Interpolation of data by smooth non-negative functions

Charles Fefferman, Arie Israel, Garving K. Luli *

March 9, 2016

Introduction

Continuing from [18], we prove a finiteness principle for interpolation of data by nonnegative C^m functions. Our result raises the hope that one can start to understand constrained interpolation problems in which e.g. the interpolating function F is required to be nonnegative.

Let us recall some notation used in [18].

We fix positive integers m, n. We write $C^m(\mathbb{R}^n)$ to denote the Banach space of all real valued locally C^m functions F on $\mathbb{R}^n$, for which the norm

$$\|F\|_{C^m(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n} \max_{|\alpha|\le m} |\partial^\alpha F(x)|$$

is finite.

We will also work with the function space $C^{m-1,1}(\mathbb{R}^n)$. A given continuous function $F : \mathbb{R}^n \to \mathbb{R}$ belongs to $C^{m-1,1}(\mathbb{R}^n)$ if and only if its distribution derivatives $\partial^\beta F$ belong to $L^\infty(\mathbb{R}^n)$ for $|\beta| \le m$. We may take the norm on $C^{m-1,1}(\mathbb{R}^n)$ to be

$$\|F\|_{C^{m-1,1}(\mathbb{R}^n)} = \max_{|\beta|\le m} \text{ess.} \sup_{x\in\mathbb{R}^n} \left|\partial^\beta F(x)\right| .$$

*The first author is supported in part by NSF grant DMS-1265524, AFOSR grant FA9550-12-1-0425, and Grant No 2014055 from the United States-Israel Binational Science Foundation (BSF). The third author is supported in part by NSF grant DMS-1355968 and a start-up fund from UC Davis.

Expressions $c(m,n)$, $C(m,n)$, $k(m,n)$, etc. denote constants depending only on m, n; these expressions may denote different constants in different occurrences. Similar conventions apply to constants denoted by $C(m,n,D)$, $k(D)$, etc.

If X is any finite set, then $\#(X)$ denotes the number of elements in X.

We are now ready to state our main theorem.

Theorem 1 *For large enough* $k^{\#} = k(m,n)$ *and* $C^{\#} = C(m,n)$ *the following hold.*

(A) C^m **FLAVOR** *Let* $f : E \to [0,\infty)$ *with* $E \subset \mathbb{R}^n$ *finite. Suppose that for each* $S \subset E$ *with* $\#(S) \leq k^{\#}$*, there exists* $F^S \in C^m(\mathbb{R}^n)$ *with norm* $\|F^S\|_{C^m(\mathbb{R}^n)} \leq 1$*, such that* $F^S = f$ *on* S *and* $F^S \geq 0$ *on* $\mathbb{R}^n$.

Then there exists $F \in C^m(\mathbb{R}^n)$ *with norm* $\|F\|_{C^m(\mathbb{R}^n)} \leq C^{\#}$*, such that* $F = f$ *on* E *and* $F \geq 0$ *on* $\mathbb{R}^n$.

(B) $C^{m-1,1}$ **FLAVOR** *Let* $f : E \to [0,\infty)$ *with* $E \subset \mathbb{R}^n$ *arbitrary. Suppose that for each* $S \subset E$ *with* $\#(S) \leq k^{\#}$*, there exists* $F^S \in C^{m-1,1}(\mathbb{R}^n)$ *with norm* $\|F^S\|_{C^{m-1,1}(\mathbb{R}^n)} \leq 1$*, such that* $F^S = f$ *on* S *and* $F^S \geq 0$ *on* $\mathbb{R}^n$.

Then there exists $F \in C^{m-1,1}(\mathbb{R}^n)$ *with norm* $\|F\|_{C^{m-1,1}(\mathbb{R}^n)} \leq C^{\#}$*, such that* $F = f$ *on* E *and* $F \geq 0$ *on* $\mathbb{R}^n$.

Our interest in Theorem 1 arises in part from its possible connection to the interpolation algorithm of Fefferman-Klartag [15, 16]. Given a function $f : E \to \mathbb{R}$ with $E \subset \mathbb{R}^n$ finite, the goal of [15, 16] is to compute a function $F \in C^m(\mathbb{R}^n)$ such that $F = f$ on E, with $\|F\|_{C^m(\mathbb{R}^n)}$ as small as possible up to a factor $C(m,n)$. Roughly speaking, the algorithm in [15, 16] computes such an F using $O(N \log N)$ computer operations, where $N = \#(E)$. The algorithm is based on an easier version [10] of Theorem 1. Our present result differs from the easier version in that we have added the hypothesis $F^S \geq 0$ and the conclusion $F \geq 0$. Accordingly, Theorem 1 raises the hope that we can start to understand constrained interpolation problems, in which e.g. the interpolant F is required to be nonnegative everywhere on $\mathbb{R}^n$.

For results related to Theorem 1, we refer the reader to our paper [18] and references therein.

In the following sections, we will set up the notation; then we will recall a main theorem in [18] and use it to prove Theorem 1.

This paper is part of a literature on extension, interpolation, and selection of functions, going back to H. Whitney's seminal work [33], and including fundamental contributions by G. Glaeser [19], Y, Brudnyi and P. Shvartsman [4, 6–9, 23–31], J. Wells [32], E. Le Gruyer [21], and E. Bierstone, P. Milman, and W. Pawłucki [1–3], as well as our own papers [10–17]. See e.g. [14] for the history of the problem, as well as Zobin [34, 35] for a related problem.

We are grateful to the American Institute of Mathematics, the Banff International Research Station, the Fields Institute, and the College of William and Mary for hosting workshops on interpolation and extension. We are grateful also to the Air Force Office of Scientific Research, the National Science Foundation, the Office of Naval Research, and the U.S.-Israel Binational Science Foundation for financial support.

We are also grateful to Pavel Shvartsman and Alex Brudnyi for their comments on an earlier version of our manuscript, and to all the participants of the Eighth Whitney Problems Workshop for their interest in our work.

1 Notation and Preliminaries

1.1 Background Notation

Fix $m, n \geq 1$. We will work with cubes in $\mathbb{R}^n$; all our cubes have sides parallel to the coordinate axes. If Q is a cube, then δ_Q denotes the sidelength of Q. For real numbers $A > 0$, AQ denotes the cube whose center is that of Q, and whose sidelength is $A\delta_Q$.

A dyadic cube is a cube of the form $I_1 \times I_2 \times \cdots \times I_n \subset \mathbb{R}^n$, where each I_ν has the form $[2^k \cdot i_\nu, 2^k \cdot (i_\nu + 1))$ for integers $i_1, \cdots, i_n$, k. Each dyadic cube Q is contained in one and only one dyadic cube with sidelength $2\delta_Q$; that cube is denoted by Q^+.

We write $B_n(x, r)$ to denote the open ball in $\mathbb{R}^n$ with center x and radius r, with respect to the Euclidean metric.

We write $\mathcal{P}$ to denote the vector space of all real-valued polynomials of degree at most $(m-1)$ on $\mathbb{R}^n$. If $x \in \mathbb{R}^n$ and F is a real-valued C^{m-1} function on a neighborhood of x, then $J_x(F)$ (the "jet" of F at x) denotes the $(m-1)^{rst}$ order Taylor polynomial of F at x, i.e.,

$$J_x(F)(y) = \sum_{|\alpha| \leq m-1} \frac{1}{\alpha!} \partial^\alpha F(x) \cdot (y-x)^\alpha .$$

Thus, $J_x(F) \in \mathcal{P}$.

