

Robustness of Interdependent Random Geometric Networks

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Abstract—We propose an interdependent random geometric graph (RGG) model for interdependent networks. Based on this model, we study the robustness of two interdependent spatially embedded networks where interdependence exists between geographically nearby nodes in the two networks. We study the emergence of the giant mutual component in two interdependent RGGs as node densities increase, and define the percolation threshold as a pair of node densities above which the giant mutual component first appears. In contrast to the case for a single RGG, where the percolation threshold is a unique scalar for a given connection distance, for two interdependent RGGs, multiple pairs of percolation thresholds may exist, given that a smaller node density in one RGG may increase the minimum node density in the other RGG in order for a giant mutual component to exist. We derive analytical upper bounds on the percolation thresholds of two interdependent RGGs by discretization, and obtain 99% confidence intervals for the percolation thresholds by simulation. Based on these results, we derive conditions for the interdependent RGGs to be robust under random failures and geographical attacks.

Index Terms—Interdependent networks, percolation, random geometric graph (RGG), robustness.

1 INTRODUCTION

Cyber-physical systems such as smart power grids and smart transportation networks are being deployed towards the design of smart cities. The integration of communication networks and physical networks facilitates network operation and control. In these integrated networks, one network depends on another for information, power, or other supplies in order to properly operate, leading to interdependence. For example, in smart grids, communication networks rely on the electric power from power grids, and simultaneously control power generators [1], [2]. Failures in one network may cascade to another network, which potentially make the interdependent networks vulnerable.

Cascading failures in interdependent networks have been extensively studied in the statistical physics literature since the seminal work in [3], where each of the two interdependent networks is modeled as a random graph. A node is functional if both itself and its interdependent node are in the giant components of their respective random graphs. After initial node failures in the first graph, their interdependent nodes in the second graph fail. Thus, a connected component in the second graph may become disconnected, and the failures of the disconnected nodes cascade back to (their interdependent) nodes in the first graph. As a result of the cascading failures, removing a small fraction of nodes in the first random graph destroys the giant components of both graphs.

To model spatially embedded networks, an interdependent lattice model was studied in [4]. Under this model,

geographical attacks may cause significantly more severe cascading failures than random attacks. Removing nodes in a finite region (*i.e.*, a zero fraction of nodes) may destroy the infinite clusters in both lattices [5].

If every node in one network is interdependent with multiple nodes in the other network, and a node is content to have at least one interdependent node, failures are less likely to cascade [6], [7]. Although the one-to-multiple interdependence exists in real-world spatially embedded interdependent networks (*e.g.*, a control center can be supported by the electric power generated by more than one power generator), it has not been previously studied using spatial graph models.

We use a random geometric graph (RGG) to model each of the two interdependent networks. RGG has been widely used to model communication networks [8]. For example, in a wireless network where the communication distance is limited by the signal to noise ratio requirement, under fixed transmission power, two users can communicate if and only if they are within a given distance. Percolation theory for RGG has been applied to study information flow in wireless networks and the robustness of networks under failures [9], [10]. In this paper, we extend percolation theory to interdependent RGGs.

The two RGGs representing two interdependent networks are allowed to have different connection distances and node densities, which can represent two networks that have different average link lengths and scales. These network properties were not captured by the lattice model in the previous literature. Moreover, the interdependent RGG model is able to capture the one-to-multiple interdependence in spatially embedded networks, and provides a more versatile framework for studying interdependent networks.

Robustness is a key design objective for interdependent networks. We study the conditions under which a positive

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fraction of nodes are functional in interdependent RGGs as the number of nodes approaches infinity. In this case, the interdependent RGGs *percolate*. Consistent with previous research [3], [4], [6], the robustness of interdependent RGGs under failures is measured by whether percolation exists after failures. To the best of our knowledge, our paper is the first to study the percolation of interdependent spatial network models using a mathematically rigorous approach.

The main contributions of this paper are as follows.

- 1) We propose an interdependent RGG model for two interdependent networks, which captures the differences in the scales of the two networks as well as the one-to-multiple interdependence in spatially embedded networks.
- 2) We derive the first analytical upper bounds on the percolation thresholds of the interdependent RGGs, above which a positive fraction of nodes are functional.
- 3) We obtain 99% confidence intervals for the percolation thresholds, by mapping the percolation of interdependent RGGs to the percolation of a square lattice where the probability that a bond in the square lattice is open is evaluated by simulation.
- 4) We characterize sufficient conditions for the interdependent RGGs to percolate under random failures and geographical attacks. In particular, if the node densities are above any upper bound on the percolation threshold obtained in this paper, the interdependent RGGs remain percolated after a geographical attack. This is in contrast with the cascading failures after a geographical attack, observed in the interdependent lattice model with one-to-one interdependence [5].
- 5) We extend our techniques to study models with more general interdependence requirement (e.g., a node in one network requires more than one supply node from the other network).

The rest of the paper is organized as follows. We state the model and preliminaries in Section 2. We derive analytical upper bounds on percolation thresholds in Section 3, and obtain confidence intervals for percolation thresholds in Section 4. In Section 5, we study the robustness of interdependent RGGs under random failures and geographical attacks. In Section 6, we extend the techniques to study graphs with more general interdependence. Section 7 concludes the paper.

2 MODEL

2.1 Preliminaries on RGG and percolation

An RGG in a two-dimensional square consists of nodes generated by a Poisson point process and links connecting nodes within a given connection distance [11]. Let $G(\lambda, d, a^2)$ denote an RGG with node density λ and connection distance d in an $a \times a$ square. The studies on RGG focus on the regime where the expected number of nodes $n = \lambda a^2$ is large. We first present some preliminaries which are useful for developing our model. The *giant component* of an RGG is a connected component that contains $\Theta(n)$ nodes. A node belongs to the giant component with a positive probability $\Theta(n)/n$ if the giant component exists. For a given connection distance, the *percolation threshold* is a node

density above which a node belongs to the giant component with a positive probability (i.e., a giant component exists) and below which the probability is zero (i.e., no giant component exists). By scaling, if the percolation threshold is λ^* under connection distance d , then the percolation threshold is $\lambda^* c^2$ under connection distance d/c . Therefore, without loss of generality, in this paper, we study the percolation thresholds represented by node densities, for given connection distances.

The RGG is closely related to the *Poisson boolean model* [12], where nodes are generated by a Poisson point process on an *infinite plane*. Let $G(\lambda, d)$ denote a Poisson boolean model with node density λ and connection distance d . The difference between $G(\lambda, d)$ and $G(\lambda, d, a^2)$ is that the number of nodes in $G(\lambda, d)$ is infinite while the expected number of nodes in $G(\lambda, d, a^2)$ is large but finite. The Poisson boolean model can be viewed as a limit of the RGG as the number of nodes approaches infinity. The percolation threshold of $G(\lambda, d)$ under a given d is defined as the node density above which a node belongs to the *infinite component* with a positive probability and below which the probability is zero. It has been shown that a node belongs to the infinite component with a positive probability if and only if an infinite component exists, and thus the percolation of $G(\lambda, d)$ can be equivalently defined as the existence of the infinite component [12]. Moreover, the percolation threshold of $G(\lambda, d)$ is identical with the percolation threshold of $G(\lambda, d, a^2)$ [11], [13].

2.2 Interdependent RGGs

Two interdependent networks are modeled by two RGGs $G_1(\lambda_1, d_1, a^2)$ and $G_2(\lambda_2, d_2, a^2)$ on the same $a \times a$ square. A node in one graph is interdependent with *all* the nodes in the other graph within the *interdependent distance* d_{dep} . See Fig. 1 for an illustration. Nodes in one graph are *supply nodes* for nodes in the other graph within d_{dep} . The physical interpretation of supply can be either electric power or information that is essential for proper operation. A node can receive supply from nearby nodes within the interdependent distance. Larger interdependent distance leads to more robust interdependent networks. The geographical nature of interdependence is observed in physical networks [1], [4].

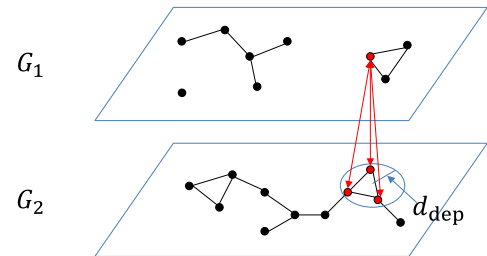


Fig. 1. Two interdependent RGGs with interdependent distance d_{dep} .

Most analysis in this paper is given in the context of two interdependent Poisson boolean models $G_{\text{IntDep}} = (G_1(\lambda_1, d_1), G_2(\lambda_2, d_2), d_{\text{dep}})$, which is the limit of two interdependent RGGs as the numbers of nodes in both graphs approach infinity.

We define a mutual component and an infinite mutual component in G_{IntDep} , in the same way as one defines a connected component and an infinite component in $G(\lambda, d)$.

Definition 1. Let V_i^0 denote nodes in a connected component in $G_i(\lambda_i, d_i)$, $\forall i \in \{1, 2\}$. If each node in $V_i \subseteq V_i^0$ has at least one supply node in $V_j \subseteq V_j^0$ within d_{dep} , $\forall i, j \in \{1, 2\}, i \neq j$, then nodes V_1 and V_2 form a mutual component of G_{IntDep} .

If, in addition, V_i contains an infinite number of nodes, $\forall i \in \{1, 2\}$, then V_1 and V_2 form an infinite mutual component.

A mutual component can be viewed as an autonomous system in the sense that nodes in a mutual component have supply nodes in the same mutual component, and in each graph, nodes that belong to a mutual component are connected regardless of the existence of nodes outside the mutual component. Note that a node can receive supply from any of its supply nodes in the same mutual component, and thus is content if it has at least one supply node. Nodes in an infinite mutual component are *functional*, since they constitute two large connected interdependent networks and can perform a given network function (e.g., data communication or power transmission to a large number of clients). This definition of functional is consistent with previous research on interdependent networks based on random graph models [3].

