Dual-rate $\mathcal{L}_1$ Adaptive Controller for Cyber-Physical Sampled-Data Systems

Hamidreza Jafarnejadsani[1], Hanmin Lee[2], Naira Hovakimyan[3], and Petros Voulgaris[4]

Abstract— **This paper proposes a dual rate output-feedback control approach for sampled-data MIMO systems with nonlinear uncertainties. Design and analysis of $\mathcal{L}_1$ adaptive controllers are extended to sampled-data systems framework. The controller is designed for detection of zero-dynamics attacks, and disturbance and uncertainty compensation. In this paper, a sufficient condition on the sampling time of the digital controller is obtained that ensures stability of the closed-loop system. It is shown that the proposed controller can recover the performance of a continuous-time reference system. A simulation study of an automatic voltage regulator is provided to validate the theoretical finding.**

I. INTRODUCTION

Many important cyber-physical systems (CPSs) such as power grids, transportation, and financial systems are subject to cyber attacks. The sampled-data (SD) nature of control systems in these infrastructures can generate additional vulnerability to stealthy attacks due to the sampling zeros in the SD system [1], [2]. From controls system perspective, the hardest to defend are the so-called zero dynamics attacks on the output-feedback systems. If the closed-loop system possesses an unstable zero, an (ultimately) unbounded actuator (or sensor) attack cannot be observed by the monitoring data.

As shown in [2], an interesting property of multirate sampling is its ability to remove certain unstable zeros of the discrete-time system when viewed in the lifted linear time-invariant (LTI) domain. Multirate sampling has been studied extensively in the context of sampled-data control, and relevant analysis and synthesis results have been reported in [3]–[5], to mention only a few. In the literature, the problem of SD output-feedback control is addressed by introducing high-gain observers to estimate the unmeasured states [6]–[8]. Disturbance compensation for SD output-feedback control of nonlinear systems is studied in [9]–[11]. Although stability analysis of SD nonlinear systems has been the focus in these papers, the study of transient performance in nonlinear systems with SD output-feedback control has not received much attention.

[1]Hamidreza Jafarnejadsani is with the Department of Mechanical Science and Engineering, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA jafarne2@illinois.edu

[2]Hanmin Lee is with the Department of Aerospace Engineering, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA hlee170@illinois.edu

[3]Naira Hovakimyan is with the Department of Mechanical Science and Engineering, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA nhovakim@illinois.edu

[4]Petros Voulgaris is with the Department of Aerospace Engineering, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA voulgari@illinois.edu

This paper aims to extend the $\mathcal{L}_1$ adaptive control theory to multirate SD systems. In continuous-time framework, the $\mathcal{L}_1$ adaptive control is known as a robust technique, for which performance bounds and robustness margins can be quantified analytically [12]–[14]. The controller compensates for uncertainties within the bandwidth of a lowpass filter. Focusing on a multirate scheme that allows attacks to be detected, the approach of this paper contributes to the body of work on SD output-feedback control of systems with nonlinear uncertainties and disturbances, by providing uniform bounds on the reference tracking errors. In addition, analytical results are derived, that provide a sufficient condition on the sampling rate of SD system to preserve the performance of the underlying continuous-time structure.

The rest of the paper is organized as follows. Section II presents the problem statement. In Section III, the structure of the dual rate controller is presented. The closed-loop SD system is analyzed in Section IV. Section V presents a simulation example. Finally, Section VI concludes the paper.

II. PROBLEM STATEMENT

A. Zero-Dynamics Attack

Consider a continuous-time LTI plant P_c, and the corresponding discrete time LTI plant $P_d = \mathcal{S}P_c\mathcal{H}$, which is defined with the standard zero order hold and sample devices $\mathcal{H}$ and $\mathcal{S}$. The relationship between P_c and P_d follows from the following definition.

Definition 1: Given an LTI system with transfer function $P_c(s)$ with its minimum realization (A_c, B_c, C_c, D_c), the equivalent step-invariant discrete-time system with z-transform $P_d[z]$ is given by the following state-space matrices:

$$A_d = e^{A_c T},\ B_d = \int_0^T e^{A_c \tau} B_c d\tau,\ C_d = C_c,\ D_d = D_c,$$

where $T > 0$ is the sampling time.

Most control systems are naturally implemented on a digital processor with the usual sample and hold elements. Even if the continuous plant has no unstable zeros, its discrete representation obtained by the sampled and hold operations may introduce unstable zeros (so-called "sampling zeros") [15].

If a system has non-minimum phase zeros, the actuator attack signal $d[i] = \epsilon {z_0}^{-i}$ can remain undetected for small enough ϵ, where z_0 is the unstable zero of the system [2]. This attack can be easily implemented in the cyber space as an additive disturbance. This unbounded signal can blow up the states of the physical system, while the observed output and control command dictate normal behavior. To deal with this problem, a multirate scheme can be applied, since it allows the attack to be detected by ensuring that there are no relevant unstable zeros in the lifted system.

Theorem 1 ([2]): Let the lifted LTI system be $\bar{P}_d = \mathcal{S}_N P_c \mathcal{H}$, where the output is sampled with the period $\frac{T}{N}$, where N is a large integer. Suppose the system P_c has full rank B_c, and $\mathcal{O} = [C_c^\top, (C_c A_c)^\top, \ldots, (C_c A_c^{N-2})^\top]^\top$ is of full rank. Then, the lifted LTI system $\bar{P}_d$ does not have any unstable zeros.

From Theorem 1, unbounded zero-dynamics attacks can be detectable, if the control system is designed in the dual rate sampled-data framework. Motivated by the work on multirate sampled-data (MSRD) control [2], this paper proposes a dual rate $\mathcal{L}_1$ adaptive controller whereby the outputs are sampled at a multiple of the hold rates for the purpose of attack detection.

B. Mathematical Formulation

Throughout this paper, $\|x_\tau\|_{\mathcal{L}_\infty}$ denotes the $\mathcal{L}_\infty$ norm of the truncated signal $x_\tau(t)$ for original $x(t) \in \mathbb{R}^n$, given as

$$\begin{aligned} x_\tau(t) &= x(t), \quad \forall t \leq \tau, \\ x_\tau(t) &= 0, \quad \text{otherwise}. \end{aligned}$$

The notation $\|.\|_p$ represents the vector or matrix p-norms with $1 \leq p \leq \infty$. Also, z denotes z-transform variables while s is used for the Laplace transform.

Consider the following MIMO system

$$\begin{aligned} &\dot{x}(t) = A_p x(t) + B_p \left(u(t) + f(t, x(t)) + d(t)\right), \, x(0) = x_0, \\ &y(t) = C_p x(t), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input signal, and $y(t) \in \mathbb{R}^m$ is the system output vector. Also, $\{A_p \in \mathbb{R}^{n\times n}, B_p \in \mathbb{R}^{n\times m}, C_p \in \mathbb{R}^{m\times n}\}$ is an observable-controllable triple. The transmission zeros of the system with the state-space realization (A_p, B_p, C_p) are assumed to be stable. The initial output signal, $y_0 \triangleq C_m x_0$, is assumed to be known, where x_0 is an unknown initial condition. Let $d(t) \in \mathbb{R}^m$ be the zero-dynamics attack on the actuator. Finally, $f(t, x) \in \mathbb{R}^m$ represents the time-varying uncertainties and disturbances subject to the following assumption.

Assumption 1: There exist constants $L_0 > 0$, $L_1 > 0$, and $L_2 > 0$ such that

$$\begin{aligned} \|f(t, x_2) - f(t, x_1)\|_\infty &\leq L_1 \|x_2 - x_1\|_\infty, \\ \|f(t, x)\|_\infty &\leq L_1 \|x\|_\infty + L_0, \\ \|d(t)\|_\infty &\leq L_2 \end{aligned}$$

hold uniformly in $t \geq 0$.

Remark 1: The boundedness of the attack signal $d(t)$ can be realized by assuming a secure software/hardware structure for the CPS (ex. Simplex architecture [16], [17]). In such structure, a backup controller will operate the system, when the normal mode controller is compromised due to a cyber attack. By switching from the normal mode to a secured backup controller, the unbounded stealthy attack can be removed (from cyber space), rendering $d(t)$ bounded.

