Scalability of Hybrid Sparse Matrix Dense Vector (SpMV) Multiplication

Brian A. Page, Peter M. Kogge
Dept. of Computer Science and Engineering
Univ. of Notre Dame, Notre Dame, IN 46556, USA
Email: (bpage1,kogge)@nd.edu

***Abstract*—SpMV, the product of a sparse matrix and a dense vector, is emblematic of a new class of applications that are memory bandwidth and communication, not flop, driven. Sparsity and randomness in such computations play havoc with conventional implementations, especially when strong, instead of weak, scaling is attempted. This paper studies improved hybrid SpMV codes that have better performance, especially for the sparsest of such problems. Issues with both data placement and remote reductions are modeled over a range of matrix characteristics. Those factors that limit strong scalability are quantified.**

***Keywords*-Scalability, Hybrid SpMV, Communication Overhead, HPC;**

I. Introduction

The product of a sparse matrix and a dense vector (**SpMV**) is a key part of many codes from disparate areas. Numerically, it makes up the bulk of the High Performance Conjugate (HPCG) [1] code that has become an alternative to LINPACK for rating supercomputers. It can also be used in machine learning apps such as SVM computations via gradient descent. When the matrix operations are changed from floating multiply and add to other non-numeric functions that form semirings, it becomes an essential part of many graph kernels [2], and is a key function in the recently released GRAPHBLAS spec [3]. There has even been a novel prototype hardware system built around such sparse operations [4].

Motivating the work presented here was an earlier study, [5], that looked at strong scaling of SpMV in a Message Passing Interface (MPI) environment for a variety of matrices of varying sparsity from a well-known repository. What caught our interest was an observed significant **dip**, not increase, in speedup versus number of processes (see Fig. 1 where the numbers on each line is the average non-zeros per row) that approached an order of magnitude for the sparsest cases, and from which recovery was slow as the compute resources increased. We have observed similar dips in other kernels and wanted to explore this phenomena more closely. Our specific goal was thus to explore the cause of the dip as a precursor for developing codes that scale better for these very sparse cases.

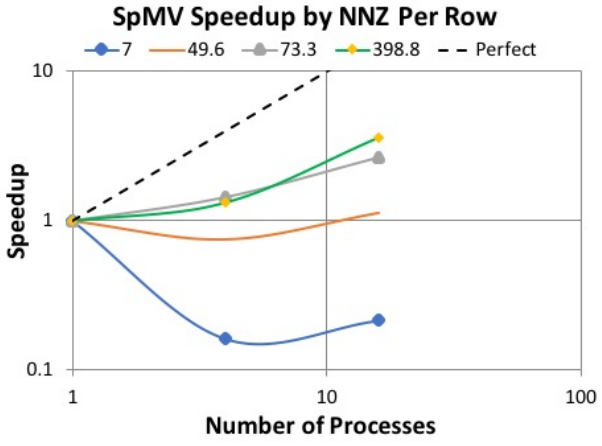


Fig. 1: Speedup from seen for 4 Sparse Matrices. This chart was generated using data seen in Table IV in [5].

In organization, Section II describes the generic hybrid algorithm. Section III describes the spectrum of sparse matrices considered. Section IV provides necessary background. Section V discusses the code developed here. Section VI evaluates the results. Section VIII concludes.

II. SpMV Hybrid Algorithm Overview

Algorithm 1 shows the simplest sequential implementation of SpMV, consisting of two nested loops, the innermost of which processes one non-zero at a time from a row in A.

Algorithm 1 Hybrid SpMV: c = Ax
A = mxm sparse matrix
m = number of rows to work over
c = result vector
x = dense input vector
a = vector of *non-zero* A element data values
$index$ = *non-zero* column id's
nnz_{row} = number of *non-zero* elements in the $i^{th} row$

```
1: procedure SpMV
2:     for i ← m do
3:         for j ← nnz_row do
4:             c[i]+= a[j] * x[index[j]]
5:         end for
6:     end for
```

Double precision floating point numbers use 8 bytes, therefore the minimal memory required by generalized SpMV is $O(nnz + 2n)$, where n is the matrix dimension.

Estimating SpMV's execution time requires knowledge about the number of operations, their type, the amount of memory accesses made, and some additional information

about the system architecture. Algorithm 1 performs 2 floating point operations for every non-zero element in the sparse matrix, along with three memory read accesses: one into the $index$ vector that identifies which column each non-zero belongs in, one into the matrix, and one into the dense vector x. This assumes each $c[i]$ is kept in a register until the row is complete. For most implementations there are very little cache hits. If each index entry can be held in 4 bytes, the memory traffic for each non-zero is approximately $8+4+8 = 20$ bytes for each two flops per non-zero. An analysis of SpMV within the HPCG code [6] demonstrates that this ratio is correct, and thus the core SpMV computations are totally memory bandwidth bound, regardless of core flop capabilities.

If the arrays in Algorithm 1 are in shared memory, then a multi-threaded implementation can compute separate rows concurrently, allowing the memory system to be fully utilized, and maximizing performance to the 2 flop per 20 bytes limit.

In contrast, the problems studied in this paper assume that SpMV is applied over large sparse matrices that must span multiple processes in a large distributed system. The matrix must be partitioned and distributed, and the pieces put back together to get a complete answer. A *hybrid algorithm* would then combine a distributed implementation with multi-threading within each process. This is the focus of this paper.

III. Matrix Benchmark Suite

We have chosen 25 matrices from the SuiteSparse Matrix Collection[1] [7]. Table I shows the dimensions, number of non-zeros, sparsity, and average number of non-zeros per row for the chosen matrices. The matrices were selected based on their average nnz_{row}, in addition to overall non-zero structure. At least two matrices from each nnz range were chosen with similar size or nnz per row but different structure.