For a fixed d_{dep} , if an infinite mutual component exists in $G_{\text{IntDep}} = (G_1(\lambda_1, d_1), G_2(\lambda_2, d_2), d_{\text{dep}})$, then an infinite mutual component exists in $G'_{\text{IntDep}} = (G_1(\lambda'_1, d_1), G_2(\lambda_2, d_2), d_{\text{dep}})$, where $\lambda'_1 > \lambda_1$. This can be explained by coupling G'_1 with G_1 as follows. By removing each node in G'_1 independently with probability $1 - \lambda_1/\lambda'_1$, the density of the remaining nodes in G'_1 is λ_1 , and an infinite mutual component exists in the interdependent graphs that consist of G_2 and the graph formed by the remaining nodes in G'_1 . Since adding nodes to a graph does not disconnect any mutual component, an infinite mutual component exists in $G'_{\text{IntDep}} = (G_1(\lambda'_1, d_1), G_2(\lambda_2, d_2), d_{\text{dep}})$. By the same analysis, an infinite mutual component also exists in $G''_{\text{IntDep}} = (G_1(\lambda_1, d_1), G_2(\lambda'_2, d_2), d_{\text{dep}})$, if $\lambda'_2 > \lambda_2$.

We define a percolation threshold of G_{IntDep} as follows.

Definition 2. A pair of node densities $(\lambda_1^*, \lambda_2^*)$ is a percolation threshold of G_{IntDep} , given connection distances d_1, d_2 and the interdependent distance d_{dep} , if an infinite mutual component exists in G_{IntDep} for $\lambda_1 > \lambda_1^*$ and $\lambda_2 > \lambda_2^*$, and no infinite mutual component exists otherwise.

For fixed d_1, d_2 and d_{dep} , there may exist multiple percolation thresholds. We show that, in most cases, the larger the node density is in one graph, the smaller the required node density is in the other graph in order for the infinite mutual component to exist. This is in contrast with the situation for a single graph $G(\lambda, d)$ where there is a unique percolation threshold λ^* for a fixed d .

There is a non-trivial phase transition in G_{IntDep} . If λ_i is smaller than the percolation threshold of a single graph $G_i(\lambda_i, d_i)$, there is no infinite component in $G_i(\lambda_i, d_i)$, and therefore there is no infinite mutual component in G_{IntDep} . Thus, $\lambda_i^* > 0$, $\forall i \in \{1, 2\}$. As we will see in the next section, there exist percolation thresholds $\lambda_i^* < \infty$, $\forall i \in \{1, 2\}$, which concludes the non-trivial phase transition.

Given that the conditions for the percolation of a random geometric graph $G_i(\lambda_i, d_i, a^2)$ and a Poisson boolean model $G_i(\lambda_i, d_i)$ are the same, the above definitions can be naturally extended to interdependent RGGs. Consider nodes $V_1 \subseteq G_1(\lambda_1, d_1, a^2)$ and $V_2 \subseteq G_2(\lambda_2, d_2, a^2)$ that form a mutual component. If V_i contains $\Theta(n_i)$ nodes, where $n_i = \lambda_i a^2$, $\forall i \in \{1, 2\}$, then V_1 and V_2 form a *giant mutual component* in interdependent RGGs. The percolation of interdependent RGGs is defined as the existence of a giant mutual component. In the rest of the paper, we sometimes use G_i to denote both $G_i(\lambda_i, d_i, a^2)$ and $G_i(\lambda_i, d_i)$. The model that it refers to will be clear from the context.

2.3 Related work

In the interdependent networks literature, the model which is closest to ours is the interdependent lattice model, first proposed in [14] and further studied in [4], [5]. In the lattice model, nodes in a network are represented by the open *sites* (nodes) of a square lattice, where every site is open independently with probability p . Network links are represented by the *bonds* (edges) between adjacent open sites. Every node in one lattice is interdependent with *one* randomly chosen node within distance r_d in the other lattice. The distance r_d indicates the geographical proximity of the interdependence. The percolation threshold of the interdependent lattice model is characterized as a function of r_d , assuming the same p in both lattices [14]. Percolation of the model where some nodes do not need to have supply nodes was studied in [4]. The analysis relies on quantities estimated by simulation and extrapolation, such as the fraction of nodes in the infinite component of a lattice for any fixed p , which cannot be computed rigorously. In contrast, we study the percolation of the interdependent RGG model using a mathematically rigorous approach.

The percolation of a single RGG (or a Poisson boolean model) has been studied in the previous literature [12], [15], [16]. The techniques employed therein involves inferring the percolation of the continuous model from the percolation of a discrete lattice model. The key is obtaining a lattice whose percolation condition is known and is related to the percolation of the original model, by discretization. The study of the percolation conditions of discrete lattice models can be found in [17], [18]. We extend the previous techniques to discretize G_{IntDep} , and obtain bounds on the percolation thresholds.

3 ANALYTICAL UPPER BOUNDS ON PERCOLATION THRESHOLDS

In this section, we study sufficient conditions for the percolation of G_{IntDep} . We provide closed-form formulas for (λ_1, λ_2) , which depend on d_1, d_2, d_{dep} , such that there exists an infinite mutual component in $G_{\text{IntDep}} = (G_1(\lambda_1, d_1), G_2(\lambda_2, d_2), d_{\text{dep}})$. The formulas provide guidelines for node densities in deploying physical interdependent networks, in order for a large number of nodes to be connected.

In G_{IntDep} , nodes in the infinite mutual component are viewed as functional while all the other nodes are not. Thus, a node is functional only if it is in the infinite component of

its own graph, and it depends on at least one node in the infinite component of the other graph. For any node b_1 in G_1 , although the number of nodes in G_2 within the interdependent distance from b_1 follows a Poisson distribution, the number of functional nodes is hard to calculate, since the probability that a node in G_2 is in the infinite component is unknown. Moreover, the nodes in the infinite component of G_2 are clustered, and thus the thinning of the nodes in G_1 due to a lack of supply nodes in G_2 is inhomogeneous. To overcome these difficulties, we consider the percolation of two graphs jointly, instead of studying the percolation of one graph with reduced node density due to a lack of supply nodes.

We now give an overview of our approach. We develop mapping techniques (discretizations) to characterize the percolation of G_{IntDep} by the percolation of a discrete model. Mappings from a model whose percolation threshold is unknown to a model with known percolation threshold are commonly employed in the study of continuum percolation. For example, one can study the percolation threshold of the Poisson boolean model $G(\lambda, d)$ by mapping it to a triangle lattice and relating the state of a site in the triangle lattice to the point process of $G(\lambda, d)$. By the mapping, the percolation of the triangle lattice implies the percolation of $G(\lambda, d)$. Consequently, an upper bound on the percolation threshold of $G(\lambda, d)$ is given by λ for which the triangle lattice percolates, a known quantity [12], [15]. In general, more than one mapping can be applied, and the key is to find a mapping that gives a good (smaller) upper bound. Following this idea, we propose different mappings that fit different conditions to obtain upper bounds on the percolation thresholds of G_{IntDep} .

In the rest of this section, we first study an example, in which the connection distances of the two graphs are the same, to understand the tradeoff between the two node densities in order for G_{IntDep} to percolate. We then develop two upper bounds on the percolation thresholds. The first bound is tighter when the ratio of the two connection distances is small, and is obtained by mapping G_{IntDep} to a square lattice with independent bond open probabilities. The second bound is tighter when the ratio of the two connection distances is large, and is obtained by mapping G_{IntDep} to a square lattice with correlated bond open probabilities.

3.1 A motivating example

To see the impact of varying the node density in one graph on the minimum node density in the other graph in order for G_{IntDep} to percolate, consider an example where $d_1 = d_2 = 2d_{\text{dep}}$. We apply a mapping similar to what is used to obtain an upper bound on the percolation threshold of $G(\lambda, d)$ in [15], to obtain upper bounds on the percolation thresholds of G_{IntDep} .

Consider a triangle lattice where each site is surrounded by a cell. The lattice bond length is determined such that any two points in adjacent cells have distance smaller than $2r$, where $2r = d_1$. The boundary of the cell consists of arcs of radius r centered at the middle of the bonds in the triangle lattice. See Fig. 2 for an illustration. The area of the cell is $A = 0.8227r^2$. A site in the triangle lattice is either *open* or

closed. If the probability that a site is open is strictly larger than $1/2$, open sites form an infinite component, and the triangle lattice percolates [15].

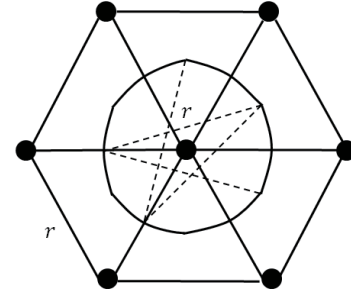


Fig. 2. A cell that contains a site in a triangle lattice.

To study the percolation of G_{IntDep} , we declare a site in the triangle lattice to be open if there is at least one node in its cell from G_1 and at least one node in its cell from G_2 . If the triangle lattice percolates, then G_{IntDep} also percolates. To see this, consider two adjacent open sites in the triangle lattice. Nodes from G_i in the two adjacent cells that contain the two open sites are connected, because they are within distance $d_i = 2r$ ($\forall i \in \{1, 2\}$). If the open sites in the triangle lattice form an infinite component, then nodes from G_i in the corresponding cells form an infinite component V_i ($\forall i \in \{1, 2\}$). Moreover, given that any pair of nodes in a cell are within distance $r \leq d_{\text{dep}}$, each node in V_i has at least one supply node in V_j within the same cell ($\forall i, j \in \{1, 2\}, i \neq j$).

