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Bounded backstepping control and robustness analysis for time-varying systems under converging-input-converging-state conditions☆

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ABSTRACT

We provide new bounded backstepping results that ensure global asymptotic stability for a large class of partially linear systems with an arbitrarily large number of integrators. We use a dynamic extension that contains one artificial delay, and a converging-input-converging-state assumption. When the nonlinear subsystem is control affine, we provide sufficient conditions for our converging-input-converging-state assumption to hold. We also show input-to-state stability with respect to a large class of model uncertainties, and robustness to delays in the measurements of the state of the nonlinear subsystem. We illustrate our result in a first example that has a nondifferentiable vector field and so is beyond the scope of classical backstepping, and then in a nonlinear example that illustrates how one can combine Lyapunov and trajectory based methods to check our assumptions.

1. Introduction

Backstepping is probably the most important, most celebrated, and most commonly used technique for constructing controls for nonlinear systems. This paper continues our group's quest (begun in [17–20,23], and [24]) for novel backstepping results that help overcome the obstacles to using classical backstepping; see [13] and [15] for traditional backstepping. Classical backstepping entails synthesizing globally asymptotically stabilizing feedback controls, by recursively building globally asymptotically stabilizing controls and corresponding Lyapunov functions for subsystems; see [8,13], and [16] for improved backstepping theory that includes nonlinearities and uncertainties, and [4,5], and [6] for backstepping applied to adaptive, aerospace, and robotic systems. However, there are significant instances that call for backstepping where the existing backstepping literature does not apply, e.g., systems with general nonlinear subsystems having bounds on the allowable sup norms of the controls, which produce challenges that we overcome in this work. In this work, we focus on systems of the form

$$\begin{cases} \dot{x}(t) = \mathcal{F}(t, x(t), z(t), \eta(t)) \\ \dot{z}_i(t) = z_{i+1}(t), \quad i \in \{1, \dots, k-1\} \\ \dot{z}_k(t) = u(t) + \sum_{j=1}^{k} v_j z_j(t) \end{cases} \tag{1}$$

with a scalar valued control u and any number k of integrators, where x is valued in $\mathbb{R}^n$ for any n, $\mathcal{F}$ is known, the v_j's are known real constants, the unknown measurable essentially bounded function η represents model uncertainty, and the nonlinear x subsystem will satisfy a converging-input-converging-state condition that we specify below. Many nonlinear systems admit changes of variables that produce the form (1); see the well known results [7, Section 9.1] for formulas for the changes of coordinates, and Section 6 for examples that illustrate the value of our theory. We write our controls as $u(t)$ to simplify notation, but they will be feedbacks that depend on t through their dependence on states of (1) and of a dynamic extension.

In most of what follows, we assume that the current values of the state are available for measurement, but our main result will still use a delay in the state values in our feedback control since this so-called artificial delay is needed to design a bounded control; see Section 5 for an extension to cases where there are also delays in the measurements of the values $x(t)$ of the nonlinear

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subsystem of (1). Our work [20] also used both the converging-input-converging-state assumption and artificial delays, but one notable improvement in the present work as compared with [20] is that here we allow an arbitrarily large number k of integrators, while [19,20] only allowed one integrator, but our results in the present work are novel and notable even in the special case where there is only one integrator. This is because we allow nonzero values of the v_i in (1) and uncertainties η that were not present in [19,20], thereby allowing uncertain nonlinear subsystems and drift terms in the z subsystem that were not allowed in [19,20]. The effects of nonzero v_i's cannot be cancelled by a change of feedback, owing to the boundedness requirements on u. Although our bounded backstepping work [23] also allowed an arbitrarily large number of integrators, a notable advantage of the present work over [23] is that we produce a globally bounded control for (1) while the controls for the original systems in [23] were not globally bounded. Also, whereas [23] required k artificial delays in the control and did not use dynamic extensions, here we only require one artificial delay, so in this sense we obtain a simpler feedback.

