Robust Distributed Estimation for Linear Systems under Intermittent Information

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***Abstract*— We provide a comprehensive solution to the estimation problem of the state for a linear time-invariant system in a distributed fashion over networks that allow only intermittent information transmission. By attaching to each node an observer that employs information received from its neighbors triggered by asynchronous communication events, we propose a distributed state observer that guarantees global exponential stability of the zero estimation error set. The design of parameters is formulated as linear matrix inequalities (LMIs). A thorough robustness analysis of the proposed observer to unmodeled dynamics, unknown communication times, as well as measurement and communication noise characterized in terms of input-to-state stability (ISS) is presented. These properties of the proposed observer are shown analytically and validated numerically.**

I. Introduction

A. Motivation and Problem Statement

In this paper, we consider the problem of robustly estimating the state of a continuous-time plant from intermittent measurements of functions of its output over a network of N agents. Under nominal conditions, the model governing the state $x \in \mathbb{R}^n$ is given by a linear time-invariant system, while, under perturbations, assumes the form

$$\dot{x} = Ax + \delta(x,t) \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is the state matrix, $\delta : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is an unknown function modeling the perturbation and $t \geq 0$ denotes ordinary time. The i-th agent in the network receives a measurement y_i at time instances t^i_s, $s \in \{1, 2, \ldots\}$, which define the incoming information events for that agent. The nominal model for $y_i \in \mathbb{R}^{p_i}$ is a linear function of the state, while the perturbed case takes the form

$$y_i = H_i x + \varphi_i(x,t) \tag{2}$$

where $\varphi_i : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^{p_i}$ is an unknown function and $H_i \in \mathbb{R}^{p_i\times n}$ is the local output matrix of the i-th agent for each $i \in \{1, 2, \ldots, N\}$. The event times t^i_s are independently defined for each agent; the only restriction imposed on communication times is that they must satisfy

$$t^i_{s+1} - t^i_s \in [T^i_1, T^i_2] \qquad \forall s \in \{1, 2, \ldots\} \tag{3}$$

where T^i_1 and T^i_2 are nominal parameters that define the lower and upper bounds, respectively, of the time allowed to elapse between consecutive communication events and are such that $T^i_2 \geq T^i_1 > 0$. Hence, the event times t^i_s can potentially be determined by a random variable taking values in the interval $[T^i_1, T^i_2]$. The parameters T^i_1 and T^i_2 are assumed to be known but are not necessarily the same for each agent. Moreover, these parameters may be further perturbed; see Section IV-C.

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The main challenges to solve this problem include the following:

1) *Asynchronous and heterogeneous communication events at unknown times:* the time instances at which agents receive information are not synchronized, meaning that each agent may receive information at different time instances. Furthermore, the amount of ordinary time elapsed between communication events for each agent can be different; for instance, an agent can receive information at a much faster rate than others. In addition, the exact event times are not known a priori.
2) *Lack of full information to reconstruct the state at each agent and at the same time:* information is not available continuously, but rather, at isolated time instances. Therefore, none of the agents may have enough information to fully reconstruct x alone. In fact, in the nominal case, (H_i, A) may not necessarily be detectable for any i. This suggests that information must be shared among agents, but, even then, the needed information to fully reconstruct x may be available at different time instances (see item 1), which, in particular, makes detectability of $((H_1, H_2, \ldots, H_N), A)$ not very useful when solving the nominal problem.
3) *Perturbations in the dynamics, parameters, and measurements:* the lack of knowledge of the actual perturbation functions δ and $\{\varphi_i\}^N_{i=1}$ (in particular, the lack of a Lipschitzness property on δ) precludes from fully compensating for their effect. Furthermore, perturbations on the parameters T^i_1 and T^i_2 could lead to diverging estimates when the estimation algorithm has little margin of robustness to such parameters.

B. Related work

State estimation in networked systems has seen increased attention recently. Several observer architectures and design methods have been proposed in the literature. Results for the case when information is continuously available include those in [1] and [2] for the estimation of the trajectories of a moving target using distributed sensor networks, and the results in [3] for robust estimation with performance guarantees. In particular, [3] provides performance guarantees for robust estimation of interconnected observers when information is available continuously, not sporadic communication as this paper considers. Discrete-time approaches with information arriving at common discrete time instances are also available. In [**?**], under nominal conditions and for linear time-invariant plants, a network of local observers communicating at the same time instances is designed to achieve an attractivity property (called omniscience) between the estimates and the state of the plant. In [5], the optimal linear estimation problem for discrete time-varying networked systems over a shared communication channel is considered.

Approaches that keep the continuous dynamics of the plant and treat the communication events as impulsive events have also been developed. An observer-based controller for a network control system modeled as a time-varying hybrid system is proposed in [6] and its design is performed using the Lyapunov-Krasovskii functional. When information is available periodically, the work in [7] established an observer-protocol pair to asymptotically reconstruct the states of a linear time-invariant plant with multiple sensors. In [8], a distributed algorithm for observer design for linear time-invariant continuous-time systems is designed by partitioning the dynamics into disjoint areas and attaching an algorithm to each area that updates the estimates over time windows with common length. In [9], an emulation-like approach is used to guarantee that a single robust continuous-time observer (when available) can be used for estimation in network control systems. Such general result is obtained using trajectory-based and small-gain arguments. In [10], for a second order system capturing the dynamics of a leader-follower problem, a distributed observer allowing for switching inter-agent topologies is proposed using switched systems theory.

In addition to the above deterministic continuous, discrete, and hybrid approaches for distributed estimation, the literature of network control and estimation is rich on stochastic approaches over a variety of time domains. These include information-oriented approaches [11], distributed Kalman-like filters with ([12]) and without communication constraints ([13], [14]), and spatially-distributed estimators [15].

The literature of observer design with no underlying network includes several results that allow for measurements being available at isolated times. Specifically, in [16], conditions guaranteeing that a continuous-time observer for a class of nonlinear systems provides states estimates with robustness to samples and delays of the measurement are provided. In particular, the main result requires an input-to-output stability of the observer-plant pair. In [17], the authors propose a method to design observers for a class of continuous-time systems with Lipschitz nonlinearities under nonuniform sampled measurements. The approach therein consists of composing a continuous-time high-gain observer with an inter-sample output predictor. Closely related to this work and also under the name "continuous-discrete observer," [18] provides a way to design observers for a class of Lipschitz continuous-time systems of small dimensions through the computation of finite-time reachable sets. A survey of recent advances in the design of continuous-discrete observers is available [19]. A hybrid observer guaranteeing global exponential stability of the zero-estimation error under sporadic measurements is available in [20]. A preliminary version of these results appeared without proofs and fewer examples in [**?**] which primarily focuses on the robust stability of the case of synchronous sporadic communication and includes a brief mention of asynchronous communication.

C. Outline of Proposed Solution

Motivated by the challenges in Section I-A, we propose a distributed hybrid observer that is capable of asymptotically reconstructing (with an exponential rate) the state of the plant x locally, at each agent, with stability and by only exchanging information from the plant and its neighbors at communication events t_s^i. In the nominal case, the algorithm guarantees global exponential stability of a set of points where the produced estimate and the state of the plant are equal. In the presence of unknown perturbations, the algorithm guarantees that such property is preserved, semiglobally and practically, when the perturbations are small enough, while when bounds on certain perturbations are known, the said global exponential stability can be guaranteed through a robust observer design procedure. Our distributed hybrid observer assures such properties by running a decentralized algorithm that, at the i-th agent, generates an estimate of the state x, which is denoted $\hat{x}_i \in \mathbb{R}^n$, and an information fusion quantity, which is denoted $\eta_i \in \mathbb{R}^n$. These state variables are continuously updated by a differential equation, which takes the general form

$$\dot{\hat{x}}_i = A\hat{x}_i + \eta_i \tag{4a}$$

$$\dot{\eta}_i = f_{oi}(\hat{x}_i, \eta_i) \tag{4b}$$

when no information is received, while when information is received, the states $\hat{x}_i$ and η_i are updated according to

$$\hat{x}_i^+ = \hat{x}_i \tag{5a}$$

$$\eta_i^+ = \sum_{k\in\mathcal{V}} g_{ik} G_{oi}^k(\hat{x}_i, \hat{x}_k, y_i, y_k) \tag{5b}$$

where $\mathcal{V} := \{1, 2, \ldots, N\}$ defines the set of all agents; g_{ik} defines a connectivity graph, namely, $g_{ik} = 1$ if the k-th agent can share information to agent i and $g_{ik} = 0$ otherwise; the map $f_{oi} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ defines the continuous evolution of the information fusion state and the map $G_{oi}^k : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{p_i} \times \mathbb{R}^{p_k} \to \mathbb{R}^n$ defines the impulsive update law when new information is collected from the plant and the k-th neighbor.[1] The information fusion state η_i is injected into the continuous dynamics of the local estimate $\hat{x}_i$ and, at communication events, injects new information impulsively – the right-hand side of (5) is the "innovation term" of the proposed observer. The continuous and discrete dynamics in (4) and (5), respectively, along with a proper autonomous model triggering the communication events, define a hybrid observer at each agent as well as hybrid dynamical system modeling the entire networked system.

D. Contributions

Distributed estimation algorithms that simultaneously cope with all of the challenges introduced in Section I-A are not yet available, being perhaps the main challenge the asynchronous and heterogeneous communication events at

[1] The actual forms of f_{oi} and G_{oi}^k are in Section III-A.

which information is available. In this paper, we propose a distributed estimation algorithm that copes with the said challenges. The main contribution of this work lies on the establishment of sufficient conditions for nominal and robust estimation over networks. The proposed design conditions guarantee reconstruction of the state with an exponential rate when information is only available at *asynchronous* and *non-periodic* time instances, cf., e.g., [3], [**?**], [7], [10]. Precisely, as shown in Section III, sufficient conditions are developed to assure global exponential stability of the zero-estimation error set when information is transmitted according to the nondeterministic law in (3). The rationale behind the choice of linear continuous-time dynamics of the proposed observers is to allow for a tractable design procedure. More precisely, in Section III-D, constructive linear matrix inequalities (LMIs) are developed based on the established sufficient conditions to efficiently determine parameters for the proposed observer. These sufficient conditions are satisfiable even if each agent may not have a measurement model such that (H_i, A) is detectable (see, e.g., Example 5.3) or if the local output measurements cannot be transmitted to all neighbors (see, e.g., Example 5.2).

Unlike previous works, and enabled by the hybrid systems approach, an in-depth robustness analysis and design procedure are presented in Section IV. In particular, we establish several key robustness properties. In Section IV-A, results on robustness with respect to perturbations emerging from unmodeled dynamics, skewed clocks, as well as measurement and communication noise are established. In Section IV-B, robustness in the form of an input to state stability (ISS) property with respect to measurement and communication noise is provided, for which an explicit ISS bound is given. A procedure to design our distributed observer with robustness to the communication parameters T_1^i and T_2^i in (3) is given in Section IV-C, while Section IV-D presents a design procedure for robustness to random packet dropouts. In Section V-B, we illustrate these results in several examples.

II. Notation and Preliminaries

A. Notation

Given a matrix A, the set $\text{eig}(A)$ contains all eigenvalues of A and $|A| := \max\{|\lambda|^{\frac{1}{2}} : \lambda \in \text{eig}(A^\top A)\}$. Given two vectors $u, v \in \mathbb{R}^n$, $|u| := \sqrt{u^\top u}$ and notation $[u^\top \ v^\top]^\top$ is equivalent to (u, v). Given a function $m : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, $|m|_\infty := \sup_{t \geq 0} |m(t)|$. $\mathbb{Z}_{\geq 1}$ denotes the set of positive integers, i.e., $\mathbb{Z}_{\geq 1} := \{1, 2, 3, \dots\}$. $\mathbb{N}$ denotes the set of natural numbers including zero, i.e., $\mathbb{N} := \{0, 1, 2, 3, \dots\}$. Given a vector $x \in \mathbb{R}^n$ and a closed set $\mathcal{A} \subset \mathbb{R}^n$, the distance from x to $\mathcal{A}$ is defined as $|x|_{\mathcal{A}} = \inf_{z \in \mathcal{A}} |x - z|$. Given a symmetric matrix P, $\overline{\lambda}(P) := \max\{\lambda : \lambda \in \text{eig}(P)\}$ and $\underline{\lambda}(P) := \min\{\lambda : \lambda \in \text{eig}(P)\}$. Given matrices A, B with proper dimensions, we define the operator $\text{He}(A, B) := A^\top B + B^\top A$; $A \otimes B$ defines the Kronecker product $\text{diag}(A, B)$ denotes a 2×2 block matrix with A and B being the diagonal entries; and $A * B$ defines the Khatri-Rao product between A and B.[2]. Denote $\star$ as the symmetric block of a matrix. Given $N \in \mathbb{Z}_{\geq 1}$, $I_N \in \mathbb{R}^{N \times N}$ defines the identity matrix and $\mathbf{1}_N$ is the vector of N ones. Given a set S, $\overline{\text{con}}\, S$ is the closed convex hull of the points in S. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class-$\mathbb{K}$ function, also written $\alpha \in \mathbb{K}$, if α is zero at zero, continuous, strictly increasing; it is said to belong to class-$\mathbb{K}_\infty$, also written $\alpha \in \mathbb{K}_\infty$, if $\alpha \in \mathbb{K}$ and is unbounded; α is positive definite, also written $\alpha \in \mathbb{PD}$, if $\alpha(s) > 0$ for all $s > 0$ and $\alpha(0) = 0$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class-$\mathbb{KL}$ function, also written $\beta \in \mathbb{KL}$, if it is nondecreasing in its first argument, nonincreasing in its second argument, $\lim_{r \to 0^+} \beta(r, s) = 0$ for each $s \in \mathbb{R}_{\geq 0}$, and $\lim_{s \to \infty} \beta(r, s) = 0$ for each $r \in \mathbb{R}_{\geq 0}$. Given $s \in \mathbb{R}$, $\lfloor s \rfloor$ denotes the largest integer that is smaller than or equal to s. A set-valued mapping $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous if its graph $\{(x, g) : x \in \mathbb{R}^n,\ g \in G(x)\}$ is closed; see [21, Lemma 5.10]. Given a closed set S, $\mathbb{T}_S(x)$ denotes its tangent cone S at x; see, e.g., [21, Definition 5.12].

[2]For more information on Kronecker and Khatri-Rao products, see [**?**].

