(k) := \int_0^h \frac{\xi_{x,t}(is)\, d\rho(s)}{s+ik}$ is trace class iff $\int_0^h d\rho(s)/s$ is bounded.

[13] In fact, boundedness alone is trivial.

4.2. Proof of the main theorem

The proof of Theorem 1.2 merely combines Theorem 4.1 and results of [18].

Proof. Take $b < 0$ and consider q_b. Below any object corresponding to q_b will be labeled by either a subscript b or a superscript (b). The KdV equation with the initial data q_b has a classical solution $u_b(x,t)$ [18] given by the Dyson formula

$$u_b(x,t) = -2\partial_x^2 \log\det\left(I + \mathbb{H}(\varphi^{(b)}_{x,t})\right), \tag{4.10}$$

where by Theorem 4.1 the operator $\mathbb{H}(\varphi^{(b)}_{x,t})$ is trace class and hence the determinant is well-defined.

Let

$$u(x,t) = -2\partial_x^2 \log\det\left(I + \mathbb{H}(\varphi_{x,t})\right), \tag{4.11}$$

where by the same theorem the operator $\mathbb{H}(\varphi_{x,t})$ is trace class. Consider

$$\begin{aligned}\Delta u &:= u - u_b \qquad (4.12)\\ &= 2\partial_x^2 \log\det\left(I + \mathbb{H}(\varphi_{x,t})\right)^{-1}\left(I + \mathbb{H}(\varphi^{(b)}_{x,t})\right)\\ &= 2\partial_x^2 \log\det\left(I + \left(I + \mathbb{H}(\varphi_{x,t})\right)^{-1}\left(\mathbb{H}(\varphi^{(b)}_{x,t}) - \mathbb{H}(\varphi_{x,t})\right)\right)\\ &= 2\partial_x^2 \log\det\left\{I + \left[I + \mathbb{H}(\varphi_{x,t})\right]^{-1}\Delta\mathbb{H}(\varphi_{x,t})\right\}.\end{aligned}$$

By Theorem 4.1 $\left[1 + \mathbb{H}\left(\varphi_{x,t}\right)\right]^{-1}$ is a bounded operator independent of b. In virtue of (4.5) and (4.7) for $\Delta\mathbb{H}(\varphi_{x,t})$ we have

$$\begin{aligned}\Delta\mathbb{H}(\varphi_{x,t}) &= \mathbb{H}\left(\varphi^{(b)}_{x,t} - \varphi_{x,t}\right) = \mathbb{H}\left(\widetilde{\varphi}^{(b)}_{x,t} - \widetilde{\varphi}_{x,t}\right)\\ &= \mathbb{H}\left(\Delta\Phi_{x,t}\right),\end{aligned}$$

where

$$\begin{aligned}\Delta\Phi_{x,t}(k) &= \frac{i}{2\pi}\int_{\mathbb{R}+ih_0} \frac{\xi_{x,t}(z)\,\Delta A(z)}{z-k} dz\\ &= \frac{i}{2\pi}\int_{\mathbb{R}+ih_0} 2iz\, f_1(z)\, \xi_{x,t}(z)\ \ \Delta f(z)\, \frac{dz}{z-k}\end{aligned}$$

and

$$\Delta f(z) := \left(m_-^{(b)}(z^2) + m_+(z^2)\right)^{-1} - \left(m_-(z^2) + m_+(z^2)\right)^{-1}.$$

By (4.8) $\partial_t^n \partial_x^m \mathbb{H}\left(\Delta\Phi_{x,t}\right)$ is trace class. We now show that for all n and m

$$\left\|\partial_t^n \partial_x^m \mathbb{H}\left(\Delta\Phi_{x,t}\right)\right\|_{\mathfrak{S}_1} \to 0,\ b \to -\infty. \tag{4.13}$$

To this end consider

$$\begin{aligned}\Delta\phi(k) &:= \partial_t^n \partial_x^m \left(\Delta\Phi_{x,t}\right) \qquad (4.14)\\ &= \frac{i}{2\pi}\int_{\mathbb{R}+ih_0} \partial_t^n \partial_x^m \xi_{x,t}(z)\, 2iz\, f_1(z)\ \ \Delta f(z)\, \frac{dz}{z-k}.\end{aligned}$$