Fig. 2 highlights the structure disparity between two of these matrices: *atmosmodd* and *parabolic_fem*. The non-zeros in *atmosmodd* cluster along the main diagonal, whereas *parabolic_fem* has a wider dispersion. We focused on structural differences to evaluate the impact of matrix structure on communication volume and message size across different load balancing methods. Additionally we chose matrices a quasi logarithmic fashion to ensure a wide spectrum of sparsity.

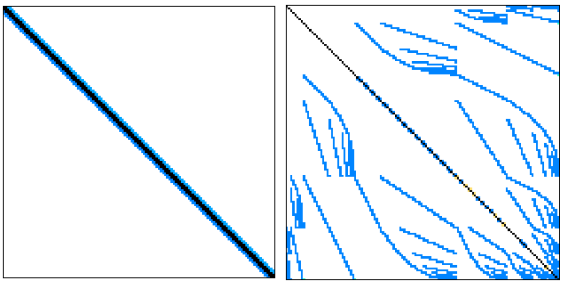
IV. Background

In this section we provide a brief overview on related work. IV-A discusses matrix nnz structure and its effects on SpMV performance. The importance of optimal load balancing along with some current methods is elaborated on in IV-B. Finally, in IV-C we look at some of the underlying implementation of MPI and its potential effects on communication overhead.

A. Matrix Structure

As input matrix size increases, the desire to leverage larger and more diverse systems facilitates the leap to distributed and hybrid parallel implementations. In such cases, the distribution of non-zero elements in the matrix has significant impact on

[1]Currently hosted at https://sparse.tamu.edu/



(a) atmosmodd (b) parbolic_fem

Fig. 2: Varying Matrix Non-Zero Structure was selected for the benchmark matrices chosen in our experiments. Distribution visualization provided from [7].

matrix	rows	nnz	nnz %	nnz row	Symmetry
atmosmodd	1270432	8814880	5.46E-06	6.93	non-symmetric
parabolic_fem	525825	3674625	1.33E-05	6.98	symmetric
rajat30	643994	6174244	1.49E-05	9.58	non-symmetric
CurlCurl_3	1219574	13544618	9.11E-06	11.10	symmetric
offshore	259789	4242673	6.29E-05	16.33	symmetric
Fem_3D_thermal2	147900	3489300	1.60E-04	23.59	non-symmetric
nlpkkt80	1062400	28192672	2.50E-05	26.53	symmetric
CO	221119	7666057	1.57E-04	34.66	symmetric
gsm_106857	589446	21758924	6.26E-05	36.91	symmetric
msdoor	415863	19173163	1.11E-04	46.10	symmetric
bmw3_2	227632	11288630	2.18E-04	49.59	symmetric
BenElechi1	245874	13150496	2.10E-04	53.48	symmetric
t3dh	79171	4352105	6.94E-04	54.97	symmetric
F2	71505	4294285	8.40E-04	60.05	symmetric
consph	83334	6010480	8.65E-04	72.12	symmetric
SiO2	155331	11283503	4.68E-04	72.64	symmetric
torso1	116158	8516500	6.31E-04	73.31	symmetric
dielFilterV3real	1102824	89306020	7.34E-05	80.97	symmetric
RM07R	381689	37464962	2.57E-04	98.15	non-symmetric
m_t1	97578	9753570	1.02E-03	99.95	symmetric
crankseg_2	63838	14148858	3.47E-03	221.63	symmetric
nd24k	72000	28715634	5.54E-03	398.82	symmetric
TSOPF_RS_b2383	38120	16171169	1.11E-02	424.21	non-symmetric
mouse_gene	45101	28967291	1.42E-02	642.27	symmetric
human_gene1	22283	24669643	4.97E-02	1107.10	symmetric

TABLE I: Benchmark Matrix Suite

SpMV performance, and subsequently affects a myriad of design decisions [8], [9], [10], [11], [12].

While the structure of a sparse matrix will undoubtedly influence the efficiency of the storage format chosen, matrix structure exhibits greater influence over communication [13], [5], [14], [15], [16], [17], [18], [19]. Hybrid implementations possess improved data locality, thread scalability, and data reusability [15]. Elevated performance is not guaranteed however, as memory contention can occur with greater frequency as thread counts increase [14]. In spite of this, the potential for significant global communication reduction, by converting much of it to local communication, has ever greater effects on performance as system size is scaled up.

B. Work Distribution

An irregular memory access pattern is the root cause of SpMV performance degradation in distributed memory systems [13]. Therefore as process and thread count increase, given a matrix of arbitrary structure, the disparity in the volume of work as well as the data locality of the individual nnz elements being computed locally can change drastically

[13], [5], [14], [15], [16], [17], [18], [19]. Many studies focusing on optimal work load balancing for distributed SpMV have been performed [17], [20], [18], [21], [22], [19] in an effort to obtain consistency in performance gains.

It should be intuitive to say that reducing communication improves overall SpMV run time by limiting latency, memory registration, and buffer copies to memory along with other factors. Several studies have shown that the key to avoiding excessive communication overhead is to determine the optimal work load distribution for a given matrix [17], [20], [18], [21], [22], [19]. These studies found that balancing work by accounting for nnz location within the input matrix allowed for reduced communication and message size.

Communication avoidance is a complex issue to resolve as it is rooted in the system hardware, interconnection hardware, as well as communication and user level protocols being used.

Furthermore, the optimal work distribution algorithm is likely to be optimal for only a subset of the matrices being evaluated. It then becomes necessary to find the appropriate level of optimality and efficiency.