Since $1 - e^{-\lambda_i A}$ is the probability that there is at least one node in the cell from G_i and the point processes in G_1 and G_2 are independent, an upper bound on the percolation thresholds of G_{IntDep} is given by (λ_1, λ_2) satisfying

$$(1 - e^{-\lambda_1 A})(1 - e^{-\lambda_2 A}) = 1/2.$$

If λ_i is large, the percolation threshold λ_j^* approaches the threshold of a single graph G_j . Intuitively, if λ_j is above the percolation threshold of G_j , disks of radius $d_j/2$ centered at nodes in G_j form a connected infinite-size region. Since λ_i is large, nodes in G_i in this region are connected and form an infinite component. Moreover, since $d_{\text{dep}} = d_j/2$, all the nodes in this region have supply nodes, and they form an infinite mutual component.

The above upper bounds on percolation thresholds are still valid if $d_{\text{dep}} > d_i/2$, because each node can depend on a larger set of nodes by increasing d_{dep} and it is easier for G_{IntDep} to percolate under the same node densities and connection distances. However, if $d_{\text{dep}} < d_i/2$, the bond length of the triangle lattice should be adjusted to $r = d_{\text{dep}}$ in order for any pair of nodes in a cell to be within d_{dep} . The percolation threshold curve (λ_1, λ_2) would shift upward. Intuitively, if d_{dep} decreases, the node density in one network should increase to provide enough supply for the other network.

3.2 Small ratio d_2/d_1

Given $G_{\text{IntDep}} = (G_1(\lambda_1, d_1), G_2(\lambda_2, d_2), d_{\text{dep}})$, without loss of generality we assume that $d_1 \leq d_2$. Moreover, we assume that $d_{\text{dep}} \geq \max(d_1/2, d_2/2) = d_2/2$ (see the remark

at the end of the section for comments on this assumption). Let $c = \lfloor d_2/d_1 \rfloor = \max\{c : d_2/d_1 \geq c, c \in \mathbb{N}\}$. For small c , we study the percolation of G_{IntDep} by mapping it to an independent bond percolation of a square lattice, and prove the following result.

Theorem 1. *If (λ_1, λ_2) satisfies*

$$(1 - e^{-\lambda_1 d_1^2/8})^c (1 - e^{-\lambda_2 c^2 d_1^2/8}) > 1/2,$$

then $G_{\text{IntDep}} = (G_1(\lambda_1, d_1), G_2(\lambda_2, d_2), d_{\text{dep}})$ percolates, where $c = \lfloor d_2/d_1 \rfloor$, $d_1 \leq d_2$, and $d_{\text{dep}} \geq d_2/2$.

Theorem 1 provides a sufficient condition for the percolation of G_{IntDep} . For node densities that satisfy the inequality, an infinite mutual component exists in G_{IntDep} . For the deployment of interdependent networks, if the node densities in the two networks are sufficiently large (characterized by Theorem 1), then a large number of nodes in the interdependent networks are functional.

Proof of Theorem 1. We first construct a square lattice as follows. Partition the plane into small squares of side length $s = d_1/2\sqrt{2}$. A large square consists of $c \times c$ small squares and has side length cs . The *diagonals* of the large squares form the bonds of a square lattice L , illustrated by the thick line segments in Fig. 3.

The state of a bond in L is determined by the point process of G_{IntDep} in the large square that contains the bond. A bond (v_1, v_2) is open if the following conditions are both satisfied.

- 1) There is at least one node from G_1 in each of the two small squares that contain the ends $(v_1$ and $v_2)$ of the bond, and they are connected through nodes from G_1 , all within the large square of side length cs .
- 2) There is at least one node from G_2 in the large square that contains the bond.

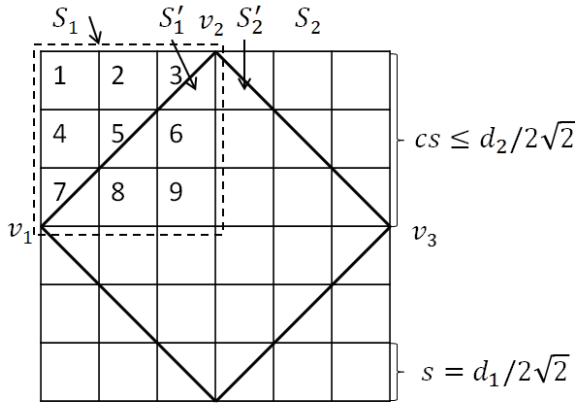


Fig. 3. Mapping to a square lattice for $c = 3$.

The first condition is satisfied if there exists a sequence of adjacent small squares, each of which contains at least one node in G_1 , from the small square that contains v_1 to the small square that contains v_2 . (Each small square is *adjacent* to its eight immediate neighbors.) In the example of Fig. 3, these sequences include 3-5-7, 3-2-4-7, and 3-6-8-7.

To obtain a closed-form formula, instead of computing the exact probability, we compute a lower bound on the

probability that the first condition is satisfied. The probability is lower bounded by the probability that the c small squares that intersect the bond each contain at least one node from G_1 , given by

$$p_1 \geq (1 - e^{-\lambda_1 d_1^2/8})^c.$$

The probability that the second condition is satisfied is

$$p_2 = 1 - e^{-\lambda_2 c^2 d_1^2/8}.$$

Given that the two Poisson point processes in G_1 and G_2 are independent, the probability that a bond is open is $p_1 p_2$.

It remains to prove that the percolation of L implies the percolation of G_{IntDep} . Consider two adjacent open bonds $(v_1, v_2), (v_2, v_3)$ in L . Let S_1 and S_2 denote the two adjacent large squares of side length cs that contain the two open bonds. Let S'_1 and S'_2 denote two adjacent small squares of side length s that contains v_2 , within S_1 and S_2 , respectively. See Fig. 3 for an illustration. Since $(v_1, v_2), (v_2, v_3)$ are open, under the second condition, nodes of G_2 exist in S_1 and S_2 and they are connected, because they are within distance $2\sqrt{2}cs \leq d_2$. Under the first condition, nodes of G_1 form a connected path from the small square (within S_1 , marked as 7 in Fig. 3) containing v_1 to S'_1 , and another path from the small square (within S_2) containing v_3 to S'_2 . Moreover, the two paths are joined, because any pair of nodes in S'_1 and S'_2 are within distance $2\sqrt{2}s = d_1$. Given that any pair of nodes within a large square have distance at most $\sqrt{2}cs \leq d_2/2 \leq d_{\text{dep}}$, all the nodes have at least one supply node inside the large square that contains an open bond. To conclude, if the open bonds in L form an infinite component, then the nodes in G_{IntDep} form an infinite mutual component.

The event that a bond is open depends on the point processes in the large square that contains the bond, and is independent of whether any other bonds are open. As long as the probability that a bond is open, $p_1 p_2$, is larger than $1/2$, which is the threshold for independent bond percolation in a square lattice [18], G_{IntDep} percolates. $\square$

The bound can be made tighter for any given $c = \lfloor d_2/d_1 \rfloor$, by computing more precisely the probability that the first condition is satisfied. We provide an example to illustrate the computation of an improved upper bound.

Example: Consider an example where $d_1 = 1, d_2 = 2d_{\text{dep}} = 3$. The probability that there is at least one node from G_2 in the large square of side length $3/2\sqrt{2}$ is $p_2 = 1 - e^{-9\lambda_2/8}$.

The probability that a small square of side length $1/2\sqrt{2}$ contains at least one node from G_1 is $p_s = 1 - e^{-\lambda_1/8}$. The probability that the first condition is satisfied is

$$p_1 = p_s^3 + (1 - p_s)p_s^4 + (1 - p_s)p_s^4 - (1 - p_s)p_s^6, \quad (1)$$

obtained by considering all the sequences of adjacent small squares. For node densities (λ_1, λ_2) that satisfy $p_1 p_2 > 1/2$, G_{IntDep} percolates. Since p_1 computed by Eq. (1) is larger than p_s^3 for any fixed p_s , the bound on λ_2 is smaller for any fixed λ_1 .

3.3 Large ratio d_2/d_1

In the mapping from G_{IntDep} to the square lattice L , the condition for a bond to be open becomes overly restrictive as

d_2/d_1 increases. A path crossing the two large squares that contain two adjacent bonds does not have to cross the small squares that contain the common end of the two bonds. In the following theorem, we give another upper bound on the percolation threshold of G_{IntDep} . This result provides an alternative sufficient condition for the existence of an infinite mutual component in G_{IntDep} . This upper bound is tighter than the bound in Theorem 1 for larger values of d_2/d_1 .

Theorem 2. *If (λ_1, λ_2) satisfies*

$$\left[1 - \frac{4}{3}(m+1)e^{m \log 3(1-p)}\right] \left[1 - \frac{4}{3}(2m+1)e^{m \log 3(1-p)}\right] p' > 0.8639,$$

then $G_{\text{IntDep}} = (G_1(\lambda_1, d_1), G_2(\lambda_2, d_2), d_{\text{dep}})$ percolates, where $p = 1 - e^{-\lambda_1 d_1^2/8}$, $p' = 1 - e^{-2D^2 \lambda_2}$, $D = \min(d_2/\sqrt{10}, d_{\text{dep}}/\sqrt{5})$, $m = \lfloor 2D/d_1 \rfloor$, $d_1 \leq d_2$, and $d_{\text{dep}} \geq d_2/2$.

This upper bound is obtained by mapping G_{IntDep} to a dependent bond percolation model L_D . The mapping from the Poisson boolean model $G(\lambda, d)$ to L_D was first proposed in [16] to study the percolation threshold of $G(\lambda, d)$, and later applied to the study of a random geometric graph under non-uniform node removals [10]. We briefly describe the method in the previous literature that uses L_D to study the percolation of $G(\lambda, d)$, and then prove Theorem 2 based on a similar method.

3.3.1 1-dependent bond percolation model L_D

In the standard bond percolation model on a square lattice L , the event that a bond is open is independent of the event that any other bond is open. If in a square lattice L_D , the event that a bond is open may depend on the event that its adjacent bond is open, but is independent of the event that any non-adjacent bond is open, then L_D is a 1-dependent bond percolation model on a square lattice. With the additional restriction that each bond is open with an identical probability, an upper bound on the percolation threshold of L_D is 0.8639 [16].

