MANIFOLD CURVATURE FROM COVARIANCE ANALYSIS

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ABSTRACT

Principal component analysis of cylindrical neighborhoods is proposed to study the local geometry of embedded Riemannian manifolds. At every generic point and scale, a high-dimensional cylinder orthogonal to the tangent space at the point cuts out a path-connected patch whose point-set distribution in ambient space encodes the intrinsic and extrinsic curvature. The covariance matrix of the points from that neighborhood has eigenvectors whose scale limit tends to the Frenet-Serret frame for curves, and to what we call the Ricci-Weingarten principal directions for submanifolds. More importantly, the limit of differences and products of eigenvalues can be used to recover curvature information at the point. The formula for hypersurfaces in terms of principal curvatures is particularly simple and plays a crucial role in the study of higher-codimension cases.

***Index Terms*—** covariance analysis, curvature, integral invariants, PCA, Riemannian manifold

1. INTRODUCTION

The ideal goal of manifold learning is the local characterization and reconstruction of manifold geometry from the study of the underlying point set.

Computations of the volume of small geodesic balls within a manifold [1], and volumes cut out by a hypersurface in a ball of the ambient space [2], establish a direct relation to extrinsic mean curvature and the intrinsic Riemann curvature. The eigenvalue decomposition of covariance matrices of spherical intersection domains on the manifold was introduced in [3], [4, 5] to obtain local adaptive Galerkin frames, which provide estimates of the dimension and the tangent space of a submanifold at every point. Covariance integral invariants have been studied [6], [7], [8, 9], [10], [11], [12], [13, 14] as a means to determine relevant local geometric information while maintaining good behavior in noise [15, 16], [17], e.g., for feature estimation of point clouds. Some of these covariance methods use the Taylor expansion of the eigenvalue decomposition of the matrices, at scale, to estimate principal directions and curvatures, primarily for curves and surfaces.

In this paper we define an integral invariant for embedded Riemannian manifolds of any dimension, based on a cylindrical covariance matrix at every point, as a function of the scale. Then its eigenvalue Taylor expansion can be computed exactly to second order. Its limit eigenvectors reproduce the Frenet-Serret frame in the case of curves and the principal directions of the Ricci-Weingarten tensor in general; its limit eigenvalues are directly related to the principal curvatures of such a tensor. For hypersurfaces this permits the partial reconstruction, up to sign choices, of the second fundamental form and thus the Riemann curvature tensor at every generic point from the statistics of the underlying point-set.

2. GEOMETRIC COVARIANCE MATRICES

The covariance matrices defined in [3], [4, 5] for general manifolds, and the integral invariants studied in [8, 9], [10], [11, 15, 16] for smooth curves and surfaces, are local moments of inertia based essentially on domains of intersection with balls.

In our work we modify this approach by considering cylindrical intersections. As motivation we may define the *spherical covariance matrix* of a euclidean n-dimensional submanifold $\mathcal{M} \subset \mathbb{R}^{n+k}$ for a $(n+k)$-ball $\mathcal{B}_p(\varepsilon)$ of radius $\varepsilon \geq 0$ at point $p \in \mathcal{M}$ as

$$\Gamma_p(\varepsilon) := \frac{1}{\mathrm{Vol}_p(\varepsilon)} \int_{\mathcal{B}_p(\varepsilon)\cap\mathcal{M}} (\boldsymbol{x} - p) \cdot (\boldsymbol{x} - p)^T \ \mathrm{dVol},$$

where dVol is the measure on the manifold induced by the euclidean measure, and $\mathrm{Vol}_p(\varepsilon)$ is the volume of $\mathcal{M} \cap \mathcal{B}_p(\varepsilon)$.