In this paper, a dual rate $\mathcal{L}_1$ adaptive controller is introduced for sampled-data systems with baseline dual rate controllers to compensate for uncertainties and to detect the zero-dynamics attacks. The overall control input, which is implemented via a zero-order hold mechanism with time period of $T_s > 0$, is given by

$$\begin{aligned} u(t) &= u_d[i], \quad t \in [iT_s, (i+1)T_s), \quad i \in \mathbb{Z}_{\geq 0}, \\ u_d[z] &= K[z]\bar{y}_d[z] + r_d[z] + u_{cd}[z], \end{aligned} \tag{2}$$

where $\bar{y}_d[z]$ is the z-transform of the output signal $\bar{y}_d[i]$

$$\bar{y}_d[i] = \left[y^\top(iT_s), \ldots, y^\top\left(\frac{(Ni+N-1)T_s}{N}\right)\right]^\top, \tag{3}$$

and $N \in \mathbb{N}$ is the ratio between the hold and sampling rates, that is, the output $y(t)$ is sampled with the period of $\frac{T_s}{N}$. Also, $r_d[z]$ is the z-transform of the discrete reference command $r_d[i]$, and $K[z]$ represents the transfer function of a baseline dual rate controller represented in the lifted domain with sampling time of T_s. The command signal is assumed to be bounded, such that $\|r_d[i]\|_\infty \leq M_r$, where M_r is a positive constant. This paper considers augmentation of the control input $u_{cd}[z]$ for the purpose of compensating uncertainties and recovering the performance of the ideal baseline system, which is defined as follows

$$\begin{aligned} \dot{x}_b(t) &= A_p x_b(t) + B_p u_b(t), \;\; x_b(0) = 0, \\ y_b(t) &= C_p x_b(t), \\ u_b(t) &= u_{bd}[i], \quad t \in [iT_s, (i+1)T_s), \quad i \in \mathbb{Z}_{\geq 0}, \\ u_{bd}[z] &= K[z]\bar{y}_{bd}[z] + r_d[z], \end{aligned} \tag{4}$$

where $\bar{y}_{bd}[z]$ is the z-transform of the ideal output signal $\bar{y}_{bd}[i]$ defined as

$$\bar{y}_{bd}[i] = \left[{y_b}^\top(iT_s), \ldots, {y_b}^\top\left(\frac{(Ni+N-1)T_s}{N}\right)\right]^\top. \tag{5}$$

Here, it is assumed that the controller with the transfer function $K[z]$ stabilizes the multi-rate sampled-data system in (4). Therefore, for bounded command signal $r_d[i]$, there exists a constant $M_x > 0$ such that

$$\|x_b(t)\|_\infty \leq M_x, \; \forall t \geq 0. \tag{6}$$

Remark 2: In [2] and [18], such dual rate controller is proposed using a linear quadratic Gaussian (LQG) structure, and it is shown that the closed-loop system is exponentially stable.

Let

$$\begin{aligned} u_{cd}[z] &= u_{ad}[z] - K[z]\bar{y}_{ad}[z], \\ u_a(t) &= u_{ad}[i], \quad t \in [iT_s, (i+1)T_s), \quad i \in \mathbb{Z}_{\geq 0}, \end{aligned} \tag{7}$$

where $\bar{y}_{ad}[z]$ is the z-transform of

$$\bar{y}_{ad}[i] \triangleq \bar{y}_d[i] - \bar{y}_{bd}[i], \tag{8}$$

and $u_{ad}[i]$ is an adaptive control input to be introduced shortly. Next, consider the system

$$\begin{aligned} \dot{x}_a(t) &= A_m x_a(t) + B_p \left(u_a(t) + \sigma(t)\right), \; x_a(0) = x_0, \\ y_a(t) &= C_p x_a(t), \end{aligned} \tag{9}$$

where $F \in \mathbb{R}^{m\times n}$ is chosen such that $A_m \triangleq A_p - B_pF$ is Hurwitz, and

$$\sigma(t) = Fx_a(t) + f\left(t, x_a(t) + x_b(t)\right) + d(t). \tag{10}$$

It can be shown that the following bound on $\sigma(t)$ holds

$$\|\sigma_t\|_{\mathcal{L}_\infty} \leq L_3 \|x_{a_t}\|_{\mathcal{L}_\infty} + L_4, \tag{11}$$

where

$$L_3 \triangleq \|F\|_1 + L_1, \quad L_4 \triangleq L_0 + L_1M_x + L_2.$$

Moreover, the sampled output of the system in (9)

$$y_{ad}[j] = y_a\left(j\frac{T_s}{N}\right), \quad \left[j\frac{T_s}{N}, (j+1)\frac{T_s}{N}\right), \quad j \in \mathbb{Z}_{\geq 0} \tag{12}$$

can be obtained from (8), where $\bar{y}_d[i]$ is given in (3). Also, $\bar{y}_{bd}[j]$ is defined in (5), in which $y_b\left(j\frac{T_s}{N}\right) = y_{bd}[j]$ and

$$x_{bd}[j+1] = e^{A_p\frac{T_s}{N}}x_{bd}[j] + \left(\int_0^{\frac{T_s}{N}} e^{A_p\tau}B_p d\tau\right)u_{Nbd}[j],$$
$$y_{bd}[j] = C_px_{bd}[j], \quad x_{bd}(0) = 0. \tag{13}$$

Notice that $u_{Nbd}[j]$ is given by

$$u_{Nbd}[j] = u_b\left(j\frac{T_s}{N}\right), \quad j \in \mathbb{Z}_{\geq 0}, \tag{14}$$

where $u_b(t)$ is defined in (4).

Given the two systems in (4) and (9), the following relations can be established: for all $t \geq 0$, $x(t) = x_a(t) + x_b(t)$, $y(t) = y_a(t) + y_b(t)$, and $u(t) = u_a(t) + u_b(t)$.

While (4) represents the ideal baseline system, the system in (9) governs the dynamics due to uncertainties and attacks. In the following, an adaptive output feedback controller, $u_{ad}[z]$, is designed to regulate the output $y_a(t)$ of the system in (9) to zero. At the same time, the augmented control input $u_{cd}[z]$ given in (7) recovers the nominal response provided by the baseline controller.

III. CONTROL DESIGN

We define a few variables of interest and design constraints. Let

$$\begin{aligned} H(s) &\triangleq (s\mathbb{I} - A_m)^{-1}B_p,\\ G(s) &\triangleq H(s)\left(\mathbb{I} - C(s)\right),\\ H_1(s) &\triangleq C(s)M^{-1}(s),\\ H_2(s) &\triangleq H(s)C(s)M^{-1}(s), \end{aligned} \tag{15}$$

where $M(s) \triangleq C_p(s\mathbb{I} - A_m)^{-1}B_p$. The design of the controller proceeds by considering a strictly proper stable transfer function $C(s)$ such that $C(0) = \mathbb{I}_m$. The selection of $C(s)$ must ensure that the following $\mathcal{L}_1$-norm condition holds

$$\|G(s)\|_{\mathcal{L}_1} L_3 < 1, \tag{16}$$

and

$$C(s)M^{-1}(s) \text{ is proper.} \tag{17}$$

Also, since $M(s)$ has stable transmission zeros, $H_1(s)$, $H_2(s)$ are stable and proper transfer matrices.