Differentiating (4.14) in k twice, one has

$$\partial_k^2 \Delta\phi(k) = \frac{1}{\pi}\int_{\mathbb{R}+ih_0} (2iz)^{3n+m+1} \xi_{x,t}(z) f_1(z)\, \Delta f(z)\ \frac{dz}{(z-k)^3}$$

and hence

$$\begin{aligned}&\sup_{k\in\mathbb{R}} \left|\partial_k^2 \Delta\phi(k)\right|\\ &\le \frac{2^{3n+m+1}}{\pi h_0^3}\int_{\mathbb{R}+ih_0} |z|^{3n+m+1}\left|\xi_{x,t}(z)\right| |f_1(z)|\, |\Delta f(z)|\ \ |dz|\\ &\le const.\int_{\mathbb{R}+ih_0} \left|z^{3n+m+1}\xi_{x,t}(z)\right| |\Delta f(z)|\ \ |dz|\ \ (\text{since } f_1 \in H^\infty_+).\end{aligned} \tag{4.15}$$

Since

$$\left|\xi_{x,t}(\alpha+ih_0)\right| = \xi_{x,t}(ih_0)\exp\left\{-24h_0t\alpha^2\right\},$$

one sees that $z^{3n+m+1}\xi_{x,t}(z)$ falls off along $\mathbb{R}+ih_0$ faster than exponential for any n, m. Split the contour $\mathbb{R}+ih_0$ into $\gamma_N = (-N+ih_0, N+ih_0)$ and $\Gamma_N = (\mathbb{R}+ih_0)\setminus\gamma_N$. Since clearly $\operatorname{Spec}\mathbb{L}_{q_b} \geq -h^2$ by Theorem 3.1 $\Delta f(z) \to 0, b \to -\infty$, uniformly on γ_N for any N and hence

$$\int_{\gamma_N}\left|z^{3n+m+1}\xi_{x,t}(z)\right| |\Delta f(z)| \ |dz| \to 0, \quad b \to -\infty.$$

Consider $\Delta f(z)$ on Γ_N. Since the m-function has a non-negative imaginary part[14] (the Herglotz property), one has

$$\begin{aligned}|\Delta f(z)| &\leq \left|\frac{1}{m_-^{(b)}(z^2)+m_+(z^2)}\right| + \left|\frac{1}{m_-(z^2)+m_+(z^2)}\right| \\ &\leq \left|\frac{1}{\operatorname{Im} m_-^{(b)}(z^2)+\operatorname{Im} m_+(z^2)}\right| + \left|\frac{1}{\operatorname{Im} m_-(z^2)+\operatorname{Im} m_+(z^2)}\right| \\ &\leq \left|\frac{2}{\operatorname{Im} m_+(z^2)}\right|.\end{aligned}$$

For our decay condition at $+\infty$ one has $\psi_+(0,z) = 1+O(1/z)$ (see e.g. [31]) and hence $m_+(z^2) = \partial_x\psi_+(0,z)/\psi_+(0,z) = iz+O(1/z)$ as $|z| \to \infty$ in $\mathbb{C}^+$. Therefore, $\left|\frac{2}{\operatorname{Im} m_+(z^2)}\right|$ is bounded on Γ_N and hence by choosing N large enough the integral

$$\begin{aligned}&\int_{\Gamma_N}\left|z^{3n+m+1}\xi_{x,t}(z)\right| |\Delta f(z)| \ |dz| \\ &\leq 2\int_{\Gamma_N}\left|z^{3n+m+1}\xi_{x,t}(z)\right| \left|\frac{dz}{\operatorname{Im} m_+(z^2)}\right|\end{aligned}$$

can be made as small as one wishes for any real x and positive t.

Thus we can conclude that $\left\|\Delta\phi''\right\|_\infty \to 0$ as $b \to -\infty$ and Proposition 2.1 implies (4.13).