A local adaptive Galerkin basis [3] for invariant manifolds of dynamical systems is obtained from the eigenvalue decomposition of this integral invariant, where the ε^2 scaling of the first n eigenvalues singles out the tangent space $T_p\mathcal{M}$ spanned by the corresponding eigenvectors, and the remaining k eigenvectors, from the eigenvalues scaling at least as ε^4, span the normal space $N_p\mathcal{M}$. This serves as a method to decompose the tangent space at p into normal and tangent directions at a given scale. In the case of surfaces, these yield expressions for the Taylor expansion of the eigenvalues in terms

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of the dimension and the principal curvatures, which can then be used as geometry descriptors at scale, e.g. [4], [15]. The other most important property is that some of these estimators are robust with respect to noise [8], [11], [17] due to the averaging nature of the integrals.

But results like these have not been generalized to manifolds of arbitrary dimension embedded in euclidean space, mainly because of the difficulty to compute integrals that require, e.g., $\mathcal{M} \cap \mathcal{B}_p(\varepsilon)$ to be parametrized. For small enough ε, $\mathcal{M}$ is locally given around p by k smooth functions $f_j : T_p\mathcal{M} \cong \mathbb{R}^n \to \mathbb{R}$, $1 \leq j \leq k$, such that $f_j(\mathbf{0}) = 0$ and $\nabla f_j(\mathbf{0}) = \mathbf{0}$; this makes the domain of integration $\|\boldsymbol{x}\|^2 + \sum_{j=1}^{k} f_j(\boldsymbol{x})^2 \leq \varepsilon^2$, for $\boldsymbol{x} \in T_p\mathcal{M}$, which renders the n-dimensional multiple integral of the spherical covariance matrix intractable in practice. We change the type of intersection by using cylinders so that the integration domain reduces to a ball on the tangent space.

We define the *cylindrical component* $\mathrm{Cyl}_p(\varepsilon, \mathbb{V})$ of radius $\varepsilon \geq 0$ of an n-manifold $\mathcal{M} \subset \mathbb{R}^{n+k}$, over the m-plane $\mathbb{V}$ in the Grassmannian $\mathrm{Gr}(m, n+k)$ at a point $p \in \mathcal{M}$, to be the component of the cylinder intersection onto $\mathbb{V}$,

$$\mathrm{Cyl}_p(\varepsilon, \mathbb{V}) := \mathcal{M} \cap \{\boldsymbol{x} \in \mathbb{R}^{n+k} : \|\mathrm{proj}_{\mathbb{V}}(\boldsymbol{x} - p)\| \leq \varepsilon\},$$

which is path-connected to the point p. Here $\mathrm{proj}_{\mathbb{V}}(\cdot)$ is the orthogonal projection of a point onto the m-plane $\mathbb{V}$. Thus, the section of $\mathcal{M}$ that is cut out only takes the points connected to p that lie inside the cylinder created by extending a ball inside $\mathbb{V}$ into the perpendicular directions $\mathbb{V}^{\perp}$. We propose this as our main object of interest.

Definition If the volume $V_p(\varepsilon)$ of $\mathrm{Cyl}_p(\varepsilon, \mathbb{V})$ is finite for an m-plane $\mathbb{V} \in \mathrm{Gr}(m, n+k)$, the *cylindrical covariance matrix* over $\mathbb{V}$ of radius ε, at point $p \in \mathcal{M}$ of an n-dimensional submanifold $\mathcal{M} \subset \mathbb{R}^{n+k}$ is

$$C_p(\varepsilon, \mathbb{V}) := \frac{1}{V_p(\varepsilon)} \int_{\mathrm{Cyl}_p(\varepsilon, \mathbb{V})} (\boldsymbol{x} - p) \cdot (\boldsymbol{x} - p)^T \ \ \mathrm{dVol},$$

where dVol is the measure on $\mathcal{M}$ from the induced metric.

It is useful to think of $C_p(\varepsilon, \mathbb{V})$ as an $(n+k) \times (n+k)$ matrix-valued function of scale ε. This function may be given an eigenvalue expansion for each ε [18], [19].