Given that $A_m \in \mathbb{R}^{n\times n}$ is Hurwitz, there exists a positive definite matrix $P \in \mathbb{R}^{n\times n}$ solving $A_m^\top P + PA_m = -Q$ for a given positive definite matrix $Q \in \mathbb{R}^{n\times n}$. Define

$$\Lambda \triangleq \begin{bmatrix} C_p \\ D\sqrt{P} \end{bmatrix}, \tag{18}$$

where $\sqrt{P}$ satisfies $P = \sqrt{P}^\top\sqrt{P}$, and $D \in \mathbb{R}^{(n-m)\times n}$ is a matrix that contains the null space of $C_p\left(\sqrt{P}\right)^{-1}$, i.e.

$$D\left(C_p\left(\sqrt{P}\right)^{-1}\right)^\top = 0\,. \tag{19}$$

Let $P_1 \in \mathbb{R}^{m\times m}$ and $P_2 \in \mathbb{R}^{(n-m)\times(n-m)}$ be positive definite matrices:

$$P_1 \triangleq C_p\sqrt{P}^{-1}\sqrt{P}^{-\top}C_p^\top, \;\; P_2 \triangleq DD^\top. \tag{20}$$

Let $T > 0$ be a given constant, and

$$\mathbf{1}_{nm} \triangleq \begin{bmatrix} \mathbb{I}_m \\ 0_{(n-m)\times m} \end{bmatrix} \in \mathbb{R}^{n\times m}. \tag{21}$$

Define

$$\begin{bmatrix} \eta_1^\top(t) & \eta_2^\top(t) \end{bmatrix} \triangleq \mathbf{1}_{nm}^\top e^{\Lambda A_m\Lambda^{-1}t}, \tag{22}$$

where $\eta_1(t) \in \mathbb{R}^{m\times m}$ and $\eta_2(t) \in \mathbb{R}^{(n-m)\times m}$, and

$$\kappa(T) \triangleq \int_0^T \left\|\mathbf{1}_{nm}^\top e^{\Lambda A_m\Lambda^{-1}(T-t)}\Lambda B_p\right\|_2 d\tau. \tag{23}$$

Further, let $\Phi(T)$ be the $n \times n$ matrix

$$\Phi(T) \triangleq \int_0^T e^{\Lambda A_m\Lambda^{-1}(T-\tau)}\Lambda d\tau. \tag{24}$$

Let $O(s) \triangleq C(s)M^{-1}(s)C_p(s\mathbb{I} - A_m)^{-1}$, and let (A_q, B_q, C_q) be a minimal state-space realization such that

$$C_q(s\mathbb{I} - A_q)^{-1}B_q \triangleq O(s). \tag{25}$$

Define the function $\Gamma(\cdot)$ as

$$\Gamma(T) \triangleq \alpha_1(T)\left\|(s\mathbb{I} - A_q)^{-1}B_q\right\|_{\mathcal{L}_1} + \alpha_2(T), \tag{26}$$

where

$$\alpha_1(T) \triangleq \max_{t\in[0,\,T]}\left\|C_q\left(e^{A_qt} - \mathbb{I}\right)\right\|_\infty,$$
$$\alpha_2(T) \triangleq \max_{t\in[0,\,T]}\int_0^t \left\|C_qe^{A_q(t-\tau)}B_q\right\|_\infty d\tau.$$

Let

$$\begin{aligned} \Upsilon(T) &= \left\|e^{-A_mT}\Phi^{-1}(T)e^{\Lambda A_m\Lambda^{-1}T}\mathbf{1}_{nm}\right\|_\infty,\\ \Psi(T) &= \left\|H_1(s)C_p(s\mathbb{I} - A_m)^{-1}\left(e^{A_mT} - \mathbb{I}\right)\right\|_{\mathcal{L}_1},\\ \mathrm{X}(T) &= \|H(s)\|_{\mathcal{L}_1}\left(N\Gamma\left(\frac{T}{N}\right) + \Psi\left(\frac{T}{N}\right)\right)\Upsilon\left(\frac{T}{N}\right)\\ &\quad + \|H_2(s)\|_{\mathcal{L}_1},\\ \Omega(T) &= \frac{\mathrm{X}(T)}{1 - \|G(s)\|_{\mathcal{L}_1}L_3},\\ \mathcal{J} &= \frac{2\sqrt{m}\left\|\Lambda^{-\top}PB_p\right\|_2}{\lambda_{\min}\left(\Lambda^{-\top}P\Lambda^{-1}\right)}\sqrt{\frac{\lambda_{\max}\left(\Lambda^{-\top}P\Lambda^{-1}\right)}{\lambda_{\max}\left(P_2\right)}}, \end{aligned} \tag{27}$$

where $H(s)$, $H_1(s)$, $H_2(s)$ are defined in (15), and P_2 is given in (20). Next, we introduce the functions

$$\beta_1(T) \triangleq \max_{t\in[0,\, T]} \|\eta_1(t)\|_2\,, \quad \beta_2(T) \triangleq \max_{t\in[0,\, T]} \|\eta_2(t)\|_2\,, \tag{28}$$

where $\eta_1(t)$ and $\eta_2(t)$ are given in (22). Also

$$\beta_3(T) \triangleq \max_{t\in[0,\, T]} \eta_3(t), \quad \beta_4(T) \triangleq \max_{t\in[0,\, T]} \eta_4(t), \tag{29}$$

where

$$\begin{aligned}\eta_3(t,T) &\triangleq \int_0^t \left\| \mathbf{1}_{nm}^\top e^{\Lambda A_m \Lambda^{-1}(t-\tau)} \Lambda \Phi^{-1}(T) e^{\Lambda A_m \Lambda^{-1} T} \mathbf{1}_{nm} \right\|_2 d\tau,\\ \eta_4(t,T) &\triangleq \int_0^t \left\| \mathbf{1}_{nm}^\top e^{\Lambda A_m \Lambda^{-1}(t-\tau)} \Lambda B_p \right\|_2 d\tau.\end{aligned} \tag{30}$$

The sampling time T_s of the digital controller is chosen such that

$$T_s \in (0,\, T_s^*]\,, \tag{31}$$

where T_s^* is an upper bound, satisfying

$$\mathcal{I}\left(\frac{T_s^*}{N}\right)\left(\sup_{T\in(0,\,T_s^*]} \Omega(T)\right) L_3 < 1, \tag{32}$$

with

$$\mathcal{I}(T) \triangleq \left(\beta_1(T) + \beta_3(T) + 1\right)\left(\beta_2(T)\mathcal{J} + \sqrt{m}\beta_4(T)\right),$$

$\Omega(T)$ and $\mathcal{J}$ being given in (27), and $\beta_1(T)$, $\beta_2(T)$, $\beta_3(T)$, $\beta_4(T)$ being defined in (28)-(29).

Remark 3: It is straightforward to verify that $\Omega(T)$ is a bounded function, as T tends to zero. In addition, we can see that $\beta_1(T)$, $\beta_2(T)$, $\beta_3(T)$, and $\beta_4(T)$ approach arbitrarily closely to zero for sufficiently small T. Therefore, there always exists a constant $T_s^* > 0$ that satisfies the condition in (31).

Next, we consider the following control laws

$$\begin{aligned} x_u[j+1] &= e^{A_q \frac{T_s}{N}} x_u[j] + A_q^{-1}\left(e^{A_q \frac{T_s}{N}} - \mathbb{I}\right) B_q e^{-A_m \frac{T_s}{N}} \hat{\sigma}_d[j],\\ u_{Nad}[j] &= -C_q x_u[j], \quad j \in \mathbb{Z}_{\geq 0}, \quad x_u[0] = 0,\\ u_{Na}(t) &= u_{Nad}[j], \quad t \in \left[j\frac{T_s}{N}, (j+1)\frac{T_s}{N}\right)\\ u_{ad}[i] &= u_{Na}(iT_s), \quad i \in \mathbb{Z}_{\geq 0}, \end{aligned} \tag{33}$$

where the system matrices $(A_q,\, B_q,\, C_q)$ satisfy (25). Also, $\hat{\sigma}_d[\cdot] \in \mathbb{R}^n$ is provided by the adaptation laws. The output predictor is given by

$$\begin{aligned} \hat{x}_{ad}[j+1] =& e^{A_m \frac{T_s}{N}} \hat{x}_{ad}[j]\\ &+ A_m^{-1}(e^{A_m \frac{T_s}{N}} - \mathbb{I})\left(B_p u_P[j] + \hat{\sigma}_d[j]\right),\\ \hat{y}_{ad}[j] =& C_p \hat{x}_{ad}[j], \quad j \in \mathbb{Z}_{\geq 0}, \quad \hat{x}_{ad}[0] = \hat{x}_0, \end{aligned} \tag{34}$$