B. Preliminaries on hybrid systems

In this paper, a hybrid system $\mathcal{H}$ has data (C, f, D, G) and is defined by

$$\begin{aligned} \dot{z} &= f(z) \quad z \in C, \\ z^+ &\in G(z) \quad z \in D, \end{aligned} \tag{6}$$

where $z \in \mathbb{R}^n$ is the state, f defines the flow map capturing the continuous dynamics and C defines the flow set on which f is effective. The map G defines the jump map and models the discrete behavior, while D defines the jump set, which is the set of points from where jumps are allowed. A solution ϕ to $\mathcal{H}$ is parametrized by $(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, where t denotes ordinary time and j denotes jump time. The domain $\text{dom}\, \phi \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a hybrid time domain if for every $(T, J) \in \text{dom}\, \phi$, the set $\text{dom}\, \phi \bigcap ([0, T] \times \{0, 1, \dots, J\})$ can be written as the union of sets $\bigcup_{j=0}^{J} (I_j \times \{j\})$, where $I_j := [t_j, t_{j+1}]$ for a time sequence $0 = t_0 \leq t_1 \leq t_2 \leq \dots \leq t_{J+1}$. The t_j's with $j > 0$ define the time instants when the state of the hybrid system jumps and j counts the number of jumps. A solution to $\mathcal{H}$ is called maximal if it cannot be extended, i.e., it is not a truncated version of another solution. It is called complete if its domain is unbounded. A solution is Zeno if it is complete and its domain is bounded in the t direction. A solution is precompact if it is complete and bounded. The set $\mathcal{S}_{\mathcal{H}}$ contains all maximal solutions to $\mathcal{H}$, and the set $\mathcal{S}_{\mathcal{H}}(\xi)$ contains all maximal solutions to $\mathcal{H}$ from ξ. See [21] for more details.

A hybrid system is said to satisfy the hybrid basic conditions if [21, Assumption 6.5] holds. The definition of global exponential stability of a closed set for $\mathcal{H}$ is given as follows.

*Definition 2.1 ([**?**, Page 1591]):* Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$. The closed set $\mathcal{A} \subset \mathbb{R}^n$ is said to be globally exponentially stable (GES) for $\mathcal{H}$ if there exist $\kappa, \alpha > 0$ such that every $\phi \in \mathcal{S}_{\mathcal{H}}$ is complete and satisfies

$$|\phi(t, j)|_{\mathcal{A}} \leq \kappa \exp(-\alpha(t + j)) |\phi(0, 0)|_{\mathcal{A}} \tag{7}$$

for each $(t, j) \in \text{dom}\, \phi$.

For a given hybrid system with inputs, we are interested in a closed set $\mathcal{A}$ being input-to-state stable as defined next. Similarly to (6), a solution to (8) is given by the solution pair (ϕ, u) with $\text{dom}\, \phi = \text{dom}\, u (= \text{dom}(\phi, u))$ that satisfies the dynamics therein with the property that, for each $j \in \mathbb{N}$, $t \mapsto \phi(t,j)$ is absolutely continuous and $t \mapsto u(t,j)$ is Lebesgue measurable and locally essentially bounded on $\{t : (t,j) \in \mathrm{dom}(\phi,u)\}$.

*Definition 2.2 ([**?**, Definition 2.1]):* Given a compact set $\mathcal{A}$, the hybrid system with state z, and input u given by

$$\begin{aligned} \dot{z} &= f(z,u) \quad (z,u) \in C \\ z^+ &\in G(z,u) \quad (z,u) \in D \end{aligned} \tag{8}$$

is input-to-state stable (ISS) with respect to $\mathcal{A}$ if there exist $\beta \in \mathbb{KL}$ and $\gamma \in \mathbb{K}$ such that, for each $\phi(0,0) \in \mathbb{R}^n$, every solution pair (ϕ,u) satisfies

$$|\phi(t,j)|_{\mathcal{A}} \leq \max\{\beta(|\phi(0,0)|_{\mathcal{A}}, t+j), \gamma(|u|_\infty)\} \tag{9}$$

for each $(t,j) \in \mathrm{dom}\,\phi$.

The $\mathcal{L}_\infty$ norm of $(t,j) \mapsto u(t,j)$ is given by

$$||u||_{(t,j)} := \max\left\{ \operatorname*{ess\,sup}_{(t',j')\in \mathrm{dom}\,u\setminus\Upsilon(u), t'+j'\leq t+j} |u(t',j')|, \sup_{(t',j')\in\Upsilon(u), t'+j'\leq t+j} |u(t',j')| \right\}$$

where $\Upsilon(u) = \{(t,j) \in \mathrm{dom}\,u : (t,j+1) \in \mathrm{dom}\,u\}$; see [**?**, Definition 2.1] for details.

C. Preliminaries on graph theory

A directed graph (digraph) is defined as $\Gamma = (\mathcal{V},\mathcal{E},\mathcal{G})$. The set of nodes of the digraph are indexed by the elements of $\mathcal{V} = \{1,2,\ldots,N\}$, and the edges are the pairs in the set $\mathcal{E} \subset \mathcal{V}\times\mathcal{V}$. Each edge directly links two nodes, i.e., an edge from i to k, denoted by (i,k), implies that agent i can receive information from agent k. The adjacency matrix of the digraph Γ is denoted by $\mathcal{G} \in \mathbb{R}^{N\times N}$, where its (i,k)-th entry g_{ik} is equal to one if $(i,k) \in \mathcal{E}$ and zero otherwise. A digraph is undirected if $g_{ik} = g_{ki}$ for all $i,k \in \mathcal{V}$. Without loss of generality, we assume that $g_{ii} = 0$ for all $i \in \mathcal{V}$. The in-degree and out-degree of agent i are defined by $d_i^{in} = \sum_{k=1}^N g_{ik}$ and $d_i^{out} = \sum_{k=1}^N g_{ki}$. The in-degree matrix $\mathcal{D}$ is the diagonal matrix with entries $D_{ii} = d_i^{in}$ for all $i \in \mathcal{V}$. The Laplacian matrix of the graph Γ, denoted by $\mathcal{L} \in \mathbb{R}^{N\times N}$, is defined as $\mathcal{L} = \mathcal{D} - \mathcal{G}$. The set of indices corresponding to the neighbors that can send information to the i-th agent is denoted by $\mathcal{N}(i) := \{k \in \mathcal{V} : (i,k) \in \mathcal{E}\}$. A digraph is said to be all-to-all connected if every pair of distinct vertices is connected by a unique edge; in that, $g_{ik} = 1$ for each $i,k \in \mathcal{V}$, $i \neq k$.

III. Distributed Hybrid Estimation Protocol and Nominal Properties

A. Hybrid model

Consider N agents that are connected via a directed graph and where each agent runs a local observer of the state x of the linear time-invariant plant in (1). Each local observer uses its own measurement and information received from its neighbors. Due to the impulsive nature of such communication mechanism, the communication events are triggered by a decreasing timer τ_i. Namely, τ_i decreases and upon reaching zero it is reset to a point in $[T_1^i, T_2^i]$. Such dynamics of τ_i at agent i can be modeled by a hybrid system given by

$$\dot{\tau}_i = -1 \qquad \tau_i \in [0, T_2^i], \tag{10a}$$

$$\tau_i^+ \in [T_1^i, T_2^i] \qquad \tau_i = 0. \tag{10b}$$

This hybrid system generates any possible sequence of time instances $\{t_s^i\}_{s=1}^\infty$ at which events occur and satisfy (3). Note that T_1^i and T_2^i may not be equal for each $i \in \mathcal{V}$, in this way the intervals which observer can update their estimates may be vastly different. [3]

In this paper, we consider the following dynamics for η_i defining a specific hybrid information fusion strategy:

$$f_{oi}(\hat{x}_i, \eta_i) = h_i \eta_i \tag{11}$$

for all $(\hat{x}_i, \eta_i) \in \mathbb{R}^n \times \mathbb{R}^n$, and, for all $(\hat{x}_i, \hat{x}_k, y_i, y_k) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{p_i} \times \mathbb{R}^{p_k}$,

$$G_{oi}^k(\hat{x}_i, \hat{x}_k, y_i, y_k) = \frac{1}{d_i^{in}} K_{ii} y_i^e + K_{ik} y_k^e + \gamma(\hat{x}_i - \hat{x}_k) \tag{12}$$

where, for each $i,k \in \mathcal{V}$, $y_i^e = H_i \hat{x}_i - y_i$ is the output estimation error; and the constants $h_i \in \mathbb{R}$, $K_{ik} \in \mathbb{R}^{n\times p_k}$, and $\gamma \in \mathbb{R}$ define the parameters of the observer. The constants g_{ik} in (5) and d_i^{in} in (12) are associated with the communication graph, which is assumed to be given. Note that due to the specific update law in (12) and the definition of g_{ik}, the second term in (12) uses the output error of each k-th agent that is a neighborhood of the i-th agent, and the third term in (12) uses the difference between the estimates $\hat{x}_i$ and $\hat{x}_k$. These are the quantities that are transmitted at communication events only. For simplicity, for the remainder of this section, we will assume that $\delta \equiv 0$ and $\varphi_i \equiv 0$ for all $i \in \mathcal{V}$, that is, the nominal case where perfect knowledge of the plant and its output is assumed – the scenario when these perturbations are nonzero is addressed in Section IV.

Remark 3.1: The nondeterministic time-invariant hybrid system model of the network in (10b) conveniently captures the event times in [9], [28], [19], which lead to a time-varying system and make analysis more difficult. Similar hybrid models were used in [**?**], [20]. When $h_i = 0$ for all $i \in \mathcal{V}$, the hybrid information fusion strategy in (11)-(12) falls into the category of zero-order sample-and-hold control; see, e.g., [22] and [23]. Note that the work in [22] and [23] pertain to single-agent systems and that robustness properties of the observer therein are not studied. □

Remark 3.2: The maps f_{oi} and G_{oi} in (11) and (12), respectively, enable other choices for the dynamics of the variable η_i. The parameter γ in (12) could be further generalized to γ_{ik}. Although not pursued in this paper, one could potentially choose sliding mode-like dynamics, such as those employed in [16]. It might also be possible to exploit the ideas in [**?**] to reduce the dimension of the estimation state, in particular, when the consensus term $\gamma(\hat{x}_i - \hat{x}_k)$ is zero. In fact, in such a case, the dimension of the state η_i could potentially be reduced using the construction in [**?**], where the augmented state z_i

[3] Consider the case of $N = 2$, $T_1^2 = T_1^1$ and $T_2^2 = 2T_2^1$. At jumps, the timer states τ_1, τ_2 are reset by $\tau_1^+ \in [T_1^1, T_2^1]$ when $\tau_1 = 0$ and $\tau_2^+ \in [T_1^1, 2T_2^1]$ when $\tau_2 = 0$, for such a jump map, τ_1 could potentially jump twice as fast as τ_2.

therein has a dimension that is potentially smaller than our η_i; see (5) therein. □

Inspired by the coordinates proposed in [25] for the study of single observers, let $e_i = \hat{x}_i - x$ and $e = (e_1, e_2 \ldots, e_N)$, $\theta = (\theta_1, \theta_2, \ldots, \theta_N)$, $\tau = (\tau_1, \tau_2, \ldots, \tau_N)$, and

$$\theta_i = K_{ii} y_i^e + \sum_{k\in\mathcal{V}} g_{ik} K_{ik} y_k^e + \gamma \sum_{k\in\mathcal{V}} g_{ik}(e_i - e_k) - \eta_i. \quad (13)$$

Then, the resulting closed-loop system is given by the interconnection between the plant in (1), all local observers and the dynamics of the information fusion states in (4)-(5). In particular, the resulting system in coordinates e, θ, and τ can be written as a hybrid system $\mathcal{H} = (C, f, D, G)$ with state $\chi = (\sigma, \tau) \in \mathcal{X} := \mathbb{R}^{nN} \times \mathbb{R}^{nN} \times \mathcal{T}$, where $\mathcal{T} := [0, T_2^1] \times [0, T_2^2] \times \cdots \times [0, T_2^N]$, $\sigma = (e, \theta)$, and data given by

$$f(\chi) := (A_{f\theta}\sigma, -\mathbf{1}_N) \quad (14)$$

for each $\chi \in C := \mathcal{X}$, and

$$G(\chi) := \{G_i(\chi) : \chi \in D_i, i \in \mathcal{V}\} \quad (15)$$

when $\chi \in D := \bigcup_{i\in\mathcal{V}} D_i$, $D_i := \{\chi \in C : \tau_i = 0\}$,

$$G_i(\chi) := \begin{bmatrix} e \\ (\theta_1, \theta_2, \ldots, \theta_{i-1}, 0, \theta_{i+1}, \ldots, \theta_N) \\ (\tau_1, \tau_2, \ldots, \tau_{i-1}, [T_1^i, T_2^i], \tau_{i+1}, \ldots, \tau_N) \end{bmatrix},$$

where

$$A_{f\theta} = \begin{bmatrix} A_\theta & -I_{nN} \\ \mathcal{K}\widetilde{A}_\theta & -\widetilde{\mathcal{K}} \end{bmatrix}, \quad (16)$$

$A_\theta = I_N \otimes A + \mathcal{K}$, $\widetilde{A}_\theta = A_\theta - H_\eta$, $\mathcal{K} = (K_g H_g) * (I_N + \mathcal{G}) + \gamma\mathcal{L} \otimes I_n$, and $\widetilde{\mathcal{K}} = \mathcal{K} - H_\eta$, the matrices $H_g = \mathrm{diag}(H_1, H_2, \ldots, H_N)$ and $H_\eta = \mathrm{diag}(h_1 I_n, h_2 I_n, \ldots, h_N I_n)$ are block diagonal and $K_g \in \mathbb{R}^{nN\times p}$ is a $N \times N$ block matrix with the (i,k)-th entry given by $K_{ik} \in \mathbb{R}^{n\times p_k}$ for all $i, k \in \mathcal{V}$ with $p = \sum_{i\in\mathcal{V}} p_i$. The matrix $K_g H_g$ is treated as an $N \times N$ block matrix for the Khatri-Rao product. As the objective of each agent is to estimate the state x of the plant, i.e., for each $i \in \mathcal{V}$, make $\hat{x}_i$ converge to x asymptotically, and since η_i approaches zero as e_i approaches zero, the set of interest is defined as

$$\mathcal{A} = \{0_{nN}\} \times \{0_{nN}\} \times \mathcal{T}. \quad (17)$$

Remark 3.3: The dynamics in (12) are quite general as they allow for the transmission of measurements y_i's and estimates $\hat{x}_i$'s through the network. As we will show later (Example 5.1), the transmission of $\hat{x}_i$'s are essential as when certain agents do not have enough information directly from measurements, the consensus term $\gamma(\hat{x}_i - \hat{x}_k)$ enables the reconstruction of the state x. As we will illustrate in Section V-A, the scenario when the transmission of y_k is not possible when $\tau_i = 0$ can be addressed by assigning $K_{ik} = 0$, $i, k \in \mathcal{V}$, $i \neq k$. □

Remark 3.4: Note that the flow set C and the jump set D of $\mathcal{H}$ are closed. Moreover, the flow map f is continuous and the jump map G is outer semicontinuous and locally bounded. Therefore, the hybrid system $\mathcal{H}$ satisfies the hybrid basic conditions. Note that satisfying the hybrid basic conditions implies that the hybrid system $\mathcal{H}$ is well-posed and asymptotic stability of a compact set is robust to small enough perturbations; see Section IV-A for more information. It turns out that the proposed hybrid observer is also robust to other important network perturbations and it can be designed to fully reject them; see Sections IV-B, IV-C, and IV-D. □

Below, we illustrate these basic properties in an example.

B. Properties of Solutions

As mentioned in Section II, solutions to general hybrid systems $\mathcal{H}$ can evolve continuously and/or discretely depending on the differential and difference equations/inclusions (and the sets where those apply) that govern the evolution of solutions.

