On the Impact of Information Disclosure on Advance Reservations: A Game-Theoretic View

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Abstract

In many branches of the economy, customers can reserve resources in advance. Yet, service providers often differ in the information they disclose to customers. In this paper, we evaluate how information about server availability impacts the strategic behavior of customers in a loss system with N servers, where each customer can either reserve a server at a certain cost, or take the risk of finding no server available. We formulate the problem as a non-cooperative game with a random number of players. Our main contributions are to establish the existence, structure, and uniqueness (or lack thereof) of pure Nash equilibria, depending on the information disclosed to customers. Specifically, we first prove that if the number of available servers is always disclosed, then there exists exactly one pure Nash equilibrium. High reservation costs lead to an equilibrium in which all servers remain unreserved, while low reservation costs lead to an equilibrium that consists of N "time-thresholds." A customer that observes n available servers, makes a reservation only if she makes her inquiry before the n-th time-threshold. Next, we consider the case where the number of available servers is disclosed only when that number falls below a certain threshold. We show that, in this game, the same types of equilibria prevail. However, multiple Nash equilibria may exist. Finally, we numerically compare the performance of the different policies and we formulate the conjecture that it is preferable for a provider to hide information about the number of available servers.

Keywords: Game theory, Queueing theory, Advance reservations, Information disclosure.

1. Introduction

Consider a service provider that lets its customers reserve resources in advance. The provider may weigh different options for disclosing information to its customers. For example, it may disclose full information about the number of available servers. This practice is common in entertainment services, where customers are allowed to choose their seats and observe the precise number of available seats before making their reservations (e.g., using Ticketmaster[1] reservation services). At the other extreme, the provider may opt to hide any information about the number of available servers. This policy is adopted by some airlines, such as Delta Air Lines[2], where customers can only choose their seats after buying their tickets. Finally, the provider may choose a middle ground solution where information about server availability is disclosed only when the number of available servers falls below some threshold. This policy has been adopted by lodging web sites, such as Booking.com[3], that alert potential customers when a few rooms are left available.

In this work, we study the impact of different information disclosure policies on the equilibrium outcomes of systems supporting advance reservation (AR). Specifically, we consider a loss system consisting of N servers. Thus, a customer leaves the system if no server is available (e.g., if all the servers have already been booked by other customers), rather than waiting in a line. This assumption is made for the sake tractability, and is reasonable for several of the aforementioned applications. The number of customers is a random variable that follows a *general* distribution. The arrival time of each customer, which represents the time

[1]See www.ticketmaster.com, accessed on 12/01/2016.
[2]See www.delta.com, accessed on 12/01/2016.
[3]See www.booking.com, accessed on 12/01/2016.

at which she makes a reservation inquiry, is also a generally distributed random variable. We assume that customers behave strategically. Each customer needs to decide whether or not to make a reservation, based on information disclosed by the provider. Making AR guarantees service but is associated with an additional fixed *cost*. This cost can be interpreted as a reservation fee [33] or as the time or resources required for making the reservation [26]. When avoiding AR, this cost is spared but may cause denial of service.

Our main contributions are as follows. We first analyze a *full-information game*. In this game, customers observe the exact number of available servers. We determine the equilibrium structure and prove that the existence of exactly one pure Nash equilibrium (for simplicity, we only focus on the existence, structure, and uniqueness of pure equilibria in this paper). High AR costs lead to an equilibrium in which all servers remain unreserved, while low AR costs lead to an equilibrium that consists of N time-thresholds. A customer that observes $1 \leq n \leq N$ available servers, makes a reservation only if she makes her inquiry before the n-th time-threshold.

We next analyze a *partial-information game*. In this game, the provider informs customers about the number of available servers only if this number falls below a certain threshold. We assume that unnotified customers realize that the number of available servers is larger than the threshold and take this fact under consideration upon making their decisions. We show that, in this game, Nash equilibria have the same structure. However, multiple equilibria may exist.

Informing customers about the number of available servers leads to a trade-off. On the one hand, customers that observe that all or almost all servers are available may be less prone to make a reservation (compared to a system where no information about server availability is disclosed). On the other hand, customers that observe that only a few servers are available may be more likely to make a reservation. To evaluate this trade-off, we resort to simulations. The simulation results indicate that, on average, the number of reservations decreases as more information is provided to customers. More specifically, the *full-information* policy yields the lowest number of reservations. In the *partial-information* policies, the number of customers making advance reservation increases as the threshold is lowered, and the largest number of reservations is achieved when no information is provided.

The rest of the paper has the following structure. In Section 2, we review related work. In Section 3, we define and analyze the *full-information* game. In Section 4, we define and analyze the *partial-information* game. In Section 5, we present simulation results that compare the performance of the different policies. The proof of Theorem 1, which establishes the structure of the equilibria, is deferred to Section 6 due to its length. Section 6 concludes the paper and points out directions for future research.

2. Related Work

In this section, we discuss how prior work in the literature relates to our paper and point out salient differences. First, we review prior work on game-theoretic analysis of customer behavior in queueing systems with and without information disclosure. Most of that prior work focuses on queueing systems with waiting lines rather than loss systems with advance reservations. Next, we review related work on algorithmic, queueing-theoretic and revenue management aspects of systems supporting advance reservation. In contrast to our paper, most of that earlier work ignores strategic customer behavior.

The application of game theory to analyze the behavior of customers in queues (also known as *queueing games*) is pioneered in [22]. In that paper, the author considers an $M/M/1$ queue where customers observe the queue length and then decide whether to join or balk (i.e., refuse to join the queue). In contrast, in [8] the authors analyze an unobservable $M/M/1$ queue, where customers decide whether to join a queue or balk, based only on knowledge of statistical parameters of the queue. Research on queueing games has been subsequently extended to address other aspects of queueing systems. For a review of results on queueing games, see [15] and [14].

In recent years, research on the impact of information on customer's behavior in queueing systems has emerged. Several studies use game theory to compare the outcomes of different information disclosure policies in $M/M/1$ queues. In [12], the authors study the effects of disclosing information about delay based on three approaches: no information is disclosed, information about the queue length is disclosed, and precise information about delay is disclosed. They find that disclosing precise delay information sometimes improves social welfare and sometimes reduces it. In [13], the author studies the impact of disclosing other types of information, such as the service rate or the quality of service (it is assumed that both parameters may change over time). The author shows that in some cases the provider is better off revealing the realizations of those parameters, while in other cases it is better off concealing them. In [27], the authors evaluate a *partial-information* disclosure policy, where customers are notified about the queue length when it falls below a certain threshold. The authors show that such a policy is never optimal. Thus, to maximize the effective arrival rate of customers, a provider should either always disclose information about queue length or always hide it (depending on the system parameters). The problem formulations and insights of our paper differ considerably from previous contributions on game-theoretic analysis of $M/M/1$ queues, which focus on delay metrics (rather than loss metrics) and do not incorporate advance reservations.

In [34], the authors evaluate the impact of disclosing inventory information on the strategic behavior of customers, assuming that the demand follows a Poisson distribution. In contrast, our analysis applies to demand that follows a general distribution and also establishes results for the case where partial information is disclosed. A comprehensive review on the impact of information in queuing games can be found in [14, Chapter 3]

AR games are introduced in [30] (see also [28, 29]). In that work, the authors consider a loss system operating under a *no-information* policy. Under this policy, decisions of customers are only based on statistical information. The authors show that, at equilibrium, either none of the customers make advance reservations or only those with arrival times smaller than some time-threshold. While based on a similar game-theoretic formulation, our paper differs from [30] by assuming that customers can observe or partially observe the number of available servers prior to making their decisions. In contrast, in [30], customers are *not* notified about the number of available servers and their decisions are only based on statistical information. This change in the model greatly affects the analysis and nature of equilibria, since the decision of one customer affects information provided to other customers.

Research on advance reservations of reusable resources can also be found in the literature of communication networks and revenue management. In the field of communication networks, most research focuses on performance evaluation and algorithmic aspects of AR systems. For example, the authors in [17] analyze an AR scheme for a loss system where customers have flexible starting time and show that this scheme leads to lower blocking probability and a higher utilization than those attainable in an inflexible scheme. In [11], the authors analyze the effect of AR on the complexity of path selection. In [32], the author evaluates the impact of advance reservation on server utilization. In [5], the authors propose algorithms for network routing that support advance reservations of network paths. For a survey of the field, see [4].

In the field of revenue management, the authors of [18] analyze a hotel reservation system with overbooking. The goal is to determine the optimal overbooking level. In [24], the authors introduce an admission control policy for a reservation system with different classes of customers. In [2], the authors propose a policy for accepting or rejecting restaurant reservations. In [23], the authors assume that customers have uncertain valuations and need to decide whether to purchase in advance before their valuations are realized. The provider, aiming to maximize the revenue, can set different prices for advance sale and spot sale. The authors of [9] derive optimal management policies for stochastic inventory systems where customers provide advance information about their demands. The work in [10] extends the analysis to the cases of multiple classes of customers and imperfect demand information. In [3], the authors analyze the benefits of acquiring demand information through advance sales to dimension service capacity. In particular, the paper establishes policies that prescribe the optimal time to stop collecting advance sales information.Most recently, the work in [7] proposes and analyzes reservation policies for inventory systems supporting multiple classes of customers and the work in [16] analyzes parking reservation policies for vehicle sharing systems. None of this prior work considers strategic customer behavior, namely, that decisions of customers are not only influenced by prices and policies set by providers but also by their beliefs about the decisions of other customers.

3. Full-information Game

3.1. Model Description

A monopoly firm offers service with a specific start time. For example, consider a sport event or a movie show. The service has limited capacity N (referred to as servers).

When deciding whether to reserve resources in advance, customers typically do not know how many other customers compete for the same set of resources. Therefore, a key element of our model is to treat the number of players as a random variable, similar to [21] and [20]. The *demand* D represents the number of

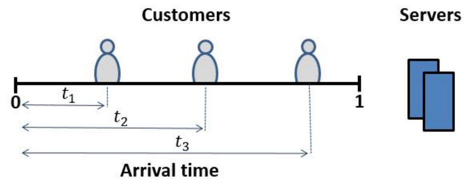


Figure 1: A service with two servers. The demand in this realization is $D = 3$, corresponding to three customers. A customer that arrives early can make an inquiry and reserve a server before a customer that arrives later (i.e., with larger arrival time).

customers requesting service (each customer requests one server). The demand is a discrete random variable that follows a *general* probability distribution P_D, supported in $[a, b]$, where $0 \leq a < N$ and $N < b \leq \infty$ (i.e, the demand has a positive probability of being smaller, equal, or larger than the number of servers). The distribution P_D is public information. We also provide explicit results for the special case where P_D is a Poisson distribution, by exploiting characteristics of Poisson games that were introduced in [21].