where the estimated initial condition $\hat{x}_0$ is given by $\hat{x}_0 = C_p^\dagger y_0$ ($C_p^\dagger$ is the pseudo-inverse of C_p). Also, the predictor control input $u_P[j]$ is defined as

$$u_P[j] = u_a\left(j\frac{T_s}{N}\right), \quad j \in \mathbb{Z}_{\geq 0}, \tag{35}$$

where $u_a(t)$ is given by (7) and (33). The adaptation laws are governed by the following update laws

$$\begin{aligned} \hat{\sigma}_d[j] &= -\Phi^{-1}\left(\frac{T_s}{N}\right)\mu[j],\\ \mu[j] &= e^{\Lambda A_m \Lambda^{-1}\frac{T_s}{N}} \mathbf{1}_{nm} \tilde{y}_{ad}[j], \quad j \in \mathbb{Z}_{\geq 0}, \end{aligned} \tag{36}$$

where $\tilde{y}_{ad}[j] = \hat{y}_{ad}[j] - y_{ad}[j]$, and $\Phi(\cdot)$ is defined in (24).

IV. ANALYSIS OF THE CLOSED-LOOP DUAL-RATE SYSTEM

In this section, the analysis of the sampled-data system is presented, and sufficient conditions for stability of the closed-loop system are obtained. We proceed by defining a few variables of interest:

$$\begin{aligned} \lambda_0 &= \left\|H_1(s)C_p(s\mathbb{I} - A_m)^{-1}(\hat{x}_0 - x_0)\right\|_{\mathcal{L}_\infty},\\ \lambda_1 &= \frac{\left\|H_2(s)C_p(s\mathbb{I} - A_m)^{-1}(\hat{x}_0 - x_0)\right\|_{\mathcal{L}_\infty}}{1 - \|G(s)\|_{\mathcal{L}_1} L_3},\\ \Theta(T) &= \|H_1(s)\|_{\mathcal{L}_1} + \left(N\Gamma\left(\frac{T}{N}\right) + \Psi\left(\frac{T}{N}\right)\right)\Upsilon\left(\frac{T}{N}\right),\\ \rho_r &= \frac{\left\|(s\mathbb{I} - A_m)^{-1}x_0\right\|_{\mathcal{L}_\infty} + \|G(s)\|_{\mathcal{L}_1} L_4}{1 - \|G(s)\|_{\mathcal{L}_1} L_3}, \end{aligned} \tag{37}$$

where $H(s)$, $H_1(s)$, and $H_2(s)$ are introduced in (15), and $\Gamma(\cdot)$ is defined in (26). Also, $\Upsilon(\cdot)$ and $\Psi(\cdot)$ are given in (27).

Let

$$\Delta(T,\, \epsilon) = L_3\left(\rho_r + \Omega(T)\epsilon + \lambda_1\right) + L_4, \tag{38}$$

where $\epsilon \in \mathbb{R}^+$. Also, let $\varsigma(T,\, \epsilon)$ and $\alpha(T,\, \epsilon)$ be defined as

$$\varsigma(T,\, \epsilon) \triangleq \left\|\eta_2\left(\frac{T}{N}\right)\right\|_2 \sqrt{\frac{\alpha(T,\, \epsilon)}{\lambda_{\max}(P_2)}} + \sqrt{m}\kappa\left(\frac{T}{N}\right)\Delta(T,\, \epsilon), \tag{39}$$

$$\begin{aligned}\alpha(T,\, \epsilon) \triangleq& \lambda_{\max}\left(\Lambda^{-\top}P\Lambda^{-1}\right)\left(\frac{2\sqrt{m}\Delta(T,\, \epsilon)\left\|\Lambda^{-\top}PB_p\right\|_2}{\lambda_{\min}\left(\Lambda^{-\top}P\Lambda^{-1}\right)}\right)^2\\ &+ \xi_0^\top \Lambda^{-\top} P \Lambda^{-1} \xi_0,\end{aligned} \tag{40}$$

where $\eta_2(T)$ is defined in (22) and $\kappa(T)$ is given in (23). Also, $\xi_0 = \Lambda\left(\hat{x}_0 - x_0\right)$, and x_0 and $\hat{x}_0$ are the unknown and estimated initial conditions, respectively.

Finally, define

$$\begin{aligned}\gamma_0(T,\, \epsilon) \triangleq& \beta_1\left(\frac{T}{N}\right)\varsigma(T,\, \epsilon) + \beta_2\left(\frac{T}{N}\right)\sqrt{\frac{\alpha(T,\, \epsilon)}{\lambda_{\max}(P_2)}}\\ &+ \beta_3\left(\frac{T_s}{N}\right)\varsigma(T,\, \epsilon) + \sqrt{m}\beta_4\left(\frac{T}{N}\right)\Delta(T,\, \epsilon).\end{aligned} \tag{41}$$

Lemma 1: For arbitrary $\xi = \begin{bmatrix} y \\ z \end{bmatrix} \in \mathbb{R}^n$, where $y \in \mathbb{R}^m$ and $z \in \mathbb{R}^{(n-m)}$, there exist positive definite $P_1 \in \mathbb{R}^{m\times m}$ and $P_2 \in \mathbb{R}^{(n-m)\times(n-m)}$ such that

$$\xi^\top\left(\Lambda^{-1}\right)^\top P \Lambda^{-1}\xi = y^\top P_1 y + z^\top P_2 z, \tag{42}$$

where Λ is given in (18). Also, P_1 and P_2 are defined in (20).

Proof: The proof of Lemma 1 is found in [13]. ■

Lemma 2: Let $O_d[z]$ denote the z-transform of a step-invariant discrete-time approximation of $O(s)$, that is defined in (25). Given a bounded discrete-time signal $r_d[j]$, define $r(t) = r_d[j]$ for $t \in [jT, (j+1)T)$, $j \in \mathbb{Z}_{\geq 0}$, where $T > 0$ is a sampling time. Then, the following holds

$$\|(\varepsilon - \varepsilon')_t\|_{\mathcal{L}_\infty} \leq \Gamma(T) \|r_t\|_{\mathcal{L}_\infty}, \tag{43}$$

where $\Gamma(\cdot)$ is defined in (26); $\varepsilon(t)$ is the signal with Laplace transform of $\varepsilon(s) = O(s)r(s)$, and $\varepsilon'(t) = \varepsilon_d[j]$, $t \in [jT, (j+1)T)$, $j \in \mathbb{Z}_{\geq 0}$, where $\varepsilon_d[j]$ is the discrete signal with the z-transform $\varepsilon_d[z] = O_d[z]r_d[z]$.

Proof: The proof is straightforward and hence omitted here. ■

Consider the following closed-loop reference system

$$\begin{aligned} &\dot{x}_{ref}(t) = A_m x_{ref}(t) + B_p\left(u_{ref}(t) + \sigma_{ref}(t)\right), \\ &u_{ref}(s) = -C(s)\sigma_{ref}(s), \\ &y_{ref}(t) = C_p x_{ref}(t), \quad x_{ref}(0) = x_0, \end{aligned} \tag{44}$$

where $\sigma_{ref}(s)$ is the Laplace transform of $\sigma_{ref}(t)$ given by

$$\sigma_{ref}(t) = F x_{ref}(t) + f\left(t, x_{ref}(t) + x_b(t)\right) + d(t). \tag{45}$$

The reference system in (44) defines *the best performance* that can be achieved by the closed-loop system given in (9), (33)-(36), where instead of the estimates the actual unknown signals are used in (44). Notice that $\sigma_{ref}(t)$ is unknown, and this reference system is used only for the analysis purposes. To prove the stability of the closed-loop sampled-data system with the digital controller proposed in (33)-(36), we introduce a condition for stability of the reference system in (44). Then, we establish uniform bounds between the closed-loop system defined by (9), (33)-(36) and the reference system.