Since using the eigenvalue decomposition of the spherical covariance to leading order singles out the dimension of the manifold and the tangent space (the cylindrical covariance eigenvectors for generic $\mathbb{V}$ do so as well), we can always choose $\mathbb{V} = T_p\mathcal{M}$, as in fig. 1. Indeed, this will result in the simplest eigenvalue expansion for $C_p(\varepsilon) := C_p(\varepsilon, T_p\mathcal{M})$ with the curvature information at second order as expected.

3. REGULAR CURVES

The case of regular curves $\gamma(s)$ in $\mathbb{R}^n$ needs to be studied independently of higher-dimensional submanifolds, since they only have extrinsic curvature. Our covariance matrix at scale ε and point t, parametrized by arclength s, reduces to

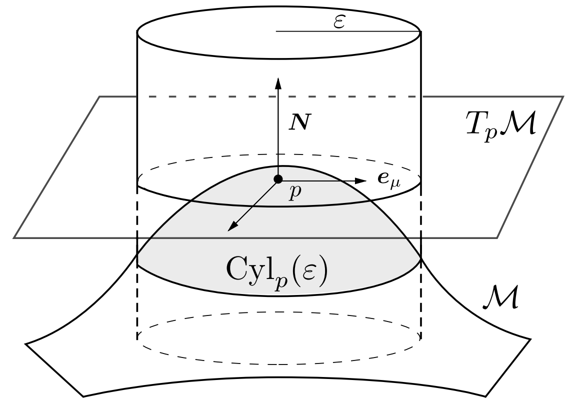

Fig. 1. The cylindrical component over the tangent space cuts out a section of the manifold whose covariance matrix defines an integral invariant at scale ε related to the curvature at p.

$$C_t(\varepsilon) = \frac{1}{2\epsilon} \int_{t-\epsilon}^{t+\epsilon} (\gamma(s) - \gamma(t)) \cdot (\gamma(s) - \gamma(t))^T ds.$$

The Frenet-Serret frame is classicaly defined as the frame obtained by the Gram-Schmidt orthonormalization of the derivative vectors $\{\gamma'(s), \gamma''(s), \ldots, \gamma^{(n)}(s)\}$ at every point. Then it is shown in [5], [20] that this frame is recovered from the limiting eigenvectors of the covariance matrix.

Proposition 3.1 *Let $\gamma : I \to \mathbb{R}^n$ be a parametric curve of class $\mathcal{C}^{n+1}$, regular of order n. Let $\boldsymbol{e}_1(t), \ldots, \boldsymbol{e}_n(t)$ denote the Frenet-Serret frame at $\gamma(t)$. Let $\boldsymbol{V}_1(t), \ldots, \boldsymbol{V}_n(t)$ denote the limit eigenvector of $C_t(\varepsilon)$ at $\gamma(t)$ for $\varepsilon \to 0$. Then for $j = 1, \ldots, n$, $\boldsymbol{e}_j(t) = \pm \boldsymbol{V}_j(t)$.*

Generalized curvatures of such a curve at point t can be defined by the scalar products $\kappa_j(s) = \langle\, \boldsymbol{e}'_j(s)\,,\, \boldsymbol{e}_{j+1}(s)\, \rangle$ for $1 \leq j \leq n-1$, and these functions are classically known to determine the curve locally up to rigid motion [21]. Now, the leading term of the eigenvalue series expansion of $C_t(\varepsilon)$ is directly related to the generalized curvatures [5]:

$$\lambda_1(\varepsilon) = \frac{1}{3}\varepsilon^2 + \mathcal{O}(\varepsilon^4),$$
$$\lambda_j(\varepsilon) = \frac{(\kappa_1 \cdots \kappa_{j-1})^2}{(j!)^2} \frac{B_j}{B_{j-1}} \varepsilon^{2j} + \mathcal{O}(\varepsilon^{2j+2}),\ \ j = 2, \ldots, n$$

where B_j are the following Hankel determinants of size $n \times n$

$$B_n = \det(A_n), \quad (A_n)_{ij} = \begin{cases} \frac{1}{i+j+1}, & \text{if } i+j \text{ is even;} \\ 0 & \text{otherwise.} \end{cases}$$