We define the *arrival time* of a customer to be the time at which she *realizes* that she will need service (that is, how long in advance before service starts). We assume customers start observing the system only after their arrival times and the arrival times are independent of the information being disclosed. The arrival times are i.i.d. random variables following a general continuous distribution $F(\cdot)$. Note that, under our model, we first determine the realization of the demand and then the arrival time of each customer. Figure 1 shows an example of the realizations of arrival times for system with a demand of 3 customers.

Upon making an inquiry, a customer observes the number of available servers $0 \leq n \leq N$. If $n \geq 1$, then she chooses one of the two actions: make AR or not make AR. Those two action are denoted AR and AR', respectively. If $n = 0$ she leaves the system with no gain or loss.

The servers are allocated in a *first-reserved-first-allocated* fashion. If $D > N$, but the number of reservations is smaller than N, then all the customers that request service but did not make AR have the same chance of being served (this assumption is common in the field of advance reservation, see [19] for an example).

All the customers have the same reward R from service. Without loss of generality, we set $R = 1$. Making an advance reservation is associated with a fixed cost C. The AR cost reflects all aspects of making an advance reservation. Analyzing a game with negative cost or cost larger than 1 is trivial. Thus, we assume that $0 < C < 1$. Customers that are not being served leave the system with no gain or loss. Thus, the payoff of each customer that chooses AR is $1 - C$. The payoff of a customer that chooses AR' is either 0 or 1 and her expected payoff is her probability of getting service.

Optimization objectives. We analyze this model as a non-cooperative game where the customers are the players. Their objective is to maximize their expected payoffs. If a customer chooses the action AR, her payoff is $1-C$, while if she chooses the action AR', her expected payoff is the probability that she is granted service (multiplied by the reward $R = 1$). That is, a customer will make a reservation only if $1-C$ is greater than her probability to get service if she does not make a reservation. As for the provider, the goal is to find out which of the information disclosure policies maximizes the average number of reservations (for any given cost C). The results of the provider's optimization will be studied numerically in Section 5.

Our goal is to characterize the existence, uniqueness (or lack thereof), and structure of the pure Nash equilibria (that is the solutions to the game, where each customer uses a deterministic strategy and no customer gains by unilaterally changing her strategy).

3.2. Preliminaries

In this section, we introduce definitions and notations that will be used throughout the paper.

Residual Life Paradox. The fact that a customer seeks service affects her estimation of the number of other customers seeking service. On the one hand, a customer is more likely to seek service when the demand is large than when the demand is small. On the other hand, she must exclude herself. This phenomenon is known as the discrete case of the waiting time paradox (or residual life paradox). We define $\tilde{D}$ as the number of customers seen by a customer beside herself. The distribution of $\tilde{D}$ is [1]

$$\mathbb{P}(\tilde{D}=i) = \mathbb{P}(D=i+1)\frac{i+1}{\mathbb{E}[D]}, \quad i \geq 0. \tag{1}$$

Normalized Arrival Time. We define the *normalized arrival time* t of a customer to be the value of the cumulative distribution function (CDF) of her arrival time x, that is, $t = F(x)$. Due to the probability integral transformation theorem [6, p. 320], the normalized arrival time of each customer is a random variable that is uniformly distributed in $[0, 1]$, where 1 is the time at which service starts. That is, a customer with normalized arrival time 0 is with probability 1 the first to make an inquiry, while a customer with arrival time 1 is with probability 1 the last to make an inquiry. Throughout the paper, and without any loss of generality, we only consider the normalized arrival time and refer to it as the *arrival time* for simplicity. Note that results for the normalized arrival time t can be converted back to the original arrival time x by applying the inverse transformation $x = F^{-1}(t)$.

Probability of Service. Since customers are statistically identical, we only consider symmetric strategies (a common assumption in the literature of queueing games [22]). These strategies correspond to a mapping σ of the arrival time t and the number of available servers n to the probability of choosing the action AR. Thus, we denote a general strategy by $\sigma(t, n)$.

Consider a tagged customer that arrives at time t. We define $\tilde{D}_{AR}$ to be the total number of reservations as seen by the tagged customer beside herself and $\tilde{D}_{AR}(t)$ to be the number of reservations made before her arrival (both are random variables). We denote by S the event that a customer gets service and by $\{S|\sigma, \tilde{D}_{AR}(t) = N - n)\}$ the event that a tagged customer arriving at time t gets service, while she observes $n \geq 1$ available servers, chooses AR', and the rest of the customers follows the strategy σ. We then refer to $\mathbb{P}(S|\sigma, \tilde{D}_{AR}(t) = N - n)$ as the *probability of service*. Note that the probability of service applies only to customers that choose AR', since service is granted to customers that choose AR upon observing $n \geq 1$ available servers.

Similarly, let $\tilde{D}_{AR'}$ represent the total number of customers choosing AR', as seen by the tagged customer beside herself. Note that $\tilde{D} = \tilde{D}_{AR} + \tilde{D}_{AR'}$.

Since customers aim to maximize their expected payoff, at equilibrium, a customer that arrives at time t and observes n available servers chooses AR only if $\mathbb{P}(S|\sigma, \tilde{D}_{AR}(t) = N - n) \leq 1 - C$ and chooses AR' only if $\mathbb{P}(S|\sigma, \tilde{D}_{AR}(t) = N - n) \geq 1 - C$.

Non-degenerate Equilibria. In theory, the game under consideration may have an infinite number of equilibria. For example, suppose that $\sigma(t, n) = 0$ for all $0 \leq t \leq 1$ and $n > 0$ and suppose that $\mathbb{P}(S|\sigma, \tilde{D}_{AR}(t) = 0) = 1 - C$ (i.e., all customers choose AR' and are indifferent between the actions AR and AR'). In this case, σ is an equilibrium strategy. Yet,

$$\sigma(t, n) = \begin{cases} 1 & \text{if } t = 0.5, \\ 0 & \text{otherwise,} \end{cases}$$

is also an equilibrium strategy, since the probability that a customer arrives exactly at $t = 0.5$ is zero. In practice, all customers choose AR' under both strategies. To eliminate such degenerate cases, we assume:

Assumption 1. *Let $n \geq 1$, $0 \leq t \leq 1$, and $0 \leq x \leq 1$. At equilibrium, if $\sigma(n, t) = x$, then there must exist a non-zero measure interval I such that $t \in I$ and $\sigma(n, t') = x$ for all $t' \in I$.*

3.3. Equilibria Analysis

In this section, we characterize the pure Nash equilibria. First, we state that there exist N time-thresholds, such that, at equilibrium, a customer that observes n available servers will make a reservation only if she arrives before the n-th time-threshold.

Definition 1. *Let $\boldsymbol{\tau^e} = \{\tau_1^e, \tau_2^e, ..., \tau_N^e\}$, where $0 \leq \tau_N^e \leq \tau_{N-1}^e \leq ... \leq \tau_1^e \leq 1$. A strategy function $\sigma(t, n)$ is a* threshold strategy *if it has the form:*

$$\sigma(t, n) = \begin{cases} 1 & \textit{if } t < \tau_n^e, \\ 0 & \textit{if } t \geq \tau_n^e. \end{cases} \tag{2}$$

Theorem 1. *At a pure equilibrium, all customers follow a threshold strategy.*

Proof. See Appendix.

This theorem implies that a customer that arrives after the n-th time-threshold τ_n^e and observes n available servers will not make a reservation. Consequently, no other customer coming afterward will make a reservation since each such customer also arrives after τ_n^e and also observes n available servers.

In the remainder of this section, we show that a unique pure equilibrium exists for any AR cost. Furthermore, we show that there is a critical value $\underline{C}$ such that costs higher than $\underline{C}$ lead to an equilibrium with no reservations.

First, we denote by $\sigma_{\boldsymbol{\tau}^{\boldsymbol{e}}}$ an equilibrium strategy constructed by a set of N time-thresholds $\boldsymbol{\tau}^{\boldsymbol{e}} = \{\tau_1^e, \tau_2^e, ..., \tau_N^e\}$. For such a strategy to be an equilibrium, the following must hold for any $n \in \{1, ..., N\}$:

(i) If $\tau_n^e > 0$, then

$$\mathbb{P}(S|\sigma_{\boldsymbol{\tau}^{\boldsymbol{e}}}, \tilde{D}_{AR}(\tau_n^e) = N - n) = 1 - C. \tag{3}$$

(ii) If $\tau_n^e = 0$, then

$$\mathbb{P}(S|\sigma_{\boldsymbol{\tau}^{\boldsymbol{e}}}, \tilde{D}_{AR}(\tau_n^e) = N - n) \geq 1 - C. \tag{4}$$

To understand this property, consider a virtual customer that arrives at time τ_n^e and observes $1 \leq n \leq N$ available servers. We refer to such a customer as the *n-th threshold customer*. If $\tau_n^e > 0$, then customers that arrive before τ_n^e choose AR, while customers that arrive after τ_n^e choose AR'. Hence, a customer that arrives exactly at the time-threshold must be indifferent between the two options. If $\tau_n^e = 0$, then all customers choose AR'. Thus, the probability of service of all customers (including the n-th threshold customer) should be no smaller than $1 - C$.

To prove that the game has exactly one pure equilibrium, we will show that for each $1 \leq n \leq N$ there is exactly one time-threshold τ_n^e for which Eq. (3) or Eq. (4) hold. For this purpose, we develop an expression of the probability of service of the n-th threshold customer as a function of the set of time-thresholds $\boldsymbol{\tau} = \{\tau_1, \tau_2, ..., \tau_N\}$. We denote this quantity by $\pi_n(\boldsymbol{\tau})$. If the n-th threshold customer chooses AR', then all customers that arrive after her also choose AR'. In this case $\tilde{D}_{AR} = N - n$. The probability of service of the n-th threshold customer is

$$\pi_n(\boldsymbol{\tau}) = \sum_{i=N-n}^{N-1} \mathbb{P}(\tilde{D} = i|\tilde{D}_{AR} = N - n) \cdot 1 + \sum_{i=N}^{b} \mathbb{P}(\tilde{D} = i|\tilde{D}_{AR} = N - n) \cdot \frac{n}{i + 1 - (N - n)}. \tag{5}$$

The first term is the probability that the total demand is smaller than N (recall that $\tilde{D}$ is the number of customers as observed by a tagged customer excluding herself). In this case, service is guaranteed to all customers. The second term is the probability to get service when the demand exceeds N. In this case, the n unreserved servers will be arbitrarily allocated among customers that chose AR'. Next, we develop an expression for $\pi_n(\boldsymbol{\tau})$ that directly relates to the system parameters (i.e., an expression that does not include $\tilde{D}_{AR}$).