Lemma 3: Let $C(s)$ and $M(s)$ verify the $\mathcal{L}_1$-norm condition. Then, the closed-loop reference system is BIBO stable and the following holds

$$\|x_{ref}\|_{\mathcal{L}_\infty} \leq \rho_r < \infty, \tag{46}$$

where ρ_r is given in (37).

Proof. The proof is similar to the proof of Lemma 4.2.3 in [13] and is omitted. ■

We consider an equivalent state-space model of the predictor dynamics in (34) given by

$$\begin{aligned} &\dot{\hat{x}}_a(t) = A_m \hat{x}_a(t) + B_p u_a(t) + \hat{\sigma}(t), \quad \hat{x}_a(0) = \hat{x}_0, \\ &\hat{y}_a(t) = C_p \hat{x}_a(t), \end{aligned} \tag{47}$$

where

$$\hat{\sigma}(t) = \hat{\sigma}_d[j], \quad , j \in \mathbb{Z}_{\geq 0}, \quad t \in \left[j\frac{T_s}{N},\ (j+1)\frac{T_s}{N}\right), \tag{48}$$

and $u_a(t)$ is given by (7) and (33). Since $\hat{\sigma}(t)$ and $u_a(t)$ are piecewise constants in (47), from (34) we have

$$\hat{y}_a\left(j\frac{T_s}{N}\right) = \hat{y}_{ad}[j], \quad j \in \mathbb{Z}_{\geq 0}. \tag{49}$$

Let $\tilde{x}_a(t) = \hat{x}_a(t) - x_a(t)$. Then the prediction error dynamics between (9) and (47) are given by

$$\begin{aligned} &\dot{\tilde{x}}_a(t) = A_m \tilde{x}_a(t) + \hat{\sigma}_a(t) - B_p\sigma(t), \quad \tilde{x}(0) = \hat{x}_0 - x_0, \\ &\tilde{y}_a(t) = C_p \tilde{x}_a(t), \end{aligned} \tag{50}$$

where $\hat{\sigma}(t)$ is defined in (48).

Lemma 4: Consider the closed-loop system defined by (9), (33)-(36), and the closed-loop reference system in (44). The following upper bounds hold

$$\left\|(x_{ref} - x_a)_t\right\|_{\mathcal{L}_\infty} \leq \Omega(T_s)\|\tilde{y}_{a_t}\|_{\mathcal{L}_\infty} + \lambda_1,$$

where $\Omega(\cdot)$ is given in (27), λ_1 is defined in (37), and $\tilde{y}_a(t)$ is the prediction error defined in (50).

Proof. Let

$$u_C(s) = -C(s)M^{-1}(s)C_p(s\mathbb{I} - A_m)^{-1}\hat{\sigma}(s), \tag{51}$$

$$u_M(s) = -C(s)M^{-1}(s)C_p(s\mathbb{I} - A_m)^{-1}e^{-A_m\frac{T_s}{N}}\hat{\sigma}(s), \tag{52}$$

It follows from (50) that

$$\begin{aligned} \tilde{y}(s) = &- M(s)\sigma(s) + C_m(s\mathbb{I} - A_m)^{-1}\hat{\sigma}(s) \\ &+ C_p(s\mathbb{I} - A_m)^{-1}(\hat{x}_0 - x_0). \end{aligned} \tag{53}$$

Letting $e(t) \triangleq x_{ref}(t) - x_a(t)$ and denoting by $d_e(s)$ the Laplace transform of

$$d_e(t) \triangleq \sigma_{ref}(t) - \sigma(t), \tag{54}$$

from (9), (7), (33), (44), (51), (52), and (53) it follows

$$\begin{aligned} e(s) = &H(s)C(s)M^{-1}(s)\tilde{y}_a(s) + H(s)\left(\mathbb{I} - C(s)\right)d_e(s) \\ &- H(s)C(s)M^{-1}(s)C_p(s\mathbb{I} - A_m)^{-1}(\hat{x}_0 - x_0) \\ &+ H(s)\left(u_C(s) - u_M(s)\right) + H(s)\left(u_M(s) - u_a(s)\right), \end{aligned} \tag{55}$$

where $H(s)$ is defined in (15). The upper bound is given by

$$\begin{aligned} &\|e_t\|_{\mathcal{L}_\infty} \leq \left\|H(s)C(s)M^{-1}(s)\right\|_{\mathcal{L}_1}\|\tilde{y}_{a_t}\|_{\mathcal{L}_\infty} \\ &+\|H(s)\|_{\mathcal{L}_1}\left\|(u_C - u_M)_t\right\|_{\mathcal{L}_\infty} + \|H(s)\|_{\mathcal{L}_1}\left\|(u_M - u_a)_t\right\|_{\mathcal{L}_\infty} \\ &+\left\|H(s)C(s)M^{-1}(s)C_p(s\mathbb{I} - A_m)^{-1}(\hat{x}_0 - x_0)\right\|_{\mathcal{L}_\infty} \\ &+\|H(s)\left(\mathbb{I} - C(s)\right)\|_{\mathcal{L}_1} L_3\|e_t\|_{\mathcal{L}_\infty}. \end{aligned} \tag{56}$$

From (12) and (49), we have

$$\tilde{y}_a\left(j\frac{T_s}{N}\right) = \tilde{y}_{ad}[j], \quad j \in \mathbb{Z}_{\geq 0}. \tag{57}$$

From (36), (48), and (57), the following relation holds

$$\left\|e^{-A_m\frac{T_s}{N}}\hat{\sigma}_t\right\|_{\mathcal{L}_\infty} \leq \Upsilon\left(\frac{T_s}{N}\right)\|\tilde{y}_{a_t}\|_{\mathcal{L}_\infty}, \tag{58}$$

where $\Upsilon(\cdot)$ is defined in (27). Notice that $u_{Nad}[j]$ given in (33) is a step-invariant discrete-time approximation of $u_M(s)$, given in (52). Therefore, using(7), (25), (33), and Lemma 2, we have

$$\|(u_M - u_a)_t\|_{\mathcal{L}_\infty} \leq N\Gamma\left(\frac{T_s}{N}\right)\Upsilon\left(\frac{T_s}{N}\right)\|\tilde{y}_{a_t}\|_{\mathcal{L}_\infty}. \tag{59}$$

Moreover, form (51), (52), and (58) one can obtain

$$\|(u_C - u_M)_t\|_{\mathcal{L}_\infty} \leq \Psi\left(\frac{T_s}{N}\right)\Upsilon\left(\frac{T_s}{N}\right)\|\tilde{y}_{a_t}\|_{\mathcal{L}_\infty}, \tag{60}$$

where $\Psi(\cdot)$ is defined in (27), and $\Gamma(\cdot)$ is introduced in (43). Also, the triple $(A_q,\, B_q,\, C_q)$ is defined in (25). From (56), (59), and (60), the following upper bound holds

$$\|e_t\|_{\mathcal{L}_\infty} \leq \Omega(T_s)\|\tilde{y}_{a_t}\|_{\mathcal{L}_\infty} + \lambda_1. \tag{61}$$

■

Lemma 5: Let T_s^* satisfy the condition in (31). Then, there exists $\bar{\epsilon} > 0$ such that

$$\gamma_0(T,\, \bar{\epsilon}) < \bar{\epsilon},\ \ \forall T \in (0,\, T_s^*]\,. \tag{62}$$

Moreover, for each $\epsilon > 0$

$$\lim_{T\to 0} \gamma_0(T,\, \epsilon) = 0, \tag{63}$$

where $\gamma_0(T,\, \epsilon)$ is given in (41).