C. MPI Overhead

MPI overhead is a well researched area and continues to be of importance as we look towards future HPC systems. As hybrid and or heterogeneous implementations of SpMV require communication at scale, message passing as well as interconnect topologies and their user level libraries play a vital role in their overall performance.

Message Volume, Size, and Latency. Message latency when using MPI is dependant on many factors such as message size, interconnect type, protocol selection, etc [23], [24]. For small messages, the Eager protocol utilizes pre-allocated buffer locations to perform communication between the two processes [24]. There are no acknowledgements or negotiation required by the Eager protocol, however it suffers from relatively low bandwidth utilization [23], [24]. If message size is greater than the threshold value in use, the Rendezvous protocol is normally utilized. The Rendezvous protocol has additional overhead costs due requiring allocation of appropriate buffer memory to receive the message, as well as perform a negotiation phase between the sender/receiver [24].

Message size varies with P (number of processes), changing how messages are sent. Optimizations such as the use of RDMA, buffer reuse, communication overlapping, and speculation all have a point at which their performance enhancement declines drastically if not disappears [23], [24].

Remote Direct Memory Access RDMA. RDMA permits zero-copy transfers of data between MPI processes by performing either an MPI_Put or MPI_Get operation[23], [24]. RDMA eliminates the need for buffers and their costly allocations, as well as the need to copy buffered data into its final memory location after transmission completion [24].

Communication Overlapping, Offloading/Onloading. Overlapping as well as offloading are designed to allow for communication during a subsequent computation phase, and are aided by the use of RDMA for increased bandwidth utilization with decreased latency. In the case of Overlapping, non-blocking MPI_Isend or MPI_Irecv are used to begin the communication phase before proceeding onto computation.

Offloading and Onloading work similarly. In Offloading communication is pushed onto the interconnect hardware for completion thereby allowing the process to continue. Conversely with onloading; communication overhead is handled by a selected number of threads, cores, or sockets in an effort to alleviate the impact of high bandwidth communication on the NIC or interconnect interface [24]. Onloading is becoming more common as many-core architectures such as Intel Xeon Phi (formerly Knights Landing) see greater use [25].

Collectives. MPI Collectives allows multiple processes to partake in a single communication event such as a reduction or gather. Several studies [26], [27], [28] have evaluated the performance and behaviour of MPI collectives in addition to implementing optimization methods. Collectives dramatically reduce communication volume and latency along with increase bandwidth utilization [26], [27], [28]. Enhanced RDMA has proven useful for such operations.

V. Experimental Implementation

To evaluate MPI overhead for our hybrid SpMV solver we implemented two different work distribution schemes. The first method distributes work to each process based on the unmodified nnz positioning of A, and is hereafter known as the "*naive*" method. In contrast the "*balanced*" method creates a near uniform work load amongst all MPI processes. Section V-A elaborates upon the sparse matrix storage format we have to chosen utilize. Section V-B briefly discusses OpenMP and MPI as they pertain to our implementations. Subsequently sections V-C and V-D elaborate upon the *naive* and *balanced* work distribution methods respectively.

A. SpMV Storage Format

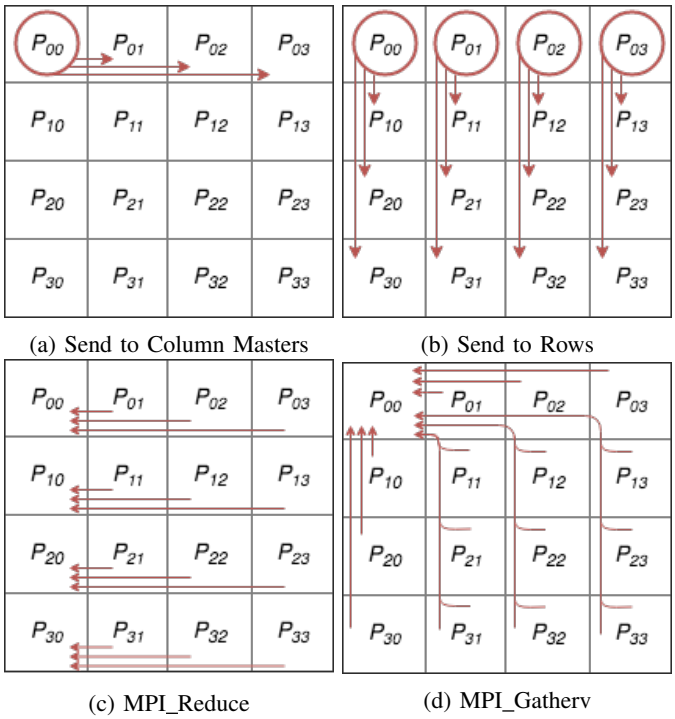
While many storage methods for sparse matrices have been devised, each has drawbacks, whether it be increased memory footprint, communication requirements, or implementation complexity [8], [29], [30], [9], [10], [11], [12]. Compressed Sparse Row (CSR) format is a commonly used method in which the non zeros within a given matrix are kept in three one-dimensional vectors or arrays: row, column, and data. CSR features a relatively simple structure and is therefore easily implemented and adapted for hybrid applications utilizing both MPI and OpenMP. Furthermore, nearly all other formats compare themselves to CSR [8], [9], [10], [11], [12]. While some formats may enable greater memory efficiency or communication reduction, the added implementation and algorithmic complexities are drawbacks that should not be overlooked. Therefore CSR was chosen here; however we acknowledge that some performance impact might be gained via the use of different storage formats.