Lemma 1. *Let $1 \leq n \leq N$. The probability of service of the n-th threshold customer is*

$$\pi_n(\boldsymbol{\tau}) = \frac{\sum\limits_{i=N-n}^{N-1} \mathbb{P}(\tilde{D} = i)\binom{i}{N-n}(1 - \tau_n)^i + \sum\limits_{i=N}^{b} \mathbb{P}(\tilde{D} = i)\binom{i}{N-n}(1 - \tau_n)^i \frac{n}{i+1-(N-n)}}{\sum\limits_{i=N-n}^{b} \mathbb{P}(\tilde{D} = i)\binom{i}{N-n}(1 - \tau_n)^i}. \tag{6}$$

Proof. First, we find an expression for $\mathbb{P}(\tilde{D} = i|\tilde{D}_{AR} = N - n)$. For brevity, let $m \triangleq N - n$. Using Bayes' theorem, we obtain

$$\begin{aligned} \mathbb{P}(\tilde{D} = i|\tilde{D}_{AR} = m) &= \frac{\mathbb{P}(\tilde{D} = i)\mathbb{P}(\tilde{D}_{AR} = m|\tilde{D} = i)}{\mathbb{P}(\tilde{D}_{AR} = m)} \\ &= \frac{\mathbb{P}(\tilde{D} = i)\mathbb{P}(\tilde{D}_{AR} = m|\tilde{D} = i)}{\sum\limits_{k=m}^{b} P(\tilde{D} = k)\mathbb{P}(\tilde{D}_{AR} = m|\tilde{D} = k)}, \quad \forall i \in [m, b]. \end{aligned} \tag{7}$$

Next, we develop an expression for $\mathbb{P}(\tilde{D}_{AR}=m|\tilde{D}=i)$. Given that i customers request service, the probability that exactly m customers choose AR is the probability that the first m customers among them choose AR (i.e., the arrival time of the j-th customer is smaller than τ_{N-j} for all $j \in \{1,...,m\}$), while the last $i-m$ customers choose AR' (i.e., the arrival times of the $m+1$ customer and on are larger than τ_n). We denote by AR_m the event that a given subset of m customers choose AR in a system that it is initially empty. The probability that the remaining $i-m$ customers choose AR' in a system with n available servers is $(1-\tau_n)^{i-m}$. Thus,

$$\mathbb{P}(\tilde{D}_{AR}\!=\!m|\tilde{D}\!=\!i) = \binom{i}{m}\mathbb{P}(AR_m)(1-\tau_n)^{i-m}. \tag{8}$$

The term $\mathbb{P}(AR_m)$ will be canceled out, and therefore we do not develop an expression for it. Substituting Eq. (7) and Eq. (8) into Eq. (5), we get

$$\pi_n(\boldsymbol{\tau}) = \frac{\sum\limits_{i=m}^{N-1}\mathbb{P}(\tilde{D}\!=\!i)\binom{i}{m}\mathbb{P}(AR_m)(1-\tau_n)^{(i-m)} + \sum\limits_{i=N}^{b}\mathbb{P}(\tilde{D}\!=\!i)\binom{i}{m}\mathbb{P}(AR_m)(1-\tau_n)^{i-m}\frac{n}{i+1-m}}{\sum\limits_{i=m}^{b}\mathbb{P}(\tilde{D}\!=\!i)\binom{i}{m}\mathbb{P}(AR_m)(1-\tau_n)^{(i-m)}}. \tag{9}$$

Note that $\mathbb{P}(AR_m)$ and $(1-\tau_n)^{-m}$ do not depend on i. Since those two terms appear in each term of the numerator and the denominator, we can cancel them out and conclude that Eq. (6) holds □.

From Lemma 1, we deduce:

Corollary 1. *$\pi_n(\boldsymbol{\tau})$ only depends on $\tau_n \in \boldsymbol{\tau}$.*

We thus redefine the function $\pi_n(\cdot)$ such that the time-threshold τ_n is its only input. This result implies that the probability of service of an *n-th threshold customer* depends only on her own time-threshold and not on the time-thresholds of other customers.
Example 1a. *Assume that the demand is uniformly distributed between* 1, 2 *and* 3. *We compute the distribution of $\tilde{D}$ using Eq.* (1)*:*

$$\mathbb{P}(\tilde{D}\!=\!i) = \begin{cases} 1/6 & \text{if } i=0, \\ 1/3 & \text{if } i=1, \\ 1/2 & \text{if } i=2. \end{cases}$$

Applying this demand distribution on Eq. (6)*, we get*

$$\pi_2(\tau_2) = \frac{1+2(1-\tau_2)+2(1-\tau_2)^2}{1+2(1-\tau_2)+3(1-\tau_2)^2}, \tag{10}$$

and

$$\pi_1(\tau_1) = \frac{2+3(1-\tau_1)}{2+6(1-\tau_1)}. \tag{11}$$

We will later return to this example. Next, we show that $\pi_n(\tau_n)$ increases with both τ_n and n, that is, the probability of service of an n-threshold customer increases as her time-threshold gets closer to the start of service. Moreover, for any time $\tau \in [0,1)$, the probability of service of an n-threshold customer with time-threshold τ is strictly larger than the probability of service of an $(n-1)$-threshold customer with the same time-threshold. We use these two properties to prove the uniqueness and existence of the equilibrium.

Lemma 2. *For any $n \in \{1,2,...,N\}$, the following holds:*

(i) $\pi_n(\tau_n)$ is continuous and strictly increasing in the range $[0,1)$.

(ii) $\pi_n(\tau) > \pi_{n-1}(\tau)$, for any $\tau \in [0,1)$.

PROOF. Set $m = N - n$ and, for $j \in [m, b]$, define

$$\alpha_j \triangleq \mathbb{P}(\tilde{D}{=}j)\binom{j}{m}, \tag{12}$$

and

$$\beta_j \triangleq \begin{cases} 1 & \text{if } j < N, \\ \frac{n}{j+1-m} & \text{if } j \geq N. \end{cases} \tag{13}$$

We rewrite Eq. (6)

$$\pi_n(\tau_n) = \frac{\sum\limits_{j=m}^{b} \alpha_j \beta_j (1-\tau_n)^j}{\sum\limits_{i=m}^{b} \alpha_i (1-\tau_n)^i}. \tag{14}$$

For proving (i), we compute the derivative of $\pi_n(\tau_n)$ with respect to τ_n and show that it is positive for any $\tau_n \in [0, 1)$. We have

$$\begin{aligned} \frac{d\pi_n}{d\tau_n} &= \frac{-\sum\limits_{j=m}^{b} \alpha_j \beta_j j (1-\tau_n)^{j-1} \sum\limits_{i=m}^{b} \alpha_i (1-\tau_n)^i + \sum\limits_{j=m}^{b} \alpha_j \beta_j (1-\tau_n)^j \sum\limits_{i=m}^{b} \alpha_i i (1-\tau_n)^{i-1}}{\left(\sum\limits_{i=m}^{b} \alpha_i (1-\tau_n)^i\right)^2} \\ &= \frac{\sum\limits_{j=m}^{b} \sum\limits_{i=m}^{b} \alpha_j \alpha_i \beta_j (1-\tau_n)^{i+j-1}(i-j)}{\left(\sum\limits_{i=m}^{b} \alpha_i (1-\tau_n)^i\right)^2}. \end{aligned} \tag{15}$$

We need to show that the numerator of Eq. (15) is positive. Let consider any element of the numerator $\{i = k, j = l\}$ and it conjugate element $\{i = l, j = k\}$. By summing these two elements we get

$$\alpha_l \alpha_k (1-\tau_n)^{l+k-1}(k-l)(\beta_l - \beta_k). \tag{16}$$

Since $\beta_i = \beta_{i-1}$ for all $i < N$ and $\beta_i < \beta_{i-1}$ for all $i \geq N$, we deduce that $\beta_l - \beta_k = 0$ if both k and l are smaller than N. Otherwise, $\beta_l - \beta_k$ has the same sign as $k - l$. Thus, for any value of k and l, Eq. (16) is either zero or positive. We conclude that the sum of the numerator of Eq. (15) is positive, and hence the derivative of $\pi_n(\tau_n)$ is positive.

Next, we show that $\pi_n(\tau_n)$ is a continuous function in $[0, 1)$. In Eq. (9), both the numerator and denominator are probabilities. Hence, although they consist of an infinite number of terms, they have a finite limit for any value of $\tau_n \in [0, 1)$. Since they are both polynomial expression of τ_n with a finite limit, they are continuous (see Cauchy's uniform convergence criterion [31, p.246]). The denominator is equal to zero only at $\tau_n = 1$. Thus, we conclude that $\pi_n(\tau_n)$ is continuous in $[0, 1)$.

For proving (ii), we show that

$$\frac{\sum\limits_{i=m}^{b} \alpha_i \beta_i (1-\tau)^i}{\sum\limits_{i=m}^{b} \alpha_i (1-\tau)^i} - \frac{\sum\limits_{j=m+1}^{b} \alpha_j \beta_j (1-\tau)^j}{\sum\limits_{j=m+1}^{b} \alpha_j (1-\tau)^j} > 0. \tag{17}$$

This is equivalent to showing that

$$
\begin{aligned}
&\sum_{i=m}^{b}\alpha_i\beta_i(1-\tau)^i\sum_{j=m+1}^{b}\alpha_j(1-\tau)^j-\sum_{i=m}^{b}\alpha_i(1-\tau)^i\sum_{j=m+1}^{b}\alpha_j\beta_j(1-\tau)^j\\
&=\sum_{i=m}^{b}\alpha_i(1-\tau)^i\left(\beta_i\sum_{j=m+1}^{b}\alpha_j(1-\tau)^j-\sum_{j=m+1}^{b}\alpha_j\beta_j(1-\tau)^j\right)\\
&=\sum_{i=m}^{b}\alpha_i(1-\tau)^i\left(\sum_{j=m+1}^{b}\alpha_j(1-\tau)^j\left(\beta_i-\beta_j\right)\right)\\
&=\sum_{i=m}^{b}\alpha_i(1-\tau)^i\left(\sum_{j=m}^{b}\alpha_j(1-\tau)^j\left(\beta_i-\beta_j\right)\right)+\sum_{i=m}^{b}\alpha_i\alpha_m(1-\tau)^{i+m}\left(\beta_m-\beta_i\right)>0. \qquad (18)
\end{aligned}
$$

In Eq. (18), the first term is 0, while the second term is positive (note that, by Eq. (13), $\beta_m = 1$).