Proof. By substituting (28)-(38) in (41), and recalling (31) and (32), one can easily show the existence of $\bar{\epsilon}$ satisfying (62). Verification of (63) is straightforward. ■

Theorem 2: Consider the system in (9) and the controller in (33)-(36) subject to conditions in (16)-(17). If $T_s \in (0,\, T_s^*]$, where T_s^* satisfies (32), then there exists a constant $\bar{\epsilon} > 0$ such that

$$\|\tilde{y}_{a_t}\|_{\mathcal{L}_\infty} < \bar{\epsilon}, \tag{64}$$

$$\|x_{ref} - x_a\|_{\mathcal{L}_\infty} \leq \gamma_1, \quad \|u_{ref} - u_a\|_{\mathcal{L}_\infty} \leq \gamma_2, \tag{65}$$

where $\tilde{y}_a(t)$ is the prediction error defined in (50), and

$$\begin{aligned} \gamma_1 &\triangleq \Omega(T_s)\bar{\epsilon} + \lambda_1, \\ \gamma_2 &\triangleq \|C(s)\|_{\mathcal{L}_1} L_3 \gamma_1 + \Theta(T_s)\bar{\epsilon} + \lambda_0. \end{aligned} \tag{66}$$

Also, $\Omega(\cdot)$ is defined in (27), and λ_0, λ_1, $\Theta(\cdot)$ are given in (37).

Remark 4: Lemma 5 indicates that arbitrarily small bound on the prediction error $\bar{\epsilon}$ can be achieved as T_s goes to zero. The terms λ_0 and λ_1 in (66) exist due to the nonzero initial condition. For zero initial condition, λ_0 and λ_1 are zero. For $\lambda_0, \lambda_1 \equiv 0$, we can show that γ_1 and γ_2 can be made arbitrarily small by selecting sufficiently small sampling time. This implies that the closed-loop sampled-data system recovers the performance of the continuous-time reference system in (44) as the sampling time goes to zero.

Proof.

Let $\bar{\epsilon}$ be a constant satisfying (62). First, we prove the bound in (64) by a contradiction argument. Since $C_m\hat{x}_0 = y_0$, i.e. $\tilde{y}(0) = 0$, and $\tilde{y}_a(t)$ is continuous, then assuming the opposite implies that there exists t' such that

$$\begin{aligned} &\|\tilde{y}_a(t)\|_\infty < \bar{\epsilon},\ \ \forall\, 0 \leq t < t', \\ &\|\tilde{y}_a(t')\|_\infty = \bar{\epsilon}, \end{aligned} \tag{67}$$

which leads to

$$\left\|\tilde{y}_{a_{t'}}\right\|_{\mathcal{L}_\infty} = \bar{\epsilon}. \tag{68}$$

Let $e(t) \triangleq x_{ref}(t) - x_a(t)$. Lemma 4 and the upper bound in (46) can be used to derive the following bound

$$\begin{aligned} \left\|x_{a_{t'}}\right\|_{\mathcal{L}_\infty} &\leq \|x_{ref\,t'}\|_{\mathcal{L}_\infty} + \|e_{t'}\|_{\mathcal{L}_\infty} \\ &\leq \rho_r + \Omega(T_s)\bar{\epsilon} + \lambda_1. \end{aligned} \tag{69}$$

Moreover,

$$\|\sigma_{t'}\|_{\mathcal{L}_\infty} \leq L_3\left\|x_{a_{t'}}\right\|_{\mathcal{L}_\infty} + L_4,$$

which implies

$$\|\sigma_{t'}\|_{\mathcal{L}_\infty} \leq \Delta(T_s,\, \bar{\epsilon}), \tag{70}$$

where $\Delta(\cdot,\, \cdot)$ is defined in (38).

Consider the state transformation

$$\tilde{\xi} = \Lambda\tilde{x}_a, \tag{71}$$

where Λ is defined in (18). From (50) and (71), it follows

$$\begin{aligned} &\dot{\tilde{\xi}}(t) = \Lambda A_m \Lambda^{-1}\tilde{\xi}(t) + \Lambda\hat{\sigma}(t) - \Lambda B_p \sigma(t), \\ &\tilde{y}_a(t) = \mathbf{1}_{nm}\tilde{\xi}(t), \quad \tilde{\xi}(0) = \Lambda\left(\hat{x}_0 - x_0\right). \end{aligned} \tag{72}$$

From (72), we have

$$\begin{aligned} \tilde{\xi}\left(j\frac{T_s}{N} + t\right) &= e^{\Lambda A_m \Lambda^{-1} t}\tilde{\xi}\left(j\frac{T_s}{N}\right) \\ &+ \int_0^t e^{\Lambda A_m \Lambda^{-1}(t-\tau)}\Lambda\left(\hat{\sigma}\left(j\frac{T_s}{N}\right) - B_p\sigma\left(j\frac{T_s}{N} + \tau\right)\right)d\tau. \end{aligned} \tag{73}$$

Since

$$\tilde{\xi}\left(j\frac{T_s}{N} + t\right) = \begin{bmatrix} \tilde{y}_a(j\frac{T_s}{N} + t) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \tilde{z}(j\frac{T_s}{N} + t) \end{bmatrix},$$

where $\tilde{z}(t) = \left[\tilde{\xi}_{m+1}(t),\, ...,\, \tilde{\xi}_n(t)\right]^\top$, and $\tilde{\xi}(j\frac{T_s}{N} + t)$ can be decomposed as

$$\tilde{\xi}\left(j\frac{T_s}{N} + t\right) = \chi\left(j\frac{T_s}{N} + t\right) + \zeta\left(j\frac{T_s}{N} + t\right), \tag{74}$$

such that

$$\begin{aligned} \chi\left(j\frac{T_s}{N} + t\right) =& e^{\Lambda A_m \Lambda^{-1} t}\begin{bmatrix} \tilde{y}_a\left(j(\frac{T_s}{N})\right) \\ 0 \end{bmatrix} \\ &+ \int_0^t e^{\Lambda A_m \Lambda^{-1}(t-\tau)}\Lambda\hat{\sigma}\left(j\frac{T_s}{N}\right)d\tau, \end{aligned} \tag{75}$$

$$\begin{aligned} \zeta\left(j\frac{T_s}{N} + t\right) =& e^{\Lambda A_m \Lambda^{-1} t}\begin{bmatrix} 0 \\ \tilde{z}(j\frac{T_s}{N}) \end{bmatrix} \\ &- \int_0^t e^{\Lambda A_m \Lambda^{-1}(t-\tau)}\Lambda B_p\sigma\left(j\frac{T_s}{N} + \tau\right)d\tau. \end{aligned} \tag{76}$$

Next, we prove that

$$\begin{aligned} &\left\|\tilde{y}\left(j\frac{T_s}{N}\right)\right\|_2 \leq \varsigma(T_s,\, \bar{\epsilon}), \\ &\tilde{z}^\top\left(j\frac{T_s}{N}\right)P_2\tilde{z}\left(j\frac{T_s}{N}\right) \leq \alpha(T_s,\, \bar{\epsilon}),\ \ \forall j\frac{T_s}{N} \leq t', \end{aligned} \tag{77}$$

where $\varsigma(T_s,\, \epsilon)$ and $\alpha(T_s,\, \epsilon)$ were defined in (39) and (40), respectively. It is straightforward to show that $\|\tilde{y}(0)\|_2 \leq \varsigma(T_s,\, \bar{\epsilon})$, $\tilde{z}^\top(0)P_2\tilde{z}(0) \leq \alpha(T_s,\, \bar{\epsilon})$. Next, for arbitrary $(k+1)\frac{T_s}{N} \leq t'$, $k \in \mathbb{Z}_{\geq 0}$, we can prove that if

$$\left\|\tilde{y}\left(k\frac{T_s}{N}\right)\right\|_2 \leq \varsigma(T_s,\, \bar{\epsilon}), \tag{78}$$

$$\tilde{z}^\top\left(k\frac{T_s}{N}\right)P_2\tilde{z}\left(k\frac{T_s}{N}\right) \leq \alpha(T_s,\, \bar{\epsilon}), \tag{79}$$

then the inequalities in (78)-(79) hold for $k+1$ as well, which would imply that the bounds in (78)-(79) hold for all $k\frac{T_s}{N} \leq t'$. The proof of this part has been omitted due to space limit.