B. Hybrid Design

Two widely known parallel methodologies, distributed memory and shared memory, offer the potential for increased

performance at the expense of elevated code complexity. MPI makes use of the distributed memory model by dispersing application functionality across different processes, thereby allowing for strong scaling through increases in system size. Increased system size comes at the expense of added communication requirements due to added intra/inter-process message count. The shared memory model has potential for localized performance enhancements from increased cache utilization and performance. Hybrid codes, typically MPI and OpenMP, take advantage of both models by implementing the shared memory model within each distributed memory element.

Our code utilizes MVAPICH2 with Infiniband support and OpenMP 4.5. Our cluster has 72 nodes, each with two 8-core Xeon E5-2650v2 @ 2.6GHz, and connected via Mellanox FDR Infiniband in non-blocking configuration. Each node is assigned two MPI processes, one per socket, with each MPI process creating one OpenMP thread per core. We insured process and thread affinity by binding processes to their assigned socket, and OpenMP threads to their respective cores.



(a) Send to Column Masters (b) Send to Rows

(c) MPI_Reduce (d) MPI_Gatherv

Fig. 3: Work Distribution Communication Pattern: (a) Work for each process column is sent to that columns process master, (b) each column master then sends each process in its column their own work portion. (c) The Naive method, described in Section V-C and [5], utilizes MPI_Reduce across each process row to compile calculated row values at the row master process. (d) In contrast, the Balanced method, described in Section V-D, uses a variable gather compile computed values on the global master process.

C. Work Load Distribution: Naive

Our first work load distribution method, based on [5], splits the matrix A into P sub-matrices $\bar{A}_{ij}$, each of which is assigned to a corresponding process P_{ij} in the process matrix. The master process, P_{00}, sends the column master process in each process column the data for all processes within that process column. Column masters, P_{0j}, distribute row, col, and data vector information to each process they govern.

Splitting up A in this manner is not only conceptually simplistic, but also limits the nnz_{row} per process that any process can receive. Any process in the process matrix will receive at most $\frac{m}{p}$ rows, with at most $\frac{m}{p}$ columns per row. Within each P_{ij}, multiple OpenMP threads are initialized and work over the process's work allotment concurrently. Once each process has completed its work allotment, all processes participate in an MPI_REDUCE with MPI_SUM. This is possible since each process in a given process row computes a sub sum of the entire row in A. The row masters, P_{i0}, now have the final result for their respective row.

Fig. 3 (a)-(c) visualizes the naive work distribution method we used. The reduction step, seen in Fig. 3c, illustrates that during the final step there are $\sqrt{p}$ concurrent reductions, each with $\sqrt{p}$ participating processes. Given that the default reduction algorithm for MPI has a time complexity of $\mathcal{O}(log_2(p))$, the *naive* method experiences the same complexity due to its use of MPI_Reduce to gather computation results.

We now look at the quality of work load distribution generated by this method. As mentioned previously the naive method is tied directly to the structure of A. It is thus possible for a process, or multiple processes, to be assigned no work to perform during the SpMV computation phase. To evaluate the quality of work load distribution between the naive and balanced methods, several metrics were employed. It is well known that the standard deviation can experience issues in illuminating the true nature of a distribution if it is not Gaussian. The balanced distribution method is designed with the intent of generating a near uniform distribution, as will be discussed in greater detail in the following section, and is therefore not Gaussian. Instead the Mean Absolute Deviation (MAD) [31] was chosen to indicate the distribution quality.

A perfectly uniform distribution would have a MAD of 0.0, meaning that the average deviation from the mean is 0.0. A higher MAD indicates a distribution with much greater variation from the mean and indicate process load imbalance. MAD values were normalized and represent % of total number of nnz for each A. The min, max, and median MADs observed for the selected matrices were 6.0E-03, 3.2E-01, 7.1E-02 and respectively. These values while appearing small are in fact relatively large, indicating significant imbalance.

D. Work Load Distribution: Balanced

The goal of the *balanced* method is to create a uniform work load distribution across all P MPI processes, thereby minimizing communication and message size. The optimal amount of nnz per process is calculated as being $\omega = \frac{nnz}{m}$, where m is the dimension of A. It is worth noting that ω may be less than ideal for computation due to cache performance. However it does allows for a quasi-uniform work distribution based on a matrices' internal structure. Where possible entire contiguous rows are assigned to a single process so that cache misses may be reduced where possible. For very large matrices

it is possible that the number of nnz in a given row may exceed that of ω, therefore ω is used as the break point for maximum row size. Should $nnz_{row_i} > \omega$, row_i is split and a new row containing all nnz of row_i after the $\omega^{th} nnz$, is created with the same row id.

Once we insure that there are no rows exceeding ω, all rows were then sorted by length. An efficient greedy bin packing algorithm assigns the sorted rows to MPI processes from largest to smallest, based on a process's current nnz assignment. Due to the desire to maintain rows in their largest possible contiguous form we allowed for the possibility of over/under assignment of work to any given process. Clearly this can generate sub-optimal work distributions. While optimal bin packing would be ideal, it is known to be an NP-complete problem [32], [33], therefore efficiency and ease of implementation took precedence in our study. The quality of distribution produced by our method is discussed later.

Similar to the *naive* method, once work load distribution has been determined, the master process, P_{00}, sends all work for a given process column to that column's column master P_{0j} (Figure 3a). Each P_{0j} then distributes the work assigned to each individual process within in its process column (Fig. 3b). Computation is then performed within all OpenMP threads of P_{ij} like the naive method, yet now each process may have differing number of rows assigned to them. MPI_Reduce requires all participating processes to a send buffer of equal size potentially requiring superfluous message padding.