We define two types of threshold equilibria:

Definition 2. *In a **none-make-AR** equilibrium, none of the customers, regardless of their arrival times, choose AR.*

Definition 3. *In a **some-make-AR** equilibrium, all customers follow a threshold strategy with a set of thresholds* $0 < \tau_N^e < \tau_{N-1}^e < ... < \tau_1^e < 1$.

We define the critical cost

$$\underline{C} \triangleq 1 - \pi_N(0). \qquad (19)$$

We are now ready to state the main result of this section:

Theorem 2. *Given any number of servers N and any demand distribution D:*

- *If $C < \underline{C}$, there exists one pure equilibrium and its type is some-make-AR.*
- *If $C \geq \underline{C}$, there exists one pure equilibrium and its type is none-make-AR.*

Proof. See Appendix.

Thus, this theorem states the existence and uniqueness of a critical reservation fee $\underline{C}$. If $C > \underline{C}$, then no customer makes a reservation because the fee is too high. If $C < \underline{C}$, then customers may make reservations depending on their arrival times and the number of available servers that they observe.

Example 1b. *Continuing the previous example, we substitute Eq.* (10) *into Eq.* (19) *and get* $\underline{C} = 1 - 5/6$. *Thus, any C larger than $1/6$ leads to a none-make-AR equilibrium. Next, assume $C = 0.1$. By solving the equalities $\pi_2(\tau_2) = 1-0.1$ and $\pi_1(\tau_1) = 1-0.1$, using Eq.* (10) *and Eq.* (11)*, we obtain that the time-thresholds at equilibrium are $\tau_1^e = 0.917$ and $\tau_2^e = 0.453$. That is, if the first customer making an inquiry arrives before $t = 0.453$, then she makes a reservation. If the demand is larger than 1 and the second customer making an inquiry arrives before 0.917, then the second server will also be reserved.*

3.4. Poisson Distributed Demand

In the previous section, we studied a game with demand that follows a general distribution. In this section, we apply the result on a system with Poisson distributed demand with mean λ. In Poisson games (i.e., games in which the number of players is Poisson distributed) the distribution of the number of players and the distribution of the number of players as seen by a tagged player, excluding herself, are identical [21] (using Eq. (1), one can indeed see that this property holds true for D and $\tilde{D}$). Furthermore, if players are randomly ascribed to different types with fixed probabilities, the number of players of each type is independent of the number of players of other types and is also Poisson distributed. Hence, from the perspective of the n-th threshold customer, the number of customers that arrive after her is a Poisson random variable with parameter $\lambda(1-\tau_n)$. If she chooses AR', then all those customers also choose AR'. Using this property, we can express the probability of service in a simpler way than in Eq. (6). Specifically,

$$\pi_n(\tau_n) = \sum_{i=0}^{N-1} \mathbb{P}(\tilde{D}_{AR'} = i) + \sum_{i=N}^{\infty} \mathbb{P}(\tilde{D}_{AR'} = i) \frac{n}{i+1}. \tag{20}$$

The first term is the probability that the number of customers choosing AR' is smaller than N. In this case, service is guaranteed to all customers. The second term is the probability to get service when the demand exceeds N. In this case, the n unreserved servers will be arbitrarily allocated to customers that chose AR'. We substitute

$$\mathbb{P}(\tilde{D}_{AR'} = i) = e^{-\lambda(1-\tau_n)} \frac{(\lambda(1-\tau_n))^i}{i!} \tag{21}$$

into Eq. (20) and obtain a closed form expression for $\pi_n(\tau_n)$, namely

$$\begin{aligned}
\pi_n(\tau_n) &= e^{-\lambda(1-\tau_n)} \sum_{i=0}^{n-1} \frac{(\lambda(1-\tau_n))^i}{i!} + e^{-\lambda(1-\tau_n)} \sum_{i=n}^{\infty} \frac{(\lambda(1-\tau_n))^i}{i!} \frac{n}{i+1} \\
&= e^{-\lambda(1-\tau_n)} \sum_{i=0}^{n-1} \frac{(\lambda(1-\tau_n))^i}{i!} + \frac{n}{\lambda(1-\tau_n)} e^{-\lambda(1-\tau_n)} \sum_{i=n+1}^{\infty} \frac{(\lambda(1-\tau_n))^i}{i!} \\
&= e^{-\lambda(1-\tau_n)} \sum_{i=0}^{n-1} \frac{(\lambda(1-\tau_n))^i}{i!} + \frac{n}{\lambda(1-\tau_n)} \left(1 - e^{-\lambda(1-\tau_n)} \sum_{i=0}^{n} \frac{(\lambda(1-\tau_n))^i}{i!}\right).
\end{aligned} \tag{22}$$

Using the upper incomplete Gamma function

$$\Gamma[s,x] \triangleq (s-1)! e^{-x} \sum_{k=0}^{s-1} \frac{x^k}{k!}, \tag{23}$$

we obtain

$$\pi_n(\tau_n) = \frac{n}{\lambda(1-\tau_n)} + \frac{\Gamma[n, \lambda(1-\tau_n)]}{(n-1)!} - \frac{\Gamma[1+n, \lambda(1-\tau_n)]}{\lambda(1-\tau_n)(n-1)!}. \tag{24}$$

Example 2. *Consider a game with $N=6$ servers, average demand $\lambda=6$ and reservation cost $C=0.15$. To find the equilibrium strategy, we first check which type of equilibrium prevails. Since $1-\pi_6(0)=0.16>0.15$, we deduce that the game has a some-make-AR equilibrium (see Theorem 2). We then solve the following set of equations:*

$$1 - 0.15 = \frac{n}{6\tau_n} + \frac{\Gamma[n, 6(1-\tau_n)]}{(n-1)!} - \frac{\Gamma[1+n, 6(1-\tau_n)]}{6(1-\tau_n)(n-1)!}, \quad \forall n \in \{1,2,3,4,5,6\}. \tag{25}$$

The solution is the set of time-thresholds $\boldsymbol{\tau^e} = \{0.944, 0.787, 0.605, 0.416, 0.223, 0.027\}$. That is, at equilibrium, a customer that observes an empty system (i.e., $n=N$) will choose AR only if she arrives before $t=0.027$, while a customer that observes one available server will choose AR if she arrives before $t=0.944$. Figure 2 shows the intersection of $1-C$ with the functions $\pi_n(\tau_n)$ for $n=1,2,...6$.

In this example, the probability that all customers arrive after τ_6^e, and hence all customers choose AR' is

$$\sum_{i=0}^{\infty} \mathbb{P}(D=i)(1-\tau_6^e)^i = e^{-6} \sum_{i=0}^{\infty} \frac{6^i}{i!}(0.973)^i = 0.850. \tag{26}$$

If the provider does not disclose information about the number of available servers, as suggested in [28], then under the same parameters there is a unique equilibrium and the probability that all customers choose AR' is smaller than 0.01.

The example illustrates the drawback of the *full-information* policy: even if a *some-make-AR* equilibrium exists, there is a high chance that no customer will make a reservation. Specifically, if the first customer arrives after τ_N^e, then neither this customer nor all following customers will make a reservation.

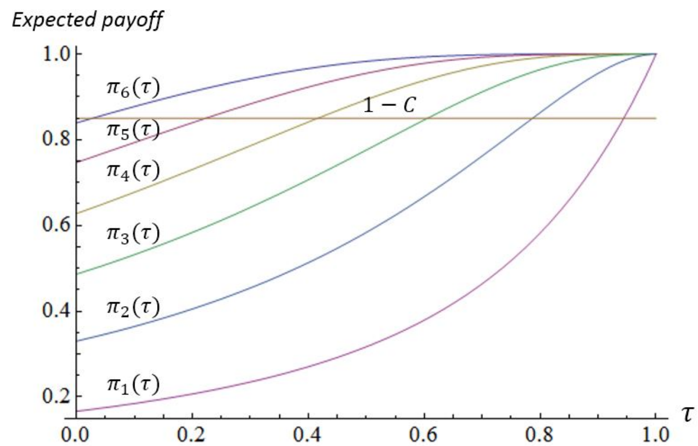


Figure 2: Illustration of Example 2. For each number of available servers $i \in 1, 2, \ldots, 6$, the probability of service π_i as a function of the time-threshold τ (which represents the expected payoff of a threshold customer that is not making AR) intersects once with the line $1 - C$ (which represents the payoff of any customer making AR), thus showing the uniqueness of the equilibrium. As the number of available servers i decreases, the time threshold at which customers stop making AR increases (i.e., customers are more likely to make AR when observing fewer available servers).

4. Partial-information Model

We next consider a *partial-information* policy. Under this policy, customers are notified about the number of available servers only if this number is below or equal to some threshold denoted M, where $M \in \{1, 2, ..., N-1\}$. Under this policy, an unnotified customer can deduce that the number of available servers is larger than M. In this section, we analyze the equilibrium structure of the *partial-information* policy, assuming that customers know that this policy is used. Note that if $M = N - 1$, then an unnotified customer knows that all servers are available, and therefore a *partial-information* policy with $M = N - 1$ is equivalent to the *full-information* policy.

A result similar to Theorem 1 also holds for the *partial-information* game. Thus, this game has the same two types of equilibria (i.e., *none-make-AR* and *some-make-AR*), as the *full-information* game. In this game, all unnotified customers use the same time-threshold, denoted by τ_u, which replaces the time-thresholds $\{\tau_{M+1}, ..., \tau_N\}$. Thus, the set of time-thresholds now contains $M + 1$ time-thresholds. Eq. (6), which describes the probability of service of the threshold customers in the *full-information* model, is still valid for the notified customers. In order to find the time-threshold followed by unnotified customers, we consider an *unnotified threshold customer*. As earlier, this represents a virtual unnotified customer that arrives exactly at the time-threshold followed by unnotified customers.