Our works [17–19] and [24] did not use converging-input-converging-state conditions or artificial delays. Moreover, our work differs from the backstepping works [18] (which uses a forwarding method to cover the one integrator case), [17] (which also only covers one integrator), [24] (which produces unbounded controls), and [31,32], and [33] (which use Lie derivatives without satisfying the input constraints that we satisfy here). Therefore, our novel combination of converging-input-converging-state conditions with artificial delays and bounded controllers for (1) is valuable. The work to follow improves on our conference version [21] by also incorporating measurement delays and input-to-state stability with respect to the uncertainties η, and allowing the nonlinear subsystem to depend on all components of the vector z. These three features were not present in [21], which was confined to cases where $\mathcal{F}$ was a function of only (t, x, z_1) and where there were no measurement delays and no input-to-state stability analysis.

The specific controllers in our main result are sums of three pieces, namely, (1) a saturation applied to a new auxiliary variable $\mathcal{M}$, (2) a second part that uses a nominal stabilizing control for an auxiliary system from our converging-input-converging-state assumption, and (3) a bounded stabilizing controller ϑ for the z subsystem from (1). This control structure is motivated by three main considerations. First, it is well known that it is not sufficient to solve the corresponding unbounded control design problem for (1) (i.e., without considering input constraints) and then to apply a saturation operator to the control that is obtained, because of the potentially destabilizing effects of states that have large norms. This motivates our applying a saturation to $\mathcal{M}$. Second, a necessary condition for globally asymptotically stabilizing (1) to 0 when $\eta = 0$ is that the $z(t)$ components of (1) converge to 0 as $t \to +\infty$, hence our use of ϑ in the control. Third, our auxiliary variable $\mathcal{M}$ is designed so that for large enough times, the z subsystem of (1) (in closed loop with the control u in our theorem) is transformed into the globally asymptotically stabilized z subsystem with the control $\vartheta(z)$, which ensures that the z components of (1) converge to 0 exponentially fast as $t \to +\infty$, and then the desired stabilization result can be obtained from our converging-input-converging-state condition.

We use standard notation and definitions. We omit arguments of functions when they are clear, and the dimensions of our Euclidean spaces are arbitrary unless otherwise noted. We use $|\cdot|$ to denote the usual Euclidean norm and the induced matrix norm, and $|\phi|_\infty$ (resp., $|\phi|_{\mathcal{I}}$) is the essential supremum (resp., supremum over any interval $\mathcal{I}$) for any bounded $\mathbb{R}^n$ valued measurable function ϕ. We set $\mathbb{N} = \{1, 2, \ldots\}$. Given any constant $T > 0$, let $C_{\rm in}$ denote the set of all continuous functions $\phi : [-T, 0] \to \mathbb{R}^a$, which we call the set of all *initial functions*. We define $\Xi_t \in C_{\rm in}$ and $\dot{\Xi}_t \in C_{\rm in}$ by $\Xi_t(s) = \Xi(t+s)$ and $\dot{\Xi}_t(s) = \Xi'(t+s)$ for all choices of Ξ, $s \le 0$, and $t \ge 0$ for which the equalities are defined. We use the convention $0! = 1$, and assume for simplicity that the initial times for our solutions are $t_0 = 0$ and that the initial functions are constant at time 0 (e.g., the states are constant on $[-T, 0]$, where T will denote the artificial delay). Let $f^{(i)}$ denote the ith derivative of a function $f : [0, +\infty) \to \mathbb{R}$ with $f^{(0)} = f$, and $\sigma_r : \mathbb{R} \to [-r, r]$ is the saturation that is defined for all constants $r > 0$ by $\sigma_r(s) = s$ for all $s \in [-r, r]$ and $\sigma_r(s) = r\text{sign}(s)$ otherwise. An integral $\int_a J(\ell){\rm d}\ell$ of a continuous column vector valued function $J = (J_1, \ldots J_L)^\top$ on an interval a is defined to be the column vector whose ith entry is $\int_a J_i(\ell){\rm d}\ell$ for all i. We use the standard definitions of global asymptotic and input-to-state stability (or ISS, which we also use to mean input-to-state stable) [13]. We also use the following standard definitions. A function $W : \mathbb{R}^n \to [0, +\infty)$ is called *positive definite* provided $W(0) = 0$ and $W(x) > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$. A function $V : [0, +\infty) \times \mathbb{R}^n \to [0, +\infty)$ is called *uniformly proper and positive definite* provided there are functions $\alpha_0 \in \mathcal{K}_\infty$ and $\alpha_1 \in \mathcal{K}_\infty$ such that the inequalities $\alpha_0(|x|) \le V(t, x) \le \alpha_1(|x|)$ hold for all $t \ge 0$ and $x \in \mathbb{R}^n$. This agrees with the *proper*ness condition in the special case where V is independent of t. Here, $\mathcal{K}_\infty$ is the set of all continuous functions $\gamma : [0, +\infty) \to [0, +\infty)$ such that $\gamma(0) = 0$, γ is strictly increasing, and $\lim_{s \to +\infty} \gamma(s) = +\infty$.