The 1-dependent bond percolation model L_D can be used to study the percolation of G' where the points are generated by homogeneous Poisson point processes. To construct a mapping from G' to L_D , consider two adjacent $D \times D$ squares S_1 and S_2 and let R be the rectangle formed by the two squares. A bond (v_1, v_2) that connects the centers of S_1 and S_2 is associated with R . Figure 4 illustrates the square lattice formed by the bonds, represented by thick line segments.

Lemma 3. *Let the state of a bond (v_1, v_2) be determined by the homogeneous Poisson point processes of G' inside R , and the conditions for a bond to be open be identical for all bonds. Then the bonds form a 1-dependent bond percolation model L_D with identical bond open probabilities.*

Proof. The event that a bond is open is not independent of the event that its adjacent bond is open, since the two events both depend on the point process in an overlapping square. However, the event that a bond is open is independent of the event that any non-adjacent bond is open, since their associated rectangles do not overlap and the point processes in the two rectangles are independent.

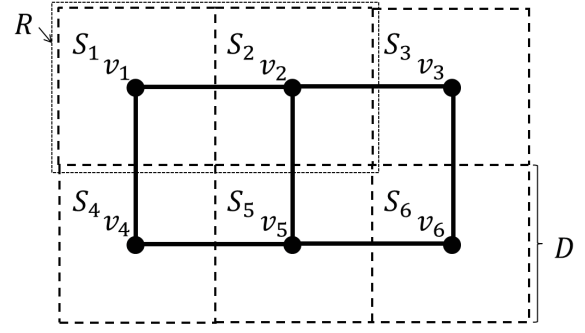


Fig. 4. Square lattice L_D formed by the bonds (v_i, v_j) .

Moreover, a Poisson point process is invariant under translation and rotation. Given that the points in G' are generated by homogeneous Poisson point processes and the conditions for a bond to be open are identical, the probability that a bond is open is identical for all bonds. $\square$

By properly setting the conditions for a bond to be open, the percolation of L_D can imply the percolation of G' . We first look at an example in [18] that studies the percolation of $G(\lambda, d)$, and then extend the technique to study G_{IntDep} .

Example [18]: Let a bond be open if a path in $G(\lambda, d)$ crosses¹ R' horizontally and another path in $G(\lambda, d)$ crosses S'_1 vertically, where R' is a $(2D - 2d) \times (D - 2d)$ rectangle that has the same center as R , and S'_1 is a $(D - 2d) \times (D - 2d)$ square that has the same center as S_1 . The reason for considering R' and S'_1 is that the existence of the two crossing paths over R' and S'_1 is determined by the point process within R , while the existence of links within distance d from the boundaries (and thus the crossings over R) may depend on nodes outside R .

If two adjacent bonds are open, the paths in $G(\lambda, d)$ in the two rectangles are joined. To see this, note that in Fig. 5, if the black and blue bonds (same direction) are both open, the crossings 1 and 2 intersect. If the black and red bonds (perpendicular) are both open, the crossings 1 and 3 intersect.

If the square lattice L_D percolates, open bonds form an infinite component. Paths in $G(\lambda, d)$ across the rectangles associated with the open bonds are connected and form an infinite component. Therefore, a node density above which L_D percolates is an upper bound on the percolation threshold of $G(\lambda, d)$.

3.3.2 Proof of Theorem 2

We map G_{IntDep} to L_D by letting a bond in L_D be open if the following three conditions are satisfied in its associated rectangle $R = S_1 \cup S_2$. The size of the rectangle satisfies $D = \min(d_2/\sqrt{10}, d_{\text{dep}}/\sqrt{5}) \geq d_2/2\sqrt{5}$.

- 1) A path from G_1 crosses R' horizontally, where R' is a $(2D - 2d_1) \times (D - 2d_1)$ rectangle that has the same center as R .

1. A path crosses a rectangle $R' = [x_1, x_2] \times [y_1, y_2]$ horizontally if the path consists of a sequence of connected nodes $v_1, v_2, \dots, v_{n-1}, v_n$, and $v_2, \dots, v_{n-1}$ are in R' , $x(v_1) \leq x_1, x(v_n) \geq x_2$, $y_1 \leq y(v_1), y(v_n) \leq y_2$, where $x(v_i)$ is the x -coordinate of v_i and $y(v_i)$ is the y -coordinate of v_i . A path crosses a rectangle vertically is defined analogously.

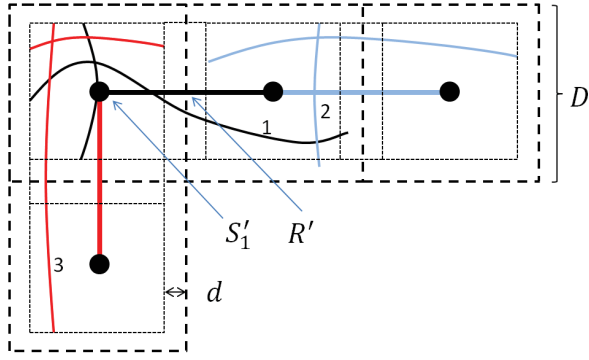


Fig. 5. Crossings over rectangles associated with two adjacent open bonds are joined.

- 2) A path from G_1 crosses S'_1 vertically, where S'_1 is a $(D - 2d_1) \times (D - 2d_1)$ square that has the same center as S_1 .
- 3) There is at least one node from G_2 in R .

To see that the percolation of L_D implies the percolation of G_{IntDep} , consider any two adjacent open bonds in L_D . In the two rectangles associated with the bonds, 1) paths from G_1 that cross one rectangle are joined with paths from G_1 that cross the other rectangle; 2) at least two nodes from G_2 , one in each rectangle, are connected by a link in G_2 , because any two nodes in adjacent rectangles are within distance $\sqrt{10}D \leq d_2$; 3) every node in G_i has at least one supply node in G_j inside the rectangle ($\forall i, j \in \{1, 2\}, i \neq j$), in which the distance between two nodes is no larger than $\sqrt{5}D \leq d_{\text{dep}}$.

If the probability p_{123} that a bond is open is above 0.8639, then L_D percolates and G_{IntDep} also percolates. An upper bound on the percolation threshold of G_{IntDep} is a pair of node densities (λ_1, λ_2) that yields $p_{123} \geq 0.8639$. In the remainder of the proof, we compute p_{123} as a function of (λ_1, λ_2) .

To determine the probability that the first and the second conditions are satisfied, we consider a discrete square lattice represented by Fig. 6. Bonds of length $d_1/2$ form a square lattice L' in a finite $md_1 \times md_1/2$ region, where $m = \lfloor 2D/d_1 \rfloor$. Let a bond in L' be open if there is at least one node from G_1 in the $d_1/2\sqrt{2} \times d_1/2\sqrt{2}$ square that contains the bond (the small square that has dashed boundaries in the figure), which occurs with probability $p = 1 - e^{-\lambda_1 d_1^2/8}$. It is clear that if the open bonds form a horizontal crossing² over L' , then nodes in G_1 form a horizontal crossing path over R' .

Let $p_x(km, m, p)$ denote the probability that there exists a horizontal crossing over the $km \times m$ square lattice L' given that each bond is open independently with probability p . A lower bound on $p_x(km, m, p)$, Eq. (2), can be derived by a standard technique in percolation theory (e.g., an extension of Proposition 2 in [9]).

$$p_x(km, m, p) \geq 1 - \frac{4}{3}(km + 1)e^{m \log 3(1-p)}. \quad (2)$$

2. A horizontal crossing of open bonds over a rectangle $R' = [x_1, x_2] \times [y_1, y_2]$ consists of a sequence of adjacent open bonds in the rectangle such that at least one bond has an endpoint with x -coordinate x_1 and at least one bond has an endpoint with x -coordinate x_2 . A vertical crossing of open bonds is defined analogously.

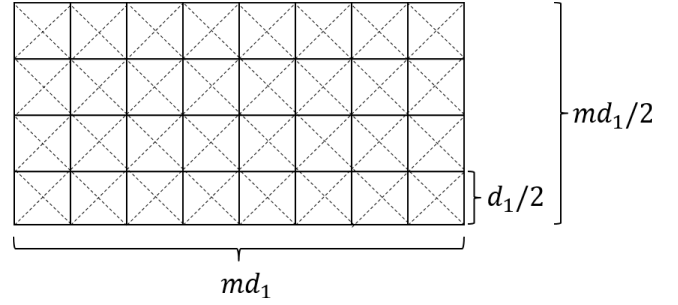


Fig. 6. Mapping the crossing in G_1 to the crossing in a square lattice L' .

The probability that the crossing exists is close to 1 if m is large and $p > 2/3$.

Finally, the probability that the first condition is satisfied is $p_1 \geq p_x(2m, m, p)$. The probability that the second condition is satisfied is $p_2 \geq p_x(m, m, p)$. Given that the existence of the two crossings are positively correlated, by the FKG inequality [18], the probability that both conditions are satisfied is lower bounded by:

$$p_{12} \geq p_1 p_2 \geq p_x(2m, m, p) p_x(m, m, p).$$

The probability that there is at least one node from G_2 in R (i.e., the third condition is satisfied) is $p_3 = 1 - e^{-2D^2\lambda_2}$. Given that the point processes in G_1 and G_2 are independent, the probability that a bond is open is $p_{123} = p_{12}p_3$. As long as $p_{123} > 0.8639$, G_{IntDep} percolates. This completes the proof.

3.3.3 An example of two RGGs with large d_2/d_1

We study two interdependent RGGs G_1 and G_2 , which have a finite number of nodes, in order to quantify d_2/d_1 as a function of the number of nodes in the graph. If $d_2 = \Omega(d_1 \log n_1)$, and $d_{\text{dep}} \geq d_2/2$, then $m = \Omega(\log n_1)$, where n_1 is the expected number of nodes in G_1 . As n_1 approaches infinity, the probability $p_x(km, m, p)$ approaches 1 if $p > 2/3$, by Eq. 2.