For each $x \in \mathbb{R}^n$, there is a natural multiplication $\odot_x$ on $\mathcal{P}$ ("multiplication of jets at x") defined by setting

$$P \odot_x Q = J_x(PQ) \text{ for } P, Q \in \mathcal{P}.$$

If F is a real-valued function on a cube Q, then we write $F \in C^m(Q)$ to denote that F and its derivatives up to m-th order extend continuously to the closure of Q. For $F \in C^m(Q)$, we define

$$\|F\|_{C^m(Q)} = \sup_{x \in Q} \max_{|\alpha| \leq m} |\partial^\alpha F(x)|.$$

The function space $C^{m-1,1}(Q)$ and the norm $\|\cdot\|_{C^{m-1,1}(Q)}$ are defined analogously.

If $F \in C^m(Q)$ and x belongs to the boundary of Q, then we still write $J_x(F)$ to denote the $(m-1)^{rst}$ degree Taylor polynomial of F at x, even though F isn't defined on a full neighborhood of $x \in \mathbb{R}^n$.

Let $S \subset \mathbb{R}^n$ be non-empty and finite. A Whitney field on S is a family of polynomials

$$\vec{P} = (P^y)_{y \in S} \text{ (each } P^y \in \mathcal{P}),$$

parametrized by the points of S.

We write $Wh(S)$ to denote the vector space of all Whitney fields on S.

For $\vec{P} = (P^y)_{y \in S} \in Wh(S)$, we define the seminorm

$$\left\|\vec{P}\right\|_{\dot{C}^m(S)} = \max_{x,y \in S, (x \neq y), |\alpha| \leq m} \frac{|\partial^\alpha (P^x - P^y)(x)|}{|x-y|^{m-|\alpha|}}.$$

(If S consists of a single point, then $\left\|\vec{P}\right\|_{\dot{C}^m(S)} = 0$.)

We also need an elementary fact about convex sets.

Helly's Theorem *Let* $K_1, \cdots, K_N \subset \mathbb{R}^D$ *be convex. Suppose that* $K_{i_1} \cap \cdots \cap K_{i_{D+1}}$ *is nonempty for any* $i_1, \cdots, i_{D+1} \in \{1, \cdots, N\}$. *Then* $K_1 \cap \cdots \cap K_N$ *is nonempty.*

See [22].

1.2 Shape Fields

Let $E \subset \mathbb{R}^n$ be finite. For each $x \in E$, $M \in (0, \infty)$, let $\Gamma(x, M) \subseteq \mathcal{P}$ be a (possibly empty) convex set. We say that $\vec{\Gamma} = (\Gamma(x, M))_{x \in E, M > 0}$ is a shape field if for all $x \in E$ and $0 < M' \leq M < \infty$, we have

$$\Gamma(x, M') \subseteq \Gamma(x, M).$$

Let $\vec{\Gamma} = (\Gamma(x, M))_{x \in E, M > 0}$ be a shape field and let $C_w, \delta_{\max}$ be positive real numbers. We say that $\vec{\Gamma}$ is $(C_w, \delta_{\max})$-convex if the following condition holds:

Let $0 < \delta \leq \delta_{\max}$, $x \in E$, $M \in (0, \infty)$, P_1, P_2, Q_1, $Q_2 \in \mathcal{P}$. Assume that

(1) $P_1, P_2 \in \Gamma(x, M)$;

(2) $|\partial^\beta (P_1 - P_2)(x)| \leq M\delta^{m - |\beta|}$ for $|\beta| \leq m - 1$;

(3) $|\partial^\beta Q_i(x)| \leq \delta^{-|\beta|}$ for $|\beta| \leq m - 1$ for $i = 1, 2$;

(4) $Q_1 \odot_x Q_1 + Q_2 \odot_x Q_2 = 1$.

Then

(5) $P := Q_1 \odot_x Q_1 \odot_x P_1 + Q_2 \odot_x Q_2 \odot_x P_2 \in \Gamma(x, C_w M)$.

1.3 Finiteness Principle for Shape Fields

We recall a main result proven in [18].

Theorem 2 *For a large enough $k^\#$ determined by m, n, the following holds. Let $\vec{\Gamma}_0 = (\Gamma_0(x, M))_{x \in E, M > 0}$ be a $(C_w, \delta_{\max})$-convex shape field and let $Q_0 \subset \mathbb{R}^n$ be a cube of sidelength $\delta_{Q_0} \leq \delta_{\max}$. Also, let $x_0 \in E \cap 5Q_0$ and $M_0 > 0$ be given. Assume that for each $S \subset E$ with $\#(S) \leq k^\#$ there exists a Whitney field $\vec{P}^S = (P^z)_{z \in S}$ such that*

$$\left\| \vec{P}^S \right\|_{\dot{C}^m(S)} \leq M_0,$$

and

$$P^z \in \Gamma_0(z, M_0) \text{ for all } z \in S.$$

Then there exist $P^0 \in \Gamma_0(x_0, M_0)$ and $F \in C^m(Q_0)$ such that the following hold, with a constant C_* determined by C_w, m, n:

- $J_z(F) \in \Gamma_0(z, C_* M_0)$ *for all* $z \in E \cap Q_0$.
- $|\partial^\beta (F - P^0)(x)| \le C_* M_0 \delta_{Q_0}^{m-|\beta|}$ *for all* $x \in Q_0$, $|\beta| \le m$.
- *In particular,* $|\partial^\beta F(x)| \le C_* M_0$ *for all* $x \in Q_0$, $|\beta| = m$.

2 C^m Interpolation by Nonnegative Functions

In this section, c, C, C', etc. denote constants determined by m and n. These symbols may denote different constants in different occurrences. For $x \in \mathbb{R}^n$ and $M > 0$, define

(1) $$\Gamma_*(x, M) = \left\{ \begin{array}{c} P \in \mathcal{P} : \text{ There exists } F \in C^m(\mathbb{R}^n) \text{ with } \|F\|_{C^m(\mathbb{R}^n)} \le M, \\ F \ge 0 \text{ on } \mathbb{R}^n,\ J_x(F) = P. \end{array} \right\}$$

It is not immediately clear how to compute Γ_*; we will return to this issue in a later section. Let $E \subset \mathbb{R}^n$ be finite, and let $f : E \to [0, \infty)$. Define $\vec{\Gamma}_f = (\Gamma_f(x, M))_{x \in E, M > 0}$, where

(2) $\Gamma_f(x, M) = \{P \in \Gamma_*(x, M) : P(x) = f(x)\}$.

Lemma 1 $\vec{\Gamma}_f$ *is a* $(C, 1)$*-convex shape field.*

Proof. It is clear that $\vec{\Gamma}_f$ is a shape field, i.e., each $\Gamma_f(x, M)$ is convex, and $M' \le M$ implies $\Gamma_f(x, M') \subseteq \Gamma_f(x, M)$. To establish $(C, 1)$-convexity, suppose we are given the following:

(3) $0 < \delta \le 1$, $x \in E$, $M > 0$;

(4) $P_1, P_2 \in \Gamma_f(x, M)$ satisfying

(5) $|\partial^\beta (P_1 - P_2)(x)| \le M \delta^{m-|\beta|}$ for $|\beta| \le m - 1$;

(6) $Q_1, Q_2 \in \mathcal{P}$ satisfying

(7) $|\partial^\beta Q_i(x)| \le \delta^{-|\beta|}$ for $|\beta| \le m - 1$, $i = 1, 2$, and

(8) $Q_1 \odot_x Q_1 + Q_2 \odot_x Q_2 = 1$.

Set

(9) $P = Q_1 \odot_x Q_1 \odot_x P_1 + Q_2 \odot_x Q_2 \odot_x P_2$.