2. Lemmas and main result

We require the following two lemmas, the first of which is shown in the appendix below:

Lemma 1. *Let $T > 0$ be a constant, and $\mu_0 : [-T, +\infty) \to \mathbb{R}$ be any continuous function, and set*

$$\zeta(t) = \int_{t-T}^{t} e^{\ell - t} Q(t, \ell, \ell + T)\mu_0(\ell){\rm d}\ell,$$

$$\Omega_j(t) = \zeta^{(j-1)}(t) \text{ and } \mu_i(t) = \frac{1}{T}\int_{t-T}^{t} e^{\ell - t}\frac{(t-\ell)^{i-1}}{(i-1)!}\mu_0(\ell){\rm d}\ell \quad (2)$$

for all $j \in \{1, \ldots, k+1\}$ and $i \in \mathbb{N}$, where $Q(t, a, b) = (t-a)^{k-1}(t-b)^{k-1}$ for all $a \in \mathbb{R}$ and $b \in \mathbb{R}$, and where $k \in \mathbb{N}$ with $k \ge 2$. Then there are constants $c_{i,j}(T) \in \mathbb{R}$ for all $i \in \{1, \ldots, 2k-1\}$ and all $j \in \{1, \ldots, k\}$, and constants $g_i(T) \in \mathbb{R}$ for all $i \in \{-1, 0, \ldots, 2k-1\}$, such that

$$\Omega_j(t) = \sum_{i=1}^{2k-1} c_{i,j}(T)\mu_i(t) \text{ and}$$
$$\Omega_{k+1}(t) = \sum_{i=0}^{2k-1} g_i(T)\mu_i(t) + g_{-1}(T)\mu_0(t-T) \quad (3)$$

hold for all $t \ge 0$.

In the next lemma (which was shown in [30]), we say that a linear system is *not exponentially unstable* provided its poles are all in the closed left-half plane:

Lemma 2. *Let $k \ge 2$ be an integer and $v = (v_1, \ldots, v_k)$ be any vector of k real constants such that*

$$\begin{cases} \dot{z}_i(t) = z_{i+1}(t), \quad i \in \{1, \ldots, k-1\} \\ \dot{z}_k(t) = u + \sum_{i=1}^{k} v_i z_i \end{cases} \quad (4)$$

is not exponentially unstable when $u = 0$. Then there is a bounded locally Lipschitz function $\vartheta : \mathbb{R}^k \to \mathbb{R}$ such that (4), in closed loop with $u = \vartheta(Z)$ where $Z = (z_1, \ldots, z_k)^\top$, is globally asymptotically and locally exponentially stable to 0.