Applying Theorem 2, by solving $p = (1 - e^{-\lambda_1 d_1^2/8}) = 2/3$, and $p_3 = 1 - e^{-2D^2\lambda_2} = 0.8639$, we obtain an upper bound on percolation threshold $\lambda_1 = 8.789/d_1^2$, $\lambda_2 = 19.94/d_2^2$. The bounds suggest that if the ratio between the connection distances of two RGGs is very large, the node density in one RGG may not affect the minimum node density in the other RGG in order for the giant mutual component to exist in the interdependent RGGs.

We conjecture that as long as the node density of each individual RGG is above the percolation threshold of the single graph, then the interdependent RGGs percolate, if $d_1 \ll d_2$ and $d_{\text{dep}} = (1 + \epsilon)d_2/2$ for $\epsilon > 0$. This can be intuitively explained as below. Let V_2^0 denote the nodes in the giant component of a single graph G_2 without considering the interdependence. Disks of radius $d_2/2$ centered at nodes in V_2^0 are connected. Disks of radius $d_{\text{dep}} > d_2/2$ centered at nodes in V_2^0 are also connected, and this region contains nodes in G_1 that have functional supply nodes. Each disk of radius d_{dep} is so large compared with d_1 , that the probability that there is a crossing formed by connected nodes in

G_1 along any direction across the disk approaches one³. Moreover, the disks of radius d_{dep} have overlaps with width and height at least $\epsilon d_2 \gg d_1$, which are sufficiently large to join the paths in G_1 across two overlapping disks. Thus, a giant component of G_1 exists near the giant component of G_2 . Nodes in the two components are interdependent and form a giant mutual component.

3.4 Numerical results

We verify the bounds in Theorem 1 by simulating G_{IntDep} in a 10×10 square. Table 1 illustrates the fraction of nodes from G_i that belong to the largest mutual component, denoted by f_i , ($\forall i \in \{1, 2\}$). The fractions are averaged over 5 instances of simulations for each combination of $(\lambda_1, \lambda_2, d_1, d_2, d_{\text{dep}})$ that satisfies the condition in Theorem 1. To verify the bounds in Theorem 2, we simulate G_{IntDep} in a 30×30 square (to simulate a sufficiently large G_2 under small node densities). Table 2 illustrates the average fraction of nodes in the largest mutual component, for $(\lambda_1, \lambda_2, d_1, d_2, d_{\text{dep}})$ given by Theorem 2. We observe that most nodes in G_1 and G_2 belong to the largest mutual component, which implies that G_{IntDep} percolates.

TABLE 1

Fraction of nodes in the largest mutual component under the condition of Theorem 1

λ_1	λ_2	d_1	d_2	d_{dep}	f_1	f_2
15	1.54	1	3	1.5	1.00	1.00
20	0.92	1	3	1.5	0.99	1.00
25	0.75	1	3	1.5	0.98	1.00
15	2.39	1	2	1	0.99	1.00
20	1.80	1	2	1	1.00	1.00
25	1.58	1	2	1	0.97	1.00

TABLE 2

Fraction of nodes in the largest mutual component under the condition of Theorem 2

λ_1	λ_2	d_1	d_2	d_{dep}	f_1	f_2
16	0.190	1	10	7.07	1.00	1.00
17	0.123	1	10	7.07	1.00	1.00
25	0.100	1	10	7.07	1.00	1.00
17	0.385	1	8	5.66	1.00	1.00
18	0.207	1	8	5.66	1.00	1.00
25	0.156	1	8	5.66	0.99	1.00

Remark: We have assumed that $d_{\text{dep}} \geq \max(d_1/2, d_2/2) = d_2/2$ throughout this section. To see that this is a reasonable assumption, note that nodes in G_1 that have at least one functional supply node are restricted in the region R_{dep} , where R_{dep} is the union of disks with radius d_{dep} centered at nodes in the infinite component of G_2 . If R_{dep} is fragmented, it is not likely for disks of radius $d_1/2 < d_2/2$ centered at random locations within R_{dep} to overlap, and it is not likely that a functional

infinite component will exist in G_1 , unless the node density in G_1 is large. Therefore, the interdependent distance d_{dep} should be large enough so that R_{dep} is a connected region, to avoid a large minimum node density in G_1 . The region R_{dep} can be made larger by increasing either λ_2 or d_{dep} . Setting $d_{\text{dep}} \geq d_2/2$ avoids increasing λ_2 high above the percolation threshold of G_2 , in order for R_{dep} to be connected. In Section 4, we develop a more general approach that does not require this assumption.

4 CONFIDENCE INTERVALS FOR PERCOLATION THRESHOLDS

In this section, we compute confidence intervals for percolation thresholds. The confidence intervals provide interval estimates for the percolation thresholds. If the node densities in G_{IntDep} are below the lower confidence bounds, then there does not exist an infinite mutual component in G_{IntDep} with high confidence. On the other hand, if the node densities are above the upper confidence bounds, then there exists an infinite mutual component in G_{IntDep} with high confidence. Compared with the analytical upper bounds in Section 3, the numerical upper confidence bounds are much tighter. Moreover, the techniques in this section apply to G_{IntDep} with general d_1, d_2, d_{dep} .

The mapping to compute confidence intervals is related to the mapping from G_{IntDep} to the 1-dependent bond percolation model L_D in Section 3.3. Both mappings satisfy the following properties: 1) the percolation of L_D implies the percolation of G_{IntDep} ; 2) the event that determines the state of a bond depends only on the point process within its associated rectangle, thus preserving the 1-dependent property. The probability that the event occurs can be computed or bounded analytically in the previous section. In contrast, in this section, we consider events whose probabilities are larger under the same point processes but can only be evaluated by simulation. Since the events that we consider in this section are more likely to occur under the same point processes, the mappings yield tighter bounds.

Our mappings from G_{IntDep} to L_D extend the mappings from $G(\lambda, d)$ to L_D proposed in [16]. For completeness, we first briefly summarize the mappings in [16] that compute upper and lower bounds on the percolation threshold of $G(\lambda, d)$.

Upper bound for $G(\lambda, d)$ [16]: Recall Fig. 4. The event that a bond $(v_1, v_2) \in L_D$ is open is determined by the point process of $G(\lambda, d)$ in the rectangle $R = S_1 \cup S_2$, where S_1 and S_2 are squares. Let V_i denote the largest component formed by the points of $G(\lambda, d)$ in S_i . If V_i is the unique largest component in S_i ($\forall i \in \{1, 2\}$) and V_1 and V_2 are connected, then the bond is open. Otherwise, the bond is closed.

If L_D percolates, open bonds form an infinite component. As a result, the largest components in the squares that intersect the open bonds are connected in $G(\lambda, d)$ and they form an infinite component. Therefore, a node density λ , above which the probability that a bond is open is larger than 0.8639, is an upper bound on the percolation threshold of $G(\lambda, d)$.

Lower bound for $G(\lambda, d)$ [16]: Let the connection process of $G(\lambda, d)$ be the union of nodes and links in $G(\lambda, d)$. Let

3. If nodes are generated by a Poisson point process with density above the percolation threshold, the probability that there is a horizontal path across a $kl \times l$ rectangle approaches one for any k as $l \rightarrow \infty$ [12].

the *complement* of the connection process be the union of the empty space that does not intersect nodes or links. If the complement of the connection process form a connected infinite region, then all the connected components in $G(\lambda, d)$ have finite sizes and $G(\lambda, d)$ does not percolate [16], [19]. Consider the complement of the connection process in rectangle R . Let a bond (in L_D) associated with rectangle R be open if the complement process forms a horizontal crossing⁴ over the rectangle R' and a vertical crossing over the square S'_1 . Recall that rectangle R' is the $(2D - 2d) \times (D - 2d)$ rectangle that has the same center as R , and square S'_1 is the $(D - 2d) \times (D - 2d)$ square that has the same center as S_1 , the left square in R . For example, in Fig. 7, the two crossings that do not intersect any nodes or links are plotted.

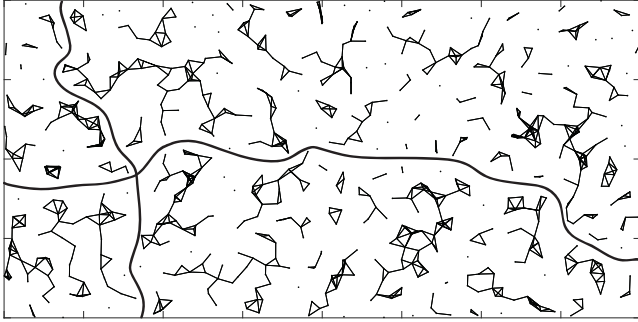


Fig. 7. The horizontal and vertical crossings from the complement of the connection process over the rectangle.

If L_D percolates, the complement process forms an infinite region and $G(\lambda, d)$ does not percolate. To conclude, a node density, under which the probability that the complement process forms the two crossings is above 0.8639, is a lower bound on the percolation threshold for $G(\lambda, d)$.

4.1 Upper bounds for G_{IntDep}

In $G(\lambda, d)$, the largest connected component that contains a node b can be computed efficiently by contracting the links (or using a breadth-first-search) starting from b . Two components are connected and form one component if there exists two nodes within distance d , one in each component. We next extend these notions to G_{IntDep} .

Let G_1 and G_2 denote the two graphs in G_{IntDep} . Let $b_1 \in G_1$ and $b_2 \in G_2$ denote two nodes within the interdependent distance d_{dep} . Algorithm 1 computes the largest mutual component $M(b_1, b_2)$ that contains b_1 and b_2 . The correctness follows from the definition of mutual component.