We must prove that

(10) $P \in \Gamma_f(x, CM)$.

Thanks to (4), we have

(11) $P_1(x) = f(x)$ and $P_2(x) = f(x)$,

and there exist functions $F_1, F_2 \in C^m(\mathbb{R}^n)$ such that

(12) $\|F_i\|_{C^m(\mathbb{R}^n)} \le M$ $(i = 1, 2)$,

(13) $F_i \ge 0$ on $\mathbb{R}^n$ $(i = 1, 2)$, and

(14) $J_x(F_i) = P_i$ $(i = 1, 2)$.

We fix F_1, F_2 as above. By (8), we have $|Q_i(x)| \ge \frac{1}{\sqrt{2}}$ for $i = 1$ or for $i = 2$. By possibly interchanging Q_1 and Q_2, and then possibly changing Q_1 to $-Q_1$, we may suppose that

(15) $Q_1(x) \ge \frac{1}{\sqrt{2}}$.

For small enough c_0, (7) and (15) yield

(16) $Q_1(y) \ge \frac{1}{10}$ for $|y - x| \le c_0\delta$.

Fix c_0 as in (16). We introduce a C^m cutoff function χ on $\mathbb{R}^n$ with the following properties.

(17) $0 \le \chi \le 1$ on $\mathbb{R}^n$; $\chi = 0$ outside $B_n(x, c_0\delta)$; $\chi = 1$ in a neighborhood of x;

(18) $\left|\partial^\beta \chi\right| \le C\delta^{-|\beta|}$ on $\mathbb{R}^n$, for $|\beta| \le m$.

We then define $\tilde{\theta}_1 = \chi \cdot Q_1 + (1 - \chi)$ and $\tilde{\theta}_2 = \chi \cdot Q_2$.

These functions satisfy the following: $\tilde{\theta}_i \in C^m(\mathbb{R}^n)$ and $\left|\partial^\beta \tilde{\theta}_i\right| \le C\delta^{-|\beta|}$ on $\mathbb{R}^n$ for $|\beta| \le m$, $i = 1, 2$; $\tilde{\theta}_1 \ge \frac{1}{10}$ on $\mathbb{R}^n$; $J_x\left(\tilde{\theta}_i\right) = Q_i$ for $i = 1, 2$; outside $B_n(x, c_0\delta)$ we have $\tilde{\theta}_1 = 1$ and $\tilde{\theta}_2 = 0$. Setting $\theta_i = \tilde{\theta}_i \cdot \left(\tilde{\theta}_1^2 + \tilde{\theta}_2^2\right)^{-1/2}$ for $i = 1$, 2, we find that

(19) $\theta_i \in C^m(\mathbb{R}^n)$ and $\left|\partial^\beta \theta_i\right| \le C\delta^{-|\beta|}$ on $\mathbb{R}^n$ for $|\beta| \le m$, $i = 1, 2$;

(20) $\theta_1^2 + \theta_2^2 = 1$ on $\mathbb{R}^n$;

(21) $J_x(\theta_i) = Q_i$ for $i = 1, 2$ (here we use (8)); and

(22) outside $B_n(x, c_0\delta)$ we have $\theta_1 = 1$ and $\theta_2 = 0$.

Now set

(23) $F = \theta_1^2 F_1 + \theta_2^2 F_2 = F_1 + \theta_2^2 (F_2 - F_1)$ (see (20)).

Clearly $F \in C^m(\mathbb{R}^n)$. By (14), we have $J_x(F_2 - F_1) = P_2 - P_1$; hence (5) yields the estimate

$$\left|\partial^\beta (F_2 - F_1)(x)\right| \le CM\delta^{m-|\beta|} \text{ for } |\beta| \le m - 1.$$

Together with (12), this tells us that

$$\left|\partial^\beta (F_2 - F_1)\right| \le CM\delta^{m-|\beta|} \text{ on } B_n(x, c_0\delta) \text{ for } |\beta| \le m.$$

Recalling (19), we deduce that

$$\left|\partial^\beta \left(\theta_2^2 \cdot (F_2 - F_1)\right)\right| \le CM\delta^{m-|\beta|} \text{ on } B_n(x, c_0\delta) \text{ for } |\beta| \le m.$$

Together with (12) and (23), this implies that

$$\left|\partial^\beta F\right| \le CM \text{ on } B_n(x, c_0\delta),$$

since $0 < \delta \le 1$ (see (3)). On the other hand, outside $B_n(x, c_0\delta)$ we have $F = F_1$ by (22), (23); hence $|\partial^\beta F| \le CM$ outside $B_n(x, c_0\delta)$ for $|\beta| \le m$, by (12). Thus, $|\partial^\beta F| \le CM$ on all of $\mathbb{R}^n$ for $|\beta| \le m$, i.e.,

(24) $\|F\|_{C^m(\mathbb{R}^n)} \le CM$.

Also, from (13) and (23) we have

(25) $F \ge 0$ on $\mathbb{R}^n$;

and (9), (14), (21), (23) imply that

(26) $J_x(F) = Q_1 \odot_x Q_1 \odot_x P_1 + Q_2 \odot_x Q_2 \odot_x P_2 = P$.

Since $F \in C^m(\mathbb{R}^n)$ satisfies (24), (25), (26), we have

(27) $P \in \Gamma_*(x, CM)$.

Moreover,

(28) $P(x) = (Q_1(x))^2 f(x) + (Q_2(x))^2 f(x) = f(x)$,

thanks to (8), (9), (11).

From (27), (28) we conclude that $P \in \Gamma_f(x, CM)$, completing the proof of Lemma 1. ∎

Lemma 2 *Let $(P^x)_{x \in E}$ be a Whitney field on the finite set E, and let $M > 0$. Suppose that*

(29) $P^x \in \Gamma_*(x, M)$ *for each* $x \in E$,

and that

(30) $\left|\partial^\beta \left(P^x - P^{x'}\right)(x)\right| \leq M\,|x - x'|^{m-|\beta|}$ *for* $x, x' \in E$ *and* $|\beta| \leq m - 1$.

Then there exists $F \in C^m(\mathbb{R}^n)$ such that

(31) $\|F\|_{C^m(\mathbb{R}^n)} \leq CM$,

(32) $F \geq 0$ *on* $\mathbb{R}^n$, *and*

(33) $J_x(F) = P^x$ *for all* $x \in E$.

Proof. We modify slightly Whitney's proof [33] of the Whitney extension theorem. We say that a dyadic cube $Q \subset \mathbb{R}^n$ is "OK" if $\#(E \cap 5Q) \leq 1$ and $\delta_Q \leq 1$. Then every small enough Q is OK (because E is finite), and no Q of sidelength $\delta_Q > 1$ is OK. Also, let Q, Q' be dyadic cubes with $5Q \subset 5Q'$. If Q' is OK, then also Q is OK. We define a Calderón-Zygmund (or CZ) cube to be an OK cube Q such that no Q' that strictly contains Q is OK. The above remarks imply that the CZ cubes form a partition of $\mathbb{R}^n$; that the sidelengths of the CZ cubes are bounded above by 1 and below by some positive number; and that the following condition holds.

(34) "Good Geometry": If $Q, Q' \in \mathrm{CZ}$ and $\frac{65}{64}Q \cap \frac{65}{64}Q' \neq \emptyset$, then $\frac{1}{2}\delta_Q \leq \delta_{Q'} \leq 2\delta_Q$.

We classify CZ cubes into three types as follows.
$Q \in \mathrm{CZ}$ is of

Type 1 if $E \cap 5Q \neq \emptyset$

Type 2 if $E \cap 5Q = \emptyset$ and $\delta_Q < 1$.