In the *balanced* distribution method it is not guaranteed and highly unlikely that all processes in each process row would posses portions of the same rows from the sparse matrix. Therefore in an effort to reduce message size we decided to implement a variable gather in the *balanced* method (Fig. 3d), as it would allow P_{00} to receive the exact number of row solutions assigned to a given process. This eliminates the padding necessary for implementing MPI_Reduce. MPI_Gatherv requires a mapping of sorts so that MPI knows the incoming buffer size for each process. This mapping was generated during work load determination and kept for use during the gather phase. Our implementation of MPI_Gatherv was such that all results were gathered at the master and a post gather insertion into the result vector was performed.

Given that the exact algorithm utilized by MVAMPICH2 for a collective is dependent on message size and other factors, we assume the variable gather implemented has a complexity of $\mathcal{O}(log_2 p)$, similar to a gather utilizing the binomial tree algorithm.

As mentioned previously it is possible for the final work distribution to be less than perfectly uniform thanks to our greedy implementation. Inspection of the MADs produced by our *balanced* distribution show a min, max, and median of 2.7E-08, 6.7E-03, and 4.74E-06 respectively. This indicates that we achieve a very high level of uniformity. Therefore our *balanced* work method can be utilized to create message sizes of greater uniformity for all processes in P as intended.

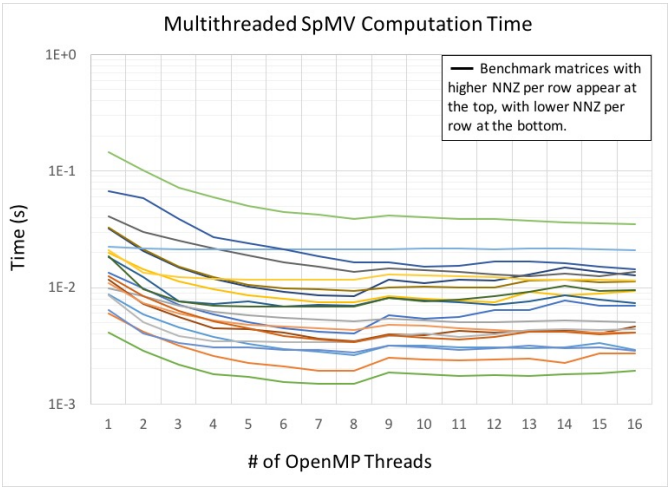


Fig. 4: Observed SpMV computation time for single MPI process with varying OpenMP thread counts. Each line represents a different benchmark matrix from the chosen suite discussed in Section III. Single node multithreaded tests include no communication since OpenMP uses a shared memory model and only one node is in use.

VI. Evaluation

A. Multithreaded SpMV Performance

To show the impact that communication from a distributed memory model has on Hybrid SpMV, we first analyzed 1 MPI process running on a single node, with increasing numbers of OpenMP threads. Fig. 4 shows that as thread count increases SpMV computation time decreases, up to a point. Each additional thread makes more memory references needed to perform additional useful computation. However, for most matrices evaluated, computation time reduction slows or stagnates completely after some value of P. This occurs approximately when the number of threads exceeds those supported by a single socket, and is likely due to cross-socket coherency traffic. Given relative best performance seen at 1 thread per core, subsequent experiments used a single MPI process per socket, and utilized all cores on that socket for computation.

B. MPI Communication Overhead

MPI communication, whether it be intra-node or inter-node, can dramatically outweigh the time requirement for useful computation. Fig. 5 shows MPI communication time as well as SpMV time for each matrix and P examined. Note that for 1 process there is no communication time. After the utilization of 4 processes MPI_Reduce time is nearly always larger than SpMV computation time, and often several times as long.

Fig. 5 clearly indicates that distributed communication, even in a hybrid implementation, is extremely high for SpMV applications. In general SpMV computation time decreases with strong scaling, and as P increases, the number of elements to be communicated by each process decreases as $\frac{m}{p}$. Fig. 6 shows the ratio of communication time to computation time from Fig. 5. On average, for a given matrix, as the number of processes increases, the ratio of comm-to-comp time increases

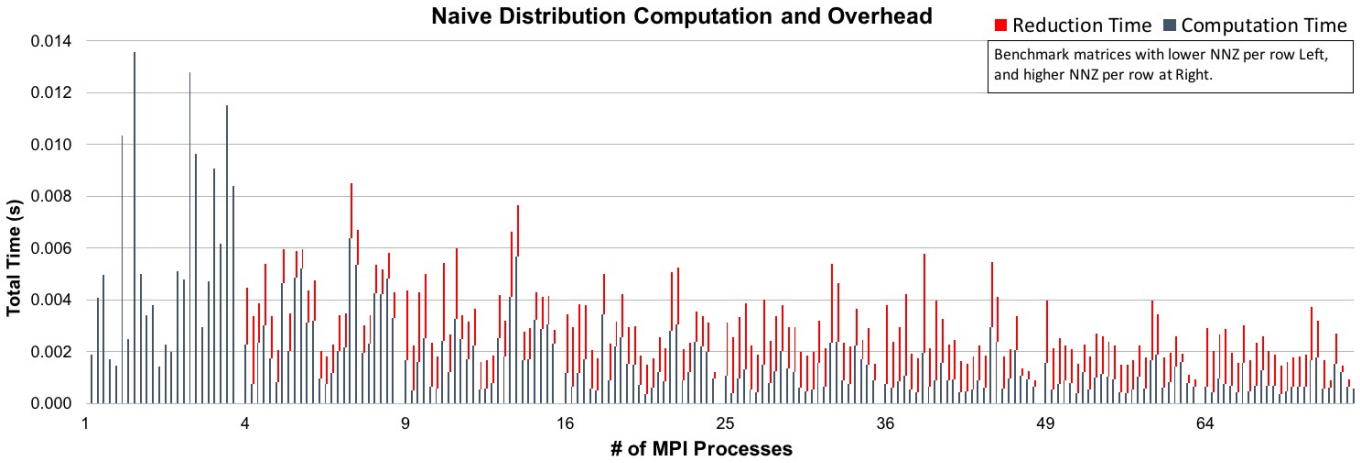


Fig. 5: Observed MPI_Reduce times using the $naive$ distribution method. Communication time sits atop spmv computation time for comparison with differing MPI process counts. Benchmark matrices are shown in ascending order by avg nnz per row.