We denote the probability of service of the unnotified threshold customer by $\pi_u(\tau_u)$. We express it, using the law of total probability, by conditioning on the number of available servers $i \in \{M+1, M+2, \ldots, N\}$, given that at least $N - M$ servers are available. As we showed in the previous section, the probability of service of the i-th threshold customer is independent of all other time-thresholds. Thus, we can construct $\pi_u(\tau_u)$ using $\pi_i(\cdot)$ as defined in Eq. (6), namely

$$\pi_u(\tau_u) = \sum_{i=M+1}^{N} \pi_i(\tau_u)\mathbb{P}(\tilde{D}_{AR} = N - i | \tilde{D}_{AR} < N - M). \tag{27}$$

The distribution of $\tilde{D}_{AR}$ can be found using the law of total probability conditioned on $\tilde{D}$. Given $\tilde{D}$, $\tilde{D}_{AR}$

is a binomial random variable with success probability τ_u. Thus,

$$\begin{aligned}\mathbb{P}(\tilde{D}_{AR}=i) &= \sum_{j=i}^{\infty}\mathbb{P}(\tilde{D}=j)\mathbb{P}(\tilde{D}_{AR}=i|\tilde{D}=j)\\ &= \sum_{j=i}^{\infty}\mathbb{P}(\tilde{D}=j)\binom{j}{i}\tau_u^i(1-\tau_u)^{j-i}.\end{aligned} \tag{28}$$

Next, we show that, unlike the probability of service of an notified threshold customer, $\pi_u(\tau_u)$ is not necessarily an increasing function of the time-threshold τ_u. In this case, the equality $\pi_u(\tau_u) = 1 - C$ may hold for more than one value of τ_u and the equilibrium will not be unique.

We prove our claim by the means of an example. Assume that the demand is Poisson distributed with parameter λ. As in the previous section, we use the properties of Poisson games to simplify the expression of $\pi_u(\tau_u)$. If all unnotified customers follow a threshold-strategy with time-threshold τ_u, then the number of customers choosing AR (i.e., $\tilde{D}_{AR}$) and the number of customers choosing AR' (i.e., $\tilde{D}_{AR'}$) are independent Poisson random variables with parameters $\lambda\tau$ and $\lambda(1-\tau_u)$, respectively. The probability of service of the unnotified threshold customer is

$$\pi_u(\tau_u) = \sum_{i=0}^{N-M-1}\mathbb{P}(\tilde{D}_{AR}=i|\tilde{D}_{AR}<N-M)\left(\sum_{j=0}^{N-i-1}\mathbb{P}(\tilde{D}_{AR'}=j) + \sum_{j=N-i}^{\infty}\mathbb{P}(\tilde{D}_{AR'}=j)\frac{N-i}{j+1}\right), \tag{29}$$

which is the sum of the probabilities that i customers choose AR, each multiplied by the probability of service given i. Note that the second term in Eq. (29) is derived in a similar way to Eq. (20).

Expressions for the inner probabilities in Eq. (29) are

$$\mathbb{P}(\tilde{D}_{AR}=i|\tilde{D}_{AR}<N-M) = \frac{\frac{(\lambda\tau_u)^i}{i!}}{\sum\limits_{k=0}^{N-M-1}\frac{(\lambda\tau_u)^k}{k!}}, \tag{30}$$

and

$$\mathbb{P}(\tilde{D}_{AR'}=j) = \frac{e^{-\lambda(1-\tau_u)}(\lambda(1-\tau_u))^j}{j!}. \tag{31}$$

The following example shows that it is possible for a game with Poisson distributed demand to have several equilibria.

Example 3. *We consider a system with the following parameters* $\lambda = 10$, $N = 10$, $C = 0.15$ *and* $M = 3$. *From Eq. (29) - Eq. (31), we get* $\pi_u(0.429) = \pi_u(0.172) = 1 - 0.15$. *Using Eq. (24), we find the n-th threshold for* $n = 1, 2, 3$ *and conclude that both* $\{\tau_1=0.966, \tau_2=0.872, \tau_3=0.763, \tau_u=0.172\}$ *and* $\{\tau_1=0.966, \tau_2=0.872, \tau_3=0.763, \tau_u=0.429\}$ *represent equilibria strategies.*

The following theorem summarizes the equilibria structure of the *partial-information* game (to simplify the presentation, the boundary cases $C = \underline{C}$ and $C = \overline{C}$ are ignored):

Theorem 3. *In the partial-information game, there exist quantities* $\underline{C}$ *and* $\overline{C} \geq \underline{C}$, *such that*

- *If* $0 < C < \underline{C}$, *there is at least one some-make-AR equilibrium.*
- *If* $\underline{C} < C < \overline{C}$, *there is a none-make-AR equilibrium and at least two some-make-AR equilibria.*
- *If* $C > \overline{C}$, *none-make-AR is the unique equilibrium.*

PROOF. From Eq. (27) and Eq. (28) we deduce that $\pi_u(0) = \pi_N(0)$. That is, if all customers choose AR' then the probability of service in both games is identical. Thus, when the reservation cost is larger than $\underline{C}$ (which is defined in Eq. (19)), the *partial-information* game has a *none-make-AR* equilibrium, similar to the *full-information* game.

Procedure 1 Simulation($N, \boldsymbol{\tau^e}, P_D$)

$j \leftarrow 1$ {index}
$D_{AR} \leftarrow 0$ {number of reservations}
$D \leftarrow$ random variable from P_D {the demand}
$\mathbf{t} \in \mathbb{R}^D \leftarrow$ vector with random values from $U[0,1]$ {arrival times}
Sort $\mathbf{t}$ in ascending order {sorting from the first customer that makes an inquiry to the last}
while $j \leq min\{D, N\}$ **do** {iterate over all customers or until all servers are reserved}
  if $t_j < \tau^e_{N-j+1}$ **then** {if the arrival time is smaller than the appropriate time-threshold}
    $\tilde{D}_{AR} \leftarrow \tilde{D}_{AR} + 1$ {choosing AR}
    $j \leftarrow j + 1$ {increase the index by 1}
  else
    break {if a customer is better off choosing AR', then all customers that arrive after her are also better off choosing AR'}
  end if
end while
return D_{AR}

In order for the *none-make-AR* equilibrium to be unique, we must have

$$\pi_u(\tau_u) > 1 - C, \quad \forall \tau_u \in [0, 1]. \tag{32}$$

We define

$$\overline{C} = 1 - \min_{0 \leq \tau_u \leq 1} \pi_u(\tau_u), \tag{33}$$

and conclude that any cost above $\overline{C}$ leads to a unique *none-make-AR* equilibrium.

Conversely, for any cost $C < \overline{C}$, there is at least one value of τ_u such that

$$1 - C = \pi_u(\tau_u). \tag{34}$$

Hence, at least one *some-make-AR* equilibrium exists. Note that $\pi_u(1) = 1$. Thus, for any reservation cost $C > 0$, an equilibrium where all customers make AR does not exist .

The number of *some-make-AR* equilibria is determined by the number of values of τ_u for which Eq. (34) holds. If $\underline{C}$ is strictly smaller than $\overline{C}$, then the following holds:

$$\pi_u(1) > \pi_u(0) > min(\pi_u(\tau_u)). \tag{35}$$

Hence $\pi_u(\tau_u)$ is not a monotonic function and any $C \in (\underline{C}, \overline{C})$ yields at least two *some-make-AR* equilibria.

5. Comparison of Information Sharing Policies

In this section, we assume that the provider is interested to persuade as many customers as possible to make reservations. We resort to simulations to find out which policy maximizes the average number of reservations. In addition to the *full-information* and *partial-information* policies, we consider also the *no-information* policy, introduced in [28]. Procedure 1 details the simulation steps at each iteration.

5.1. Simulation Results

5.1.1. Geometric Demand

Consider a movie-theater with 100 seats that charges \$10 per ticket and offers advance reservations through its website for an additional one dollar. Let assume that the customers payoff upon getting a ticket equals the ticket cost. Hence, we can normalize the reservation fee to $C = 0.1$. Let also assume that the demand follows a geometric distribution with mean $\lambda = 100$.

We evaluate seven policies: *full-information*, *partial-information* with $M = \{10, 20, 30, 40, 50\}$ and *no-information*. We run each policy for $10,000$ times and compute the average number of reservations for each policy. The results indicate that the average number of reservations decreases as more information is shared. Over 50% of seats are reserved when no information is not shared, but only 25% of seats are reserved when full information is shared. The simulation results are summarized in Table 1.

Policy	Average number of reservations
Full-information	25.3
Partial-information with $M = 50$	38.2
Partial-information with $M = 40$	41.6
Partial-information with $M = 30$	45.8
Partial-information with $M = 20$	49.5
Partial-information with $M = 10$	54
No-information	57.7

Table 1: Simulation results for geometric demand.

5.1.2. Poisson Demand

To further validate the simulation results, we next consider Poisson distributed demand. We first examine an overloaded system where the mean demand is $\lambda = 15$, while the number of servers is $N = 10$. Using Eq. (24), we get that $\underline{C} = 1 - \pi_N(0) = 0.342$. We evaluate the performance of the different policies for seven different reservation costs between 0 and $\underline{C}$.

For all costs in this range, all policies have a unique *some-make-AR* equilibrium. Each point is computed by averaging out results over $10,000$ iterations. As shown in Figure 3(a), for each AR cost, the policy that maximizes the number of reservations is the *no-information* policy. The *full-information* policy performs the worst, and the performance of the *partial-information* policy performs better as M (the threshold at which information starts to be disclosed) decreases.

The simulation results also indicate that the gap between the outcomes of the different policies increases as the AR cost increases. When the cost is low, the motivation to make reservations is high and almost all servers are reserved, regardless of the policy. As the cost increases, the different information disclosure policies have larger impact on customers' decisions. For example, when the cost is $C = 0.34$, the average number of reservation is 20 times higher with the no-information policy that with the the *full-information* policy.