Type 3 if $E \cap 5Q = \emptyset$ and $\delta_Q = 1$.

Let $Q \in \mathrm{CZ}$ be of Type 1. Since Q is OK, we have $\#(E \cap 5Q) \leq 1$. Hence $E \cap 5Q$ is a singleton, $E \cap 5Q = \{x_Q\}$. Since $P^{x_Q} \in \Gamma_*(x_Q, M)$, there exists $F_Q \in C^m(\mathbb{R}^n)$ such that

(35) $\|F_Q\|_{C^m(\mathbb{R}^n)} \leq M$, $F_Q \geq 0$ on $\mathbb{R}^n$, $J_{x_Q}(F_Q) = P^{x_Q}$.

We fix F_Q as in (35).

Let $Q \in \mathrm{CZ}$ be of Type 2. Then $\delta_{Q^+} \leq 1$ but Q^+ is not OK; hence $\#(E \cap 5Q^+) \geq 2$. We pick $x_Q \in E \cap 5Q^+$. Since $P^{x_Q} \in \Gamma_*(x_Q, M)$, there exists $F_Q \in C^m(\mathbb{R}^n)$ satisfying (35). We fix such an F_Q.

Let $Q \in \mathrm{CZ}$ be of Type 3. Then we set $F_Q = 0$. In place of (35), we have the trivial results

(36) $\|F_Q\|_{C^m(\mathbb{R}^n)} = 0$ and $F_Q \geq 0$ on $\mathbb{R}^n$.

Thus, we have defined F_Q for all $Q \in \mathrm{CZ}$, and we have defined $x_Q \in E \cap 5Q^+$ for all Q of Type 1 or Type 2. Note that

(37) $J_x(F_Q) = P^x$ for all $x \in E \cap 5Q$.

Indeed, if Q is of Type 1, then (37) follows from (35) since $E \cap 5Q = \{x_Q\}$. If Q is of Type 2 or Type 3, then (37) holds vacuously since $E \cap 5Q = \emptyset$. Now suppose $Q, Q' \in \mathrm{CZ}$ and $\frac{65}{64}Q \cap \frac{65}{64}Q' \neq \emptyset$. We will show that

(38) $\left|\partial^\beta (F_Q - F_{Q'})\right| \leq CM\delta_Q^{m-|\beta|}$ on $\frac{65}{64}Q \cap \frac{65}{64}Q'$ for $|\beta| \leq m$.

To see this, suppose first that Q or Q' is of Type 3. Then δ_Q or $\delta_{Q'}$ is equal to 1, hence $\delta_Q \geq \frac{1}{2}$ by (34). Consequently, (38) asserts simply that

(39) $\left|\partial^\beta (F_Q - F_{Q'})\right| \le CM$ on $\frac{65}{64}Q \cap \frac{65}{64}Q'$ for $|\beta| \le m$,

and (39) follows at once from (35), (36). Thus, (38) holds if Q or Q' is of Type 3. Suppose that neither Q nor Q' is of Type 3. Then $x_Q \in E \cap 5Q^+$, $x_{Q'} \in E \cap 5(Q'^+)$, $\frac{65}{64}Q \cap \frac{65}{64}Q' \neq \emptyset$, $\frac{1}{2}\delta_Q \le \delta_{Q'} \le 2\delta_Q$. Consequently,

(40) $|x_Q - x_{Q'}| \le C\delta_Q$, and

(41) $|x - x_Q|, |x - x_{Q'}| \le C\delta_Q$ for all $x \in \frac{65}{64}Q \cap \frac{65}{64}Q'$.

Applying (35) to Q and to Q', we find that

(42) $\left|\partial^\beta (F_Q - P^{x_Q})(x)\right| \le CM|x - x_Q|^{m-|\beta|} \le CM\delta_Q^{m-|\beta|}$, and

(43) $\left|\partial^\beta (F_{Q'} - P^{x_{Q'}})(x)\right| \le CM|x - x_{Q'}|^{m-|\beta|} \le CM\delta_Q^{m-|\beta|}$,

for $x \in \frac{65}{64}Q \cap \frac{65}{64}Q'$, $|\beta| \le m$.
Also, (30), (40), (41) imply that

(44) $\left|\partial^\beta (P^{x_Q} - P^{x_{Q'}})(x)\right| \le CM\delta_Q^{m-|\beta|}$ for $x \in \frac{65}{64}Q \cap \frac{65}{64}Q'$, $|\beta| \le m$.

(Recall, $P^{x_Q} - P^{x_{Q'}}$ is a polynomial of degree at most $m-1$.)

Estimates (42), (43), (44) together imply (38) in case neither Q nor Q' is of Type 3. Thus, (38) holds in all cases.

Next, as in Whitney [33], we introduce a partition of unity

(45) $1 = \sum_{Q \in CZ} \theta_Q$ on $\mathbb{R}^n$,

where each $\theta_Q \in C^m(\mathbb{R}^n)$, and

(46) support $\theta_Q \subset \frac{65}{64}Q$, $\left|\partial^\beta \theta_Q\right| \le C\delta_Q^{-|\beta|}$ for $|\beta| \le m$, $\theta_Q \ge 0$ on $\mathbb{R}^n$.

We define

(47) $F = \sum_{Q \in CZ} \theta_Q F_Q$ on $\mathbb{R}^n$.

Thus, $F \in C^m_{loc}(\mathbb{R}^n)$ since CZ is a locally finite partition of $\mathbb{R}^n$, and $F \ge 0$ on $\mathbb{R}^n$ since $\theta_Q \ge 0$ and $F_Q \ge 0$ for each Q. Let $\hat{x} \in \mathbb{R}^n$, and let $\hat{Q}$ be the one and only CZ cube containing $\hat{x}$. Then for $|\beta| \le m$, we have

(48) $\partial^\beta F(\hat{x}) = \partial^\beta F_{\hat{Q}}(\hat{x}) + \sum_{Q \in CZ} \partial^\beta \left(\theta_Q \cdot \left(F_Q - F_{\hat{Q}}\right)\right)(\hat{x})$.

A given $Q \in CZ$ enters into the sum in (48) only if $\hat{x} \in \frac{65}{64}Q$; there are at most C such cubes Q, thanks to (34). Moreover, for each $Q \in CZ$ with $\hat{x} \in \frac{65}{64}Q$, we learn from (38) and (46) that

$$\left|\partial^\beta \left(\theta_Q \cdot \left(F_Q - F_{\hat{Q}}\right)\right)(\hat{x})\right| \leq CM\delta_Q^{m-|\beta|} \leq CM \text{ for } |\beta| \leq m, \text{ since } \delta_Q \leq 1.$$

Since also $\left|\partial^\beta F_{\hat{Q}}(\hat{x})\right| \leq CM$ for $|\beta| \leq m$ by (35), (36), it now follows from (48) that $\left|\partial^\beta F(\hat{x})\right| \leq CM$ for all $|\beta| \leq m$. Here, $\hat{x} \in \mathbb{R}^n$ is arbitrary. Thus, $F \in C^m(\mathbb{R}^n)$ and $\|F\|_{C^m(\mathbb{R}^n)} \leq CM$.