Next we show that $u(x,t)$ given by (4.11) is differentiable three time in x and once in t. We have

$$\begin{aligned}u(x,t) &= -2\partial_x^2 \log\det\left\{1+\mathbb{H}(\varphi_{x,t})\right\} = -2\partial_x^2\log\det\left\{1+\mathbb{H}(\widetilde{\varphi}_{x,t})\right\} \\ &= -2\partial_x^2\log\det\left\{1+\mathbb{H}(\varphi_{x,t}^0+\Phi_{x,t})\right\} \text{ (by (4.7))} \\ &= -2\partial_x^2\log\det\left\{1+\mathbb{H}(\varphi_{x,t}^0)+\mathbb{H}\left(\Phi_{x,t}\right)\right\} \\ &= -2\partial_x^2\log\det\left\{1+\mathbb{H}(\varphi_{x,t}^0)\right\} \\ &\quad -2\partial_x^2\log\det\left\{1+\left[1+\mathbb{H}(\varphi_{x,t}^0)\right]^{-1}\mathbb{H}\left(\Phi_{x,t}\right)\right\} \\ &= u_0(x,t)+U(x,t) \text{ (by (4.10) with } b=0),\end{aligned}$$

where

$$U(x,t) := -2\partial_x^2\log\det\left\{1+\left[1+\mathbb{H}(\varphi_{x,t}^0)\right]^{-1}\mathbb{H}\left(\Phi_{x,t}\right)\right\}.$$

The well-known differentiation formula

$$(\log\det(1+A))' = \operatorname{tr}(1+A)^{-1}A',$$

(4.8) and (4.9) imply that $U(x,t)$ is differentiable three time in x and once in t. Since $u_0(x,t)$ is the classical solution to (1.1) with $q=q_0$ by definition $u_0(x,t)$ has the same property and thus so is $u=u_0+U$. It follows from (4.12), (4.13), and Theorem 4.1 that for $0 \leq m \leq 3$

$$\partial_x^m u_b(x,t) \to \partial_x^m u(x,t), \quad \partial_t u_b(x,t) \to \partial u(x,t). \tag{4.16}$$

[14] Note that z^2 is in $\mathbb{C}^+$ if z is in the first quadrant. If z is in the second quadrant, then $\operatorname{Im} m(z^2) \leq 0$.

Finally, it only remains to show that u indeed solves the KdV equation. To this end, represent $u = u_b + \Delta u$ where as above $\Delta u = u - u_b$. We have

$$\begin{aligned}&\partial_t u - 6u\partial_x u + \partial_x^3 u \\ &= \partial_t\Delta u + 3\partial_x\left[(\Delta u - 2u)\Delta u\right] + \partial_x^3\Delta u \\ &\to 0, \quad b \to -\infty \text{ (due to (4.16))}\end{aligned} \tag{4.17}$$

and the proof is complete. □

The conditions of Hypothesis 1.1 are very general and admit the case of $|R(k)| = 1$ for almost all real k that has never been considered in the literature before. In the quantum mechanical sense, such q's are repulsive for plane waves coming from $+\infty$. Examples include (1) *Gaussian white noise* on a left half line (like the stock market), (2) *Pearson blocks* (certain sparse sequences of bumps), (3) *Kotani potentials* (certain random slowly decaying at $x \to -\infty$ functions [46]), and (4) functions growing at $-\infty$ (not quite physical), to mention just four.

Remark 4.3. As a by-product, we have shown that the operator-valued function $(x,t) \to \left[1+\mathbb{H}(\xi_{x,t}R_0)\right]^{-1}\mathbb{H}\left(\varphi_{x,t}-\xi_{x,t}R_0\right)$ is continuously differentiable in trace norm five times in x and at least once in t. Analogous statements for $(x,t) \to \mathbb{H}\left(\varphi_{x,t}\right)$ will be studied jointly with S. Grudsky elsewhere. We only mention here that Proposition 2.1 is no longer useful and our arguments are based upon Peller's subtle characterization of all trace class Hankel operators [35] and preliminary results to this effect is to appear in [47].