We can now fix functions Λ_j such that the Ω_j's from Lemma 1 can be written as

$$\Omega_j(t) = \int_{t-T}^{t} \Lambda_j(\ell, t)\mu_0(\ell)\mathrm{d}\ell \text{ for } 1 \le j \le k \text{ and all } t \ge 0 \tag{5}$$

for all choices of the continuous function $\mu_0 : [-T, +\infty) \to \mathbb{R}$, where we omit the dependence of the Λ_j's on T for brevity. By a simple induction on the index j that we omit, we can prove that each function $\Lambda_j(\ell, t)$ can be written as a function $\mathcal{D}_j(t-\ell, t-\ell-T)$ of the differences $t-\ell$ and $t-\ell-T$. For instance, we have

$$\begin{aligned}\Lambda_1(\ell,t) &= \mathcal{D}_1(t-\ell, t-\ell-T)\\ &= e^{\ell-t}(t-\ell)^{k-1}(t-\ell-T)^{k-1} \text{ and}\\ \Lambda_2(\ell,t) &= \mathcal{D}_2(t-\ell, t-\ell-T)\\ &= e^{\ell-t}\big[-(t-\ell)^{k-1}(t-\ell-T)^{k-1}\\ &\quad + (k-1)(t-\ell)^{k-2}(t-\ell-T)^{k-1}\\ &\quad + (k-1)(t-\ell)^{k-1}(t-\ell-T)^{k-2}\big]\end{aligned} \tag{6}$$

so we can choose $\mathcal{D}_1(a,b) = e^{-a}a^{k-1}b^{k-1}$ and $\mathcal{D}_2(a,b) = e^{-a}[-a^{k-1}b^{k-1} + (k-1)a^{k-2}b^{k-1} + (k-1)a^{k-1}b^{k-2}]$, and the formulas for the other Λ_j's and $\mathcal{D}_j$'s can be computed from Lemma 1. Note for later use that for each $T > 0$ and j, the function $\sup\{|\Lambda_j(\ell,t)| : t-T \le \ell \le t\}$ is a bounded function of $t \ge 0$. For instance, when $k = 2$,

$$\max\{|\Lambda_j(\ell,t)| : 1 \le j \le 2, \ell \in [t-T, t], t \ge 0\} \le T(T+2). \tag{7}$$

We will assume the following, where $\Lambda = (\Lambda_1, \dots, \Lambda_k)^\top$:

Assumption 1. (i) The function $\mathcal{F}$ in (1) is continuous in t and η, globally Lipschitz in (x, z), and satisfies

$$\mathcal{F}(t, 0, 0, 0) = 0 \text{ for all } t \ge 0. \tag{8}$$

(ii) There are a globally Lipschitz bounded function $\omega : \mathbb{R}^n \to [-\bar{\omega}, \bar{\omega}]$ having some bound $\bar{\omega} > 0$ such that $\omega(0) = 0$ and a constant $T > 0$ such that for each continuous function $\delta : [0, +\infty) \to \mathbb{R}^k$ that exponentially converges to 0, the following is true: All solutions $\xi : [0, +\infty) \to \mathbb{R}^n$ of the system

$$\dot{\xi}(t) = \mathcal{F}\left(t, \xi(t), \int_{t-T}^{t} \Lambda(\ell,t)\omega(\xi(\ell))\mathrm{d}\ell + \delta(t), 0\right) \tag{9}$$

satisfy $\lim_{t\to+\infty} \xi(t) = 0$.