For all $j\frac{T_s}{N}+t \leq t'$, and $t \in \left[0,\, \frac{T_s}{N}\right]$, using the expression from (73), we obtain

$$\begin{aligned}\tilde{y}_a\left(j\frac{T_s}{N}+t\right) =& \mathbf{1}_{nm}^{\top} e^{\Lambda A_m \Lambda^{-1} t}\tilde{\xi}\left(j\frac{T_s}{N}\right) \\ &+ \mathbf{1}_{nm}^{\top}\int_0^t e^{\Lambda A_m \Lambda^{-1}(t-\tau)}\Lambda\hat{\sigma}\left(j\frac{T_s}{N}\right)d\tau \\ &- \mathbf{1}_{nm}^{\top}\int_0^t e^{\Lambda A_m \Lambda^{-1}(t-\tau)}\Lambda B_p\sigma\left(j\frac{T_s}{N}+\tau\right)d\tau.\end{aligned}$$

The upper bound in (70) and the expressions of $\eta_1(\cdot)$, $\eta_2(\cdot)$, $\eta_3(\cdot,\,\cdot)$, and $\eta_4(\cdot,\,\cdot)$, given in (22) and (30), lead to

$$\begin{aligned}&\left\|\tilde{y}_a\left(j\frac{T_s}{N}+t\right)\right\|_2 \leq \|\eta_1(t)\|_2\left\|\tilde{y}_a(j\frac{T_s}{N})\right\|_2 + \|\eta_2(t)\|_2\left\|\tilde{z}(j\frac{T_s}{N})\right\|_2 \\ &+\eta_3\left(t,\,\frac{T_s}{N}\right)\left\|\tilde{y}_a(j\frac{T_s}{N})\right\|_2 + \eta_4\left(t,\,\frac{T_s}{N}\right)\sqrt{m}\Delta(T_s,\,\bar{\epsilon}).\end{aligned}$$

Consider (77) and $\beta_1(\cdot)$, $\beta_2(\cdot)$, $\beta_3(\cdot)$, $\beta_4(\cdot)$ defined in (28)-(29). For arbitrary nonnegative integer j subject to $j\frac{T_s}{N}+t \leq t'$ and for all $t \in \left[0,\, \frac{T_s}{N}\right]$, we have

$$\begin{aligned}\left\|\tilde{y}\left(j\frac{T_s}{N}+t\right)\right\|_2 \leq& \beta_1\left(\frac{T_s}{N}\right)\varsigma(T_s,\,\bar{\epsilon}) + \beta_2\left(\frac{T_s}{N}\right)\sqrt{\frac{\alpha(T_s,\,\bar{\epsilon})}{\lambda_{\max}(P_2)}} \\ &+ \beta_3\left(\frac{T_s}{N}\right)\varsigma(T_s,\,\bar{\epsilon}) + \sqrt{m}\beta_4\left(\frac{T_s}{N}\right)\Delta(T_s,\,\bar{\epsilon}).\end{aligned}$$

Since the right-hand side coincides with the definition of $\gamma_0(T_s,\,\epsilon)$ in (41), we have the bound

$$\|\tilde{y}_a(t)\|_2 \leq \gamma_0(T_s,\,\bar{\epsilon}),\;\; \forall t \in [0,\, t'],$$

which, along with the design constraint on T_s introduced in (62), yields

$$\|\tilde{y}_{a_{t'}}\|_{\mathcal{L}_\infty} < \bar{\epsilon}.$$

This clearly contradicts the statement in (68). Therefore, $\|\tilde{y}_a\|_{\mathcal{L}_\infty} < \bar{\epsilon}$, which proves (64). Further, it follows from Lemma 4 that

$$\|e_t\|_{\mathcal{L}_\infty} \leq \Omega(T_s)\bar{\epsilon} + \lambda_1,$$

which holds uniformly for all $t \geq 0$ and therefore leads to the first upper bound in (65).

To prove the second bound in (65), from (1), (33), (44), (51), and (52), (53), it follows

$$\begin{aligned}u_{ref}(s) - u_a(s) =& C(s)M^{-1}(s)\tilde{y}_a(s) - C(s)d_e(s) \\ &- C(s)M^{-1}(s)C_p(s\mathbb{I}-A_m)^{-1}(\hat{x}_0 - x_0) \\ &+ (u_C(s) - u_M(s)) + (u_M(s) - u_a(s)),\end{aligned} \tag{80}$$

where $d_e(s)$ is the Laplace transform of $d_e(t)$ defined in (54). Then, it leads to

$$\begin{aligned}&\|u_{ref}(s) - u_a(s)\|_{\mathcal{L}_\infty} \leq \left\|C(s)M^{-1}(s)\right\|_{\mathcal{L}_1}\|\tilde{y}_a\|_{\mathcal{L}_\infty} \\ &+\|(u_C(s) - u_M(s))\|_{\mathcal{L}_\infty} + \|(u_M(s) - u_a(s))\|_{\mathcal{L}_\infty} \\ &+\left\|C(s)M^{-1}(s)C_p(sI-A_m)^{-1}(\hat{x}_0 - x_0)\right\|_{\mathcal{L}_\infty} \\ &+\|C(s)\|_{\mathcal{L}_1}L_3\|e\|_{\mathcal{L}_\infty}.\end{aligned} \tag{81}$$

Combining (59), (60), (64), (65), and (81) leads to

$$\|u_{ref}(s) - u_a(s)\|_{\mathcal{L}_\infty} \leq \Theta(T_s)\bar{\epsilon} + \|C(s)\|_{\mathcal{L}_1}L_3\gamma_1 + \lambda_0. \tag{82}$$

This concludes the proof. ■

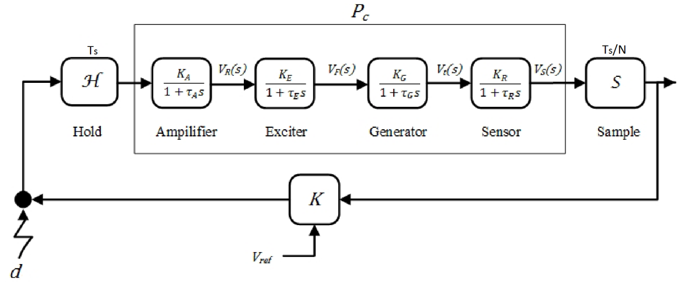


Fig. 1: AVR with system parameters $K_A = 10, \tau_A = 0.1, K_E = 1, \tau_e = 0.4, K_g = 1, \tau_g = 1, K_r = 1, \tau_r = 0.05$.

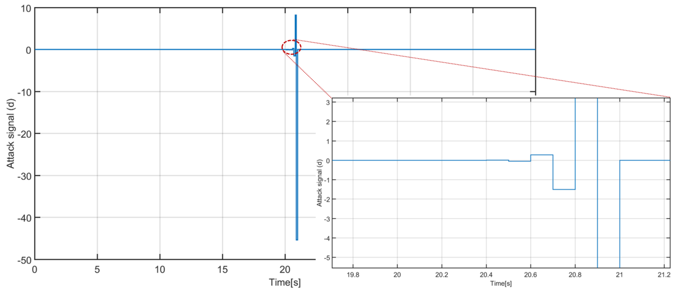
V. Simulation Example

In this section, simulation study of a sampled-data automatic voltage regulator (AVR) system is considered. AVR specifies the terminal voltage magnitude of a synchronous generator by controlling the reactive power. A simplified block diagram of a linearized AVR is shown in Figure 1. The open loop system has unstable zero at $z_0 = -5.49$, if it is sampled at the single rate of $T_s = 0.1sec$. Next, we consider an actuator attack of the form $d[i] = \epsilon z_0^{-i}$, where $\epsilon = 0.00001$ and z_0 is the unstable zero of the system. By faster sampling rate of $\frac{T_s}{N} = 0.05sec$ $(N = 2)$ at the output, the multirate closed-loop system does not have any unstable zeros. Therefore, the attack on the multirate system becomes detectable. Here, the LQG controller of [2], designed for dual rate systems, is adopted as baseline controller. Then, a dual rate $\mathcal{L}_1$ adaptive controller is augmented to detect the stealthy attack and improve the performance of the closed-loop in the presence of disturbance.