Next, let $x \in E$. For any $Q \in CZ$ such that $x \in \frac{65}{64}Q$, we have $J_x(F_Q) = P^x$, by (37). Since support $\theta_Q \subset \frac{65}{64}Q$ for each $Q \in CZ$, it follows that $J_x(\theta_Q F_Q) = J_x(\theta_Q) \odot_x P^x$ for each $Q \in CZ$, and consequently,

$$J_x(F) = \sum_{Q \in CZ} J_x(\theta_Q F_Q) = \left[\sum_{Q \in CZ} J_x(\theta_Q)\right] \odot_x P^x = P^x, \text{ by (45).}$$

Thus, $F \in C^m(\mathbb{R}^n)$, $\|F\|_{C^m(\mathbb{R}^n)} \leq CM$, $F \geq 0$ on $\mathbb{R}^n$, and $J_x(F) = P^x$ for each $x \in E$.

The proof of Lemma 2 is complete. ■

Theorem 3 (Finiteness Principle for Nonnegative C^m Interpolation) *There exist constants $k^\#$, C, depending only on m, n, such that the following holds.*

Let $E \subset \mathbb{R}^n$ be finite, and let $f : E \to [0, \infty)$. Let $M_0 > 0$. Suppose that for each $S \subset E$ with $\#(S) \leq k^\#$, there exists $\vec{P}^S = (P^x)_{x \in S} \in Wh(S)$ such that

- $P^x \in \Gamma_f(x, M_0)$ *for each* $x \in S$, *and*
- $|\partial^\beta(P^x - P^y)(x)| \leq M_0 |x - y|^{m-|\beta|}$ *for* $x, y \in S$, $|\beta| \leq m - 1$.

Then there exists $F \in C^m(\mathbb{R}^n)$ such that

- $\|F\|_{C^m(\mathbb{R}^n)} \leq CM_0$,
- $F \geq 0$ *on* $\mathbb{R}^n$, *and*
- $F = f$ *on* E.

Proof. Suppose first that $E \subset \frac{1}{2}Q_0$ for a cube Q_0 of sidelength $\delta_{Q_0} = 1$. Pick any $x_0 \in E$. (If E is empty, our theorem holds trivially.)

Let $S \subset E$ with $\#(S) \leq k^{\#}$.

Our present hypotheses supply the Whitney field $\vec{P}^S$ required in the hypotheses of Theorem 2.

Hence, recalling Lemma 1 and applying Theorem 2, we obtain

(49) $P^0 \in \Gamma_f(x_0, CM_0)$

and

(50) $F^0 \in C^m(Q_0)$

such that

(51) $J_x(F^0) \in \Gamma_f(x, CM_0)$ for all $x \in E \cap Q_0 = E$

and

(52) $|\partial^\beta(P^0 - F^0)| \leq CM_0$ on Q_0, for $|\beta| \leq m$.

From (1), (2), (49), we have $|\partial^\beta P^0(x_0)| \leq CM_0$ for $|\beta| \leq m-1$.

Since P^0 is a polynomial of degree at most $m-1$, and since $x_0 \in E \subset Q_0$ with $\delta_{Q_0} = 1$, it follows that $|\partial^\beta P^0| \leq CM_0$ on Q_0 for $|\beta| \leq m$.

Together with (52), this tells us that

(53) $|\partial^\beta F^0| \leq CM_0$ on Q_0 for $|\beta| \leq m$.

Note that F^0 needn't be nonnegative.

Set $P^x = J_x(F^0)$ for $x \in E$. Then

(54) $P^x \in \Gamma_f(x, CM_0)$ for $x \in E$, and

(55) $\left|\partial^\beta (P^x - P^y)(x)\right| \leq CM_0 |x-y|^{m-|\beta|}$ for $x, y \in E$, $|\beta| \leq m-1$.

By Lemma 2, there exists $F \in C^m(\mathbb{R}^n)$ such that

(56) $\|F\|_{C^m(\mathbb{R}^n)} \leq CM_0$,

(57) $F \geq 0$ on $\mathbb{R}^n$, and

(58) $J_x(F) = P^x$ for each $x \in E$.

From (54) and (2), we have $P^x(x) = f(x)$ for each $x \in E$; hence, (58) implies that

(59) $F(x) = f(x)$ for each $x \in E$.

Our results (56), (57), (59) are the conclusions of our theorem. Thus, we have proven Theorem 3 in the case in which $E \subset \frac{1}{2}Q_0$ with $\delta_{Q_0} = 1$.

To pass to the general case (arbitrary finite $E \subset \mathbb{R}^n$), we set up a partition of unity $1 = \sum_\nu \chi_\nu$ on $\mathbb{R}^n$, where each $\chi_\nu \in C^m(\mathbb{R}^n)$ and $\chi_\nu \geq 0$ on $\mathbb{R}^n$, $\|\chi_\nu\|_{C^m(\mathbb{R}^n)} \leq C$, support $\chi_\nu \subset \frac{1}{2}Q_\nu$, with $\delta_{Q_\nu} = 1$, and with any given point of $\mathbb{R}^n$ belonging to at most C of the Q_ν.

For each ν, we apply the known special case of our theorem to the set $E_\nu = E \cap \frac{1}{2}Q_\nu$ and the function $f_\nu = f|_{E_\nu}$. Thus, we obtain $F_\nu \in C^m(\mathbb{R}^n)$, with $\|F_\nu\|_{C^m(\mathbb{R}^n)} \leq CM_0$, $F_\nu \geq 0$ on $\mathbb{R}^n$, and $F_\nu = f$ on $E \cap \frac{1}{2}Q_\nu$.

Setting $F = \sum_\nu \chi_\nu F_\nu \in C^m_{loc}(\mathbb{R}^n)$, we verify easily that $F \in C^m(\mathbb{R}^n)$, $\|F\|_{C^m(\mathbb{R}^n)} \leq CM_0$, $F \geq 0$ on $\mathbb{R}^n$, and $F = f$ on E.

This completes the proof of Theorem 3. ■

Remark *Conversely, we make the following trivial observation: Let* $E \subset \mathbb{R}^n$ *be finite, let* $f : E \to [0, \infty)$, *and let* $M_0 > 0$. *Suppose* $F \in C^m(\mathbb{R}^n)$ *satisfies* $\|F\|_{C^m(\mathbb{R}^n)} \leq M_0$, $F \geq 0$ *on* $\mathbb{R}^n$, $F = f$ *on* E. *Then for each* $x \in E$, *we have*

- $P^x = J_x(F) \in \Gamma_f(x, M_0)$ *by* (1), (2); *and*
- $|\partial^\beta(P^x - P^y)(x)| \leq CM_0|x - y|^{m-|\beta|}$ *for* $x, y \in E$, $|\beta| \leq m - 1$.

Therefore, for any $S \subset E$, *the Whitney field* $\vec{P}^S = (P^x)_{x \in S} \in Wh(S)$ *satisfies*

- $P^x \in \Gamma_f(x, CM_0)$ *for* $x \in S$, *and*
- $|\partial^\beta(P^x - P^y)(x)| \leq CM_0|x - y|^{m-|\beta|}$ *for* $x, y \in S$, $|\beta| \leq m - 1$.

Note that Theorem 1 (A) follows easily from Theorem 3.

3 Computable Convex Sets

In this section, we discuss computational issues regarding the convex set

(1) $\Gamma_*(x, M) = \left\{ J_x(F) : F \in C^m(\mathbb{R}^n),\ \|F\|_{C^m(\mathbb{R}^n)} \leq M,\ F \geq 0 \text{ on } \mathbb{R}^n \right\}.$

We write c, C, C', etc., to denote constants determined by m and n. These symbols may denote different constants in different occurrences.