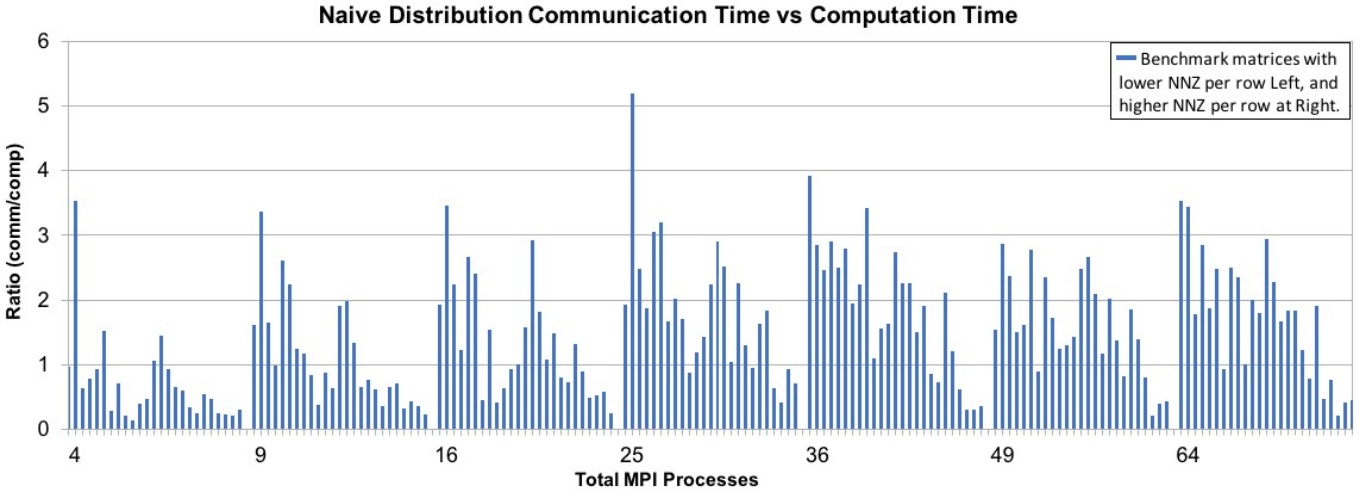


Fig. 6: Communication-to-Computation Time Ratio: This chart shows the ratio of communication time to SpMV computation time for the evaluated process counts. Matrices are listed in order ascending according to nnz_{row}. Matrices with the lowest nnz_{row} experience the worst $\frac{comm}{comp}$ ratio, while higher nnz_{row} matrices see considerably less.

as well. Additionally we notice that the matrices with fewer nnz per row experienced some of the highest ratios, while matrices with higher avg nnz per row experienced some of the lowest. Furthermore, in an effort to eliminate the impact of cluster node selection on communication paths all tests were run 50 times with the averaged results used in analysis.

While Fig. 5 and 6 prove that communication time is high, these charts only include data for our tests over Mellanox Infiniband interconnects. Infiniband's performance increases over Ethernet are well documented and beyond the scope of this paper. We did however examine the communication time for the entirety of the benchmark suite using both distribution methods, and observed comm-to-comp ratios of 10x or more.

The *balanced* method is designed to reduce communication by providing a balanced workload. Fig. 7 shows the comm-to-comp ratios observed using the Infiniband interconnect. It illustrates the same pattern in which matrices with low average nnz per row experience higher comm-to-comp ratios while the most dense matrices received the lowest ratios. In Fig. 7(b) at $P = 81$ there is a considerable jump in communication overhead somewhere in the range of 10x compared to $P = 64$. Our system consists of 72 dual socket nodes, connected to two Inifiband switches. The jump from 64 to 81 processes corresponds with a jump in total node utilization as well. At 81 processes our system size now spans two switches meaning that messages accross the additional hardware will have increased latency. This is the reason for the observed increase, as after this initial threshold communication times increase at a $log_2(p)$ rate similar to $P < 81$.

It is clear that as P increases MPI communication becomes the dominant factor in overall runtime, even for the densest matrices in the benchmark suite.

C. Communication Overhead Impact

To compare communication time against theoretical expectations Fig. 8 plots the normalized observed time against a $log_2(p)$ curve. The solid line beginning at 1 represents $log_2(p)$, the simplified time complexity for MPI_Gather [26], while the second solid line beginning at 81 represents $\gamma log_2(p)$