Concrete examples and the explicit computation of the recursion formula for these determinants is carried out in [20] by using the theory of monic orthogonal polynomials. Solving the Stieltjes moment problem for a particular Hankel sequence that includes the case of interest, produces an explicit expression for the determinants, using Selberg's integral formula. This yields the main result of covariance analysis for regular curves [20]:

Theorem 3.2 *Let $\gamma : I \to \mathbb{R}^n$ be a parametric curve of class $\mathcal{C}^{n+1}$, regular of order n for any $n \in \mathbb{N}$. Let $\kappa_j(t)$ denote the j^{th} curvature function of γ evaluated at t, and let $\lambda_j(\varepsilon)$ be the j^{th} eigenvalue of $C_t(\varepsilon)$. For each $t \in I$ and each $j < n$,*

$$\lim_{\varepsilon\to 0} \frac{\lambda_{j+1}}{\lambda_1 \lambda_j} = A_j \kappa_j^2 \quad \textit{with} \quad A_{j-1} = \left(\frac{j+(-1)^j}{j}\right)^2 \frac{3}{4j^2-1}.$$

Given a sign choice, the curvature functions κ_j could in principle be estimated by this method, applying a covariance analysis for decreasing scales around every point. Implementation of this for a fine sample of points would constitute a scheme at scale for local characterization of the curve up to rigid motion.

4. RIEMANNIAN MANIFOLDS

For a Riemannian manifold $(\mathcal{M}, \langle\cdot,\cdot\rangle)$ the intrinsic curvature tensor can be defined [22], [23], [24], [25] as a measure of the non-commutativity of the Levi-Civita connection:

$$\boldsymbol{R}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{z} = (\nabla_{\boldsymbol{x}}\nabla_{\boldsymbol{y}} - \nabla_{\boldsymbol{y}}\nabla_{\boldsymbol{x}} - \nabla_{[\boldsymbol{x},\boldsymbol{y}]})\boldsymbol{z},$$

for vector fields $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ of the bundle $T\mathcal{M}$. The Ricci tensor is the result of tracing out over a tangent basis $\{\boldsymbol{e}_\mu\}_{\mu=1}^n$:

$$\boldsymbol{\mathcal{R}ic}(\boldsymbol{x},\boldsymbol{y}) = \sum_{\mu=1}^{n} \langle \boldsymbol{R}(\boldsymbol{e}_\mu,\boldsymbol{x})\boldsymbol{y},\, \boldsymbol{e}_\mu\rangle,$$

and intuitively corresponds to an average of sectional curvatures. When embedded in euclidean space with directional derivative $\overline{\nabla}$, the extrinsic curvature is given in terms of the second fundamental form $\mathbf{II}(\boldsymbol{x},\boldsymbol{y}) = (\overline{\nabla}_{\boldsymbol{x}}\,\boldsymbol{y})^\perp$, where $^\perp$ is the orthogonal projection onto the normal bundle of $\mathcal{M}$. For a hypersurface, it conceptually measures how the normal vector turns tangentially. All these are symmetric bilinear forms that smoothly vary with the point, and as such, there are corresponding linear operators on $T_p\mathcal{M}$ associated by metric pairing over a tangent basis [24]:

$$\langle \widehat{\boldsymbol{\mathcal{R}}}\boldsymbol{e}_\mu, \boldsymbol{e}_\nu\rangle = \boldsymbol{\mathcal{R}ic}(\boldsymbol{e}_\mu,\boldsymbol{e}_\nu),$$

$$\langle \widehat{\boldsymbol{S}}_{\boldsymbol{N}}\boldsymbol{e}_\mu, \boldsymbol{e}_\nu\rangle = \langle \mathbf{II}(\boldsymbol{e}_\mu,\boldsymbol{e}_\nu), \boldsymbol{N}\rangle,$$