Two mutual components $M = V_1 \cup V_2$ and $\hat{M} = \hat{V}_1 \cup \hat{V}_2$ form one mutual component if and only if V_i and $\hat{V}_i$ are connected in G_i ($\forall i \in \{1, 2\}$). The necessity of the condition is obvious. To see that this condition is sufficient, note that every node in the connected component formed by V_i and $\hat{V}_i$ has at least one supply node that belongs to the connected component formed by V_j and $\hat{V}_j$ ($\forall i, j \in \{1, 2\}, i \neq j$). The

4. The complement of a connection process forms a horizontal crossing over a rectangle if a curve in the rectangle touches the left and right boundaries of the rectangle and the curve does not intersect any nodes or links. The vertical crossing of the complement process is defined analogously.

Algorithm 1 Computing the largest mutual component that contains two specified nodes $b_i \in G_i$ within d_{dep} ($\forall i \in \{1, 2\}$).

- 1) Find all the nodes $V_i^0(b_i)$ that are connected to b_i (either directly or through a sequence of links) in G_i ($\forall i \in \{1, 2\}$).
- 2) Remove nodes in $V_i^0(b_i)$ that do not have any supply nodes in $V_j^0(b_j)$ ($\forall i, j \in \{1, 2\}, i \neq j$). Among the remaining nodes, find the nodes $V_i^1(b_i) \subseteq V_i^0(b_i)$ that are connected to b_i ($\forall i \in \{1, 2\}$).
- 3) Repeat step 2 until $V_i^{k+1}(b_i) = V_i^k(b_i)$ ($\forall i \in \{1, 2\}$). Let $M(b_1, b_2) = V_1^k(b_1) \cup V_2^k(b_2)$.

condition can be generalized naturally for more than two mutual components to form one mutual component.

The method of obtaining an upper bound on the percolation threshold of $G(\lambda, d)$ can be modified to obtain an upper bound on the percolation threshold of G_{IntDep} , by declaring a bond to be open if the unique largest mutual components in the two adjacent $D \times D$ squares S_1 and S_2 are connected. However, computing the largest mutual component of G_{IntDep} in S_i is not as straightforward as computing the largest component of $G(\lambda, d)$ in S_i . In $G(\lambda, d)$, a node belongs to exactly one (maximal) connected component. All the components can be obtained by contracting the links, and the largest component can be obtained by comparing the sizes of the components. However, in G_{IntDep} , a node may belong to multiple mutual components. For example, let b_1 and b_2 be two isolated nodes in G_1 , and let b_3 and b_4 be two connected nodes in G_2 . If both b_1 and b_2 are within the interdependent distance from b_3 and b_4 , $\{b_1, b_3, b_4\}$ and $\{b_2, b_3, b_4\}$ are two mutual components. An algorithm that computes the largest mutual component of G_{IntDep} in a square 1) selects a pair of nodes, one from each graph, and computes the largest mutual component that contains the two nodes by Algorithm 1, and then 2) chooses the largest mutual component over all pairs of nodes in the square within the interdependent distance. Thus, it requires much more computation than finding the largest component of $G(\lambda, d)$ in a square.

Instead of optimizing the algorithm and obtaining the largest mutual component in square S , a mutual component $M^{\text{greedy}}(S)$ can be computed by Algorithm 2. This algorithm has good performance in finding a large mutual component when the square size is large. In particular, if the square had infinite size, this algorithm would find an infinite mutual component if one exists.

Let a bond (v_1, v_2) in L_D be open if the two components $M^{\text{greedy}}(S_1)$ and $M^{\text{greedy}}(S_2)$ form one mutual component. Since $M^{\text{greedy}}(S_i)$ is unique in any square S_i , a connected component in L_D implies that $\{M^{\text{greedy}}(S_i)\}$ form one mutual component in G_{IntDep} , where S_i are the squares that intersect the open bonds in the connected component in L_D . If the probability that a bond is open is larger than 0.8639, L_D percolates and G_{IntDep} also percolates.

An alternative condition for a bond to be open is that nodes in $M^{\text{greedy}}(R)$ form a horizontal crossing over rectangle R' and a vertical crossing over square S'_1 in both graphs (recall Fig. 5 and the condition for two mutual components

Algorithm 2 An algorithm that greedily computes a mutual component $M^{\text{greedy}}(S)$ in region S .

- 1) Find the largest connected component $V_i^0(S)$ in $G_i(S)$, where $G_i(S)$ consists of the nodes and links of G_i in region S . If there is more than one largest connected component, apply any deterministic tie-breaking rule (e.g., choose the component that contains a node with the smallest x -coordinate).
- 2) Remove nodes in $V_i^0(S)$ that do not have supply nodes in $V_j^0(S)$ ($\forall i, j \in \{1, 2\}, i \neq j$). Find the largest connected component $V_i^1(S)$ formed by the remaining nodes in $V_i^0(S)$ ($\forall i \in \{1, 2\}$), and apply the same tie-breaking rule.
- 3) Repeat step 2 until $V_i^{k+1}(S) = V_i^k(S)$ ($\forall i \in \{1, 2\}$). Let $M^{\text{greedy}}(S) = V_1^k(S) \cup V_2^k(S)$.

to form one mutual component). In order for the existence of the two crossings to only depend on the point processes in R , in the definition of the $(2D - 2d) \times (D - 2d)$ rectangle R' and the $(D - 2d) \times (D - 2d)$ square S'_1 , $d = \max(d_1, d_2) + d_{\text{dep}}$.

An upper bound on the percolation threshold can be obtained by either approach. The smaller bound obtained by the two approaches is a better upper bound on the percolation threshold for G_{IntDep} .

4.2 Lower bounds for G_{IntDep}

In G_{IntDep} , the connection process consists of nodes and links in mutual components. To avoid the heavy computation of mutual components, we study another model in which the connection process $\tilde{P}_i$ of G_i in the new model *dominates*⁵ the connection process P_i of G_i in G_{IntDep} ($\forall i \in \{1, 2\}$). As a consequence, the complement of the connection process $\tilde{P}_i^c$ of G_i in the new model is dominated by P_i^c ($\forall i \in \{1, 2\}$). If $\tilde{P}_i^c$ percolates, then P_i^c percolates and P_i does not percolate (i.e., all the components in P_i have finite sizes). If either P_1 or P_2 does not percolate, then G_{IntDep} does not percolate. Thus, node densities under which at least one of $\tilde{P}_1^c$ and $\tilde{P}_2^c$ percolates are lower bounds on the percolation thresholds of G_{IntDep} .

The new model can be viewed to have a *relaxed* supply requirement. In this model, every node (as opposed to nodes in the same mutual component) is viewed as a valid supply node for nodes in the other graph. A node b_i in G_i is removed if and only if there is no node in G_j within the interdependent distance d_{dep} from b_i ($\forall i, j \in \{1, 2\}, i \neq j$). After all such nodes are removed, the remaining nodes in G_i are connected if their distances are within the connection distance d_i . The computation of the connection process $\tilde{P}_i$ is efficient and avoids the computation of mutual components in G_{IntDep} through multiple iterations.

The connection process $\tilde{P}_i$ in the new model dominates P_i in the original model G_{IntDep} . On the one hand, for any realization, all the links in P_i are present in $\tilde{P}_i$, because all the nodes in a mutual component have supply nodes, and links between these nodes are present in the new model

5. One connection process dominates another if the nodes and links in the first process form a superset of the nodes and links in the second process, for any realization of G_i .

as well. On the other hand, in the new model, nodes in a connected component $\tilde{V}_i$ in G_i may depend on nodes in multiple components in G_j . In contrast, in G_{IntDep} , the nodes in $\tilde{V}_i$ may be divided into several mutual components, and links do not exist between two disjoint mutual components.

An algorithm that computes a lower bound on the percolation threshold of G_{IntDep} is as follows. First, compute the connection process $\tilde{P}_i$ in the new model. Next, in the $2D \times D$ rectangle R , consider the complement of the connection process $\tilde{P}_i^c$. Let p_i denote the probability that there is a horizontal crossing over R' and a vertical crossing over S'_1 in the complement process $\tilde{P}_i^c$, where R' and S'_1 are the same as before. A lower bound on the percolation threshold of G_{IntDep} is given by node densities under which $\max(p_1, p_2) \geq 0.8639$.

4.3 Confidence intervals

The probability that a bond is open can be represented by an integral that depends on the point processes in the rectangle R . However, direct calculation of the integral is intractable; so instead the integral is evaluated by simulation. In every trial of the simulation, nodes in G_1 and G_2 are randomly generated by the Poisson point processes with densities λ_1 and λ_2 , respectively. The events that a bond is open are independent in different trials. Let the probability that a bond is open be p given (λ_1, λ_2) . The probability that a bond is closed in k out of N trials follows a binomial distribution. The interval $[0.8639, 1]$ is a 99.5% confidence interval [20] for p , given that $N = 100$ and $k = 5$. If $k < 5$, $p \in [0.8639, 1]$ with a higher confidence. This suggests that if $k \leq 5$, with 99.5% confidence, $p \geq 0.8639$ and the 1-dependent bond percolation model L_D percolates given (λ_1, λ_2) .

Based on this method, with 99.5% confidence, an upper bound on the percolation threshold of G_{IntDep} can be obtained by declaring a bond to be open using the method in Section 4.1, and a lower bound can be obtained by declaring a bond to be open using the method in Section 4.2. For a fixed λ_2^* , a 99% confidence interval for λ_1^* is given by the interval between the upper and lower bounds. Confidence intervals for different percolation thresholds can be obtained by changing the value of λ_2^* and repeating the computation. We make a similar remark as in [16]. The confidence intervals are rigorous, and the only uncertainty is caused by the stochastic point processes in the $2D \times D$ rectangle. This is in contrast with the confidence intervals obtained by estimating whether G_{IntDep} percolates based on extrapolating the observations of simulations in a finite region (which is usually not very large because of limited computational power).

4.4 Numerical results

The simulation-based confidence intervals are much tighter than the analytical bounds. Given that $d_1 = d_2 = 2d_{\text{dep}} = 1$, and $\lambda_2^* = 2$, the upper and lower bounds on λ_1^* are 2.25 and 1.80, respectively, both with 99.5% confidence. In contrast, even if $\lambda_2^* \rightarrow \infty$, the analytical upper bound on λ_1^* is no less than 3.372, which is the best available analytical upper bound for a single G_1 [15]. Confidence intervals for

the percolation thresholds are plotted in Fig. 8, where the intervals between bars are 99% confidence intervals.