Remark 4.4. The first condition in Hypothesis 1.1 cannot be relaxed as the following simple argument suggests. Consider a sequence of soliton type bumps $q_n(x)$ of height $-\varkappa_n^2$ located on $(-\infty, 0)$ with some phases γ_n. Under the KdV flow all q_n start moving to the right with velocities $2\varkappa_n^2$. We can choose $(\varkappa_n), (\gamma_n) \to \infty$ so that all $q_n(x,t)$ would meet at a fixed point x_0 at a fixed time t_0. Apparently, this means that a *blow-up solution* develops in time t_0 which can be made arbitrarily small. The operator $\mathbb{L}_q$ is clearly unbounded below. Note that our approach breaks down in a crucial way if we relax the semiboundedness condition. We do not plan to pursue this issue any further as this situation looks physical irrelevant.

Remark 4.5. The second condition in Hypothesis 1.1 can be somewhat relaxed but the statement becomes weaker. For example, the condition $\int^\infty |xq(x)|\,dx < \infty$ will guarantee the existence of $\det\left(1+\mathbb{H}(\varphi_{x,t})\right)$ but classical differentiability becomes a serious issue. Further relaxation of the decay at $+\infty$ is a big open problem. It follows from the famous 1993 Bourgain result [17] that the problem (1.1) will remain well-posed (although in a much weaker sense) if q is square integrable at $+\infty$ but it is currently unknown if (1.1) is completely integrable for $q \in L^2$. Note that even the particular case of Wigner–von Neumann initial profiles is still an open problem [15]. But as opposed to Remark 4.4, such initial profiles are physically relevant as they may be used to model rogue waves [15].

We emphasize that as we have shown the solution (1.7) is classical. That is, the solution is at least three-times continuously differentiable in x and at least once in t while we do not assume any smooths of the initial data. Thus the *KdV flow* instantaneously smoothens any (integrable) singularities of $q(x)$. This effect, commonly called now *dispersive smoothing*, was first proven in 1978 by Cohen [48] for box shaped initial data with much of effort. In [43] we prove it for any initial data with the decay

$$q(x) = O\left(\exp\left\{-Cx^\delta\right\}\right), \quad x \to \infty, \tag{4.18}$$

with some positive C and δ. If $\delta > 1/2$ then $q(x,t)$ is meromorphic with respect to x on the whole complex plane (with no real poles) for any $t > 0$. If $\delta = 1/2$ then $q(x,t)$ is meromorphic in a strip around the x-axis widening proportionally to $\sqrt{t}$. For $0 < \delta < 1/2$ the solution need not be analytic but is at least Gevrey smooth. We also prove in [16] that under the extra condition (4.18) the singular numbers of $\mathbb{H}(\varphi_{x,t})$ have subexponential decay uniformly on compacts of (x,t). The latter means that the determinant in (1.7) rapidly converges suggesting that (1.7) could be used for numerical computations (cf. recent [49] for new numerical techniques for Fredholm determinants).

We also note that our approach can handle [40] nonintegrable singularities like Dirac $\delta-$functions, Coulomb potentials, etc. and the strong smoothing effect takes place even in this very singular, although not quite physical, setting.

Note that our solutions do not in general satisfy conservations laws. It would be interesting to find an analog of $\int u^2(x,t)\,dx$ under our conditions. Certain regularizations of conservation laws in a highly singular setting were considered in our [50]. We also bypassed answering such questions as well-posedness of the one-sided inverse scattering problem or direct proof of time evolution of scattering quantities. It is the limiting procedure that allowed us to detour such delicate questions.

5. Conclusions

We have given a partial answer to Zakharov's question stated in [51]: "In spite of all these brilliant achievements, the theory of the KdV equation is not yet developed to a level which would satisfy a pragmatic physicist, who may ask the following question: What happens if the initial data in the KdV equation is neither decaying at infinity nor periodic? Suppose that the initial data is a bounded function