called respectively the Ricci operator and the Weingarten operator in the normal direction $\boldsymbol{N}$. We finally recall the notion of mean curvature vector, generalizing the mean curvature of hypersurfaces: $\boldsymbol{H} = \sum_{\mu=1}^n \mathbf{II}(\boldsymbol{e}_\mu,\boldsymbol{e}_\mu)$, i.e., the trace of the Weingarten operator. Similarly, the scalar curvature $\mathcal{R}$ is defined to be the trace of the Ricci operator.

Let us define the *Ricci-Weingarten operator* as the endomorphism of $T_p\mathcal{M}$ given by

$$\widehat{\boldsymbol{\mathcal{W}}} := \widehat{\boldsymbol{S}}_{\boldsymbol{H}} - \widehat{\boldsymbol{\mathcal{R}}},$$

and we call its eigenvectors and eigenvalues the Ricci-Weingarten principal directions and principal curvatures.

Proposition 4.1 *The asymptotic volume ratio between the cylindrical component over the tangent space of $\mathcal{M}$ and the euclidean ball of the same dimension is:*

$$\frac{\mathrm{Vol}(\mathrm{Cyl}_p(\varepsilon))}{\mathrm{Vol}(B_p(\varepsilon))} = 1 + \frac{\varepsilon^2}{2(n+2)}(\|\boldsymbol{H}\|^2 - \mathcal{R}) + \mathcal{O}(\varepsilon^4),$$

where in fact $\|\boldsymbol{H}\|^2 - \mathcal{R} = \mathrm{tr}\,\widehat{\boldsymbol{\mathcal{W}}}$.

The cylindrical covariance analysis eigenvectors tend to the generalized principal directions.

Proposition 4.2 *Let $\mathcal{M}$ be an n-dimensional Riemannin submanifold of $\mathbb{R}^{n+k}$. Then for every generic point $p \in \mathcal{M}$ the limit eigenvectors $\{\boldsymbol{V}_i(0)\}_{i=1}^{n+k}$ of the tangent cylindrical covariance matrix $C_p(\varepsilon)$ yield a local adapted orthonormal frame of $T_p\mathcal{M} \oplus N_p\mathcal{M}$. The tangent basis consists of the Ricci-Weingarten principal directions $\{\boldsymbol{e}_\mu\}_{\mu=1}^n$ corresponding to n eigenvalues that scale as ε^2, whereas the normal basis $\{\boldsymbol{N}_j\}_{j=1}^k$ corresponds to k eigenvalues that scale as ε^4.*

For Einstein manifolds the Ricci curvature vanishes at orthonormal vectors which amounts to diagonalizing just the Weingarten operator and thus having the classical principal directions for the mean curvature as limit eigenvectors. When the manifold is minimal the mean curvature is zero so the diagonalization reduces to that of the Ricci operator and its principal directions. The general tangent frame selected by the limit eigenvectors of the covariance matrix is thus an intermediate set of perpendicular directions between these two extreme cases. The main result of the present work is a version of theorem 3.2 for general manifolds in terms of the eigenvalue decomposition of $\widehat{\boldsymbol{\mathcal{W}}}$.

Theorem 4.3 *The tangent eigenvalues of $C_p(\varepsilon)$ satisfy*

$$\lim_{\varepsilon\to 0} \frac{\lambda_\mu(\varepsilon) - \lambda_\nu(\varepsilon)}{\lambda_\mu(\varepsilon)\lambda_\nu(\varepsilon)} = \frac{n+2}{n+4}\,(\,W_\mu - W_\nu\,)\,,$$

where W_μ are the eigenvalues of $\widehat{\boldsymbol{\mathcal{W}}}$ at p. Also, the normal eigenvalues satisfy:

$$\lim_{\varepsilon\to 0} \frac{1}{\lambda_\mu(\varepsilon)\lambda_\nu(\varepsilon)} \sum_{j=n+1}^{n+k} \lambda_j(\varepsilon) = \frac{n+2}{n+4}\left(\frac{3}{4}\|\boldsymbol{H}\|^2 - \frac{1}{2}\mathcal{R}\right)$$

for any $\mu, \nu = 1, \ldots, n$.