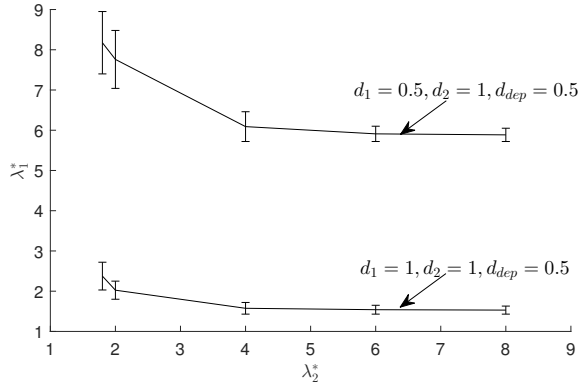


Fig. 8. The 99% confidence intervals for percolation thresholds of G_{IntDep} with different connection distances.

To verify the confidence intervals, we simulate G_{IntDep} within a 20×20 square, for $d_1 = d_2 = 2d_{\text{dep}} = 1$. Nodes in the largest mutual component are colored black, while the remaining nodes are colored blue. In Fig. 9, the node densities are at the upper confidence bound ($\lambda_1 = 2.25, \lambda_2 = 2.00$), and there exists a mutual component that consists of a large fraction of nodes. In Fig. 10, the node densities are at the lower confidence bound ($\lambda_1 = 1.80, \lambda_2 = 2.00$), and the size of the largest mutual component is small.

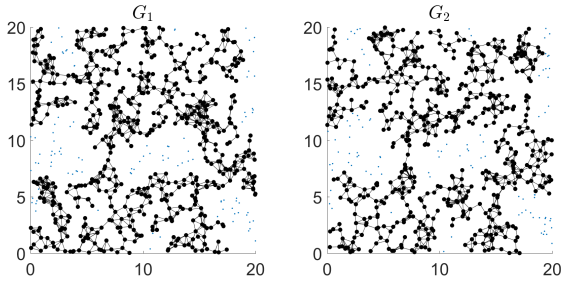


Fig. 9. The largest mutual component for $\lambda_1 = 2.25, \lambda_2 = 2.00, d_1 = d_2 = 2d_{\text{dep}} = 1$.

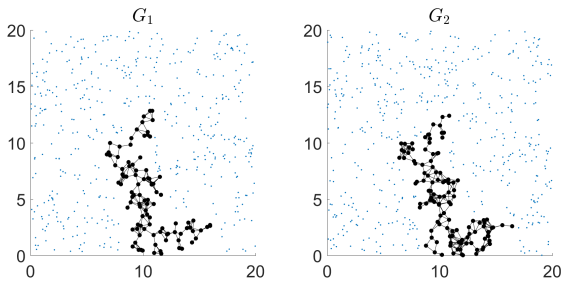


Fig. 10. The largest mutual component for $\lambda_1 = 1.80, \lambda_2 = 2.00, d_1 = d_2 = 2d_{\text{dep}} = 1$.

We next study the impact of interdependent distance d_{dep} on the percolation thresholds. Given d_1, d_2, λ_2^* , a

smaller d_{dep} leads to a higher λ_1^* , since the probability that a node in G_1 has at least one supply nodes from G_2 decreases for a smaller d_{dep} . The effect is more significant when the number of nodes in G_2 is small. This is consistent with Fig. 11, where the increase of λ_1^* is more significant as d_{dep} decreases when λ_2^* is small.

The confidence intervals confirm that the reduced node density due to a lack of supply nodes is not sufficient to characterize the percolation of one of the interdependent graphs. The average density of nodes in G_1 that have at least one node within d_{dep} in G_2 is $\tilde{\lambda}_1 = \lambda_1(1 - e^{-\lambda_2 \pi d_{\text{dep}}^2})$, given that $e^{-\lambda_2 \pi d_{\text{dep}}^2}$ is the probability that there is no node in G_2 within a disk area πd_{dep}^2 . If $\lambda_2^* = 1.8$, with 99% confidence, $\lambda_1^* \in [2.03, 2.72]$ when $d_{\text{dep}} = 0.5$, and $\lambda_1^* \in [7.50, 11.20]$ when $d_{\text{dep}} = 0.25$. We observe that the ranges of $\tilde{\lambda}_1^*$ are different: $\tilde{\lambda}_1^* \in [1.54, 2.06]$ when $d_{\text{dep}} = 0.5$, and $\tilde{\lambda}_1^* \in [2.23, 3.33]$ when $d_{\text{dep}} = 0.25$. Intuitively, nodes in G_1 that have at least one supply nodes are clustered around the nodes in G_2 , smaller d_{dep} leads to a more clustered point process. The critical node density of a clustered point process is not the same as the critical node density of the homogeneous Poisson point process for percolation. More detailed study on the percolation of a clustered point process can be found in [21].

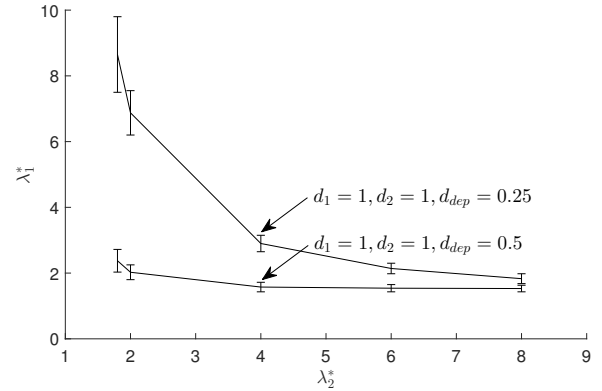


Fig. 11. The 99% Confidence intervals for percolation thresholds of two G_{IntDep} with different interdependence distances.

5 ROBUSTNESS OF INTERDEPENDENT RGGs UNDER RANDOM AND GEOGRAPHICAL FAILURES

Removing nodes independently at random with the same probability in one graph is equivalent to reducing the node density of the Poisson point process. To study the robustness of G_{IntDep} under random failures, the first step is to obtain the upper and lower bounds on percolation thresholds. With the bounds, we can determine which graph is able to resist more random node removals, by comparing the gap between the node density λ_i and the percolation threshold λ_i^* given λ_j ($i, j \in \{1, 2\}, i \neq j$). The graph that can resist a smaller fraction of node removals is the bottleneck for the robustness of G_{IntDep} . Moreover, we are able to compute the maximum fraction of nodes that can be randomly removed from two graphs while guaranteeing G_{IntDep} to be percolated.

We next show that G_{IntDep} still percolates after a geographical attack that removes nodes in a finite connected region, if the node densities of the two graphs before the attack are above any *upper bound* on the percolation thresholds obtained in this paper (either analytical or simulation-based). Recall that we obtained upper bounds on the percolation thresholds of G_{IntDep} by mapping the percolation of G_{IntDep} to either the independent bond percolation on a square lattice L or the 1-dependent bond percolation on a square lattice L_D . Under both mappings, the event that a bond e is open is entirely determined by the point processes in a finite region R_e that contains the bond. After removing nodes of G_{IntDep} in a connected finite geographical region, the state of a bond e may change from open to closed only if R_e intersects the attack region. Let R_f be the union of R_e that intersects the attack region. The region R_f is also a connected finite region. As long as L or L_D still percolates after setting bonds in R_f to be closed, G_{IntDep} percolates.

Results from the percolation theory indeed indicate that setting all the bonds in a finite region R_f to be closed does not affect the percolation of L or L_D . For any percolated L , the probability that there exists a horizontal crossing of open bonds over a $kl \times l$ rectangle approaches 1 for any integer $k > 1$, as $l \rightarrow \infty$ (Lemma 8 on Page 64 of [18]). The percolation of L (after setting all bonds in R_f to be closed) is justified by the fact that the connected open bonds across rectangles form a square annulus that does not intersect R_f (shown in Fig. 12), which is a standard approach to prove the percolation of L [18]. Moreover, the percolation of L_D after all bonds in R_f are set closed can be proved in the same approach, by noting that the probability that open bonds of L_D form a horizontal crossing over a rectangle approaches 1 as the rectangle size increases to infinity [16].

If the $kl \times l$ rectangle is large but finite, the probability that a horizontal crossing formed by open bonds exists is close to 1 if L or L_D percolates. Therefore, the same analysis demonstrates the robustness of two finite interdependent RGGs under a geographical attack that removes the nodes in a disk region of size βa^2 , where $0 < \beta < 1$.

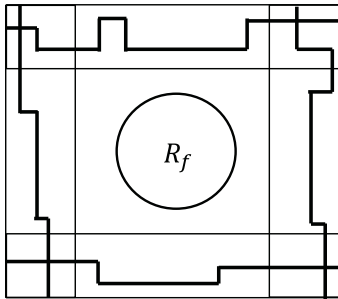


Fig. 12. Open bonds form a connected path across rectangles around R_f .

The robustness of interdependent RGGs under geographical failures is illustrated in Fig. 13. Nodes and links in the giant mutual component are colored black. The interdependent RGGs still percolate after all the nodes in a disk region are removed. This is in contrast with the cascading failures observed in [5] in the interdependent lattice model after an initial disk attack. One reason may be that every node can have more than one supply node

in our model, while every node has only one supply node in [5]. The multiple localized interdependence helps the interdependent RGGs to resist geographical attacks.

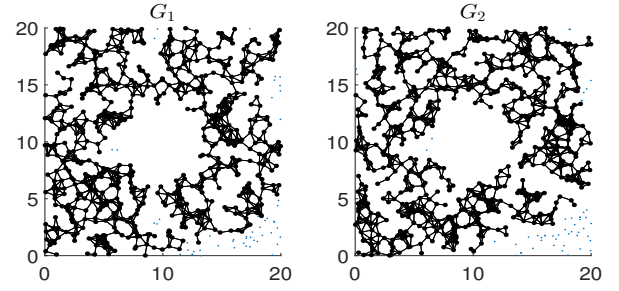


Fig. 13. Interdependent RGGs with the same connection distance $d_1 = d_2 = 1$ and $d_{\text{dep}} = 0.5$.