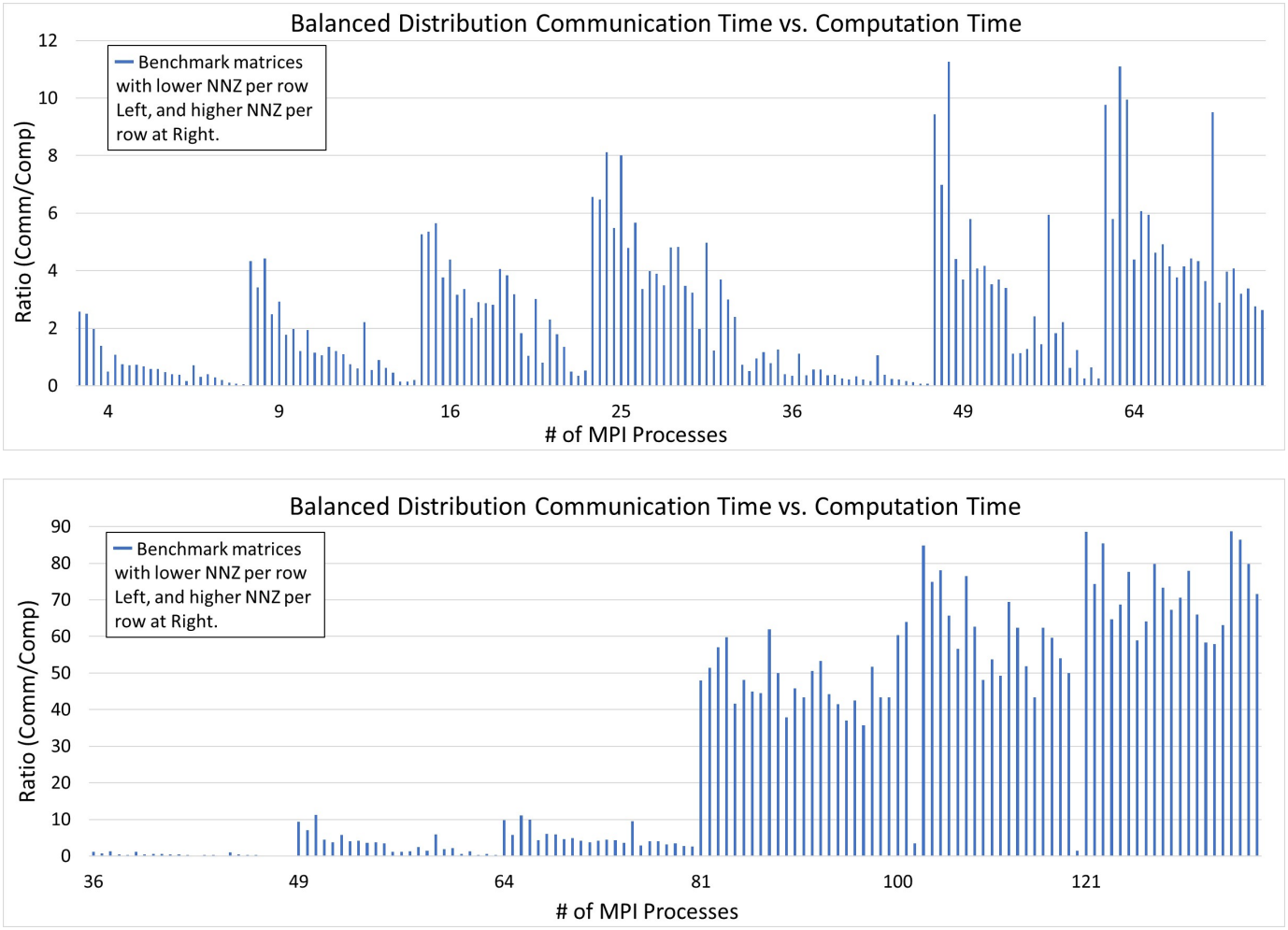


Fig. 7: Ratio of observed communication time to observed SpMV computation time for the *balanced* distribution method. (a) Illustrates the calculated ratio for all benchmark matrices using between 0 and 64 MPI processes. (b) Shows the same ratio using 36 to 121 MPI processes. Benchmark matrix results are in ascending order left-to-right based on the average nnz per row.

in which γ is the observed scaling factor caused by our network hardware layout. Some variance was expected when producing the fit shown in Fig. 8 as a single latency and effective bandwidth value was calculated for the two regions, $P = 0 - 64$ and $P = 81 - 121$. A tighter fit is possible if latency and effective bandwidth are accounted for each value of P when normalizing and plotting.

What can be seen is that for most P, the observed MPI communication time is in the range of the expected MPI_Gather time complexity. However for 36-49 processes there are significant dips in the communication time for nearly all matrices. In fact when each matrices values were plotted and fit individually, we noticed a similar dip for all matrices.

The message being transmitted is the computed value for a particular row, or multiple rows depending on the amount assigned to any arbitrary process. As P increases $\frac{m}{P}$ decreases, meaning the number of rows that must be sent per process decreases correspondingly. MPI implements a message size threshold with which it determines when to utilize the Eager or Rendezvous protocol for sending messages. Once message size drops below this threshold MPI will select the Eager protocol, which experiences very low latency due to the elimination of: memory buffer allocation, negotiation between processes, and message receipt acknowledgements.

While there are other causes for this communication performance increase, such as cache effects or better work balancing, we believe that the transition between protocols provides the greatest impact at these values of P. Shortly after these local minima, we saw an increase in communication time as P increased. This was expected due to the nature of the default communication algorithm for MPI_Gather. Similar results were found when we analyzed the $naive$ distribution method, which also has time complexity of $\mathcal{O}(log_2(p))$.