The second formula above provides a closed-form expression for the average of the curvature integrals in the normal eigenvalues of [4, 5], generalizing the explicit result for surfaces [4].

Example Let $\mathcal{S}$ be a hypersurface in $\mathbb{R}^{n+1}$. At every generic point p, the limit eigenvectors of $C_p(\varepsilon)$ yield a local adapted orthonormal frame $\langle \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n \rangle \oplus \langle \boldsymbol{N} \rangle \cong T_p\mathbb{R}^{n+1}$. If the principal curvatures are of different absolute value, this basis exactly consists of the classical principal directions. Moreover, in this case the Ricci-Weingarten eigenvalues simplify to result in

$$\lim_{\varepsilon\to 0} \frac{\lambda_\mu(\varepsilon) - \lambda_\nu(\varepsilon)}{\lambda_\mu(\varepsilon)\lambda_\nu(\varepsilon)} = \frac{n+2}{n+4}(\,\kappa_\mu^2 - \kappa_\nu^2\,),$$

and

$$\lim_{\varepsilon\to 0} \frac{\lambda_{n+1}(\varepsilon)}{\lambda_\mu(\varepsilon)\lambda_\nu(\varepsilon)} = \frac{n+2}{n+4}\left(\frac{3}{4}H^2 - \mathcal{K}\right),$$

for any $\mu, \nu = 1, \ldots, n$. Here κ_μ, H, $\mathcal{K}$ are respectively the principal, mean and Gaussian curvatures of $\mathcal{S}$ at p. This yields the same factor as our other formula for planar curves.

Within a general ambient Riemannian manifold $\mathcal{N}$ there are no global cartesian coordinates available but for scales smaller than the injectivity radius r_p at each point, geodesic coordinates (Riemann normal coordinates) are available via the exponential map of $\mathcal{N}$, since in that case it is a diffeomorphism from the tangent space. Then the cylindrical component of radius $0 \leq \varepsilon \leq r_p$ at a point p of an n-dimensional submanifold $\mathcal{M}$ of an $(n+k)$-dimensional Riemannian manifold $\mathcal{N}$, over the m-plane $\mathbb{V}$, can be defined to be:

$$\mathrm{Cyl}_p(\varepsilon, \mathbb{V}) = \mathcal{M} \cap \{q \in \mathcal{N} : \|\mathrm{proj}_{\mathbb{V}}(\exp_p^{-1}(q))\| \leq \varepsilon \leq r_p\},$$

and with this, a cylindrical geodesic covariance matrix can in turn still be meaningful:

$$C_p(\varepsilon, \mathbb{V}) = \frac{1}{V_p(\varepsilon)} \int_{\mathrm{Cyl}_p(\varepsilon,\mathbb{V})} [\exp_p^{-1}(q) \otimes \exp_p^{-1}(q)] \;\; \mathrm{dVol},$$

where dVol is the induced measure on $\mathcal{M}$ from the metric of $\mathcal{N}$. The entries of the matrix are given by choosing a frame at p that establishes an isomorphism $T_p\mathcal{N} \cong \mathbb{R}^{n+k}$ in which the $\exp$ map acquires vector components through the normal coordinates. In this context, our analysis can be generalized as long as the extra ambient curvature operator is taken into account: $\widehat{\boldsymbol{\mathcal{W}}} = \widehat{\boldsymbol{S}}_{\boldsymbol{H}} + \widehat{\boldsymbol{\mathcal{R}}}_{\mathcal{N}} - \widehat{\boldsymbol{\mathcal{R}}}_{\mathcal{M}}$.