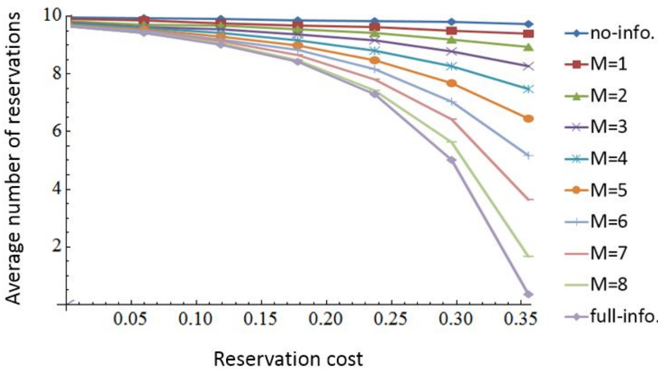
Next, we consider an underloaded system with mean demand $\lambda = 8$. This time, we use six different prices between 0 and $\underline{C} = 0.053$. Figure 3(b) shows similar results as Figure 3(a).

Finally, we set the mean demand to $\lambda = 10$ and explore the case of multiple equilibria (i.e., AR costs larger than $\underline{C}$). In this case, the *full-information* policy yields zero reservations (see Theorem 2). We run simulation to compare between the performance of the *no-information* and the *partial-information* policies. We choose different AR costs in the range $[0.215, 0.228]$. In order to compare performance when multiple equilibria exist, we always choose the one leading to the largest expected number of reservations. As shown in Fig. 3(c), the results follow the same pattern as in the previous simulations, namely as less information is disclosed, the number of reservations increases.

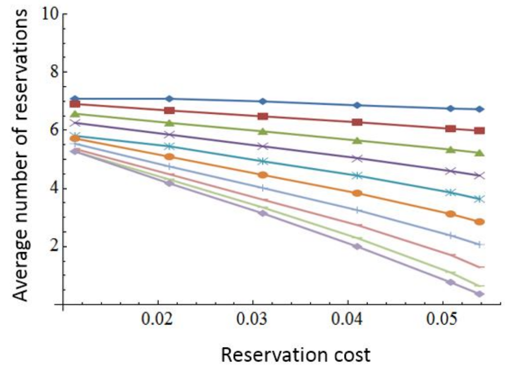
5.2. Numerical Example

To better understand the gap between the outcomes of the *full-information* and *no-information* policies, we return to Example 1.

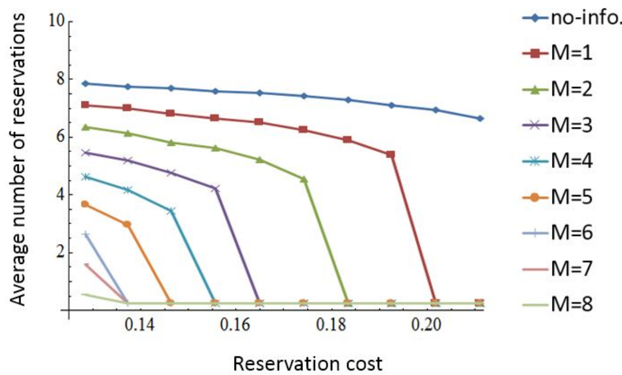
Example 1c. *In the no-information model of [28], customers are only informed whether servers are available or not. The equilibrium structure of the game resembles the equilibrium structure of the partial-information game, but since all customers have the same information, they all use the same time-threshold.*



(a) Average demand $\lambda = 15$.



(b) Average demand $\lambda = 8$.



(c) Average demand of $\lambda = 10$.

Figure 3: A comparison between the number of customers making AR in a system with $N = 10$ servers and Poisson demand. As less information is disclosed, more customers make AR on average.

For the distribution of demand $\tilde{D}$ given in Example 1, the probability of service of the threshold customer in the no-information model is

$$\pi_{no-info}(\tau) = \frac{3 + 3(1-\tau) - (1-\tau)^2}{3 + 6(1-\tau) - 3(1-\tau)^2}. \tag{36}$$

In Figure 4, we plot $\pi_2(\tau)$, $\pi_1(\tau)$ and $\pi_{no-info}(\tau)$. We can see in the graph that for any AR cost in $(0, 0.167)$, at equilibrium, τ^e (which is the fraction of customers making AR in a game with no information) is much closer to τ_2^e than to τ_1^e. That means that when disclosing information, we dramatically decrease the probability that the first customer making an inquiry will choose AR, while we only slightly increase the probability that the second customer will choose AR (assuming the first one chose AR). Thus, we can expect that the average number of reservations will be higher when no information is disclosed.

Based on the simulation results and this example, we make the following conjecture:

Conjecture 1. *In AR games, hiding information about the number of available servers maximizes the average number of reservations.*

6. Conclusions

In this paper, we study the impact of information disclosure on the strategic behavior of customers in a loss system that support advance reservations. Our results apply to general forms of demand and customer arrival distributions. We first derive the equilibrium structure under a *full-information* policy, where the precise number of available servers is disclosed to the customers. We prove the uniqueness of a pure Nash

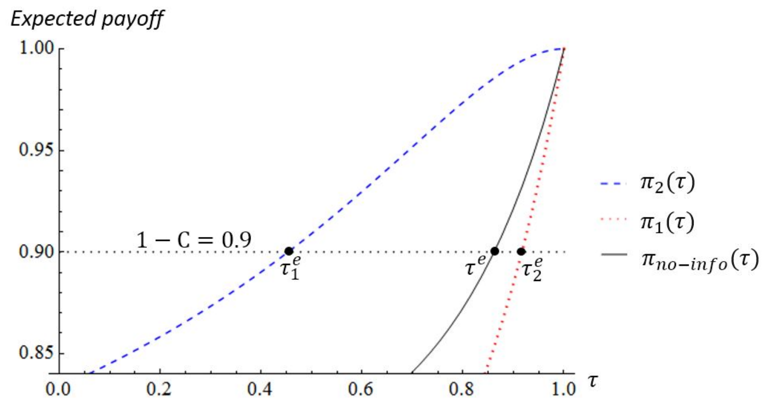


Figure 4: The probability of service of the threshold customers as a function of the time-thresholds. The fact that τ^e is much closer to τ_2^e than to τ_1^e explains why the number of reservations is larger with the *no-information* policy than with the *full-information* policy.

equilibrium in that case. We determine a critical value $\underline{C}$, such that if the AR cost satisfies $C < \underline{C}$, then there exists a unique *some-make-AR* equilibrium, and if $C > \underline{C}$, then there exists a unique *none-make-AR* equilibrium. The *some-make-AR* equilibrium implies that the decision of customers that observe $1 \leq n \leq N$ available servers is determined by the n-th time-threshold (i.e., a customer makes a reservation only if she arrives before the the n-th time-threshold).

Next, we consider a *partial-information* policy and find its equilibrium structure. Under this policy, customers are notified about the number of available servers only if this number is below or equal to some threshold $M < N$. We show the existence of two critical values $\underline{C}$ and $\overline{C}$. If $0 < C < \underline{C}$, then there exists at least one *some-make-AR* equilibrium. If $\underline{C} < C < \overline{C}$, then there is a *none-make-AR* equilibrium and at least two *some-make-AR* equilibria. If $C > \overline{C}$, *none-make-AR* is the unique equilibrium. Here, each *some-make-AR* equilibrium consists of $M + 1$ time-thresholds.

Using simulation, we show that the policy that maximizes the fraction of customers making AR is a *no-information* policy [28], where customers are not notified at all about the number of available servers. From the simulation results, we also observe that customers become more sensitive to the information disclosure policy as the reservation cost increases. Thus, providing information should be done only when the reservation cost is low. This result is in line with common practice. Namely, airlines tend to have high reservation costs (cancellations are heavily fined) and they tend to hide availability information. In contrast, reservation costs of movie theaters are relatively low and information is typically shared. Furthermore, the results indicate that in order for the *full-information* policy to be beneficial, a provider should encourage customers to make reservations when all or almost all servers are available. For example, in entertainment events, customers have such an incentive since they can choose better seats as they buy their tickets earlier.

Many interesting questions remain about the impact of information disclosure on strategic customer behavior in systems supporting advance reservations. For instance, evaluating the impact of policies disclosing ambiguous or imprecise information such as “the system is almost full”. The analysis of more complex models that incorporate time-varying AR costs, heterogeneous customers with different valuations of service, and cancellations should be of interest as well.

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Appendix

In this Appendix, we prove Theorem 1 and Theorem 2. First, we provide technical lemmas that we will use to prove Theorem 1.

Lemma 3. *Let X_t be a non-negative random variable with parameter t. Let $g(x)$ be a non-negative function of x whose derivative is positive with respect to x for $x \in [k, l]$ and non-negative elsewhere. If the complementary cumulative distribution function (CCDF) $\bar{F}_{X_t}(x)$ strictly increases with t for any $x \in [k, l]$ and increases with t elsewhere, then $\mathbb{E}[g(X_t)]$ strictly increases with t.*

Proof. Let $x \in [k, l]$, $\gamma = g(x)$ and $t_2 > t_1$. Then,

$$\mathbb{P}(g(X_{t_2}) > \gamma) = \mathbb{P}(X_{t_2} > x) > \mathbb{P}(X_{t_1} > x) = \mathbb{P}(g(X_{t_1}) > \gamma), \quad \forall x \in [k, l]. \tag{37}$$

Using the well-known formula for the expectation of a non-negative random variable [25, Chapter 9]

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X > \gamma) d\gamma, \tag{38}$$

we obtain

$$\mathbb{E}[g(X_{t_1})] = \int_0^{g(k)} \mathbb{P}(g(X_{t_1})>\gamma)d\gamma + \int_{g(k)}^{g(l)} \mathbb{P}(g(X_{t_1})>\gamma)d\gamma + \int_{g(l)}^{\infty} \mathbb{P}(g(X_{t_1})>\gamma)d\gamma \tag{39}$$

and

$$\mathbb{E}[g(X_{t_2})] = \int_0^{g(k)} \mathbb{P}(g(X_{t_2})>\gamma)d\gamma + \int_{g(k)}^{g(l)} \mathbb{P}(g(X_{t_2})>\gamma)d\gamma + \int_{g(l)}^{\infty} \mathbb{P}(g(X_{t_2})>\gamma)d\gamma. \tag{40}$$

From Eq. (37), we know that the middle term in the RHS of Eq. (39) is strictly smaller than that in Eq. (40), while the two other terms in Eq. (39) are no larger than the corresponding terms in Eq. (40). Thus, we deduce that $\mathbb{E}[g(X_{t_2})] > \mathbb{E}[g(X_{t_1})]$.