6 EXTENSIONS TO MORE GENERAL INTERDEPENDENCE

In the previous sections, we studied a model where every node in G_i is content to have at least one supply node in G_j in the same mutual component ($\forall i, j \in \{1, 2\}, i \neq j$). The techniques can be extended to study models where every node in G_i must have at least K_j supply nodes from G_j to receive enough supply, where K_j can be either a constant or a random variable ($\forall i, j \in \{1, 2\}, i \neq j$). We briefly discuss the extensions to models with more general supply requirement using the example in Section 3.1, where $d_1 = d_2 = 2d_{\text{dep}}$.

6.1 Deterministic supply requirement

The extension is straightforward if K_i is a constant, $\forall i \in \{1, 2\}$. By the same discretization technique, the state of a site in the triangle lattice is determined by the point processes in a cell of area A (recall Fig. 2). Declare a site to be open if there are at least K_i nodes from G_i in the cell that contains the site ($\forall i, j \in \{1, 2\}, i \neq j$). For each open site, every node from G_i in the cell has at least K_j supply nodes from G_j in the same cell, satisfying the supply requirement. Following the same analysis as that in Section 3.1, the percolation of the triangle lattice implies the percolation of G_{IntDep} .

For a Poisson point process of density λ_j , the probability that there are at least K_j nodes in a cell of area A is $1 - \sum_{l=0}^{K_j-1} (\lambda_j A)^l e^{-\lambda_j A} / l!$. An upper bound on the percolation thresholds is given by (λ_1, λ_2) that satisfies:

$$\left[1 - \sum_{l=0}^{K_1-1} \frac{(\lambda_1 A)^l e^{-\lambda_1 A}}{l!} \right] \left[1 - \sum_{l=0}^{K_2-1} \frac{(\lambda_2 A)^l e^{-\lambda_2 A}}{l!} \right] = \frac{1}{2}.$$

6.2 Random supply requirement

Some extra work is necessary if K_i is a random variable, $\exists i \in \{1, 2\}$. For simplicity, we first consider the case where $K_1 \geq 1$ is a constant and K_2 is a discrete random variable with a cumulative distribution function $F_{K_2}(x)$, $x \in \mathbb{N}$. Furthermore, we assume that the number of supply nodes needed by every node in G_1 is independent. After

the discretization, a site in the triangle lattice is open if the following two conditions are satisfied for at least one integer-valued $k_2 \geq 1$.

- 1) There are exactly k_2 nodes from G_2 in the cell.
- 2) There are at least K_1 nodes from G_1 in the cell, each of which needs no more than k_2 supply nodes.

If both conditions are satisfied, at least K_1 nodes from G_1 and the k_2 nodes from G_2 each have enough supply. It is easy to see that the percolation of the triangle lattice still implies the percolation of G_{IntDep} .

Next we compute the probability that the two conditions are satisfied. The probability that there are k_2 nodes from G_2 in the cell is:

$$\Pr(N_2 = k_2) = (\lambda_2 A)^{k_2} e^{-\lambda_2 A} / k_2!.$$

The probability that there are l nodes from G_1 in the cell is:

$$\Pr(N_1 = l) = (\lambda_1 A)^l e^{-\lambda_1 A} / l!.$$

The probability that a node in G_1 needs no more than k_2 supply nodes is $F_{K_2}(k_2)$. Since the number of supply nodes needed by every node in G_1 is independent, the probability that at least K_1 out of the l nodes in G_1 each need no more than k_2 supply nodes is:

$$\Pr(K_2^{(K_1)} \leq k_2 | N_1 = l) = \sum_{t=K_1}^l \binom{l}{t} [F_{K_2}(k_2)]^t [1 - F_{K_2}(k_2)]^{l-t},$$

for $K_1 \leq l$, and $\Pr(K_2^{(K_1)} \leq k_2 | N_1 = l) = 0$ for $K_1 > l$. By the law of total probability, for a given k_2 , the probability that there exist at least K_1 nodes from G_1 in the cell that each need no more than k_2 supply nodes is:

$$\Pr(K_2^{(K_1)} \leq k_2) = \sum_{l \geq K_1} \Pr(N_1 = l) \Pr(K_2^{(K_1)} \leq k_2 | N_1 = l).$$

Since the events that there are exactly k_2 nodes from G_2 in the cell are mutually exclusive for distinct values of k_2 , using the law of total probability again, the probability that both conditions are satisfied is:

$$p_{12} = \sum_{k_2 \geq 1} \Pr(N_2 = k_2) \Pr(K_2^{(K_1)} \leq k_2).$$

Any (λ_1, λ_2) that satisfies $p_{12} \geq 1/2$ is an upper bound on the percolation threshold of G_{IntDep} .

Finally, we consider the case where both K_1 and K_2 are discrete random variables. Suppose that N_i nodes from G_i are in the cell of area A . If there exist integers $k_i^* \leq N_i$, such that at least k_i^* nodes from G_i each need no more than k_j^* supply nodes, then the k_i^* nodes from G_i all have enough supply ($\forall i, j \in \{1, 2\}, i \neq j$). However, it is difficult to obtain a clean formula of the probability that (k_1^*, k_2^*) exists (to satisfy the condition). The events that (k_1^*, k_2^*) exists are not mutually exclusive for distinct values of k_1^* and k_2^* . While it is possible to compute this probability using the inclusion-exclusion formula, the computation is expensive, since the number of choices of (k_1^*, k_2^*) can be large and each term in the inclusion-exclusion formula requires the computation of order statistics.

A practical approach to estimate the probability that nodes have enough supply is by simulation. In each trial

of the simulation, N_i nodes are randomly generated in area A , where N_i follows a Poisson distribution of rate $\lambda_i A$ ($\forall i \in \{1, 2\}$). Then, each of the N_i nodes is tagged with a realization of the random variable K_j , which indicates the number of required supply nodes ($\forall i, j \in \{1, 2\}, i \neq j$). Let I indicate whether there exist (k_1^*, k_2^*) such that at least k_i^* nodes among the N_i nodes all have tags no more than k_j^* ($\forall i, j \in \{1, 2\}, i \neq j$). The value of I can be computed by Algorithm 3.

Algorithm 3 An algorithm that determines whether nodes have enough supply.

Initialization:

Sort the N_i realizations of the random variable K_j in the ascending order. Let $K_j^{(t)}, t = 1, \dots, N_i$ be the sorted list ($\forall i, j \in \{1, 2\}, i \neq j$). Let $t_1 = t_2 = 1$.

Main loop:

```

while  $I$  is not determined do
     $t'_1 \leftarrow K_1^{(t'_2)}, t'_2 \leftarrow K_2^{(t'_1)}$ .
    if  $t'_1 \leq t_1$  and  $t'_2 \leq t_2$  then
         $I \leftarrow 1$ .
    end if
    if  $t'_1 > N_1$  or  $t'_2 > N_2$  then
         $I \leftarrow 0$ .
    end if
     $t_1 \leftarrow \max(t_1, t'_1), t_2 \leftarrow \max(t_2, t'_2)$ .
end while
    
```

We now prove the correctness of Algorithm 3. For easy presentation, the N_i nodes are referred to as nodes in G_i ($\forall i \in \{1, 2\}$). Initially, among the nodes in G_i , the algorithm chooses one node that needs the smallest number of supply nodes. To support this node, at least $t'_j = K_j^{(t'_j)}$ nodes need to be in G_j . If $t'_1 \leq 1$ and $t'_2 \leq 1$, one node from G_1 and one node from G_2 suffice to support each other. Otherwise, if $t'_j > 1$, at least t'_j nodes need to be in G_j . The t'_j nodes must be supported by $K_i^{(t'_j)}$ nodes from G_i . If $K_i^{(t'_j)}$ is larger than the total number of nodes in G_i , then there are not enough supporting nodes in G_i and $I = 0$. If $K_1^{(t'_2)} \leq t'_1$ and $K_2^{(t'_1)} \leq t'_2$, then t'_1 nodes from G_1 support t'_2 nodes from G_2 , and vice versa. Note that t'_1 and t'_2 never decrease in the iterations, and at least one of them strictly increases in an iteration where I is not determined. If there exists at least one pair (k_1^*, k_2^*) , the algorithm terminates with $I = 1$ at the smallest pair for both coordinates, which can be easily shown by contradiction. If no such pair (k_1^*, k_2^*) exists, the algorithm terminates with $I = 0$.

Given (λ_1, λ_2) , by repeating a sufficiently large number of trials, the probability that $I = 1$ can be estimated within a small multiplicative error with high confidence using Monte Carlo simulation. As long as this probability is at least $1/2$, G_{IntDep} percolates with high confidence.

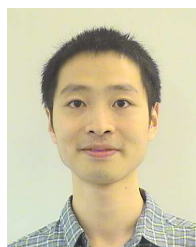
7 CONCLUSION

We developed an interdependent RGG model for interdependent spatially embedded networks. We obtained analytical upper bounds and confidence intervals for the percolation thresholds. The percolation thresholds of two

interdependent RGGs form a curve, which shows the trade-off between the two node densities in order for the interdependent RGGs to percolate. The curve can be used to study the robustness of interdependent RGGs to random failures. Moreover, if the node densities are above any upper bound on the percolation thresholds obtained in this paper, then the interdependent RGGs remain percolated after a geographical attack. Finally, we extended the techniques to models with more general interdependence. The study of percolation thresholds in this paper can be used to design robust interdependent networks.

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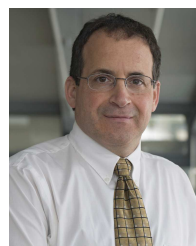
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