5. CURVATURE RECONSTRUCTION

Using the volume comparison formula for a hypersurface $\mathcal{S}$, we can determine at every point the number

$$A := H^2 - 2\mathcal{K} = \lim_{\varepsilon\to 0} \frac{2(n+2)}{\varepsilon^2}\left[\frac{\mathrm{Vol}(\mathrm{Cyl}_p(\varepsilon, T_p\mathcal{S}))}{\mathrm{Vol}(B_p(\varepsilon))} - 1\right].$$

Calling $A_{\mu\nu} = \kappa_\mu^2 - \kappa_\nu^2$, these can be computed from the eigenvalue limits of our theorem in the hypersurface case, under the same generic conditions, so that the principal curvatures can be determined:

$$\kappa_\mu^2 = \frac{1}{n}(A + \sum_{\rho=1}^{n} A_{\mu\rho}), \kappa_{\nu\neq\mu}^2 = \frac{1}{n}(A + \sum_{\rho=1}^{n} A_{\mu\rho}) - A_{\mu\nu}.$$

Since the orientation of the normal vector changes all signs of the curvature, we have 2^{n-1} sign choices, locally maintained around p by the n functions κ. In principle these functions and principal directions at every point determine the second fundamental form locally, so by the hypersurface characterization theorem [25], there exist 2^{n-1} hypersurfaces unique up to rigid motion corresponding to the given underlying pointset via this covariance analysis.

Hence, all the information to recover the extrinsic and intrinsic curvatures of $\mathcal{M}$, up to $2^{k(n-1)}$ sign choices, is in the eigenvalue decomposition of the matrices $C_p(\varepsilon)$ of the k local hypersurfaces $\mathcal{S}_j$, created by projecting $\mathcal{M}$ to the linear spaces $T_p\mathcal{M} \oplus \langle \boldsymbol{N}_j \rangle$. Precisely, if $[V_j]$ are the matrices of the principal direction vectors of $\mathcal{S}_j$, i.e. the limit eigenvectors of $C_p(\varepsilon)$ as columns in a chosen orthonormal frame $\{\boldsymbol{u}_\mu\}_{\mu=1}^n \cup \{\boldsymbol{v}_j\}_{j=1}^k$ for $T_p\mathcal{M} \oplus N_p\mathcal{M}$, and $[K_j]$ are the diagonal matrices of principal curvatures, then the second fundamental form is:

$$\mathbf{II}(\boldsymbol{u}_\mu, \boldsymbol{u}_\nu) = \sum_{j=1}^{k} [V_j K_j V_j^T]_{\mu\nu}\, \boldsymbol{v}_j \;, \qquad \mu, \nu = 1, \ldots, n.$$

Therefore, by Gauss equation, the Riemann curvature tensor components at p in this frame are $\langle \boldsymbol{R}(\boldsymbol{u}_\mu, \boldsymbol{u}_\nu)\boldsymbol{u}_\alpha,\, \boldsymbol{u}_\beta \rangle =$

$$\sum_{j=1}^{k} \left(V_j K_j V_j^T|_{\mu\beta} V_j K_j V_j^T|_{\nu\alpha} - V_j K_j V_j^T|_{\mu\alpha} V_j K_j V_j^T|_{\nu\beta}\right)$$

The next natural step would be the study of special geometric properties from covariance eigenvalues.

6. CONCLUSION

We have proposed a cylindrical covariance matrix, defined at the scale of the cylinder radius, as a descriptor to be estimated from the point set of a Riemannian submanifold. The eigenvectors of this covariance matrix converge to a basis for a tangent frame, and this basis defines generalized principal directions. The eigenvalues of the covariance matrix converge to functions of extrinsic and intrinsic curvature. This extends the use of multiscale integral invariants to manifolds of arbitrary dimension embedded in euclidean space, and can be generalized to other ambient Riemannian manifolds by using geodesic normal coordinates through the exponential map.

7. ACKNOWLEDGMENT

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