Definition 4. *Let $\{a_k, .., a_l\}$ be a set of positive real numbers, t be a real number and $h(t)$ be a general positive function of t whose derivative is strictly negative with respect to t. A discrete non-negative random variable X_t supported in $[k, l]$ is said to belong to the distributions family $\mathscr{F}$ if it has the following CDF:*

$$F_{X_t}(x) \triangleq \mathbb{P}(X_t \leq x) = \frac{\sum_{j=k}^{x} a_j \left(h(t)\right)^j}{\sum_{j=k}^{l} a_j \left(h(t)\right)^j}, \quad \forall x \in \{k, ..., l\}, \tag{41}$$

where x is an integer.

Lemma 4. *Suppose X_t is a discrete random variable, supported in $[k, l]$ with a CDF F_{X_t}. If $F_{X_t} \in \mathscr{F}$, then for any $x \in [k, l)$, $F_{X_t}(x)$ strictly increases with t.*

Proof. We compute the derivative of F_{X_t} (as defined in Eq. (41)) with respect to t and show that it is positive:

$$\begin{aligned}
\frac{\partial F_{X_t}}{\partial t} &= \frac{-\sum_{i=k}^{x} a_i i |h'(t)| \left(h(t)\right)^{i-1} \sum_{j=k}^{l} a_j \left(h(t)\right)^j + \sum_{i=k}^{x} a_i \left(h(t)\right)^i \sum_{j=k}^{l} a_j j |h'(t)| \left(h(t)\right)^{j-1}}{\left(\sum_{j=k}^{l} a_j \left(h(t)\right)^j\right)^2} \\
&= \frac{\sum_{i=k}^{x} \left(a_i \left(h(t)\right)^{i-1} \left(-i|h'(t)| \sum_{j=k}^{l} a_j \left(h(t)\right)^j + \sum_{j=k}^{l} a_j j |h'(t)| \left(h(t)\right)^j\right)\right)}{\left(\sum_{j=k}^{l} a_j \left(h(t)\right)^j\right)^2} \\
&= \frac{\sum_{i=k}^{x} a_i \left(h(t)\right)^{i-1} \sum_{j=k}^{l} a_j \left(h(t)\right)^j |h'(t)| (-i+j)}{\left(\sum_{j=k}^{l} a_j \left(h(t)\right)^j\right)^2} \\
&= \frac{\sum_{i=k}^{x} a_i \left(h(t)\right)^{i-1} \sum_{j=k}^{x} a_j \left(h(t)\right)^j |h'(t)| (-i+j)}{\left(\sum_{j=k}^{l} a_j \left(h(t)\right)^j\right)^2} + \frac{\sum_{i=k}^{x} a_i \left(h(t)\right)^{i-1} \sum_{j=x+1}^{l} a_j \left(h(t)\right)^j |h'(t)| (-i+j)}{\left(\sum_{j=k}^{l} a_j \left(h(t)\right)^j\right)^2}.
\end{aligned} \tag{42}$$

In the numerator of Eq. (42), the first term is canceled out due to symmetry and the second term is positive for any $x \in [k, l)$.

Next, we show that for any strategy σ, the number of reservations (stochastically) increases with the demand.

Lemma 5. *If all customers follow an arbitrary strategy σ, then, $\mathbb{P}(\tilde{D}_{AR}>\tilde{d}_{AR}|\sigma,\tilde{D}=\tilde{d})$ increases with $\tilde{d}$.*

Proof. The proof is based on a coupling argument [25, Chapter 9]. Consider a realization R_1 with demand $\tilde{d}$ and set of arrival times $T=\{t_1,t_2,...,t_d\}$. Consider a second realization R_2 which is identical to R_1 but with an additional customer with arrival time t' that observes n' available servers. We respectively denote $\tilde{D}^1_{AR}(t)$ and $\tilde{D}^2_{AR}(t)$ as the number of reservations made by time t in realizations R_1 and R_2.

If $\sigma(t',n')=0$ (i.e., the additional customer does not make AR), then R_1 and R_2 are identical in terms of the number of reservations. Otherwise, $\tilde{D}^2_{AR}(t')>\tilde{D}^1_{AR}(t')$. Let $T'=\{t \mid t\in T, t>t'\}$. If there is an arrival point $t''\in T'$ such that $\tilde{D}^2_{AR}(t'')=\tilde{D}^1_{AR}(t'')$, then R_1 and R_2 merge (i.e., the decisions of all customers that arrive after t'' are identical in both realizations). In this case, the number of reservations in both realizations are equal. If there is no such t'', then the total number of reservations in R_2 is larger than in R_1. We conclude that, in any case, the number of reservations $\tilde{D}_{AR}$ cannot decrease with the demand.

Proof of Theorem 1

We prove the theorem by induction. First, we show that, at equilibrium, a threshold strategy is followed by customers that observe N available servers. Then, we show that if all customers that observe $n\in[k,k+1,\dots,N]$ available servers follow a threshold strategy, then a threshold strategy is also followed by customers that observe $k-1$ available servers.

Base case. Since we only consider pure strategies, we know that, at equilibrium, the strategy followed by all customers that observe a certain number of available servers is a set of intervals where in each interval either all customers choose AR or all choose AR'. Consider the intervals $I_1=(t_1,t_2)$ and $I_2=(t_2,t_3)$, where $t_3>t_2>t_1$. Let assume by contradiction that there is an equilibrium strategy σ such that $\sigma(t,N)=0$ for all $t\in I_1$ and $\sigma(t,N)=1$ for all $t\in I_2$. In this case, the following must hold:

$$\mathbb{P}(S|\sigma,\tilde{D}_{AR}(t)=0)\geq 1-C,\quad \forall t\in I_1 \tag{43}$$

and

$$\mathbb{P}(S|\sigma,\tilde{D}_{AR}(t)=0)\leq 1-C,\quad \forall t\in I_2. \tag{44}$$

Next, we show that $\mathbb{P}(S|\sigma,\tilde{D}_{AR}(t)=0)$ strictly increases with t within I_2, and hence the payoff when choosing AR along the interval I_2 must be strictly larger than $1-C$, which leads to a contradiction.

Using the law of total probability, conditioned on the demand, the probability of service can be written

$$\mathbb{P}(S|\sigma,\tilde{D}_{AR}(t)=0)=\sum_{i=a}^{b}\mathbb{P}(S|\sigma,\tilde{D}=i,\tilde{D}_{AR}(t)=0)\mathbb{P}(\tilde{D}=i|\sigma,\tilde{D}_{AR}(t)=0). \tag{45}$$

Using Lemma 3, we next show that $\mathbb{P}(S|\sigma,\tilde{D}_{AR}(t)=0)$ strictly increases with t. To do so, we show that

(i) For any $\tilde{d}\in[a,b)$, $\mathbb{P}(\tilde{D}>\tilde{d}|\sigma,\tilde{D}_{AR}(t)=0)$ strictly decreases with t (the conditional random variable $\tilde{D}|\{\sigma,\tilde{D}_{AR}(t)=0\}$ corresponds to X_t in Lemma 3).

(ii) Within the range $\tilde{d}\in[N,b]$, $\mathbb{P}(S|\sigma,\tilde{D}=\tilde{d},\tilde{D}_{AR}(t)=0)$ (which corresponds to $g(x)$ in Lemma 3) strictly decreases with $\tilde{d}$ (which corresponds to x in Lemma 3).

Starting with (i), we define $T_{AR'}(t)$ to be the sum of length of intervals within customers that observe N available servers choose AR' prior to t under strategy σ. From Bayes' Theorem, the distribution of $\tilde{D}|\{\sigma,\tilde{D}_{AR}(t)=0\}$ is

$$\mathbb{P}(\tilde{D}=i|\sigma,\tilde{D}_{AR}(t)=0)\ =\frac{\mathbb{P}(\tilde{D}=i,\tilde{D}_{AR}(t)=0|\sigma)}{\mathbb{P}(\tilde{D}_{AR}(t)=0|\sigma)},\quad \forall i\in[a,b]. \tag{46}$$

The term $\mathbb{P}(\tilde{D}=i,\tilde{D}_{AR}(t)=0|\sigma)$ is the probability that i customers arrive, and each customer arrives either within intervals where customers do not make AR or after t. Since the arrival time of each customer is independent of others, the probability that a customer arrives either within intervals where customers do not make AR or after t is $T_{AR'}(t) + (1 - t)$. Thus, for any $i \in [a, b]$,

$$\frac{\mathbb{P}(\tilde{D} \le i, \tilde{D}_{AR}(t)=0|\sigma)}{\mathbb{P}(\tilde{D}_{AR}(t)=0|\sigma)} = \frac{\sum_{j=a}^{i} \mathbb{P}(\tilde{D}=j)\mathbb{P}(\tilde{D}_{AR}(t)=0)|\sigma, \tilde{D}=j)}{\sum_{j=a}^{b} \mathbb{P}(\tilde{D}=j)\mathbb{P}(\tilde{D}_{AR}(t)=0|\sigma, \tilde{D}=j)} = \frac{\sum_{j=a}^{i} \mathbb{P}(\tilde{D}=j)(T_{AR'}(t)+1-t)^j}{\sum_{j=a}^{b} \mathbb{P}(\tilde{D}=j)(T_{AR'}(t)+1-t)^j}. \tag{47}$$

Since $T_{AR'}(t)$ is a constant within the interval I_2, the term $T_{AR'}(t) + 1 - t$ decreases with t. By Definition 4, the conditional random variable $\tilde{D}|\{\sigma, \tilde{D}_{AR}(t)=0\}$ belongs to the family of distributions $\mathscr{F}$, where $\mathbb{P}(\tilde{D}=j)$ corresponds to a_j and $T_{AR'}(t) + 1 - t$ corresponds to $h(t)$. Hence, by Lemma 4, for any $\tilde{d} \in [a, b)$, $\mathbb{P}(\tilde{D} > \tilde{d}|\sigma, \tilde{D}_{AR}(t)=0)$ strictly decreases with t.