We will define convex sets $\tilde{\Gamma}_*(x, M) \subset \mathcal{P}$, prove that

(2) $\tilde{\Gamma}_*(x, cM) \subset \Gamma_*(x, M) \subset \tilde{\Gamma}_*(x, CM)$ for all $x \in \mathbb{R}^n$, $M > 0$,

and explain how (in principle) one can compute $\tilde{\Gamma}_*(x, M)$.

We may then use

(3) $\tilde{\Gamma}_f(x, M) = \left\{ P \in \tilde{\Gamma}_*(x, M) : P(x) = f(x) \right\}$

in place of $\Gamma_f(x, M)$ in the statement of Theorem 3. (The assertion in terms of $\tilde{\Gamma}_f$ follows trivially from (2) and the original assertion in terms of Γ_f.)

To achieve (2), we will define

(4) $\tilde{\Gamma}_*(x, M) = \left\{ MP(\cdot + x)) : P \in \tilde{\Gamma}_0 \right\}$, for a convex set $\tilde{\Gamma}_0$.

We will prove that

(5) $\Gamma_*(0, c) \subset \tilde{\Gamma}_0 \subset \Gamma_*(0, C)$.

Property (2) then follows at once from (1), (4), and (5).

Thus, our task is to define a convex set $\tilde{\Gamma}_0$ satisfying (5), and explain how (in principle) one can compute $\tilde{\Gamma}_0$.

Recall that $\mathcal{P}$ is the vector space of $(m-1)$-jets. We will work in the space of m-jets. In this section, we let $\mathcal{P}^+$ denote the vector space of real-valued polynomials of degree at most m on $\mathbb{R}^n$, and we write $J_x^+(F)$ to denote the m^{th}-degree Taylor polynomial of F at x, i.e.,

$$J_x^+(F)(y) = \sum_{|\alpha| \leq m} \frac{1}{\alpha!} (\partial^\alpha F(x)) \cdot (y - x)^\alpha.$$

We define

$$(6)\quad \Gamma_0^+ = \left\{ \begin{array}{c} P \in \mathcal{P}^+ : \left|\partial^\beta P(0)\right| \leq 1 \text{ for } |\beta| \leq m;\ P(x) + |x|^m \geq 0 \text{ for all } x \in \mathbb{R}^n; \\ \text{and for every } \epsilon > 0 \text{, there exists } \delta > 0 \text{ such that} \\ P(x) + \epsilon |x|^m \geq 0 \text{ for } |x| \leq \delta. \end{array} \right\}.$$

Later, we will discuss how Γ_0^+ may be computed in principle.

We next establish the following result.

Lemma 3 *For small enough* c *and large enough* C*, the following hold.*

(A) *If* $F \in C^m(\mathbb{R}^n)$*,* $\|F\|_{C^m(\mathbb{R}^n)} \leq c$*,* $F \geq 0$ *on* $\mathbb{R}^n$*, then* $J_0^+(F) \in \Gamma_0^+$*.*

(B) *If* $P \in \Gamma_0^+$*, then there exists* $F \in C^m(\mathbb{R}^n)$ *such that* $\|F\|_{C^m(\mathbb{R}^n)} \leq C$*,* $F \geq 0$ *on* $\mathbb{R}^n$*, and* $J_0^+(F) = P$*.*

Proof. (A) follows trivially from Taylor's theorem. We prove (B).

Let $P \in \Gamma_0^+$ be given. We introduce cutoff functions $\varphi, \chi \in C^m(\mathbb{R}^n)$ with the following properties.

(7) $\|\chi\|_{C^m(\mathbb{R}^n)} \leq C$, $\chi = 1$ in a neighborhood of 0, $\chi = 0$ outside $B_n(0, 1/2)$, and $0 \leq \chi \leq 1$ on $\mathbb{R}^n$.

(8) $\|\varphi\|_{C^m(\mathbb{R}^n)} \leq C$, $\varphi = 1$ for $1/2 \leq |x| \leq 2$, $\varphi \geq 0$ on $\mathbb{R}^n$,

and $\varphi(x) = 0$ unless $1/4 < |x| < 4$.

For $k \geq 0$, let

(9) $\varphi_k(x) = \varphi\left(2^k x\right)$ $(x \in \mathbb{R}^n)$.

Thus,

(10) $\|\varphi_k\|_{C^m(\mathbb{R}^n)} \leq C 2^{mk}$, $\varphi_k \geq 0$ on $\mathbb{R}^n$, $\varphi_k(x) = 1$ for $2^{-1-k} \leq |x| \leq 2^{1-k}$, $\varphi_k(x) = 0$ unless $2^{-2-k} \leq |x| \leq 2^{2-k}$.

Also, for $k \geq 0$, we define a real number b_k as follows.

(11) $b_k = 0$ if $P(x) \geq 0$ for $|x| \leq 2^{-k}$; $b_k = -\min\left\{P(x) : |x| \leq 2^{-k}\right\}$ otherwise.

Since $P \in \Gamma_0^+$, the b_k satisfy the following:

(12) $0 \leq b_k \leq 2^{-mk}$ for all $k \geq 0$.

(13) $b_k \cdot 2^{mk} \to 0$ as $k \to \infty$.

By definition of the b_k, we have also for each $k \geq 0$ that

(14) $P(x) + b_k \geq 0$ for $|x| \leq 2^{-k}$.

We define a function $\tilde{F}$ on the closed unit ball $\overline{B_n(0,1)}$ by setting

(15) $\tilde{F}(x) = P(x) + \sum_{k=0}^{\infty} b_k \varphi_k(x)$ for $x \in \overline{B_n(0,1)}$.

(The sum contains at most C nonzero terms for any given x.)

We will check that

(16) $\tilde{F} \geq 0$ on $\overline{B_n(0,1)}$.

Indeed, $\tilde{F}(0) = P(0) \geq 0$ since each $\varphi_k(0) = 0$ and $P \in \Gamma_0^+$. For $\hat{x} \in \overline{B_n(0,1)} \setminus \{0\}$ we have $2^{-1-\hat{k}} \leq |\hat{x}| \leq 2^{-\hat{k}}$ for some $\hat{k} \geq 0$.

We then have $\varphi_{\hat{k}}(\hat{x}) = 1$ by (10), hence $P(\hat{x}) + b_{\hat{k}}\varphi_{\hat{k}}(\hat{x}) \geq 0$ by (14). Since also $b_k\varphi_k(\hat{x}) \geq 0$ for all k, it follows that

$$\tilde{F}(\hat{x}) = [P(\hat{x}) + b_{\hat{k}}\varphi_{\hat{k}}(\hat{x})] + \sum_{k \neq \hat{k}} b_k \varphi_k(x) \geq 0,$$

completing the proof of (16).

Next, we check that

(17) $\tilde{F} \in C^m\left(\overline{B_n(0,1)}\right)$, $\left\|\tilde{F}\right\|_{C^m\left(\overline{B_n(0,1)}\right)} \leq C$, $J_0^+\left(\tilde{F}\right) = P$.

To see this, let

(18) $\tilde{F}_K = P + \sum_{k=0}^{K} b_k \varphi_k$ for $K \geq 0$.

Since $P \in \Gamma_0^+$, we have $\left|\partial^\beta P(0)\right| \leq 1$ for $|\beta| \leq m$, hence

(19) $\|P\|_{C^m\left(\overline{B_n(0,1)}\right)} \leq C$.