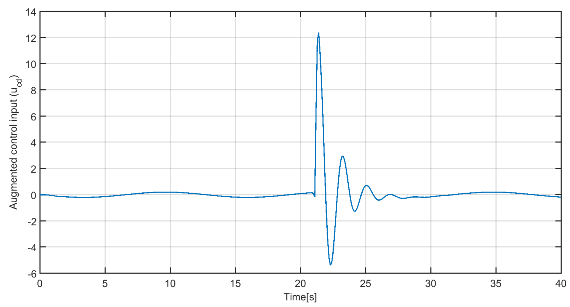
In the simulation, an external unknown disturbance of the form $f(x(t), t) = 0.2\sin(0.5t)$ is applied at the input of the AVR system. Figure 2a shows the zero-dynamics attack signal, which is activated at $t = 20sec$ and is removed at $t = 21sec$. After being detected, the attack signal can be removed by switching to a secured computing platform, which performs as a backup for the compromised controller software. Figure 2b shows the augmented $\mathcal{L}_1$ control input. Figure 2c illustrates the output of the AVR system with/without the augmented dual rate $\mathcal{L}_1$ adaptive controller. It can be seen that the baseline LQG controller cannot compensate for the sinusoidal disturbance at the input of the system, while the augmented $\mathcal{L}_1$ control mitigates the effect of disturbance, efficiently. The adaptation variable $\hat{\sigma}(t)$ is plotted in Figure 2d. Fast rate of change of $\hat{\sigma}(t)$ can be observed prior to removal of attack signal at $t = 21sec$. Therefore, a criteria based on the fast adaptation loop in the dual rate $\mathcal{L}_1$ control structure, such as $\|\hat{\sigma}[i] - \hat{\sigma}[i-1]\|_2 > \Delta_{threshold}$, is a feasible choice for timely attack detection.

VI. Conclusion

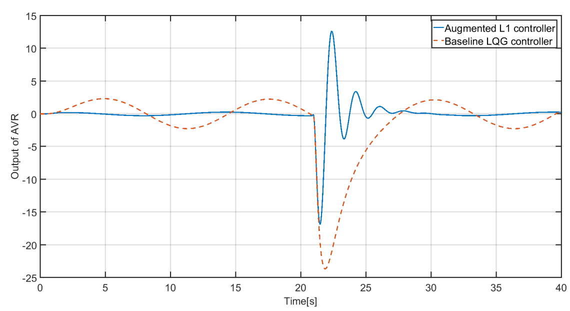
A dual rate output-feedback control approach is proposed for sampled-data MIMO systems with nonlinear uncertainties. A sufficient condition on the sampling time of the digital controller is obtained that ensures stability of the closed-loop system. A simulation study of an automatic voltage



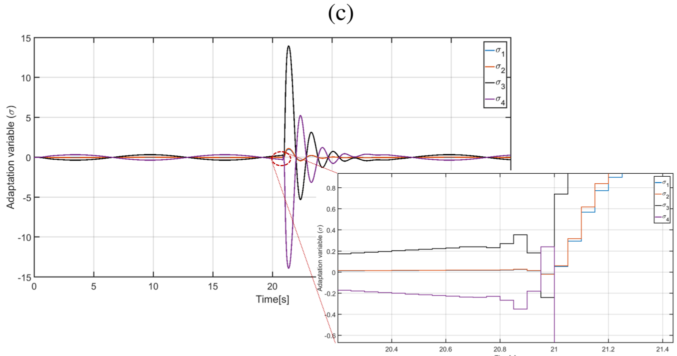
(a)



(b)



(c)



(d)

Fig. 2: Response of the closed-loop AVR system in the presence of the disturbance input $f(x(t), t) = 0.2\sin(0.5t)$ and zero-dynamics attack of the form $d[i] = z_0^{-i}$. The baseline LQG controller is implemented with hold rate of $0.1sec$ and sampling rate of $0.05sec$. (a) Zero-dynamics attack signal. (b) The augmented $\mathcal{L}_1$ control input. (c) Output of AVR with/without augmented $\mathcal{L}_1$ controller. (d) Adaptation variable $\hat{\sigma}(t)$.

regulator is provided to show that the fast estimation loop in the $\mathcal{L}_1$ control structure can detect zero-dynamics attacks. Also, the augmented dual rate $\mathcal{L}_1$ controller compensates for disturbance and uncertainties.

VII. Acknowledgments

This work has been supported in part by NSF and AFOSR.

References

[1] T. Chen and B. Francis, *Optimal sampled-data control systems*, 1995.

[2] M. Naghnaeian, N. Hirzallah, and P. G. Voulgaris, "Dual rate control for security in cyber-physical systems," in *2015 54th IEEE Conference on Decision and Control*, December 2015, pp. 1415–1420.

[3] P. G. Voulgaris, M. A. Dahleh, and L. S. Valavani, "H_∞ and H_2 optimal controllers for periodic and multi-rate systems," in *Proceedings of the 30th IEEE Conference on Decision and Control*, December 1991, pp. 214–216.

[4] P. G. Voulgaris and B. Bamieh, "Optimal H_∞ control of hybrid multirate systems," in *Proceedings of the 31st IEEE Conference on Decision and Control*, December 1992, pp. 457–462.

[5] L. Qiu and T. Chen, "H_2 optimal design of multirate sampled-data systems," *IEEE Transactions on Automatic Control*, vol. 39, no. 12, pp. 2506–2511, December 1994.

[6] H. K. Khalil, "Performance recovery under output feedback sampled-data stabilization of a class of nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 49, no. 12, pp. 2173–2184, 2004.

[7] J. H. Ahrens, X. Tan, and H. K. Khalil, "Multirate sampled-data output feedback control with application to smart material actuated systems," *IEEE Transactions on Automatic Control*, vol. 54, no. 11, pp. 2518–2529, 2009.

[8] S. Ahmed Ali, N. Langlois, and M. Guermouche, "Sampled-data disturbance observer for a class of nonlinear systems," in *19th World Congress*, vol. 19, no. 1, 2014, pp. 3346–3351.

[9] C. Zhang and J. Yang, "Semi-global sampled-data output feedback disturbance rejection control for a class of uncertain nonlinear systems," *International Journal of Systems Science*, pp. 1–12, 2016.

[10] W. Liu, Q. Ma, and X. Jia, "Sampled-data output feedback control for a class of nonlinear systems with mismatched disturbances," in *35th Chinese Control Conference (CCC)*, July 2016, pp. 380–385.

[11] H. Chu, C. Qian, J. Yang, S. Xu, and Y. Liu, "Almost disturbance decoupling for a class of nonlinear systems via sampled-data output feedback control," *International Journal of Robust and Nonlinear Control*, 2015.

[12] C. Cao and N. Hovakimyan, "Design and analysis of a novel $\mathcal{L}_1$ adaptive control architecture with guaranteed transient performance," *IEEE Transactions on Automatic Control*, vol. 53, no. 2, pp. 586–591, March 2008.

[13] N. Hovakimyan and C. Cao, *$\mathcal{L}_1$ Adaptive Control Theory: Guaranteed Robustness with Fast Adaptation*. Philadelphia, PA: Society for Industrial and Applied Mathematics, 2010.

[14] C. Cao and N. Hovakimyan, "$\mathcal{L}_1$ adaptive output-feedback controller for non-strictly-positive-real reference systems: Missile longitudinal autopilot design," *Journal of guidance, control, and dynamics*, vol. 32, no. 3, pp. 717–726, 2009.

[15] T. Chen and B. A. Francis, *Optimal sampled-data control systems*. Springer Science & Business Media, 2012.

[16] L. Sha, "Dependable system upgrade," in *The 19th IEEE Real-Time Systems Symposium*, December 1998, pp. 440–448.

[17] ——, "Using simplicity to control complexity," *IEEE Software*, vol. 18, no. 4, pp. 20–28, Jul 2001.

[18] P. Voulgaris, "Control of asynchronous sampled data systems," *IEEE Transactions on Automatic Control*, vol. 39, no. 7, pp. 1451–1455, Jul 1994.