Due to the fact that the timer variables being zero is the only trigger of the jump map, properties on the domain of solutions can be characterized in the following results.

Lemma 3.5: Let $0 < T_1^i \le T_2^i$ be given for all $i \in \mathcal{V}$. Every $\phi \in \mathcal{S}_\mathcal{H}$ satisfies the following:

1) *ϕ is complete, i.e., $\mathrm{dom}\,\phi$ is unbounded;*
2) *for each $(t, j) \in \mathrm{dom}\,\phi$,*
$$\left(\frac{j}{N} - 1\right) T_1^{\min} \le t \le \frac{j}{N} T_2^{\max},$$
where $T_1^{\min} := \min_{i\in\mathcal{V}} T_1^i$ and $T_2^{\max} := \max_{i\in\mathcal{V}} T_2^i$;
3) *for all $j \in \mathbb{Z}_{\ge 1}$ such that $(t_{(j+1)N}, (j+1)N), (t_{jN}, jN) \in \mathrm{dom}\,\phi$,*
$$t_{(j+1)N} - t_{jN} \in \left[T_1^{\min}, T_2^{\max}\right].$$

The proof of Lemma 3.5 can be found in Appendix A.

C. A Sufficient Condition for Global Exponential Stability

In this section, we establish sufficient conditions that guarantee the GES property of the set $\mathcal{A}$ for $\mathcal{H}$. With the change of coordinates in (13), our choice of a Lyapunov function candidate is given by

$$V(\chi) := e^\top P e + \theta^\top \widetilde{Q}(\tau)\theta \qquad \forall \chi \in \mathcal{X} \quad (18)$$

where P and $\widetilde{Q}$ are symmetric and positive definite for all $\tau \in \mathcal{T}$, and precisely defined below. Note that this is a proper choice since it satisfies $V(\chi) = 0$ for each $\chi \in \mathcal{A}$, while for any $\chi \in \mathcal{X} \setminus \mathcal{A}$, $V(\chi)$ is positive. More importantly, intuitively, regardless of which timer τ_i triggers a jump, this function satisfies the useful property that $V(\chi^+) - V(\chi)$ is upper bounded by a nonpositive function of θ_i for all $\chi \in D$. Such a property is possible due to the convenient choice of the update law of the observer used at jumps, which, in the coordinates in (13), leads to e being mapped by the identity and θ_i to zero. The injection of η_i in the flows of the local estimate in (4) and the continuous dynamics of η_i with flow map (11) further permit a decrease of V during flows, which conveniently uses exponential functions in the definition of $\widetilde{Q}$. These properties are exploited in the following result, which are also illustrated in several examples in Section V-A.

Theorem 3.6: Let $0 < T_1^i \le T_2^i$ be given for all $i \in \mathcal{V}$. Suppose N agents are connected via a digraph $\Gamma = (\mathcal{V}, \mathcal{E}, \mathcal{G})$. Moreover, suppose there exist $\delta > 0$ and matrices $K_g \in$

$\mathbb{R}^{nN\times p}$, $P\in\mathbb{R}^{nN\times nN}, Q_i\in\mathbb{R}^{n\times n}$ *satisfying* $P=P^\top>0$, $Q_i=Q_i^\top>0$ *for all* $i\in\mathcal{V}$, *and*

$$\mathcal{M}(\nu):=\begin{bmatrix}\mathrm{He}(A_\theta,P) & -P+\widetilde{A}_\theta^\top\mathcal{K}^\top\widetilde{Q}(\nu)\\ \star & -\delta\widetilde{Q}(\nu)-\mathrm{He}(\widetilde{\mathcal{K}},\widetilde{Q}(\nu))\end{bmatrix}<0 \quad (19)$$

for all $\nu=(\nu_1,\nu_2,\dots,\nu_N)\in\mathcal{T}$, *where* $\mathcal{K}$ *is defined in* (16), $\widetilde{Q}(\nu)=\mathrm{diag}\left(\widetilde{Q}_1(\nu_1),\widetilde{Q}_2(\nu_2),\dots,\widetilde{Q}_N(\nu_N)\right)$ *and* $\widetilde{Q}_i(\nu_i)=\exp(\delta\nu_i)Q_i$ *for each* $i\in\mathcal{V}$. *Then, the set* $\mathcal{A}$ *is GES for the hybrid system* $\mathcal{H}$. *Furthermore, every solution* $\phi\in\mathcal{S}_\mathcal{H}$ *satisfies the bound in* (7) *with*

$$\begin{aligned}\alpha&=\frac{-\max_{\nu\in\mathcal{T}}\overline{\lambda}(\mathcal{M}(\nu))}{2\alpha_2}\min\left\{\epsilon,(1-\epsilon)\frac{T_1^{\min}}{N}\right\}\\ \kappa&=\sqrt{\frac{\alpha_2}{\alpha_1}}\exp\left(\frac{\beta}{2\alpha}(1+\epsilon)T_1^{\min}\right)\end{aligned}\quad (20)$$

where $\epsilon\in(0,1)$,

$$\alpha_1=\min\left\{\underline{\lambda}(P),\underline{\lambda}\left(\widetilde{Q}(0)\right)\right\}, \quad (21)$$

$$\alpha_2=\max\left\{\overline{\lambda}(P),\overline{\lambda}\left(\widetilde{Q}(\overline{T}_2)\right)\right\}, \quad (22)$$

and $\overline{T}_2=(T_2^1,T_2^2,\dots,T_2^N)$.

Proof Consider the Lyapunov function candidate $V:\mathcal{X}\to\mathbb{R}_{\geq 0}$ given by (18). It follows that

$$\alpha_1|\chi|_\mathcal{A}^2\leq V(\chi)\leq\alpha_2|\chi|_\mathcal{A}^2 \quad (23)$$

for all $\chi\in C$, where α_1 and α_2 are in (21) and (22), respectively.

Then, for each $\chi\in C$, we have

$$\begin{aligned}\langle\nabla V(\chi),f(\chi)\rangle&=\dot{e}^\top Pe+e^\top P\dot{e}+\dot{\theta}^\top\widetilde{Q}(\tau)\theta+\theta^\top\widetilde{Q}(\tau)\dot{\theta}\\&\quad+\theta^\top\dot{\widetilde{Q}}(\tau)\theta\\&=e^\top\mathrm{He}(A_\theta,P)e-2e^\top P\theta+2\theta^\top\widetilde{Q}(\tau)\mathcal{K}\widetilde{A}_\theta e\\&\quad-\theta^\top\mathrm{He}(\widetilde{\mathcal{K}},\widetilde{Q}(\tau))\theta-\delta\theta^\top\widetilde{Q}(\tau)\theta\\&=\sigma^\top\mathcal{M}\sigma\end{aligned}$$

since $\sigma=(e,\theta)$, where $\mathcal{M}$ is the matrix in (19). Therefore, by using inequality (19), we have

$$\langle\nabla V(\chi),f(\chi)\rangle\leq-\beta|\chi|_\mathcal{A}^2,\quad\forall\chi\in C \quad (24)$$

where $\beta=-\max_{\nu\in\mathcal{T}}\overline{\lambda}(\mathcal{M}(\nu))$. Moreover, for each $\chi=(e,\theta,\tau)\in D$ and for each $g\in G(\chi)$, there exists at least one component of τ, say, the i-th component, such that $\tau_i=0$. Then, at jumps we have

$$V(g)-V(\chi)\leq-\theta_i^\top Q_i\theta_i\leq 0. \quad (25)$$

Now, for each $\phi\in\mathcal{S}_\mathcal{H}$, pick any $(t,j)\in\mathrm{dom}\,\phi$ and let $0=t_0\leq t_1\leq\cdots\leq t_{j+1}\leq t$ satisfy

$$\mathrm{dom}\,\phi\bigcap([0,t_{j+1}]\times\{0,1,\dots,j\})=\bigcup_{s=0}^{j}\left([t_s,t_{s+1}]\times\{s\}\right).$$

For each $s\in\{0,1,\dots,j\}$ and almost all $r\in[t_s,t_{s+1}]$, $\phi(r,s)\in C$. Then, (19) and (24) imply that, for each $s\in\{0,1,\dots,j\}$ and for almost all $r\in[t_s,t_{s+1}]$,

$$\frac{d}{dr}V(\phi(r,s))\leq-\beta|\phi(r,s)|_\mathcal{A}^2\leq-\frac{\beta}{\alpha_2}V(\phi(r,s)).$$

Integrating both sides of this inequality yields

$$V(\phi(t_{s+1},s))\leq\exp\left(-\frac{\beta}{\alpha_2}(t_{s+1}-t_s)\right)V(\phi(t_s,s)) \quad (26)$$

for each $s\in\{0,1,\dots,j\}$. Similarly, for each $s\in\{1,2,\dots,j\}$, $\phi(t_s,s-1)\in D$, and using (25), we get

$$V(\phi(t_s,s))-V(\phi(t_s,s-1))\leq 0\quad\forall s\in\{1,2,\dots,j\}. \quad (27)$$

From inequalities (26) and (27), we have that

$$V(\phi(t,j))\leq\exp\left(-\frac{\beta}{\alpha_2}t\right)V(\phi(0,0)). \quad (28)$$

Therefore, by using (23), for any $(t,j)\in\mathrm{dom}\,\phi$,

$$|\phi(t,j)|_\mathcal{A}\leq\sqrt{\frac{\alpha_2}{\alpha_1}}\exp\left(-\frac{\beta}{2\alpha_2}t\right)|\phi(0,0)|_\mathcal{A}.$$

From Lemma 3.5, we have that $t\geq\left(\frac{j}{N}-1\right)T_1^{\min}$, which implies that

$$\begin{aligned}|\phi(t,j)|_\mathcal{A}&\leq\sqrt{\frac{\alpha_2}{\alpha_1}}\exp\left(-\frac{\beta}{2\alpha_2}t\right)|\phi(0,0)|_\mathcal{A}\\&\leq\sqrt{\frac{\alpha_2}{\alpha_1}}\exp\left(-\frac{\beta}{2\alpha_2}\left(\epsilon t+(1-\epsilon)\left(\frac{j}{N}-1\right)T_1^{\min}\right)\right)|\phi(0,0)|_\mathcal{A}\\&\leq\kappa\exp(-\alpha(t+j))|\phi(0,0)|_\mathcal{A}.\end{aligned}$$

where $\epsilon\in(0,1)$ and we used the property that $t=\epsilon t+(1-\epsilon)t\geq\epsilon t+(1-\epsilon)\left(\frac{j}{N}-1\right)T_1^{\min}$. Along with the fact that every maximal solution to $\mathcal{H}$ is complete, this bound implies GES of $\mathcal{A}$ for $\mathcal{H}$. ■

Remark 3.7: Note that when $N=1$, condition (19) reduces to the condition in [20, Theorem 1]. Moreover, this condition is tight as it governs the growth of the estimation error between jumps. □

Remark 3.8: The matrix inequality in (19) arises from the asymptotic stability analysis in the proposed new coordinates $\chi=(e,\theta,\tau)$, namely, the analysis during flows; see (24). However, this approach introduces conservativeness as the seminegativity of the change of V at the events (see (25)) is not necessarily exploited. Another source of conservativeness in Theorem 3.6 comes from the bounding techniques used in the derivation of the upper bound on the change of V during flows. The strict inequality (19) could be further relaxed to a less than or equal to inequality, in which case, the invariance principle for hybrid systems in [21, Corollary 8.9] can be applied. In such a case, GES will not be guaranteed, but rather global asymptotic stability of $\mathcal{A}$ for $\mathcal{H}$ as in [21, Definition 3.6] can be assured. □

Condition (19) needs to be checked over the compact set $\mathcal{T}$, which might be a numerically challenging task. The following result relaxes this requirement.

Proposition 3.9: Let $0<T_1^i\leq T_2^i$ *be given for all* $i\in\mathcal{V}$. *The inequality in* (19) *holds if there exist* $\delta>0$ *and matrices* $K_g\in\mathbb{R}^{nN\times p}$, $P\in\mathbb{R}^{nN\times nN}, Q_i\in\mathbb{R}^{n\times n}$ *satisfying* $P=$

$P^\top > 0$, $Q_i = Q_i^\top > 0$ *for all* $i \in \mathcal{V}$ *such that*

$$\begin{bmatrix} \mathrm{He}(A_\theta, P) & -P+\widetilde{A}_\theta^\top \mathcal{K}^\top Q \\ \star & -\delta Q - \mathrm{He}(\widetilde{\mathcal{K}}, Q) \end{bmatrix} < 0, \qquad (29)$$

$$\begin{bmatrix} \mathrm{He}(A_\theta, P) & -P+\widetilde{A}_\theta^\top \mathcal{K}^\top \overline{E} Q \\ \star & -(\delta \overline{E} Q+\mathrm{He}(\widetilde{\mathcal{K}}, \overline{E} Q)) \end{bmatrix} < 0, \qquad (30)$$

where $\overline{E} = \mathrm{diag}(\exp(\delta T_2^1), \exp(\delta T_2^2), \ldots, \exp(\delta T_2^N)) \otimes I_n$ *and* $Q = \mathrm{diag}(Q_1, Q_2, \ldots, Q_N)$.

The proof of Proposition 3.9 can be found in Appendix A.

We can further relax the conditions in Proposition 3.9 by noting that, by definition of $T_2^{\max}$ in Lemma 3.5, each $\tau_i \in [0, T_2^i] \subset [0, T_2^{\max}]$. This leads to the following result.

Proposition 3.10: Let $T_2^i > 0$ *be given for each* $i \in \mathcal{V}$. *The inequality in* (19) *holds if there exists* $\delta > 0$ *and matrices* $K_g \in \mathbb{R}^{nN\times p}$, $P \in \mathbb{R}^{nN\times nN}, Q_i \in \mathbb{R}^{n\times n}$ *satisfying* $P = P^\top > 0$, $Q_i = Q_i^\top > 0$ *for all* $i \in \mathcal{V}$ *such that*

$$\begin{bmatrix} \mathrm{He}(A_\theta, P) & -P+\widetilde{A}_\theta^\top \mathcal{K}^\top Q \\ \star & -\delta Q - \mathrm{He}(\widetilde{\mathcal{K}}, Q) \end{bmatrix} < 0, \quad (31)$$

$$\begin{bmatrix} \mathrm{He}(A_\theta, P) & -P+\exp(\delta T_2^{\max})\, \widetilde{A}_\theta^\top \mathcal{K}^\top Q \\ \star & -\exp(\delta T_2^{\max})\, (\delta Q+\mathrm{He}(\widetilde{\mathcal{K}}, Q)) \end{bmatrix} < 0, \quad (32)$$

where $Q = \mathrm{diag}(Q_1, Q_2, \ldots, Q_N)$ *and* $T_2^{\max} = \max_{i\in\mathcal{V}} T_2^i$.