Also, (10) and (12) give

$$\|b_k \varphi_k\|_{C^m\left(\overline{B_n(0,1)}\right)} \leq C \text{ for each } k.$$

Since any given $x \in \overline{B_n(0,1)}$ belongs to at most C of the supports of the φ_k, it follows that

(20) $\left\|\sum_{k=0}^{K} b_k \varphi_k\right\|_{C^m\left(\overline{B_n(0,1)}\right)} \leq C.$

From (18), (19), (20), we see that

(21) $\tilde{F}_K \in C^m\left(\overline{B_n(0,1)}\right)$ and $\left\|\tilde{F}\right\|_{C^m\left(\overline{B_n(0,1)}\right)} \leq C.$

Also, (10) and (18) tell us that

(22) $J_0^+\left(\tilde{F}_K\right) = P$ for each K.

Furthermore for $K_1 < K_2$, (18) gives $\tilde{F}_{K_2} - \tilde{F}_{K_1} = \sum_{K_1<k\leq K_2} b_k \varphi_k$. Let $\epsilon > 0$. From (10) and (13) we see that

$$\max_{K_1<k\leq K_2} \|b_k \varphi_k\|_{C^m\left(\overline{B_n(0,1)}\right)} < \epsilon \text{ if } K_1 \text{ is large enough.}$$

Since any given point lies in support φ_k for at most C distinct k, it follows that

$$\left\|\sum_{K_1<k\leq K_2} b_k \varphi_k\right\|_{C^m\left(\overline{B_n(0,1)}\right)} \leq C\epsilon \text{ if } K_2 > K_1 \text{ and } K_1 \text{ is large enough.}$$

Thus, $(\tilde{F}_K)_{K\geq 0}$ is a Cauchy sequence in $C^m(\overline{B_n(0,1)})$. Consequently, $\tilde{F}_K \to \tilde{F}_\infty$ in $C^m(\overline{B_n(0,1)})$-norm for some $\tilde{F}_\infty \in C^m(\overline{B_n(0,1)})$. From (21) and (22), we have

$$\left\|\tilde{F}_\infty\right\|_{C^m\left(\overline{B_n(0,1)}\right)} \leq C \text{ and } J_0^+\left(\tilde{F}_\infty\right) = P.$$

On the other hand, comparing (15) to (18), and recalling that any given x belongs to support θ_k for at most C distinct k, we conclude that $\tilde{F}_K \to \tilde{F}$ pointwise as $K \to \infty$.

Since also $\tilde{F}_K \to \tilde{F}_\infty$ pointwise as $K \to \infty$, we have $\tilde{F}_\infty = \tilde{F}$.

Thus, $\tilde{F} \in C^m\left(\overline{B_n(0,1)}\right)$, $\left\|\tilde{F}\right\|_{C^m\left(\overline{B_n(0,1)}\right)} \leq C$, and $J_0^+\left(\tilde{F}\right) = P$, completing the proof of (17).

Finally, we recall the cutoff function χ from (7), and define $F = \chi\tilde{F}$ on $\mathbb{R}^n$.

From (16), (17), and the properties (7) of χ, we conclude that $F \in C^m(\mathbb{R}^n)$, $\|F\|_{C^m(\mathbb{R}^n)} \leq C$, $F \geq 0$ on $\mathbb{R}^n$, and $J_0^+(F) = P$.

Thus, we have established (B).

The proof of Lemma 3 is complete. ■

Now let $\pi : \mathcal{P}^+ \to \mathcal{P}$ denote the natural projection from m-jets at 0 to $(m-1)$-jets at 0, namely, $\pi P = J_0(P)$ for $P \in \mathcal{P}^+$.

We then set $\tilde{\Gamma}_0 = \pi \Gamma_0^+$.

From the above lemma, we learn the following.

(A′) Let $F \in C^m(\mathbb{R}^n)$ with $\|F\|_{C^m(\mathbb{R}^n)} \leq c$, $F \geq 0$ on $\mathbb{R}^n$. Then $J_0(F) \in \tilde{\Gamma}_0$.

(B′) Let $P \in \tilde{\Gamma}_0$. Then there exists $F \in C^m(\mathbb{R}^n)$ such that $\|F\|_{C^m(\mathbb{R}^n)} \leq C$, $F \geq 0$ on $\mathbb{R}^n$, and $J_0(F) = P$.

Recalling the definition (1), we conclude from (A′), (B′) that $\Gamma_*(0, c) \subset \tilde{\Gamma}_0 \subset \Gamma_*(0, C)$.

Thus, our $\tilde{\Gamma}_0$ satisfies the key condition (5).

We discuss briefly how the convex set $\tilde{\Gamma}_0$ may be computed in principle. Recall [20] that a semialgebraic set is a subset of a vector space obtained by taking finitely many unions, intersections, and complements of sets of the form $\{P > 0\}$ for polynomials P. Any subset of a vector space V defined by $E = \{x \in V : \Phi(x) \text{ is true}\}$, where Φ is a formula of first-order predicate calculus (for the theory of real-closed fields) is semialgebraic; moreover, there is an algorithm that accepts Φ as input and exhibits E as a Boolean combination of sets of the form $\{P > 0\}$ for polynomials P. For any given m, n, we see, by inspection of the definitions of Γ_0^+ and $\tilde{\Gamma}_0$, that $\Gamma_0^+ \subset \mathcal{P}^+$ is defined by a formula of first-order predicate calculus; hence, the same holds for $\tilde{\Gamma}_0 \subset \mathcal{P}$.

Therefore, in principle, we can compute $\tilde{\Gamma}_0$ as a Boolean combination of sets of the form $\{P \in \mathcal{P} : \Pi(P) > 0\}$, where Π is a polynomial on $\mathcal{P}$.

In practice, we make no claim that we know how to compute $\tilde{\Gamma}_0$.

It would be interesting to give a more practical method to compute a convex set satisfying (5).

4 $C^{m-1,1}$ Interpolation by Nonnegative Functions

In this section we will establish Theorem 1 (B) and discuss computational issues for $C^{m-1,1}$ interpolation by nonnegative functions.

We note that the derivatives $\partial^\beta F$ of $F \in C^{m-1,1}(\mathbb{R}^n)$ of order $|\beta| \le m-1$ are continuous. Also, Taylor's theorem holds in the form

$$\left| \partial^\beta F(y) - \sum_{|\beta|+|\gamma| \le m-1} \frac{1}{\gamma!} \left[\partial^{\gamma+\beta} F(x)\right] \cdot (y-x)^\gamma \right| \le C \|F\|_{C^{m-1,1}(\mathbb{R}^n)} \cdot |y-x|^{m-|\beta|}$$

for $x, y \in \mathbb{R}^n$.

Similar remarks apply to $C^{m-1,1}(Q)$ and $C^m(Q)$ for cubes $Q \subset \mathbb{R}^n$.

Therefore, we may repeat the proofs [18] of Lemmas 1 and 2 in Section 2, to derive the following results.

Lemma 4 *For* $x \in \mathbb{R}^n$, $M > 0$, *let*

$$\Gamma'_*(x, M) = \left\{ \begin{array}{c} P \in \mathcal{P} : \exists F \in C^{m-1,1}(\mathbb{R}^n) \textit{ such that} \\ \|F\|_{C^{m-1,1}(\mathbb{R}^n)} \le M, F \ge 0 \textit{ on } \mathbb{R}^n, J_x(F) = P \end{array} \right\}.$$

Let $f : E \to [0, \infty)$, *where* $E \subset \mathbb{R}^n$ *is finite. For* $x \in E$, $M > 0$, *let*

$$\Gamma'_f(x, M) = \{P \in \Gamma'_*(x, M) : P(x) = f(x)\}.$$

Then $\vec{\Gamma}'_f := (\Gamma'_f(x, M))_{x \in E, M > 0}$ *is a* $(C, 1)$*-convex shape field, where* C *depends only on* m, n.