After evaluating communication time for larger process counts, we then sought to evaluate the overall performance by combining SpMV computation time with its corresponding communication time. As can be seen in Fig. 9 the speedup seen for all matrices can be split up into three general performance bands. Obtaining the best performance are those matrices with the highest nnz per row, receiving an increase in performance

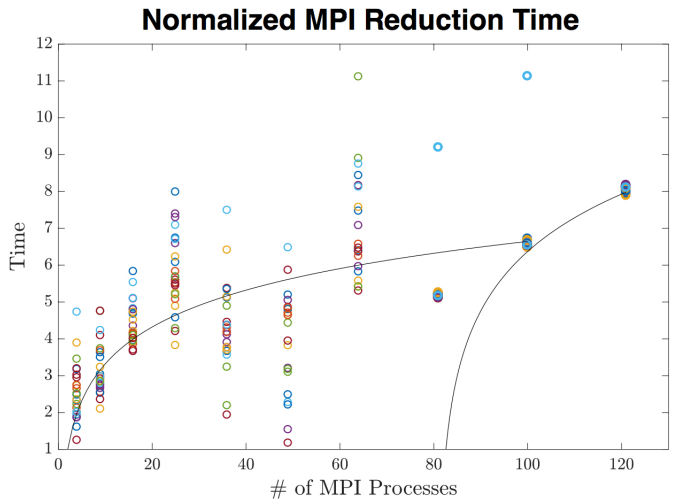


Fig. 8: Observed MPI Collective Times: shows the normalized time measurements for MPI_Gatherv used within the *balanced* method plotted along the expected $\mathcal{O}(log_2(p)$ curves for both small and large values of P. Each circle represents a different benchmark matrix evaluated at the given process count.

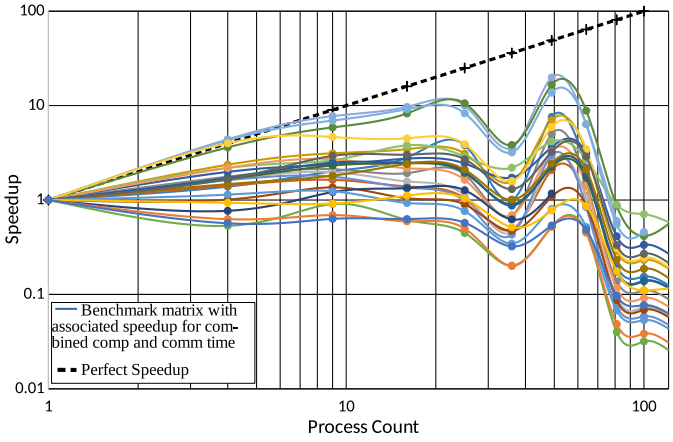


Fig. 9: Combined performance for SpMV computation and Communication of result, in GFlops. Each line represents a different benchmark matrix for the given number of MPI Processes, using the *balanced* distribution method.

until 16 processes. Matrices with approximately 10 to 100 nnz per row form the center band exhibiting mild speedup until 4 processes before flattening out. Here any additional performance gains seen as process count increases is likely offset by the accompanying increase in communication overhead. Finally the lower band consisting of matrices with fewer than 10 nnz per row never experience an increase speedup.

Regardless of overall matrix sparsity or nnz distribution pattern, all matrices tested exhibited similar behavior as process count increased. The dip in performance seen between 25 and 36 processes is likely due to crossing the switch capacity boundary for the Inifiband fat tree configuration. The peak at 49 processes corresponds to a switch between the Eager and Rendezvous message transfer protocols within MPI thus providing a boost in performance as message size decreased below the threshold. Degradation in performance after 49 processes is due to several factors including compounding switch latency, rapidly declining number of non-zeros per process, and increased message volume.

Another important observation is one of maximum peak performance. From Fig. 9 it is obvious that some matrices drastically outperformed others. In general, the more dense a matrix, the higher its maximum performance, while the matrices with higher sparsity experienced performance flattening for lower values of P. The results suggest an upper bound for performance based on communication overhead and its accompanying process count.

VII. Future Work

While our work has shown that performance degradation created by communication overhead exists for extremely sparse problems on conventional computing clusters, further exploration is needed. Sparse problems will be evaluated on additional architectures such as GPUs and Intel's Xeon Phi, as will the communication impact these architectures experience in a multi-node environment. Similarly we would like to investigate resource over-subscription in an effort to further explain potential NUMA effects brought on by the cross socket coherency traffic complexities we believe were seen in Fig 4.

Our tests thus far have required that the entirety of a sparse matrix and its accompanying dense vector fit on a single node. The issues associated with larger data sets incapable of residing on individual nodes, as well as the ability for streaming updates to the sparse matrix and or the dense vector are also of interest. These concepts are worthy of further analysis as they will likely generate communication issues for work load distribution and computation.

Also, as mentioned in the Introduction, SpMV-like kernels have key use in non-numeric applications such as GraphBLAS. A valuable study would be to generalize the algorithm for non-floating point operation suites and data, such as might be found in calculations around adjacency matrices. In addition, new data compression methods will be evaluated in an attempt to further reduce communication between processes and nodes. Finally, it would be of significant value to explore variations in the x vector, from being sparse itself, to multiple vectors, to matrices (both dense and sparse).

VIII. Conclusions

This study evaluated hybrid SpMV implementations with particular focus on strong scaling where the problem is fixed and additional computing resources are added. We demonstarte that dingle process performance does improve as thread counts increase, but flattens out once the memory system is saturated, with some degradation in performance once the process covers more than one socket. In addition, this paper demonstrates that overall scalability in a message passing environment is driven by its communication pattern. Our findings suggest SpMV possesses an inability to strong scale beyond some point, even with cutting edge network interconnects such as Inifiniband,

and that this scaling limit is tightly tied to the level of sparsity in the matrix. Further, the current focus in the literature on reducing SpMV computation time (such as by smart partitioning) does not address this dominant communication overhead factor at scale. Manycore architectures are thought to be the answer to this issue, however our studies show that as $P \to \infty$ communication overhead is the dominant factor and cannot be discarded. For extremely sparse matrices we saw an impact due to inter-node communication with as few as 2 nodes (4 processes). This would suggest a need to shift efforts away from reducing SpMV computation time in favor of attempting to mitigate or eliminate conventional communication within such applications. Alternative schemes for remote aggregation will be central for new algorithms that perform better than the current paradigm.

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