Proof First, note that if the inequality in (19) holds for all $\nu = (\nu_1, \nu_2, \ldots, \nu_N) \in [0, T_2^{\max}]^N$, it holds for all $\nu = (\nu_1, \nu_2, \ldots, \nu_N) \in \mathcal{T}$. Then, the result follows by applying Proposition 3.9. ∎

Remark 3.11: In practice, one may want to search for parameters to maximize the allowable value for $T_2^{\max}$ (often called "maximum allowable transfer interval" (MATI); see [9]) that satisfies (32). By doing so, information is allowed to be transmitted at low frequency, which, in turn, would reduce the amount of energy consumed but may take longer for the estimates to converge to the state of the plant. □

D. Nominal Design of Parameters via LMIs

The conditions in (29) and (30) discussed in the previous section guarantee exponential stability of the set $\mathcal{A}$ in (17). However, because of the existence of the nonlinear cross term $\mathcal{K}\mathcal{K}$ from the multiplication of $\mathcal{K}A_\theta$ in (29) and (30), these two conditions are not computationally tractable. In this section, we focus on a decomposition of this cross term and propose design methods in terms of LMIs that can be efficiently solved numerically. Below, we will use the fact that the matrix $A_{f\theta}$ in (16) can be written in the form $A_{f\theta} = \widetilde{A}_1\widetilde{A}_2$, where

$$\widetilde{A}_1 = \begin{bmatrix} I & 0 \\ \mathcal{K} & I \end{bmatrix}, \quad \widetilde{A}_2 = \begin{bmatrix} A_\theta & -I_{nN} \\ -\mathcal{K}H_\eta & H_\eta \end{bmatrix}. \qquad (33)$$

We have the following result with its proof in Appendix A.

Proposition 3.12: Let $0 < T_1^i \le T_2^i$ *be given for all* $i \in \mathcal{V}$. *The positive definite symmetric matrices* $P, Q \in \mathbb{R}^{nN\times nN}$, *the constants* $\delta > 0$, $\gamma \in \mathbb{R}$, *and the matrices* H_η, *and* $K_g \in \mathbb{R}^{nN\times p}$ *satisfy conditions* (29) *and* (30) *if and only if there exist matrices* $\mathcal{O}_i, \mathcal{M}_i \in \mathbb{R}^{nN\times nN}, i \in \mathbb{N}, 1 \le i \le 6$, *such that*

$$\begin{bmatrix} \mathrm{He}(\Omega_1, I_{2nN}) & \Omega_2 + \widetilde{P}_1 \\ \star & \widetilde{Y} + \mathrm{He}(\Omega_3, I_{2nN}) \end{bmatrix} < 0, \qquad (34)$$

$$\begin{bmatrix} \mathrm{He}(\Lambda_1, I_{2nN}) & \Lambda_2 + \widetilde{P}_2 \\ \star & \widetilde{Z} + \mathrm{He}(\Lambda_3, I_{2nN}) \end{bmatrix} < 0, \qquad (35)$$

where

$$\begin{aligned} \widetilde{P}_1 &= \mathrm{diag}(P, Q), \quad \widetilde{Z} = \mathrm{diag}(0_{nN}, -\delta \overline{E} Q), \\ \widetilde{P}_2 &= \mathrm{diag}(P, \overline{E} Q), \quad \widetilde{Y} = \mathrm{diag}(0_{nN}, -\delta Q), \end{aligned} \qquad (36)$$

and

$$\Omega_1 = \begin{bmatrix} -\mathcal{O}_1 + \mathcal{K}^\top \mathcal{O}_4 & -\mathcal{O}_2 + \mathcal{K}^\top \mathcal{O}_5 \\ -\mathcal{O}_4 & -\mathcal{O}_5 \end{bmatrix}$$

$$\Omega_2 = \begin{bmatrix} \mathcal{O}_1^\top A_\theta - \mathcal{O}_3 + \mathcal{K}^\top \mathcal{O}_6 - \mathcal{O}_4^\top \mathcal{K} H_\eta & -\mathcal{O}_1^\top + \mathcal{O}_4^\top H_\eta \\ \mathcal{O}_2^\top A_\theta - \mathcal{O}_6 - \mathcal{O}_5^\top \mathcal{K} H_\eta & -\mathcal{O}_2^\top + \mathcal{O}_5^\top H_\eta \end{bmatrix}$$

$$\Omega_3 = \begin{bmatrix} A_\theta^\top \mathcal{O}_3 - H_\eta^\top \mathcal{K}^\top \mathcal{O}_6 & 0 \\ -\mathcal{O}_3 + H_\eta \mathcal{O}_6 & 0 \end{bmatrix}$$

$$\Lambda_1 = \begin{bmatrix} -\mathcal{M}_1 + \mathcal{K}^\top \mathcal{M}_4 & -\mathcal{M}_2 + \mathcal{K}^\top \mathcal{M}_5 \\ -\mathcal{M}_4 & -\mathcal{M}_5 \end{bmatrix}$$

$$\Lambda_2 = \begin{bmatrix} \mathcal{M}_1^\top A_\theta - \mathcal{M}_3 + \mathcal{K}^\top \mathcal{M}_6 - \mathcal{M}_4^\top \mathcal{K} H_\eta & -\mathcal{M}_1^\top + \mathcal{M}_4^\top H_\eta \\ \mathcal{M}_2^\top A_\theta - \mathcal{M}_6 - \mathcal{M}_5^\top \mathcal{K} H_\eta & -\mathcal{M}_2^\top + \mathcal{M}_5^\top H_\eta \end{bmatrix}$$

$$\Lambda_3 = \begin{bmatrix} A_\theta^\top \mathcal{M}_3 - H_\eta^\top \mathcal{K}^\top \mathcal{M}_6 & 0 \\ -\mathcal{M}_3 + H_\eta \mathcal{M}_6 & 0 \end{bmatrix}.$$

By the transformation in Proposition 3.12, it can be seen that, in the new set of inequalities (34) and (35), the cross term $\mathcal{K}\mathcal{K}$ from the multiplication of $\mathcal{K}A_\theta$ in (29) and (30) vanishes. In fact, the conditions (34) and (35) lead to several designs by choosing the matrices $\mathcal{O}_i, \mathcal{M}_i \in \mathbb{R}^{nN\times nN}$ properly. The following result illustrates one particular design when the graph is all-to-all connected.

Corollary 3.13: Let $0 < T_1^i \le T_2^i$ *be given for all* $i \in \mathcal{V}$. *The set* $\mathcal{A}$ *is GES for* $\mathcal{H}$ *if* N *agents are connected via an all-to-all connectivity graph and there exist* $\delta > 0$, $\gamma \in \mathbb{R}$ *and positive definite symmetric matrices* $P \in \mathbb{R}^{nN\times nN}, Q_i \in \mathbb{R}^{n\times n}$ *for all* $i \in \mathcal{V}$, $\mathcal{Q} \in \mathbb{R}^{p\times nN}$, *and* $\mathcal{R} \in \mathbb{R}^{nN\times nN}$ *such that*

$$\begin{bmatrix} \mathrm{He}(\mathcal{Z}_1, I_{2nN}) & \mathcal{Z}_2 + \widetilde{P}_1 \\ \star & \widetilde{Y} + \mathrm{He}(\mathcal{Z}_3, I_{2nN}) \end{bmatrix} < 0, \qquad (37)$$

$$\begin{bmatrix} \mathrm{He}(\mathcal{Z}_1, I_{2nN}) & \mathcal{Z}_2 + \widetilde{P}_2 \\ \star & \widetilde{Z} + \mathrm{He}(\mathcal{Z}_3, I_{2nN}) \end{bmatrix} < 0, \qquad (38)$$

where $Q = \mathrm{diag}(Q_1, \ldots, Q_N)$, *and* $\widetilde{P}_1, \widetilde{P}_2, \widetilde{Y}, \widetilde{Z}$ *are in* (36), *and*

$$\mathcal{Z}_1 = \begin{bmatrix} -\mathcal{R} + H_g^\top \mathcal{Q} + \widetilde{\mathcal{L}}^\top \mathcal{R} & H_g^\top \mathcal{Q} + \widetilde{\mathcal{L}}^\top \mathcal{R} \\ -\mathcal{R} & -\mathcal{R} \end{bmatrix},$$

$$\mathcal{Z}_2 = \begin{bmatrix} \mathcal{R}^\top \widetilde{A} + \mathrm{He}(\mathcal{Q}, H_g) + \mathrm{He}(\mathcal{R}, \widetilde{\mathcal{L}}) - \widetilde{\mathcal{R}} & -\mathcal{R}^\top (I - H_\eta) \\ -\widetilde{\mathcal{R}} & \mathcal{R}^\top H_\eta \end{bmatrix},$$

$$\mathcal{Z}_3 = \begin{bmatrix} \widetilde{A}^\top \mathcal{R} + H_g^\top \mathcal{Q} + \widetilde{\mathcal{L}}^\top \mathcal{R} - H_\eta H_g^\top \mathcal{Q} - H_\eta \widetilde{\mathcal{L}}^\top \mathcal{R} & 0 \\ -\mathcal{R} + H_\eta \mathcal{R} & 0 \end{bmatrix},$$

$\widetilde{\mathcal{R}} = \mathcal{R} + \mathcal{Q}^\top H_g H_\eta + \mathcal{R}^\top \widetilde{\mathcal{L}} H_\eta$, $\widetilde{A} = I_N \otimes A$, $\widetilde{\mathcal{L}} = \gamma \mathcal{L} \otimes I_n$, $K_g^\top = \mathcal{Q}\mathcal{R}^{-1}$.

Proof When the graph is all-to-all connected, $(K_g H_g) * (I_N + \mathcal{G}) = K_g H_g$. By choosing $\mathcal{O}_1 = \mathcal{O}_3 = \mathcal{O}_4 = \mathcal{O}_5 = \mathcal{O}_6 = \mathcal{R}$

and $\mathcal{M}_1 = \mathcal{M}_3 = \mathcal{M}_4 = \mathcal{M}_5 = \mathcal{M}_6 = \mathcal{R}$, and $K_g^\top = \mathcal{R}^{-1}\mathcal{Q}$, $\mathcal{O}_2 = \mathcal{M}_2 = 0$, conditions (34) and (35) reduce to (37) and (38), respectively. ■

Remark 3.14: The design for the case when N agents are connected via a generic graph can be treated similarly. The inequalities in (31) and (32) can be transformed into LMIs similarly as done in Proposition 3.12 and Corollary 3.13. □

IV. Robustness Properties and Design

In this section, we consider the effect of general perturbations and unmodeled dynamics on the plant. In such a setting, the perturbed plant is given in (1) and the measurement taken by agent i are given by (2), where the functions $\delta : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ and $\varphi_i : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^{p_i}$ are unknown functions that capture unmodeled dynamics, disturbances and measurement noise. Moreover, the values of $\hat{x}_k$ and y_k received from its k-th neighbor ($k \in \mathcal{N}(i)$) may also be affected by channel noise, i.e., the i-th agent receives $\tilde{x}_k^i(t_s) = \hat{x}_k(t_s) + c_i^x(t_s)$ and $\tilde{y}_k^i(t_s) = y_k(t_s) + c_i^y(t_s)$ at event time t_s, where $c_i = (c_i^x, c_i^y) : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n+p_i}$ models channel noise. Furthermore, the timers triggering communication events at each node may be governed by the perturbation of (10) given by

$$\dot{\tau}_i = -1 + \varsigma_i \qquad \tau_i \in [0, T_2^i + \vartheta_2^i], \tag{39a}$$

$$\tau_i^+ \in [T_1^i + \vartheta_1^i, T_2^i + \vartheta_2^i] \qquad \tau_i = 0. \tag{39b}$$

where $\varsigma_i \in (-\infty, 1)$ is a constant modeling a possible skew on the timer dynamics for τ_i, while $\vartheta_i = (\vartheta_1^i, \vartheta_2^i)$ is a constant that satisfies $0 < T_1^i + \vartheta_1^i \leq T_2^i + \vartheta_2^i$ and models perturbations on the known nominal values of T_1^i and T_2^i.

Following (11)-(12), the perturbed versions of the dynamics of the proposed observer are

$$\dot{\eta}_i = h_i \eta_i \tag{40}$$

when $\tau_i \in [0, T_2^i + \vartheta_i]$, and, at every event time,

$$\eta_i^+ = K_{ii} H_i e_i + \sum_{k \in \mathcal{V}} g_{ik}(K_{ik} H_k e_k + \gamma e_i - \gamma e_k) + \zeta_i \tag{41}$$

when $\tau_i = 0$, where

$$\zeta_i(x,t) = -K_{ii}\varphi_i(x,t) + \sum_{k \in \mathcal{V}} g_{ik} K_{ik}(H_k c_i^x - c_i^y - \varphi_k(x,t)) - \gamma \sum_{k \in \mathcal{V}} g_{ik} c_i^x,$$

where, for simplicity, we drop the arguments of some of the perturbations. Then, following the definition of θ_i in (13), which without perturbations is given by

$$\begin{aligned} \theta_i &= K_{ii} y_i^e + \sum_{k \in \mathcal{V}} g_{ik} K_{ik} y_k^e + \gamma \sum_{k \in \mathcal{V}} g_{ik}(e_i - e_k) - \eta_i \\ &= K_{ii} H_i e_i + \sum_{k \in \mathcal{V}} g_{ik}(K_{ik} H_k e_k + \gamma e_i - \gamma e_k) - \eta_i, \end{aligned}$$

the resulting perturbed hybrid system $\widetilde{\mathcal{H}} = (\widetilde{C}, \widetilde{f}, \widetilde{D}, \widetilde{G})$ with state $\chi = (\sigma, \tau) = (e, \theta, \tau)$ is given by

$$\widetilde{f}(\chi) := f(\chi) + (-\mathbf{1}_N \otimes \delta(x,t), -\widetilde{\mathcal{K}}(\mathbf{1}_N \otimes \delta(x,t)), \varsigma)$$

when $\chi \in \widetilde{C} := \mathbb{R}^{nN} \times \mathbb{R}^{nN} \times \widetilde{\mathcal{T}}$, where

$$\widetilde{\mathcal{T}} := [0, T_2^1 + \vartheta_2^1] \times [0, T_2^2 + \vartheta_2^2] \times \cdots \times [0, T_2^N + \vartheta_2^N],$$

and $\varsigma = (\varsigma_1, \varsigma_2, \ldots, \varsigma_N)$. Moreover, when $\chi \in \widetilde{D} := \bigcup_{i \in \mathcal{V}} \widetilde{D}_i$, $\widetilde{D}_i = \{\chi \in \widetilde{C} : \tau_i = 0\}$,

$$\widetilde{G}(\chi, \zeta) := \{\widetilde{G}_i(\chi, \zeta) : \chi \in \widetilde{D}_i, i \in \mathcal{V}\}$$

and

$$\widetilde{G}_i(\chi, \zeta) := \begin{bmatrix} e \\ (\theta_1, \theta_2, \ldots, \theta_{i-1}, -\zeta_i(x,t), \theta_{i+1}, \ldots, \theta_N) \\ (\tau_1, \tau_2, \ldots, \tau_{i-1}, \Xi_i, \tau_{i+1}, \ldots, \tau_N) \end{bmatrix},$$

where $\Xi_i = [T_1^i + \vartheta_1^i, T_2^i + \vartheta_2^i]$,

$$\begin{aligned} \zeta(x,t) &= (\zeta_1(x,t), \zeta_2(x,t) \ldots, \zeta_N(x,t)), \\ \varphi(x,t) &= (\varphi_1(x,t), \varphi_2(x,t), \ldots, \varphi_N(x,t)), \end{aligned}$$

and

$$\begin{aligned} \zeta(x,t) &= K_m \varphi(x,t) + K_c c, \\ K_m &= -K_g * (I + \mathcal{G}), \\ K_c &= \left[(K_g H_g - \gamma I) * \mathcal{G} \quad -K_g * \mathcal{G} \right], \end{aligned} \tag{42}$$

$c = (c^x, c^y)$, $c^x = (c_1^x, c_2^x, \ldots, c_N^x)$, and $c^y = (c_1^y, c_2^y, \ldots, c_N^y)$.[4]

A. Robustness properties with respect to small perturbations

Motivated by Remark 3.4, in this section, we focus on the generic robustness property to small perturbations. Below, given a set $S_x \subset \mathbb{R}^n$, $\mathcal{R}^x(S_x)$ denotes the infinite horizon reachable set of $\dot{x} = Ax + \delta(x,t)$ from S_x.