Next we prove (ii). Using the law of total probability, conditioned on the number of reservations, the probability of service can be written

$$\mathbb{P}(S|\sigma, \tilde{D}=\tilde{d}, \tilde{D}_{AR}(t)=0) = \sum_{i=a}^{b} \mathbb{P}(S|\tilde{D}=\tilde{d}, \tilde{D}_{AR}=i)\mathbb{P}(\tilde{D}_{AR}=i|\sigma, \tilde{D}=\tilde{d}, \tilde{D}_{AR}(t)=0). \tag{48}$$

From the definition of the model,

$$\mathbb{P}(S|\tilde{D}=\tilde{d}, \tilde{D}_{AR}=\tilde{d}_{AR}) = \begin{cases} 1 & \text{if } \tilde{d} < N, \\ \frac{N-\tilde{d}_{AR}}{\tilde{d}+1-\tilde{d}_{AR}} & \text{if } \tilde{d} \ge N \text{ and } \tilde{d}_{AR} < N, \\ 0 & \text{otherwise.} \end{cases} \tag{49}$$

From Eq. (49), one can see that $\mathbb{P}(S|\tilde{D} = \tilde{d}, \tilde{D}_{AR} = \tilde{d}_{AR})$ decreases with both $\tilde{d}$ and $\tilde{d}_{AR}$. Furthermore, Lemma 5 showed that the number of reservations cannot decrease as the demand increases. Thus, for any demand realizations $\tilde{d}_1 > \tilde{d}_2$ the following holds:

$$\mathbb{P}(S|\sigma, \tilde{D}=\tilde{d}_1, \tilde{D}_{AR}(t)=0) = \sum_{i=a}^{b} \mathbb{P}(S|\tilde{D}=\tilde{d}_1, \tilde{D}_{AR}=i)\mathbb{P}(\tilde{D}_{AR}=i|\sigma, \tilde{D}=\tilde{d}_1, \tilde{D}_{AR}(t)=0) \tag{50}$$

$$\le \sum_{i=a}^{b} \mathbb{P}(S|\tilde{D}=\tilde{d}_2, \tilde{D}_{AR}=i)\mathbb{P}(\tilde{D}_{AR}=i|\sigma, \tilde{D}=\tilde{d}_1, \tilde{D}_{AR}(t)=0) \tag{51}$$

$$\le \sum_{i=a}^{b} \mathbb{P}(S|\tilde{D}=\tilde{d}_2, \tilde{D}_{AR}=i)\mathbb{P}(\tilde{D}_{AR}=i|\sigma, \tilde{D}=\tilde{d}_2, \tilde{D}_{AR}(t)=0) \tag{52}$$

$$= \mathbb{P}(S|\sigma, \tilde{D}=\tilde{d}_2, \tilde{D}_{AR}(t)=0). \tag{53}$$

Moreover, from Eq. (49), it follows that if $\tilde{d}_1 \ge N$, then Eq. (50) is strictly smaller than Eq. (51).

We showed that both items 1 and 2 hold. Hence, $\mathbb{P}(S|\sigma, \tilde{D}_{AR}(t)=0)$ strictly increases with t. Therefore, at equilibrium, an interval within which customers choose AR' cannot be followed by an interval within which customers choose AR. Hence, at equilibrium, the strategy followed by customers that observe N available servers must be a threshold strategy.

Inductive step. We assume that a threshold strategy with time-thresholds $\tau_k \ge \tau_{k+1}, \ge ... \ge \tau_N$ is followed by all customers that observe $n \in [k, k+1, \ldots, N]$ available servers for $1 < k < N$. We show that, at equilibrium, a threshold strategy with time-threshold $\tau_{k-1} \ge \tau_k$ must be followed by customers that observe $k - 1$ available servers. We split the proof into two parts. In the first part, we show that a customer with arrival time $t' < \tau_k$ that observes $k - 1$ available servers is better off choosing AR. Thus, there is a time-threshold $\tau_{k-1} \geq \tau_k$ such that all customers that arrive before τ_{k-1} and observe $k-1$ available servers choose AR. In the second part, we show that all customers that arrive after τ_{k-1} and observe $k-1$ available servers choose AR'.

Consider a scenario where a customer arrives at time $t < \tau_k$ and observes k available servers and a second scenario where a customer arrives at the same time and observes $k-1$ available servers. We need to show that

$$\mathbb{P}(S|\sigma, \tilde{D}_{AR}(t)=N-k) \geq \mathbb{P}(S|\sigma, \tilde{D}_{AR}(t)=N-k+1). \tag{54}$$

From Eq. (50)-Eq. (53), it follows that the probability of service increases with the demand, regardless of the number of available servers observed upon arrival (i.e., item 2 in the base case holds also for $\tilde{D}_{AR}(t) \neq 0$). Thus, we only need to show that

$$\mathbb{P}(\tilde{D}>\tilde{d}|\sigma, \tilde{D}_{AR}(t)=N-k+1) \geq \mathbb{P}(\tilde{D}>\tilde{d}|\sigma, \tilde{D}_{AR}(t)=N-k). \tag{55}$$

From the induction's assumption, a customer that observes k available servers knows that exactly $N-k$ customers arrived earlier. Likewise, a customer that observes $k-1$ available servers knows that at least $N-k+1$ customers arrived earlier. Since the system is sampled at the same time and the same strategy is followed in both cases, and since arrivals are i.id, observing more reservations makes it more likely that the demand will be larger. Thus, Eq. (55) holds.

Next, we prove the second part. We do so by showing that both items 1 and 2 in the base case also hold when observing $k-1$ available servers. Starting with item 1, we show that the conditional random variable $\tilde{D}|\{\sigma, \tilde{D}_{AR}(t)=N-k+1\}$ belongs to $\mathscr{F}$. We denote by AR_{N-k+1} the event that the first $N-k+1$ customers choose AR (i.e., the i-th customer arrives before τ_i, for $i = 1, 2, \ldots, N-k+1$). Let $T_{AR'}(t)$ be the sum of length of intervals within which customers that observe $k-1$ available servers choose AR' under strategy σ. The CDF of $\tilde{D}|\{\sigma, \tilde{D}_{AR}(t)=N-k+1\}$ is

$$\begin{aligned}
\mathbb{P}(\tilde{D}\leq j|\sigma, \tilde{D}_{AR}(t)=N-k+1) &= \frac{\sum\limits_{j=N-k+1}^{i} \mathbb{P}(\tilde{D}=j, \tilde{D}_{AR}(t)=N-k+1|\sigma)}{\mathbb{P}(\tilde{D}_{AR}(t)=N-k+1)} \\
&= \frac{\sum\limits_{j=N-k+1}^{i} \mathbb{P}(\tilde{D}=j)\binom{j}{N-k+1}\mathbb{P}(AR_{N-k+1})(T_{AR'}(t)+1-t)^{j-(N-k+1)}}{\sum\limits_{j=N-k+1}^{b} \mathbb{P}(\tilde{D}=j)\binom{j}{N-k+1}\mathbb{P}(AR_{N-k+1})(T_{AR'}(t)+1-t)^{j-(N-k+1)}} \\
&= \frac{\sum\limits_{j=N-k+1}^{i} \mathbb{P}(\tilde{D}=j)\binom{j}{N-k+1}(T_{AR'}(t)+1-t)^{j}}{\sum\limits_{j=N-k+1}^{b} \mathbb{P}(\tilde{D}=j)\binom{j}{N-k+1}(T_{AR'}(t)+1-t)^{j}}, \quad \forall i \geq N-k+1.
\end{aligned} \tag{56}$$

By Definition 4, this distribution belongs to $\mathscr{F}$, where $\mathbb{P}(\tilde{D}=j)\binom{j}{N-k+1}$ corresponds to a_j and $T_{AR'}(t)+1-t$ corresponds to $h(t)$. Thus, item 1 holds. As explained earlier in the proof, item 2 also holds for $D_{AR}(t) = N-k+1$. We conclude that for $n = k-1$ it is also true that an interval within which customers choose AR cannot follow an interval within which customers choose AR'. Hence, a threshold strategy is followed by all customers that observe $k-1$ available servers.

Proof of Theorem 2

From Theorem 1, we know that, at equilibrium, all customers follow a threshold strategy. First, we show that, at equilibrium, all time-thresholds are smaller than 1 (i.e., an equilibrium where all customers choose AR regardless of their arrival time does not exist). Let assume by contradiction that, for some $n \in \{1, ..., N\}$, $\sigma(t, n) = 1$ for all $t \in [0, 1]$. In this case, the n-th threshold customer (which arrives at time 1) knows that all customers that arrive earlier chose AR and since no customer will arrive after her, her service is granted. That is,

$$\pi_n(1) = 1, \quad \forall n \in \{1, ..., N\}. \tag{57}$$

Hence, the n-th threshold customer is better off choosing AR' which leads to a contradiction.

Next, assume that $C < \underline{C}$. From Lemma 2, we deduce that, in this case, there is exactly one set of thresholds $0 < \tau_N^e < \tau_{N-1}^e < ... < \tau_1^e < 1$ such that, for any $n \in \{1, ..., N\}$, $1 - C = \pi_n(\tau_n^e)$. We will now show this set represents a *some-make-AR* equilibrium. Assume that all customers follow this set of thresholds. Consider a customer with arrival time $t < \tau_n^e$ that observes n available servers. Her probability of service is smaller than $\pi_n(t)$ since there is a positive probability that reservations will be made between t and τ_n^e, while $\pi_n(t)$ is the probability of service given that no reservations will be made after time t. From the first part of Lemma 2, we know that $\pi_n(\tau_n^e) > \pi_n(t)$. Thus, her probability of service is smaller than the probability of service of the n-th threshold customer and she is better off choosing AR.

Now, consider a customer with arrival time $t > \tau_n^e$ that observes n available servers. Since all customers that arrive later than τ_n^e and observe n available servers choose AR', observing n available servers at time t is equivalent to observing n available servers at time τ_n^e. Thus, this customer (just like the n-th threshold customer) is indifferent between the two actions and has no motivation to deviate. We conclude that this set of thresholds represents a *some-make-AR* equilibrium. A *none-make-AR* equilibrium does not exist, since if all customers choose AR', thir expected payoff is $1 - \underline{\text{C}}$ which is smaller than $1 - C$.

Finally, assume that $C \geq \underline{C}$. If all customers that observe N available servers choose AR', then they all have the same probability of service $1 - \underline{\text{C}}$ which is greater than $1 - C$. Thus, none will deviate and *none-make-AR* is an equilibrium. A *some-make-AR* equilibrium does not exist, since there is no value of τ_N for which $1 - C = \pi_N(\tau_N)$.