$u(x) = u(x,0),\ |u(x)| < c.$

Can we extend the IST to this case, which has great practical importance?" Theorem 1.2 gives the affirmative answer to this question under the extra assumption that the initial profile decays fast enough at $+\infty$. However, only boundedness[15] from below is actually required. This can be viewed as a very strong manifestation of unidirectional nature of the KdV equation: no condition at $-\infty$ and a decay condition at $+\infty$. A complete answer to Zakharov's question requires the study of the influence of $+\infty$ on the KdV solution. By Bourgain's theorem a decay slower than $O\left(x^{-1/2}\right), x \to +\infty$, will cause some major issues as the problem (1.1) may fail to be well-posed. But even if it is well-posed, we need not have even one-sided scattering in this situation and would have to deal the spectral problem instead. The latter becomes very complicated and the time evolution of the spectral data need not be simple. The Lax pair representation of the KdV equation does not appear to be any easier than the KdV equation itself. Due to complexity of the spectrum the solution may have such a complicated structure that tracking it may be impossible and examples of such situations are already known. It happens in the study of the so-called *soliton gas*, a random distribution of infinitely many solitons. The underlying physics of this situation suggests that statistical description is much more suitable. Such approach was pioneered by Zakharov [52] back in 1971 and recently received renewed interest in the connection with *integrable turbulence* considered in [53]. The theory is under construction. We only mention [51] where certain type of soliton gas is described as a closure of the set of reflectionless rapidly decaying potentials of the Schrödinger operator. The resulting solutions are bounded, but neither periodic nor vanishing as $x \to \pm\infty$. (see also Gesztesy et al. [54]). A different approach to soliton gas and integrable turbulence was put forward by El (see, e.g. [55,56] and the literature cited therein). His approach is based on a closure of finite band potentials. Physical examples of integrable turbulence include coastal areas of seas, and effects occurring in optical fibers.

As we have already mentioned, step like initial profiles were first considered during the initial boom in the 1970s. The case of q's attaining different limits at $\pm\infty$ was considered first by Gurevich–Pitaevski [57] in 1973 and has been further developed by Hruslov [23] in 1976, Cohen [58] in 1984, Venakides [59] in 1986, and many others. The most complete asymptotic analysis of this case was recently done by Teschl and his collaborators in [25,28] (which also contain the expensive literature on the subject). The treatment is based upon the scattering theory for step potentials and somewhat similar to the rapidly decaying case but with serious complications coming from the negative continuous spectrum. The main feature of this case is that the initial step will emit solitons which are asymptotically twice as high as the original step followed by a nearly periodic "washboard". Another physically interesting case of a profile rapidly decaying at one end and approaching a periodic function at the other was first considered by Kotlyarov–Hruslov [24] in 1994. The study of such initial profiles recently culminated in [26] where two crystals fused together were considered.

Save [26] our class of step-type initial data is much more general. However the important problem of finding asymptotics of our solutions given in Theorem 1.2 is yet to be solved. The main challenge is that it is not clear at all how to adapt the powerful machinery of the Riemann–Hilbert problem so effectively used since the seminal 1993 paper [32] by Deift–Zhou to our setting. The above mentioned 2016 papers [25,28] do not suggest an easy solution.

Another important recent breakthrough is related to the 2008 question due to Deift [14]. He conjectures that, as in the periodic case, the solution will be almost periodic in time emphasizing that its existence even for small time is not known. A partial affirmative answer was recently given by Binder et al. [60].

Note that there are classes of explicit solutions to the KdV equation which are neither rapidly decaying nor periodic (quasi periodic). Many such solutions come from considering specific tau-functions in (1.4). This way a very important class of positon solutions was discovered by Matveev (see e.g. [61]). Such solutions are parametrized by a finite number of constants and have some interesting properties. However they are all singular and cannot be described within a suitable IST. It has also been long known (see e.g. the books [12] and [62]) that certain (formal) substitutions parametrized by some functions solve the KdV equation but again neither rapidly decaying nor (quasi) periodic. However as Marchenko says [13] "It has not been found yet whether it is possible (and if possible, then by what means) to determine these parameters so as to obtain the solution satisfying the initial data $u(x,0) = q(x)$, i.e., to solve the Cauchy problem". A partial answer is given in the same paper [13] in terms of a closure of certain specific types of potentials. The membership in such classes is hard to verify. Since time evolution in all these formulas is a priori given and simple, such solutions are very specific.

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[15] In fact, only essential boundedness from below is required [16].

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