Theorem 4.1: Let $0 < T_1^i \leq T_2^i$ be given for all $i \in \mathcal{V}$. Suppose the parameters K_g, γ, h_i for all $i \in \mathcal{V}$ are chosen such that the set $\mathcal{A}$ is GES for the unperturbed hybrid system $\mathcal{H}$. Then, there exists $\beta \in \mathbb{KL}$ such that for every compact sets $S_e \subset \mathcal{X}$ and $S_x \subset \mathbb{R}^n$, and every $\varepsilon > 0$, there exists $\rho^\star \in (0, \infty)$ such that if

$$\max\{\overline{\rho}_1, \overline{\rho}_2, |c|_\infty, |\varsigma|_\infty\} \leq \rho^\star, \tag{43}$$

where

$$\begin{aligned} \overline{\rho}_1 &= \sup_{(x,t) \in \mathcal{R}^x(S_x) \times \mathbb{R}_{\geq 0}} |\varphi(x,t)|, \\ \overline{\rho}_2 &= \sup_{(x,t) \in \mathcal{R}^x(S_x) \times \mathbb{R}_{\geq 0}} |\delta(x,t)|, \end{aligned}$$

then, every $\phi \in \mathcal{S}_{\widetilde{\mathcal{H}}}(S_e)$ under the effect of solutions to (1) *from S_x satisfies*

$$|\phi(t,j)|_{\mathcal{A}} \leq \beta(|\phi(0,0)|_{\mathcal{A}}, t+j) + \varepsilon$$

for all $(t,j) \in \operatorname{dom} \phi$.

Proof Consider the hybrid system $\mathcal{H}$ and a continuous function $\rho : \mathbb{R}^{nN} \times \mathbb{R}^{nN} \times \mathcal{T} \to \mathbb{R}_{\geq 0}$, the ρ-perturbation of $\mathcal{H}$, denoted $\mathcal{H}_\rho$, is the hybrid system

$$\begin{cases} \chi \in C_\rho & \dot{\chi} \in F_\rho(\chi) \\ \chi \in D_\rho & \chi^+ \in G_\rho(\chi) \end{cases}$$

[4]Note that the Khatri-Rao product $-K_g * (I + \mathcal{G})$ is such that the (i,k)-th entry K_{ik} of K_g is multiplied by the (i,k)-th scalar entry of the matrix $I + \mathcal{G}$ for all $i, k \in \mathcal{V}$.

where

$$C_\rho = \{\chi \in \mathcal{X} : (\chi + \rho(\chi)\mathbb{B}) \cap C \neq \emptyset\},$$
$$F_\rho(\chi) = \overline{\text{con}} f((\chi + \rho(\chi)\mathbb{B}) \cap C) + \rho(\chi)\mathbb{B} \quad \forall \chi \in \mathcal{X},$$
$$D_\rho = \{\chi \in \mathcal{X} : (\chi + \rho(\chi)\mathbb{B}) \cap D \neq \emptyset\},$$
$$G_\rho(\chi) = \{v \in \mathcal{X} : v \in g + \rho(g)\mathbb{B}, g \in G((\chi + \rho(\chi)\mathbb{B}) \cap D)\} \quad \forall \chi \in \mathcal{X}.$$

Since the set $\mathcal{A}$ is GES for $\mathcal{H}$, it is also UGAS for $\mathcal{H}$.[5] Since the hybrid system $\mathcal{H}$ satisfies the hybrid basic conditions, by [21, Theorem 6.8], $\mathcal{H}_\rho$ is nominally well-posed and, moreover, by [21, Proposition 6.28] is well-posed. Then, [21, Theorem 7.20] implies that $\mathcal{A}$ is semiglobally practically robustly $\mathbb{KL}$ asymptotically stable for $\mathcal{H}$. Namely, for every compact $S_e \subset \mathcal{X}$ and every $\varepsilon > 0$, there exists $\widetilde{\rho} \in (0,1)$ such that every $\phi \in \mathcal{S}_{\mathcal{H}_{\widetilde{\rho}\rho}}(S_e)$ satisfies $|\phi(t,j)|_\mathcal{A} \leq \beta(|\phi(0,0)|_\mathcal{A}, t+j) + \varepsilon$ for all $(t,j) \in \operatorname{dom}\phi$. Then, the result follows by picking $\rho^\star > 0$ such that

$$\max\left\{1, |K_m| + |K_c|, |\widetilde{\mathcal{K}}|\right\} \rho^\star \leq \widetilde{\rho}$$

and relating solutions to $\widetilde{\mathcal{H}}$ and solutions to $\mathcal{H}_{\widetilde{\rho}\rho}$. ■

In simple terms, the above result establishes that the solutions to the hybrid system ($\widetilde{\mathcal{H}}$) with small enough perturbations do not differ much from those of the unperturbed system ($\mathcal{H}$).

Remark 4.2: If $\delta(x,t) = \delta(t)$ for all $t \in \mathbb{R}_{\geq 0}$, the arguments in Theorem 4.1 apply without using the reachable set $\mathcal{R}^x(S_x)$. A typical form of $\delta(x,t)$ is $\Delta A x$ with $\Delta A \in \mathbb{R}^{n\times n}$, which captures linear unmodeled dynamics. Furthermore, rather than requiring boundedness of $\delta(x,t)$ over the reachable set from S_x, one could state the $\mathbb{KL}$-bound in Theorem 4.1 under boundedness of the solutions to the plant (1) that correspond to the solutions to the perturbed system $\widetilde{\mathcal{H}}$, which are in error coordinates. □

Remark 4.3: In general, though not pursued here, the result in Theorem 4.1 is also applicable to the case when the measurement and communication noise have specific statistical properties, such as Gaussian noise. Furthermore, if the proposed observer includes an estimate of the perturbation δ, which is possible by adding $\hat{\delta}(\hat{x})$ to the right-hand side of equation (4), then under typical assumptions in the literature (see, e.g., those in [28, Lemma 2.1]), we can bound the difference between δ and $\hat{\delta}$ by $|\delta(x) - \hat{\delta}(x + x^\star)| \leq L|x^\star| + L_0$ for all $x, x^\star \in \mathbb{R}^n$, where $L, L_0 \in \mathbb{R}_{\geq 0}$. Moreover, for such robustified observer, a result guaranteeing asymptotic stability of the zero estimation error follows similar to the proof in Theorem 3.6. □

B. Robustness to measurement and communication noise

In this section, we consider the hybrid system $\widetilde{\mathcal{H}}$ in Section IV where only the communication noise and channel noise are present. Namely, $\delta \equiv 0$, $\varsigma_i \equiv 0$, $\vartheta_i \equiv 0$, for all $i \in \mathcal{V}$, and the function φ_i reduces to a function m_i, i.e., $m_i(t) := \varphi_i(x,t)$ for all $t \in \mathbb{R}_{\geq 0}$. Then, we have the following result.

[5]We consider the definition of uniform global asymptotic stability (UGAS) for a set given in [21, Definition 3.6].

Theorem 4.4: Let $0 < T_1^i \leq T_2^i$ be given for all $i \in \mathcal{V}$. Suppose there exist matrices $K_g \in \mathbb{R}^{nN\times p}$ and $P \in \mathbb{R}^{2nN\times 2nN}$ such that $P = P^\top > 0$ and condition (19) holds. Then, the set $\mathcal{A}$ is ISS with respect to measurement noise and communication noise, in particular, for each $\phi \in \mathcal{S}_{\widetilde{\mathcal{H}}}$ and for any $(t,j) \in \operatorname{dom}\phi$,

$$|\phi(t,j)|_\mathcal{A} \leq \max\left\{\sqrt{2}\kappa \exp\left(-\alpha(t+j)\right)|\phi(0,0)|_\mathcal{A}, \widetilde{\gamma}_m|m|_\infty, \widetilde{\gamma}_c|c|_\infty\right\},$$

where α and κ are defined in (20), α_1 and α_2 are given in (21) and (22) respectively, $\widetilde{\beta} = \frac{-\overline{\lambda}(\mathcal{M})}{\alpha_2}$ and

$$E = \frac{N\exp\left(\widetilde{\beta}T_1^{\min}\right)}{\exp\left(\widetilde{\beta}T_1^{\min}\right) - 1}, \quad \widetilde{\gamma}_m = 2\sqrt{\frac{E}{\alpha_1}\overline{\lambda}(Q)}|K_m|,$$
$$\widetilde{\gamma}_c = 2\sqrt{\frac{E}{\alpha_1}\overline{\lambda}(Q)}|K_c|$$

with K_m and K_c as in (42) and $\mathcal{M}$ as in (19).

Proof Consider the Lyapunov function candidate $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$ given by (18). It follows that V satisfies (23) for all $\chi \in \widetilde{C}$, where α_1 and α_2 are given by (21) and (22). Then, as in the proof of Theorem 3.6, during flows, by using (19), we have

$$\langle \nabla V(\chi), \widetilde{f}(\chi)\rangle \leq -\beta|\chi|_\mathcal{A}^2, \quad \forall \chi \in \widetilde{C},$$

where $\beta = -\overline{\lambda}(\mathcal{M})$ and $\mathcal{M}$ is given in (19). Moreover, for each $\chi = (e,\theta,\tau) \in \widetilde{D}$ and for each $g \in \widetilde{G}(\chi)$, there exists at least one component of τ, say, the i-th component, such that $\tau_i = 0$. Then, at jumps we have

$$\begin{aligned} V(g) - V(\chi) &\leq -\theta_i^\top Q_i \theta_i + \zeta_i^\top Q_i \zeta_i \\ &\leq \zeta^\top Q \zeta. \end{aligned}$$

Now, pick $\phi \in \mathcal{S}_{\widetilde{\mathcal{H}}}$. By item 3 in Lemma 3.5, we have $t_{(j+1)N} - t_{jN} \in [T_1^{\min}, T_2^{\max}]$ for all $j \in \mathbb{Z}_{\geq 1}$ such that $(t_{(j+1)N}, (j+1)N), (t_{jN}, jN) \in \operatorname{dom}\phi$. Then, from Proposition B.1 with $a = \frac{\beta}{\alpha_2}$, and $\widetilde{T}_2 = T_2^{\max}$, we have that

$$V(\phi(t,j)) \leq \exp\left(-\frac{\beta}{\alpha_2}t\right)V(\phi(0,0)) + N\overline{\lambda}(Q)|\zeta|_\infty^2 \sum_{s=0}^{\tilde{j}} \exp\left(-\frac{\beta}{\alpha_2}sT_1^{\min}\right),$$

where $\tilde{j} = \left\lfloor \frac{j}{N} \right\rfloor$. Note that for each $n \in \mathbb{N}$

$$\sum_{s=0}^{n} \exp\left(-\frac{\beta}{\alpha_2}sT_1^{\min}\right) = \frac{\exp\left(\frac{\beta}{\alpha_2}T_1^{\min}\right) - \exp\left(-\frac{\beta}{\alpha_2}nT_1^{\min}\right)}{\exp\left(\frac{\beta}{\alpha_2}T_1^{\min}\right) - 1}.$$

Then, by the definition of E and with (23), we have that

$$|\phi(t,j)|_\mathcal{A}^2 \leq \frac{\alpha_2}{\alpha_1}\exp\left(-\frac{\beta}{\alpha_2}t\right)|\phi(0,0)|_\mathcal{A}^2 + \frac{1}{\alpha_1}\overline{\lambda}(Q)E|\zeta|_\infty^2.$$

Since $|\zeta|_\infty^2 \leq 2|K_m|^2|m|_\infty^2 + 2|K_c|^2|c|_\infty^2$, it follows that, for each $(t,j) \in \operatorname{dom}\phi$,

$$|\phi(t,j)|_{\mathcal{A}} \leq \max\left\{\sqrt{2\frac{\alpha_2}{\alpha_1}}\exp\left(-\frac{1}{2}\frac{\beta}{\alpha_2}t\right)|\phi(0,0)|_{\mathcal{A}},\right.$$
$$\left. 2\sqrt{\frac{E}{\alpha_1}\overline{\lambda}(Q)}|K_m||m|_\infty, 2\sqrt{\frac{E}{\alpha_1}\overline{\lambda}(Q)}|K_c||c|_\infty\right\}.$$

We can conclude the proof by following similar arguments as in the proof of Theorem 3.6. ■

C. Robustness properties with respect to T_1^i and T_2^i perturbations

In a real-world setting, the bounds on update communication times T_1^i and T_2^i may not be explicitly known and may be affected by a perturbation $\vartheta_i = (\vartheta_1^i, \vartheta_2^i)$, as modeled in (39). In this section, we consider the case of $\widetilde{\mathcal{H}}$ with $\delta = 0$, $\varphi_i = 0$, $\varsigma_i = 0$, and with ϑ_i being a constant perturbation parameter. We have the following results for this particular case. Its proof follows from an application of Proposition 3.6.

Corollary 4.5: Let $0 < T_1^i \leq T_2^i$ be given for each $i \in \mathcal{V}$. Suppose N agents are connected via a digraph $\Gamma = (\mathcal{V}, \mathcal{E}, \mathcal{G})$. The set $\{0_{nN}\} \times \{0_{nN}\} \times \widetilde{\mathcal{T}}$ is GES for the hybrid system $\widetilde{\mathcal{H}}$ if there exists $\delta > 0$ and matrices $K_g \in \mathbb{R}^{nN\times p}, P = P^\top \in \mathbb{R}^{nN\times nN}, Q_i = Q_i^\top \in \mathbb{R}^{n\times n}$ for each $i \in \mathcal{V}$ such that (19) holds for each $\nu = (\nu_1, \nu_2, \ldots, \nu_N) \in \widetilde{\mathcal{T}}$.

From Corollary 4.5, in particular, we have that the global exponential stability property of the nominal system $\mathcal{H}$ is robust to perturbations on T_2^i, which, in turn, implies robustness to perturbations on $T_2^{\max}$. Employing the idea used in Proposition 3.10, which, in the nominal case, allows to only check the pertaining inequalities at $T_2^{\max}$, we propose the following result to check the condition in Corollary 4.5 over the larger range of values due to perturbations. Its proof follows from an application of Proposition 3.10.