Lemma 5 *Let* E, f, $\Gamma'_*(x, M)$ *be as in Lemma 4, and let* $M > 0$, $\vec{P} = (P^x)_{x \in E} \in Wh(E)$. *Suppose we have* $P^x \in \Gamma'_*(x, M)$ *for all* $x \in E$, *and* $\left|\partial^\beta (P^x - P^y)(x)\right| \le M |x-y|^{m-|\beta|}$ *for* $x, y \in E$, $|\beta| \le m-1$. *Then there exists* $F \in C^{m-1,1}(\mathbb{R}^n)$ *such that* $J_x(F) = P^x$ *for all* $x \in E$, *and* $\|F\|_{C^{m-1,1}(\mathbb{R}^n)} \le CM$, *where* C *depends only on* m, n.

Similarly, by making small changes in the proof [18] of Theorem 3, we obtain the following result.

Lemma 6 *There exist* $k^\#$, C, *depending only on* m, n *for which the following holds.*

Let $E \subset \mathbb{R}^n$ *be finite, let* $f : E \to [0, \infty)$, *and let* $M_0 > 0$. *Suppose that for each* $S \subset E$ *with* $\#(S) \le k^\#$ *there exists* $\vec{P}^S = (P^x)_{x \in S} \in Wh(S)$ *such that* $P^x \in \Gamma'_f(x, M_0)$ *for all* $x \in S$, *and* $\left|\partial^\beta (P^x - P^y)\right| \le M_0 |x-y|^{m-|\beta|}$ *for* $x, y \in S$, $|\beta| \le m-1$.

Then there exists $F \in C^{m-1,1}(\mathbb{R}^n)$ *such that* $\|F\|_{C^{m-1,1}(\mathbb{R}^n)} \le CM_0$, $F \ge 0$ *on* $\mathbb{R}^n$, *and* $F = f$ *on* E.

Now we can easily deduce the following result.

Theorem 4 (Finiteness Principle for Nonnegative $C^{m-1,1}$-Interpolation) *There exists constants* $k^{\#}$, C, *depending only on* m, n *for which the following holds.*

Let $f : E \to [0, \infty)$, *with* $E \subset \mathbb{R}^n$ *arbitrary (not necessarily finite). Let* $M_0 > 0$. *Suppose that for each* $S \subset E$ *with* $\#(S) \leq k^{\#}$ *there exists* $\vec{P} = (P^x)_{x \in S} \in Wh(S)$ *such that*

- $P^x \in \Gamma_f'(x, M_0)$ *for all* $x \in S$,
- $\left|\partial^\beta (P^x - P^y)(x)\right| \leq M_0 |x - y|^{m-|\beta|}$ *for* $x, y \in S$, $|\beta| \leq m - 1$.

Then there exists $F \in C^{m-1,1}(\mathbb{R}^n)$ *such that*

- $\|F\|_{C^{m-1,1}(\mathbb{R}^n)} \leq CM_0$,
- $F \geq 0$, *and*
- $F = f$ *on* E.

Proof. Suppose first that $E \subset Q$ for some cube $Q \subset \mathbb{R}^n$. Then by Ascoli's theorem,

$$\left\{F \in C^{m-1,1}(Q) : \|F\|_{C^{m-1,1}(Q)} \leq CM_0,\ F \geq 0 \text{ on } Q\right\} \equiv X$$

is compact in the $C^{m-1}(Q)$-norm topology.

For each finite $E_0 \subset E$, Lemma 6 tells us that there exists $F \in X$ such that $F = f$ on E_0.

Consequently, there exists $F \in X$ such that $F = f$ on E. That is,

(1) $F \in C^{m-1,1}(Q)$, $\|F\|_{C^{m-1,1}(Q)} \leq CM_0$, $F \geq 0$ on Q, $F = f$ on E.

We have achieved (1), assuming that $E \subset Q$.

Now suppose $E \subset \mathbb{R}^n$ is arbitrary.

We introduce a partition of unity $1 = \sum_\nu \theta_\nu$ on $\mathbb{R}^n$, with $\theta_\nu \geq 0$ on $\mathbb{R}^n$, $\theta_\nu \in C^m(\mathbb{R}^n)$, $\|\theta_\nu\|_{C^m(\mathbb{R}^n)} \leq C$, support $\theta_\nu \subset Q_\nu$ for a cube $Q_\nu \subset \mathbb{R}^n$, with (say) $\delta_{Q_\nu} = 1$, and such that any given $x \in \mathbb{R}^n$ has a neighborhood that intersects at most C of the Q_ν. (Here C depends only on m, n.)

Applying our result (1) to $f|_{E \cap Q_\nu} : E \cap Q_\nu \to [0, \infty)$ for each ν, we obtain functions $F_\nu \in C^{m-1,1}(Q_\nu)$ such that $\|F_\nu\|_{C^{m-1,1}(Q_\nu)} \leq CM_0$, $F_\nu \geq 0$ on Q_ν, $F_\nu = f$ on $E \cap Q_\nu$.

(Here C depends only on m, n.)

We define $F = \sum_\nu \theta_\nu F_\nu$ on $\mathbb{R}^n$. One checks easily that $\|F\|_{C^{m-1,1}(\mathbb{R}^n)} \leq C' M_0$ with C' determined by m, n; $F \geq 0$ on $\mathbb{R}^n$; and $F = f$ on E.

This completes the proof of Theorem 4. ■

Note that Theorem 4 easily implies Theorem 1 (B).

As in the case of nonnegative C^m-interpolation, we want to replace $\Gamma_f'(x, M)$ by something easier to calculate. In the $C^{m-1,1}$-setting, it is enough to make the following observation.

Define

$$\tilde{\Gamma}_0' = \left\{ \begin{array}{c} P \in \mathcal{P} : \left|\partial^\beta P(0)\right| \leq 1 \text{ for } |\beta| \leq m-1 \text{ and} \\ P(x) + |x|^m \geq 0 \text{ for all } x \in \mathbb{R}^n \end{array} \right\}.$$

Then

(2) $\Gamma_*'(0, c) \subset \tilde{\Gamma}_0' \subset \tilde{\Gamma}_*'(0, C)$ with c, C depending only on m, n.

Indeed, the first inclusion in (2) is immediate from the definitions and Taylor's theorem. To prove the second inclusion, we let $P \in \tilde{\Gamma}_0'$ be given, and set $F(x) = \chi(x)(P(x) + |x|^m)$, where χ is a nonnegative C^m function with norm at most C_* (depending only on m, n), satisfying $J_0(\chi) = 1$ and support $\chi \subset B_n(0, 1)$.

We then have $F \in C^{m-1,1}(\mathbb{R}^n)$, $\|F\|_{C^{m-1,1}(\mathbb{R}^n)} \leq C$ (depending only on m, n), $F \geq 0$ on $\mathbb{R}^n$, $J_0(F) = P$. Hence, $P \in \Gamma_*'(0, C)$, completing the proof of (2).

This concludes our discussion of interpolation by nonnegative $C^{m-1,1}$ functions.

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Charles Fefferman
Affiliation: Department of Mathematics, Princeton University, Fine Hall
Washington Road, Princeton, New Jersey, 08544, USA
Email: cf@math.princeton.edu

Arie Israel
Affiliation: The University of Texas at Austin, Department of Mathematics,
2515 Speedway Stop C1200, Austin, Texas, 78712-1202, USA
Email: arie@math.utexas.edu

Garving K. Luli
Affiliation: Department of Mathematics, University of California, Davis,
One Shields Ave, Davis, California, 95616, USA
Email: kluli@@math.ucdavis.edu