Corollary 4.6: Let $0 < T_1^i \leq T_2^i$ be given for all $i \in \mathcal{V}$. Suppose the assumptions in Proposition 3.9 hold. The set $\{0_{nN}\} \times \{0_{nN}\} \times \widetilde{\mathcal{T}}$ is GES for the hybrid system $\widetilde{\mathcal{H}}$ if there exists $\delta > 0$ and matrices $K_g \in \mathbb{R}^{nN\times p}, P = P^\top \in \mathbb{R}^{nN\times nN}, Q_i = Q_i^\top \in \mathbb{R}^{n\times n}$ for each $i \in \mathcal{V}$ and

$$\begin{bmatrix} \operatorname{He}(A_\theta, P) & -P + \exp(\delta\nu)\widetilde{A}_\theta^\top\mathcal{K}^\top Q \\ \star & -\exp(\delta\nu)(\delta\widetilde{Q} + \operatorname{He}(\widetilde{\mathcal{K}}, Q)) \end{bmatrix} < 0 \qquad (44)$$

for $\nu = T_2^{\max} + \max_{i\in\mathcal{V}}\vartheta_2^i$, where $Q = \operatorname{diag}(Q_1, Q_2, \ldots, Q_N)$, A_θ, $\widetilde{A}_\theta$ and $\widetilde{\mathcal{K}}$ are given in (16).

D. Robustness to information dropout

In the proposed model, at each communication event, a data packet containing the information $(y_k, \hat{x}_k)$ for all $k \in \mathcal{N}(i)$ is received by agent i. In this section, we study the robustness of the exponential stability of the set $\mathcal{A}$ to information dropout, i.e., the situation when some of such data packets are lost. We assume the following properties.

Assumption 4.7: A Bernoulli random variable b_k indicates whether the packet with index k is successfully received. If it is received successfully, then $b_k = 1$; otherwise, $b_k = 0$. For each $k \in \mathbb{Z}_{\geq 1}$, b_k is I.I.D. with probability distribution $\mathcal{P}(b_k = 1) = d_r$ and $P(b_k = 0) = 1 - d_r$, where $d_r \in (0, 1)$. Note that this model is one of the simplest and often used to model for information dropouts in large-scale networks, see, e.g., [29, Section 2]. A simulation of the proposed observer subject to this information dropout model is discussed in Example 5.6.

On the other hand, if further information about dropouts is available, for example, if one knows that for the overall network, at most $k^\star$ packets could be dropped consecutively, then, we can use the following result to robustify the design.

Corollary 4.8: Let $0 < T_1^i \leq T_2^i$ be given for all $i \in \mathcal{V}$. Moreover, suppose at most $k^\star$ packets could be dropped consecutively. Then, the set $\mathcal{A}$ is GES if there exists $\delta > 0$ and matrices $K_g \in \mathbb{R}^{nN\times p}, P = P^\top \in \mathbb{R}^{nN\times nN}, Q_i = Q_i^\top \in \mathbb{R}^{n\times n}$ for each $i \in \mathcal{V}$ such that if there exists $\delta > 0$ and matrices $K_g \in \mathbb{R}^{nN\times p}$, $P \in \mathbb{R}^{nN\times nN}, Q_i \in \mathbb{R}^{n\times n}$ satisfying $P = P^\top > 0$, $Q_i = Q_i^\top > 0$ for all $i \in \mathcal{V}$ such that

$$\begin{bmatrix} \operatorname{He}(A_\theta, P) & -P + \widetilde{A}_\theta^\top\mathcal{K}^\top Q \\ \star & -\delta Q - \operatorname{He}(\widetilde{\mathcal{K}}, Q) \end{bmatrix} < 0, \qquad (45)$$

$$\begin{bmatrix} \operatorname{He}(A_\theta, P) & -P + \exp(\delta\nu)\widetilde{A}_\theta^\top\mathcal{K}^\top Q \\ \star & -\exp(\delta\nu)(\delta Q + \operatorname{He}(\widetilde{\mathcal{K}}, Q)) \end{bmatrix} < 0, \qquad (46)$$

where $\nu = (k^\star + 1)T_2^{\max}$, $Q = \operatorname{diag}(Q_1, \ldots, Q_N)$, $T_2^{\max} = \max_{i\in\mathcal{V}} T_2^i$, A_θ, $\widetilde{A}_\theta$ and $\widetilde{\mathcal{K}}$ are given in (16).

Proof Under the assumption that the maximum number of consecutive packets dropouts are bounded by $k^\star$, it follows that the resulting maximum time interval between two successfully received data packets are $T_2^{\max} + k^\star T_2^{\max}$. Then, the proof follows from applying Proposition 3.10. ■

V. Numerical Examples

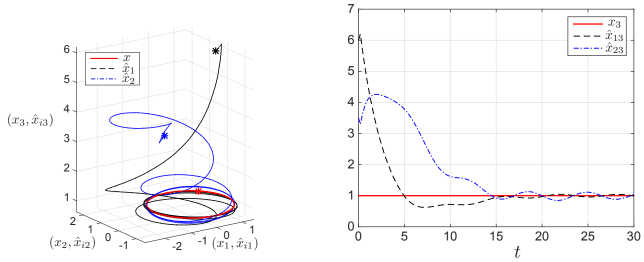
A. Examples for the Nominal Case

In this section, we exemplify the main results developed for the nominal case. Namely, we consider multiple examples showcasing the key attributes of our estimation algorithm, in particular, the fact that the estimates converge exponentially to the state of the plant when communication is asynchronous under general graphs, and, potentially, when the full state is not measurable at each agent. First, we showcase an example pertaining to two agents for which, if communication between them is not possible, they cannot individually estimate the state of the plant. In that example, when information is shared between them, our observer guarantees that the state of each agent converges exponentially to the state of the plant.

Example 5.1: Consider a plant with state $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ that has oscillatory dynamics for (x_1, x_2) and trivial dynamics for x_3, in particular,

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \qquad (47)$$

Consider the case of two agents that are all-to-all connected, i.e., $\mathcal{G} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and measure x according to (2) with $H_1 = [0 \quad 0 \quad 1], H_2 = [1 \quad 1 \quad 0]$. Since the pairs (H_1, A)



(a) Phase plots of $x, \hat{x}_1, \hat{x}_2$ from initial conditions denoted $*$.

(b) Estimation error for the third state component.

Fig. 1. Phase portrait and estimation errors for the third state component for the system in Example 5.1, where $x = (x_1, x_2, x_3)$, $\hat{x}_i = (\hat{x}_{i1}, \hat{x}_{i2}, \hat{x}_{i3})$, $e_{ik} = \hat{x}_{ik} - x_k$, for $i \in \{1, 2\}$, $k \in \{1, 2, 3\}$. Initial conditions are $x(0, 0) = (1, 1, 1)$, $\hat{x}_1(0, 0) = (1, 0, 6)$, $\eta_1(0, 0) = (1, 1, 1)$, $\hat{x}_2(0, 0) = (-1, 0, 3.5)$, $\eta_2(0, 0) = (-1, -1, -1)$, $\tau_1(0, 0) = 0.2$ and $\tau_2(0, 0) = 0$.

and (H_2, A) are not detectable, neither agent can estimate the full state of the plant by running an observer that does not use information from the other agent. However, when communication between agents is allowed, our observer is able to reconstruct x. In fact, given $T_1 = 0.2$ and $T_2 = 0.4$, by solving conditions in (29) and (30), we obtain the following parameters: $K_{11} = \begin{bmatrix} -0.5 & -0.2 & -0.1 \end{bmatrix}^\top$, $K_{12} = \begin{bmatrix} -0.2 & -0.2 & -0.5 \end{bmatrix}^\top$, $K_{21} = \begin{bmatrix} 0.2 & 0.3 & 0.3 \end{bmatrix}^\top$, $K_{22} = \begin{bmatrix} -0.1 & -0.5 & 0.2 \end{bmatrix}^\top$, $h_1 = h_2 = 0$, $\delta = 10$, and gain $\gamma = -0.4$. The simulation shown in Figure 1 indicates that the estimates $\hat{x}_1$ and $\hat{x}_2$ converge to x exponentially. Figure 1(a) shows the trajectory of $x = (x_1, x_2, x_3)$, and the observer states $\hat{x}_1 = (\hat{x}_{11}, \hat{x}_{12}, \hat{x}_{13})$, $\hat{x}_2 = (\hat{x}_{21}, \hat{x}_{22}, \hat{x}_{23})$. Figure 1(b) shows their third components, denoted x_3, $\hat{x}_{13}$ and $\hat{x}_{23}$. [6] Note that even though the data in this example would satisfy the conditions in [**?**], in our setting, the information is arriving at different time instances, which makes the reconstruction of the state not possible with the results therein. △

Unlike the previous example, the next example considers the case when the pair $(H_i + H_k, A)$ for each $(i, k) \in \mathcal{E}$ may not be observable. Due to the consensus terms, when information is shared between them, our observer guarantees that the state of each agent converges exponentially fast to the state of the plant.

Example 5.2: Consider a plant with state $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ that has unstable dynamics, in particular,

$$A = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}. \tag{48}$$

Consider the case of three agents that are connected via a graph with

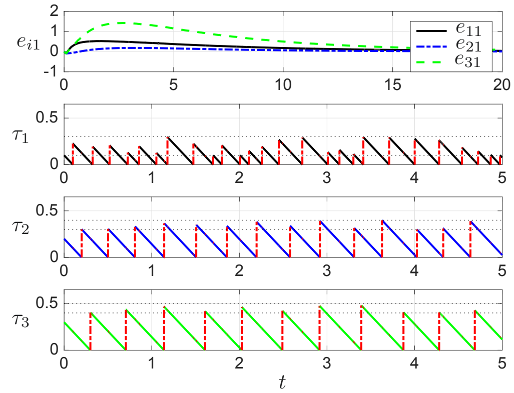
$$\mathcal{G} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

and measure x according to (2) with $H_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, $H_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$, $H_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$. Since, for each $i \in \mathcal{V}$, the (H_i, A) pair is not detectable, no single agent can estimate the full state of the plant by running an observer that does not use information from the other agent. Note that, even though agent 1 can access both y_1 and y_2 when $\tau_1 = 0$, the combination of measurements would not allow for local state reconstruction since $(H_1 + H_2, A)$ is also not detectable (a similar argument applies for the other agents). However, when communication between agents is allowed over the network with adjacency matrix $\mathcal{G}$, each agent would be able to reconstruct x. In fact, given $T_1^2 = 0.3$, $T_2^2 = 0.6$, and $T_3^2 = 0.5$, by solving conditions in (29) and (30), we obtain the following parameters[7]:

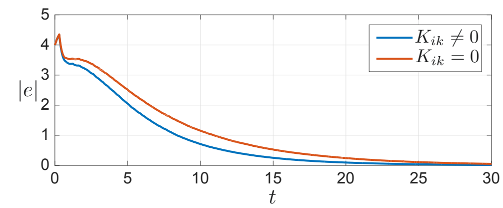
$$K_{11} = \begin{bmatrix} -2.5 & 0 & 0 \end{bmatrix}^\top, \quad K_{12} = \begin{bmatrix} 0 & -0.2 & 0 \end{bmatrix}^\top,$$
$$K_{22} = \begin{bmatrix} 0 & -3 & 0 \end{bmatrix}^\top, \quad K_{23} = \begin{bmatrix} 0.2 & -0.3 & -0.4 \end{bmatrix}^\top,$$
$$K_{31} = \begin{bmatrix} -0.2 & 0 & -0.3 \end{bmatrix}^\top, \quad K_{33} = \begin{bmatrix} 0 & 0 & -2.5 \end{bmatrix}^\top,$$

$h_1 = h_2 = h_3 = -2.1$, $\delta = 4$, and gain $\gamma = -0.7$. The simulation shown in Figure 2(a) indicates that the estimates $\hat{x}_i$'s converge to x exponentially. Furthermore, note that our observer exploits the parameters h_i to update the estimates in between events.

More interestingly, in this case, conditions (29) and (30) (and hence, condition (19)) can be satisfied when y_k is not communicated between agents; namely, $K_{ik} = 0$ for each $k \in \mathcal{N}(i)$ and each $i \in \mathcal{V}$. It can be verified that the same set of parameters with $K_{12} = K_{23} = K_{31} = 0$ also satisfies



(a) The first component of the estimation error for each agent (top) and timer states trajectories (bottom three). The red dashed lines indicate the jumps of τ_i.

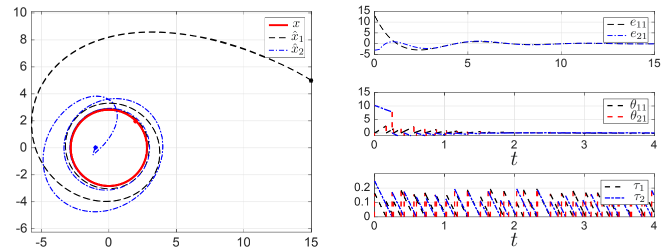


(b) The estimation error $|e|$ for the unstable three agent system as in Example 5.2.

Fig. 2. Numerical simulation over flow time for the cases of K_{ik} in Example 5.2, for $i, k \in \{1, 2, 3\}$, $i \neq k$. Note that the update times, as shown by τ_i, occur at different intervals (as indicated by the dashed black lines). Initial conditions are $x(0, 0) = (1, 1, 1)$, $\hat{x}_1(0, 0) = (1, 1, 1)$, $\hat{x}_2(0, 0) = (-1, 1, -1)$, $x_3(0, 0) = (-1, -1, -1)$, $\eta_1(0, 0) = (1, 1, 1)$, $\eta_2(0, 0) = (-1, -1, -1)$, $x_3(0, 0) = (-1, 1, -1)$, $\tau_1(0, 0) = 0$, $\tau_2(0, 0) = 0.2$ and $\tau_3(0, 0) = 0.3$.

[6] Code at https://github.com/HybridSystemsLab/ObsSyncTimes3rdAsyn.

[7] Code at https://github.com/HybridSystemsLab/ObsAyncTimesNoObsNeig.



(a) Phase plots of $x, \hat{x}_1, \hat{x}_2$ from initial conditions that are denoted by $*$.

(b) Estimation error and timer states, where $x = (x_1, x_2)$, $\hat{x}_i = (\hat{x}_{i1}, \hat{x}_{i2})$, $e_{i1} = \hat{x}_{i1} - x_1$, $e_{i2} = \hat{x}_{i2} - x_2$, for $i \in \{1, 2\}$. The red dashed lines indicate the jumps of θ_{i1} and τ_i.

Fig. 3. Phase plots of $x, \hat{x}_1$ and $\hat{x}_2$ as well as the first component of the estimation errors e_1 and e_2, the first component of the θ_i coordinates and the timer state τ_i for two all-to-all connected agents using the observer in Example 5.3. Initial conditions are $x(0,0) = (2,2)$, $\hat{x}_1(0,0) = (15,5)$, $\hat{x}_2(0,0) = (-1,0)$, $\eta_1(0,0) = (1,1)$, $\eta_2(0,0) = (-1,-1)$, $\tau_1(0,0) = 0.1$ and $\tau_2(0,0) = 0.25$.

the conditions in (29) and (30). A simulation of this scenario compared with the case when K_{ik} may not be zero for $i \neq k$ (as in the previous example) is in Figure 2(b).

Next, we illustrate Corollary 3.13. We consider the above system when the graph is all-to-all connected. Given that $T_2^i = 0.3$ for each $i \in \mathcal{V}$, it can be verified that the linear matrix inequalities given by (37) and (38) can be solved by choosing $h_1 = h_2 = 0.2$ and $\gamma = -0.4$, with the resulting gain matrix $K_g^\top$ being

$$-\begin{bmatrix} 0.68 & 0 & 0 & 1.02 & 0 & 01.02 & 0 & 0 & 0 \\ 0 & 1.02 & 0 & 0 & 0.68 & 0 & 0 & 1.02 & 0 \\ 0 & 0 & 1.02 & 0 & 0 & 1.02 & 0 & 0 & 0.68 \end{bmatrix}$$

△

The following example illustrates the result in Proposition 3.9.

Example 5.3: Consider the case of two agents that are all-to-all connected, namely, $\mathcal{G} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Let the measurements of x at agent 1 be $y_1 = H_1 x, \quad H_1 = [1 \quad 0]$ and the measurements at agent 2 be $y_2 = H_2 x, \quad H_2 = [0 \quad 1]$. By solving inequalities[8] (29) and (30) in Proposition 3.9, we obtain the following parameters: $K_{11} = \begin{bmatrix} -0.5 & -0.2 \end{bmatrix}^\top$, $K_{12} = \begin{bmatrix} -0.2 & -0.2 \end{bmatrix}^\top$, $K_{21} = \begin{bmatrix} 0.2 & 0.3 \end{bmatrix}^\top$, $K_{22} = \begin{bmatrix} -0.1 & -0.5 \end{bmatrix}^\top$, with $h_1 = h_2 = 0$, $\gamma = -0.1$, $T_1^1 = T_1^2 = 0.1$, $T_2^1 = T_2^2 = 0.2$ and $\delta = 10$. A simulation with these parameters is shown in Figure 3. The estimates $\hat{x}_1$ and $\hat{x}_2$ converge to x exponentially as guaranteed by Proposition 3.9.[9] Moreover, as seen in Figure 3(b), as expected the novel coordinate θ_i jumps to zero when $\tau_i = 0$ and flows away from zero during flows. This tendency to move away from zero during flows is precisely the reason behind the choice of the Lyapunov function V in (18) which compensates this flow action by utilizing the negativity of the flow map of the timer to ensure negativity of V during flows.

More interestingly, consider the scenario where agent 1

[8]Note that the inequality in (29) and (30) are not linear. The tool developed in [26] provides a way to solve it.

[9]Code at https://github.com/HybridSystemsLab/ObsSyncTimes2ndAsyn.

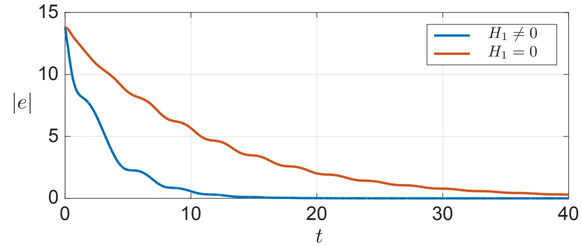


Fig. 4. The estimation error $|e|$ for two agents connected all-to-all as in Example 5.3 are compared for two cases of H_1 with identical initial conditions to those in Figure 3. Specifically, in Example 5.3, the case when $H_1 = [1 \ 0]$ (in blue) and when $H_1 = 0$ (in red) is considered.

loses the capability of receiving measurements, i.e., $H_1 = 0$. A simulation with same initial conditions and gains as those in Figure 3 is shown in Figure 4. As suggested from the simulation, it can be seen that even though the agent has measurement $y_1 \equiv 0$, through the consensus-like term (the third term in (12)), the components $\hat{x}_1$ and $\hat{x}_2$ get close to each other first and then converge to x exponentially. This highlights further a benefit of the third term in the dynamics of η in (12). In fact, the third term enforces consensus between the estimates $\hat{x}_1$ and $\hat{x}_2$ which can be seen from the dynamics of the error between $\hat{x}_1$ and $\hat{x}_2$. More precisely, due to the fact that the parameters satisfy the conditions in Theorem 3.6, the error state exponentially converges to zero. This implies that $\hat{x}_1$, $\hat{x}_2$ converge to the state of the plant x exponentially and, therefore, converge to each other. As argued in Section I, a distributed observer guaranteeing such properties under such a complex setting is not available in the literature.

Next, we illustrate Corollary 3.13. We consider this systems dynamics with $T_2^i = 0.4$ for each $i \in \mathcal{V}$, it can be verified that the inequalities (37) and (38) can be solved by choosing $h_1 = h_2 = 0$, $\delta = 10$ and $\gamma = -0.2$, which lead to

$$K_g^\top \approx \begin{bmatrix} -1.11 & -0.43 & -0.41 & -0.09 \\ 0.09 & -0.41 & 0.43 & -1.11 \end{bmatrix}.$$

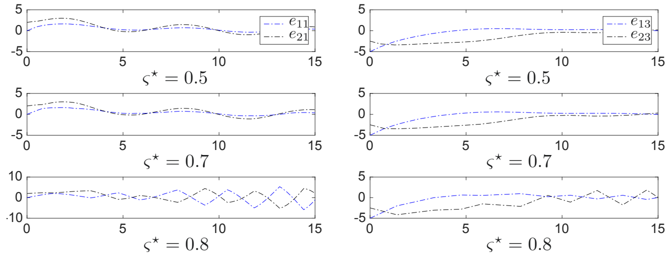
Note that (37) and (38) are LMIs that can be solved efficiently.

△

B. Examples for the Robustness Cases

In this section, we showcase the main results of the particular cases of robustness we consider. The following example revisits the system in Example 5.1 under the effect of skewed clocks.

Example 5.4: Consider the system in Example 5.1 with same set of parameters. Furthermore, we consider a particular perturbation where $\delta(x,t) \equiv 0$, $\varphi_1(x,t) = \varphi_2(x,t) \equiv 0$, $\vartheta_i(\tau_i) \equiv 0$ for all $\tau_i \in [0, T_2^i]$ and $\varsigma_1 = \varsigma_2 = \varsigma^\star$ where $\varsigma^\star \in (-\infty, 1)$ is a constant. Figure 5 shows numerical simulations to $\widetilde{\mathcal{H}}$ when $\varsigma \in \{0.5, 0.7, 0.8\}$ and all other perturbations are zero. Following this process, Table I shows the average relative error for varying ς^* with respect to the nominal case $\varsigma^* = 0$. Due to the non-deterministic nature of solutions we simulate 40 solutions from the same initial conditions as those in Figure 5 for each case of ς^*. The average error was found for the last 10 seconds of flow time for each of



(a) Estimation error for the first state component

(b) Estimation error for the third state component.

Fig. 5. Estimation error under different choice of perturbation $\varsigma^\star$.

ς^*	0	0.10	0.50	0.70	0.73	0.76	**0.77**	0.78
relative error	0	0.01	0.13	0.28	0.32	1.98	**5.72**	10.01

TABLE I
COMPARISON BETWEEN THE AVERAGE STEADY STATE ERROR e_i FOR VARYING ς^* RELATIVE TO THE NOMINAL CASE, ($\varsigma^* = 0$). WHEN THE TIMER SKEW CONSTANT $\varsigma^* > 0.77$, THE RESULTING ESTIMATION ERROR TENDS TO DIVERGE, AS SHOWN IN FIGURE 5.

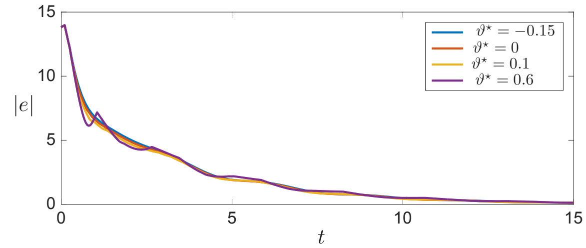


Fig. 6. The estimation error $|e|$ for the two agent all-to-all connected system in Example 5.3. Initial conditions are $x(0,0) = (2,2)$, $\hat{x}_1(0,0) = (15,5)$, $\hat{x}_2(0,0) = (-1,0)$, $\eta_1(0,0) = (1,1)$, $\eta_2(0,0) = (-1,-1)$, $\tau_1(0,0) = 0$, $\tau_2(0,0) = 0.3$.

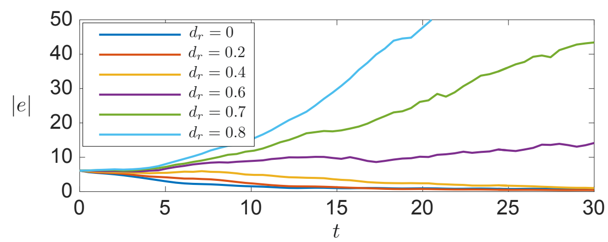


Fig. 7. The relationship between the dropout rate d_r and the corresponding average estimation error $|e|$ for 20 solutions and projected onto the flow time t.

the 40 simulations. The overall average was found across the 40 simulations and the relative error to the nominal is given in Table I. Note that for $\varsigma^* > 0.77$ solutions to $\mathcal{H}$ may no longer converge to zero as indicated by the large relative error and the bottom plots in Figure 5. $\triangle$

Example 5.5: Consider the system and parameters in Example 5.3 with $T_1^i = T_2^i = 0.2$ for each $i \in \{1,2\}$ and the perturbation $\vartheta_1^i = \vartheta_2^i = \vartheta^\star$. From Corollary 4.6, we can show that (44) is satisfied when $\vartheta^\star = 0.17$. A series of simulations are shown in Figure 6. In particular, these simulations compare the nominal case when $\vartheta^* = 0$ with several perturbed cases from the same initial condition; namely, for values of $\varsigma^* \in \{-0.15, 0.1, 0.6\}$. Note that when $\varsigma^\star = 0.6$, condition (44) cannot be satisfied with the parameters, but the estimation error still converges to zero. This is due to the fact that the condition in Corollary 4.6 is only sufficient but not necessary. $\triangle$

Example 5.6: Consider the system in Example 5.1 with same set of parameters as in Example 5.1. For each dropout rate $d_r \in \{0, 0.2, 0.4, 0.6, 0.7, 0.8\}$, under the same initial conditions as used in Example 5.1, the projection of average estimation error $|e|$ from twenty simulations on the t direction is shown in Figure 7. Note that in this particular study, larger dropout rate degrades convergence and when the dropout rate d_r is larger than 0.6, the average estimation error diverges. $\triangle$

VI. CONCLUSION

We presented a comprehensive solution to the problem of robustly estimating the state of a plant in a distributed fashion and under intermittent information communication. In contrast to classic observers for linear time-invariant systems, with enough information from its neighbors, but likely obtained at different time instances, an agent in a network can estimate the plant state even without detectability or even taking measurements of the plant output. Sufficient conditions that guarantee global exponential stability of the convergence of estimation error to zero are presented through the use of novel hybrid system models and analysis tools. Moreover, design methods involving LMI-like conditions are also proposed, enabling for efficient numerical design of the observers. The proposed observer is also shown to be robust to a wide range of perturbations encountered in real-world settings of estimation.

Future research directions include robustness to variations of the graph structure and to information delays, which will require the maturity of tools for time-varying and infinite-dimensional hybrid systems currently under development.

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APPENDIX A
SELECTED PROOFS OF MAIN RESULTS

Proof of Lemma 3.5 Let $\phi \in \mathcal{S}_{\mathcal{H}}$ be arbitrary. Note that, by the definition of the dynamics of τ_i in (10), within any window of flow with length $T_1^{\min}$, at most all of the N timers can jump twice. Moreover, once any of them jumps, the amount of time to its next jump is less or equal than $T_1^{\min}$. Then, for each $(t,j) \in \operatorname{dom}\phi$ $j \leq N\left(\frac{t}{T_1^{\min}}+1\right)$ which leads to $\left(\frac{j}{N}-1\right)T_1^{\min} \leq t$. Similarly, within any window of flow with length $T_2^{\max}$, all N timers jump at least once. Then, $t \leq \frac{j}{N}T_2^{\max}$ for each $(t,j) \in \operatorname{dom}\phi$. Moreover, for $(t_N, N) \in \operatorname{dom}\phi$, the next N jumps take at least $T_1^{\min}$ flow time, i.e., $t_{2N} \geq t_N + T_1^{\min}$, and they take at most $T_2^{\max}$ flow time, i.e., $t_{2N} \leq t_N + T_2^{\max}$. Therefore, we have $T_1^{\min} \leq t_{2N} - t_N \leq T_2^{\max}$. In fact, for any $(t_{jN}, jN) \in \operatorname{dom}\phi$ and $j \in \mathbb{Z}_{\geq 1}$, by using similar arguments, $T_1^{\min} \leq t_{(j+1)N} - t_{jN} \leq T_2^{\max}$, which leads to the property in item 3.

Now we show completeness of each $\phi \in \mathcal{S}_{\mathcal{H}}$. First, note that for any $\chi \in C \setminus D$, we have $\mathbb{T}_C(\chi) \cap f(\chi) \neq \emptyset$. Moreover, when $\chi \in C \cap D$, solutions cannot be extended via flow. Due to the fact that the flow map is linear, finite escape time during flows cannot occur. Furthermore, it is easy to check that $G(D) \subset C \cup D$. Therefore, according to [21, Proposition 6.10], each maximal solution ϕ to $\mathcal{H}$ is complete. ■

Proof of Proposition 3.9 Given $T_2^i \geq T_1^i > 0$ for all $i \in \mathcal{V}$ and $\delta > 0$, for each $i \in \mathcal{V}$, define the function $r_i : [0, T_2^i] \to [0,1]$ as $r_i(\nu_i) = \frac{\exp(\delta\nu_i) - \exp\left(\delta T_2^i\right)}{1-\exp\left(\delta T_2^i\right)}$ for all $\nu_i \in [0, T_2^i]$. Then, it can be verified that for any $\nu_i \in [0, T_2^i]$

$$\exp(\delta\nu_i) = r_i(\nu_i) + (1 - r_i(\nu_i))\exp\left(\delta T_2^i\right). \qquad (49)$$

Therefore, for each $\nu = (\nu_1, \nu_2, \ldots, \nu_N) \in \mathcal{T}$,

$$\widetilde{Q}(\nu) = \overline{R}(\nu)\widetilde{Q}(0) + (I - \overline{R}(\nu))\widetilde{Q}(\overline{T}_2), \qquad (50)$$

where $\overline{T}_2 = (T_2^1, T_2^2, \ldots, T_2^N)$ and

$$\overline{R}(\nu) = \operatorname{diag}(r_1(\nu_1), \ldots, r_N(\nu_N)) \otimes I_n. \qquad (51)$$

In light of (50) and (51), the inequality (19) can be rewritten as

$$\begin{bmatrix} \operatorname{He}(A_\theta, P) & -P + \widetilde{A}_\theta^\top \mathcal{K}^\top Q(\nu) \\ -P + Q(\nu)\mathcal{K}\widetilde{A}_\theta & -\delta Q(\nu) - \operatorname{He}(\widetilde{\mathcal{K}}, Q(\nu)) \end{bmatrix} = \overline{R}(\nu)E_1 + (I - \overline{R}(\nu))E_2, = \mathcal{B}_1(\nu)E_1\mathcal{B}_1(\nu) + \mathcal{B}_2(\nu)E_2\mathcal{B}_2(\nu)$$

where

$$E_1 = \begin{bmatrix} \operatorname{He}(A_\theta, P) & -P + \widetilde{A}_\theta^\top \mathcal{K}^\top \widetilde{Q}(0) \\ -P + \widetilde{Q}(0)\mathcal{K}\widetilde{A}_\theta & -\delta\widetilde{Q}(0) - \operatorname{He}(\widetilde{\mathcal{K}}, \widetilde{Q}(0)) \end{bmatrix},$$

$$E_2 = \begin{bmatrix} \operatorname{He}(A_\theta, P) & -P + \widetilde{A}_\theta^\top \mathcal{K}^\top \widetilde{Q}(\overline{T}_2) \\ -P + \widetilde{Q}(\overline{T}_2)\mathcal{K}\widetilde{A}_\theta & -(\delta\widetilde{Q}(\overline{T}_2) + \operatorname{He}(\widetilde{\mathcal{K}}, \widetilde{Q}(\overline{T}_2))) \end{bmatrix},$$

$$\mathcal{B}_1(\nu) = \operatorname{diag}\left(\sqrt{r_1(\nu_1)}, \sqrt{r_2(\nu_2)}, \ldots, \sqrt{r_N(\nu_N)}\right) \otimes I_n,$$

$$\mathcal{B}_2(\nu) = \operatorname{diag}\left(\sqrt{1-r_1(\nu_1)}, \sqrt{1-r_2(\nu_2)}, \ldots, \sqrt{1-r_N(\nu_N)}\right) \otimes I_n.$$

By using (29) and (30), $E_1 < 0$ and $E_2 < 0$, hence (19) holds for each $\nu \in \mathcal{T}$. ■

Proof of Proposition 3.12 Let

$$N_{\mathcal{W}} = \begin{bmatrix} A_{f\theta} \\ I \end{bmatrix} \quad \widetilde{\mathcal{Y}} = \begin{bmatrix} 0 & \widetilde{P}_1 \\ \star & \widetilde{Y} \end{bmatrix} \quad \widetilde{\mathcal{Z}} = \begin{bmatrix} 0 & \widetilde{P}_2 \\ \star & \widetilde{Z} \end{bmatrix} \qquad (52)$$

$$N_{\mathcal{Y}} = \begin{bmatrix} 0_{nN} & 0_{nN} & 0_{nN} & I_{nN} \end{bmatrix}^\top. \qquad (53)$$

Then, inequality (29) can be written as

$$E_1 = N_{\mathcal{W}}^\top \widetilde{\mathcal{Y}} N_{\mathcal{W}} < 0, \quad N_{\mathcal{Y}}^\top \widetilde{\mathcal{Y}} N_{\mathcal{Y}} < 0, \qquad (54)$$

and inequality (30) can be written as

$$E_2 = N_{\mathcal{W}}^\top \widetilde{\mathcal{Z}} N_{\mathcal{W}} < 0, \quad N_{\mathcal{Y}}^\top \widetilde{\mathcal{Z}} N_{\mathcal{Y}} < 0. \qquad (55)$$

Moreover, the columns of $N_{\mathcal{W}}$ (respectively, $N_{\mathcal{Y}}$) form the basis of the null space of $\mathcal{W}$ (respectively, $\mathcal{Y}$), where $\mathcal{W}$ and $\mathcal{Y}$ are given by

$$\mathcal{W} = \begin{bmatrix} -\widetilde{A}_1^{-1} & \widetilde{A}_2 \end{bmatrix} = \begin{bmatrix} -I & 0 & A_\theta & -I_{nN} \\ \mathcal{K} & -I & -\mathcal{K}H_\eta & H_\eta \end{bmatrix}, \quad (56)$$

and

$$\mathcal{Y} = \begin{bmatrix} I_{nN} & 0_{nN} & 0_{nN} & 0_{nN} \\ 0_{nN} & I_{nN} & 0_{nN} & 0_{nN} \\ 0_{nN} & 0_{nN} & I_{nN} & 0_{nN} \end{bmatrix}. \quad (57)$$

Then, using the projection lemma [27], inequalities (54) and (55) are equivalent to the existence of two matrices $\mathcal{O}, \mathcal{M} \in \mathbb{R}^{2nN \times 3nN}$ such that $\widetilde{\mathcal{Y}} + \mathcal{W}^\top \mathcal{O} \mathcal{Y} + \mathcal{Y}^\top \mathcal{O}^\top \mathcal{W} < 0$, and $\widetilde{\mathcal{Z}} + \mathcal{W}^\top \mathcal{M} \mathcal{Y} + \mathcal{Y}^\top \mathcal{M}^\top \mathcal{W} < 0$. Therefore, by letting

$$\mathcal{O} = \begin{bmatrix} \mathcal{O}_1 & \mathcal{O}_2 & \mathcal{O}_3 \\ \mathcal{O}_4 & \mathcal{O}_5 & \mathcal{O}_6 \end{bmatrix}, \quad \mathcal{M} = \begin{bmatrix} \mathcal{M}_1 & \mathcal{M}_2 & \mathcal{M}_3 \\ \mathcal{M}_4 & \mathcal{M}_5 & \mathcal{M}_6 \end{bmatrix}, \quad (58)$$

we have

$$\mathcal{W}^\top \mathcal{O} \mathcal{Y} + \mathcal{Y}^\top \mathcal{O}^\top \mathcal{W} + \widetilde{\mathcal{Y}} = \begin{bmatrix} \mathrm{He}(\Omega_1, I) & \Omega_2 + \widetilde{P}_1 \\ \star & \widetilde{Y} + \mathrm{He}(\Omega_3, I) \end{bmatrix},$$

which leads to (29). The proof of (30) follows similarly. ∎

Appendix B
An ISS property for a perturbed hybrid system

The following property is used to prove the ISS result in Theorem 4.4.

Proposition B.1: Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ *with state* $x \in \mathbb{R}^n$. *Suppose there exists a continuous function* $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *which is continuously differentiable on a neighborhood of* C *and is such that*

$$\langle \nabla V(x), \widetilde{f} \rangle \leq -aV(x) \qquad \forall x \in C, \widetilde{f} \in F(x) \quad (59)$$

$$V(g) \leq V(x) + b \qquad \forall x \in D, g \in G(x) \quad (60)$$

where $a, b \geq 0$. *Moreover, suppose there exist scalars* $0 < \widetilde{T}_1 \leq \widetilde{T}_2$ *and* $\widetilde{N} \in \mathbb{Z}_{\geq 1}$ *such that each* $\phi \in \mathcal{S}_{\mathcal{H}}$ *satisfies*

$$t_{(j+1)\widetilde{N}} - t_{j\widetilde{N}} \in [\widetilde{T}_1, \widetilde{T}_2] \quad (61)$$

for all $j \in \mathbb{Z}_{\geq 1}$ *such that* $(t_{(j+1)\widetilde{N}}, (j+1)\widetilde{N}), (t_{j\widetilde{N}}, j\widetilde{N}) \in \operatorname{dom} \phi$. *Then, for each* $\phi \in \mathcal{S}_{\mathcal{H}}$, *we have*

$$V(\phi(t,j)) \leq \exp(-at) V(\phi(0,0)) + \widetilde{N} b \sum_{s=0}^{\tilde{j}} \exp(-as\widetilde{T}_1),$$

for all $(t,j) \in \operatorname{dom} \phi$, *where* $\tilde{j} = \left\lfloor \frac{j}{\widetilde{N}} \right\rfloor$.

Proof Given a $\phi \in \mathcal{S}_{\mathcal{H}}$, pick any $(t,j) \in \operatorname{dom} \phi$ and let $0 = t_0 \leq t_1 \leq \cdots \leq t_{j+1} \leq t$ satisfy

$$\operatorname{dom} \phi \bigcap ([0, t_{j+1}] \times \{0, \ldots, j\}) = \bigcup_{s=0}^{j} [t_s, t_{s+1}] \times \{s\}.$$

For each $s \in \{0, \ldots, j\}$ and almost all $r \in [t_s, t_{s+1}]$, $\phi(r,i) \in C$. Then, (19) implies that, for each $s \in \{0, \ldots, j\}$ and for almost all $r \in [t_s, t_{s+1}]$, from (59) we have that

$$\frac{d}{dr} V(\phi(r,i)) \leq -\beta |\phi(r,i)|_{\mathcal{A}}^2 \leq -aV(\phi(r,i))$$

while integrating both sides leads to $V(\phi(t_{s+1}, s)) \leq (\exp(-a(t_{s+1} - t_s))) V(\phi(t,s))$ for each $s \in \{0, \ldots, j\}$. Similarly, for each $s \in \{1, \ldots, j\}$, $\phi(t_s, s-1) \in D$, and thus $V(\phi(t_s, s)) \leq V(\phi(t_s, s-1)) + b$. Due to the last two displayed inequalities, we have, for each $(t,j) \in \operatorname{dom} \phi$ with $t \geq t_j$, that

$$\begin{aligned} V(\phi(t,j)) &\leq \exp(-at) V(\phi(0,0)) + b \sum_{s=1}^{j} \exp(-a(t - t_s)) \\ &\leq \exp(-at) V(\phi(0,0)) + b \sum_{s=1}^{j} \exp(-a(t_j - t_s)). \end{aligned}$$

Due to the increasing sequence of times $t_1 \leq t_2 \leq t_3 \leq \cdots \leq t_j$, we have that there must exist a integer $\tilde{j}$ which defines the maximum multiple of $\widetilde{N}$, i.e., $\tilde{j} = \left\lfloor \frac{j}{\widetilde{N}} \right\rfloor$. Then, we can group the sum $\sum_{s=1}^{j} \exp(-a(t_j - t_s))$ as follows:

$$\begin{aligned} \sum_{s=1}^{j} \exp(-a(t_j - t_s)) &= \sum_{s=0}^{\tilde{j}-1} \sum_{k=1}^{\widetilde{N}} \exp(-a(t_j - t_{s\widetilde{N}+k})) \\ &\quad + \sum_{k=\tilde{j}\widetilde{N}+1}^{j} \exp(-a(t_j - t_k)). \end{aligned}$$

Note that for each $s \in \{0, \ldots, \tilde{j}-1\}$, we have

$$\begin{aligned} \max_{t_{s\widetilde{N}+k}, k \in \{1, \ldots, \widetilde{N}-1\}} \sum_{k=1}^{\widetilde{N}} \exp(-a(t_j - t_{s\widetilde{N}+k})) \\ = \sum_{k=1}^{\widetilde{N}} \exp(-a(t_j - t_{(s+1)\widetilde{N}})) \\ = \widetilde{N} \exp(-a(t_j - t_{(s+1)\widetilde{N}})), \end{aligned}$$

which corresponds to the maximizer satisfying $t_{s\widetilde{N}+k} = t_{(s+1)\widetilde{N}}$ for all $k \in \{1, \ldots, \widetilde{N}-1\}$. Therefore, it follows that

$$\begin{aligned} &\sup_{\substack{\phi \in \mathcal{S}_{\mathcal{H}} \\ (t_j, j) \in \operatorname{dom} \phi}} \sum_{s=1}^{j} \exp(-a(t_j - t_s)) \\ &\leq \sup_{\substack{\phi \in \mathcal{S}_{\mathcal{H}} \\ (t_j, j) \in \operatorname{dom} \phi}} \sum_{s=0}^{\tilde{j}-1} \max_{t_{s\widetilde{N}+k}, k \in \{1, \ldots, \widetilde{N}-1\}} \sum_{k=1}^{\widetilde{N}} \exp(-a(t_j - t_{s\widetilde{N}+k})) \\ &\qquad + \sup_{\substack{\phi \in \mathcal{S}_{\mathcal{H}} \\ (t_j, j) \in \operatorname{dom} \phi}} \sum_{k=\tilde{j}\widetilde{N}+1}^{j} \exp(-a(t_j - t_k)) \\ &\leq \sup_{\substack{\phi \in \mathcal{S}_{\mathcal{H}} \\ (t_j, j) \in \operatorname{dom} \phi}} \sum_{s=0}^{\tilde{j}-1} \widetilde{N} \exp(-a(t_j - t_{(s+1)\widetilde{N}})) + \widetilde{N}, \\ &\leq \sup_{\substack{\phi \in \mathcal{S}_{\mathcal{H}} \\ (t_j, j) \in \operatorname{dom} \phi}} \sum_{s=0}^{\tilde{j}-1} \widetilde{N} \exp(-a(t_{\tilde{j}\widetilde{N}} - t_{(s+1)\widetilde{N}})) + \widetilde{N}, \end{aligned}$$

where we used the property that $j - \tilde{j}\widetilde{N} < \widetilde{N}$. From the assumption on the hybrid time domain in (61), it follows that $t_{(s+1)\widetilde{N}} - t_{s\widetilde{N}} \in \left[\widetilde{T}_1, \widetilde{T}_2\right]$, for all $(t_{s\widetilde{N}}, s\widetilde{N}), (t_{(s+1)\widetilde{N}}, (s+1)\widetilde{N}) \in \operatorname{dom} \phi$ with $\phi \in \mathcal{S}_{\mathcal{H}}$, which implies that for each $s \in \{0, 1, \ldots, \tilde{j}-1\}$, $t_{\tilde{j}\widetilde{N}} - t_{s\widetilde{N}} \in \left[(\tilde{j}-s)\widetilde{T}_1, (\tilde{j}-s)\widetilde{T}_2\right]$.

Therefore,

$$\sup_{\substack{\phi \in \mathcal{S}_{\mathcal{H}} \\ (t_j, j) \in \operatorname{dom} \phi}} \widetilde{N} \sum_{s=0}^{\tilde{j}-1} \exp(-a(t_{\tilde{j}\widetilde{N}} - t_{(s+1)\widetilde{N}})) + \widetilde{N}$$

$$\leq \sum_{s=1}^{\tilde{j}} \exp(as\widetilde{T}_1) + \widetilde{N} = \widetilde{N} \sum_{s=0}^{\tilde{j}} \exp(as\widetilde{T}_1)$$

where $\tilde{j} = \left\lfloor \frac{j}{\widetilde{N}} \right\rfloor$. Therefore, for each $(t, j) \in \operatorname{dom} \phi$, V satisfies

$$V(\phi(t, j)) \leq \exp(-at) V(\phi(0, 0)) + \widetilde{N} b \sum_{s=0}^{\tilde{j}} \exp(-as\widetilde{T}_1)$$

for $\tilde{j} = \left\lfloor \frac{j}{\widetilde{N}} \right\rfloor$ which concludes the proof. ■

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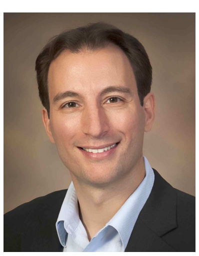

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