We refer to part (ii) of Assumption 1 as our *converging-input-converging-state assumption*; see Section 5 for a generalization involving measurement delays in the ξ measurements in the function ω. An important special case is where $\mathcal{F}$ has the form $\mathcal{F}(t,x,z,\eta) = \mathcal{F}_d(t,x) + \mathcal{F}_c(t,x)[z+\eta]$ for some drift term $\mathcal{F}_d$ and some control term $\mathcal{F}_c$, i.e., affineness with respect to z and η. In this special case, our condition (8) is the requirement that $\mathcal{F}_d(t,0) = 0$ for all $t \ge 0$, and (9) has the form

$$\dot{\xi}(t) = \mathcal{F}_d(t, \xi(t)) + \mathcal{F}_c(t, \xi(t))\left[\int_{t-T}^{t} \Lambda(\ell,t)\omega(\xi(\ell))\mathrm{d}\ell + \delta(t)\right].$$

See Section 3 for readily checked sufficient conditions for the required converging-input-converging-state condition in the preceding affine case. The system (9) differs from the nonlinear subsystem of (1) because the third argument of $\mathcal{F}$ in (1) has been replaced by the sum of an integral term and $\delta(t)$, and because η has been set to 0. In terms of the Jordan matrix

$$J_{2k-1} = \begin{bmatrix} -1 & 1 & 0 & \dots & 0 \\ 0 & -1 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -1 & 1 \\ 0 & \dots & \dots & 0 & -1 \end{bmatrix} \in \mathbb{R}^{(2k-1)\times(2k-1)}, \tag{10}$$

our main result is as follows; see Section 3 for sufficient conditions for ISS of (15).

Theorem 1. *Let $k \ge 2$, and $T > 0$ and $\mathcal{F}$ and ω be such that Assumption 1 holds, where $k \in \mathbb{N}$. Let ϑ and ν satisfy the requirements from Lemma 2. Consider the (x, Z, Y) system, consisting of (1) and*

$$\dot{Y}(t) = J_{2k-1}Y(t) + \frac{e_{2k-1}}{T}\omega(x(t)) \tag{11}$$

where $e_{2k-1} = (0, 0, \dots, 1)^\top \in \mathbb{R}^{2k-1}$ is the $(2k-1)$-st standard basis vector, in closed loop with the control

$$\begin{aligned} u(Z(t), Y_t, x_t) &= \sigma_{\bar{c}}(\mathcal{M}(Y_t)) + g_0(T)\omega(x(t)) \\ &\quad + g_{-1}(T)\omega(x(t-T)) + \vartheta(Z_\star(t)) \end{aligned} \tag{12}$$

with the saturation level

$$\bar{c} = \left|\sum_{j=1}^{k} \nu_j \mathcal{C}_j(T) - \mathcal{G}(T)\right| e^{|J_{2k-1}|T}\bar{\omega} \tag{13}$$

where $Z_\star(t) = (z_1(t) - \mathcal{C}_1(T)\Psi(Y_t), \dots, z_k(t) - \mathcal{C}_k(T)\Psi(Y_t))^\top$, $\Psi(Y_t) = Y(t) - e^{TJ_{2k-1}}Y(t-T)$, $\mathcal{G}(T) = [g_{2k-1}(T), \dots, g_1(T)]$, and

$$\mathcal{M}(Y_t) = \left(\mathcal{G}(T) - \sum_{j=1}^{k} \nu_j \mathcal{C}_j(T)\right)\Psi(Y_t) \text{ and}$$

$$\mathcal{C}_j(T) = [c_{2k-1,j}(T), \dots, c_{1,j}(T)],\ 1 \le j \le k \tag{14}$$

and where the constants $c_{i,j}$ and g_i satisfy the requirements from Lemma 1 for the function $\mu_0(t) = \omega(x(t))$. Then all maximal solutions $(x, Z, Y)(t)$ of the (x, Z, Y) system, consisting of (1) and (11) and with (12) as the control, satisfy $\lim_{t\to+\infty}(x, Z, Y)(t) = 0$ when $\eta = 0$. If, in addition, the system

$$\dot{\xi}(t) = \mathcal{F}\left(t, \xi(t), \int_{t-T}^{t} \Lambda(\ell,t)\omega(\xi(\ell))\mathrm{d}\ell + \delta(t), \eta(t)\right) \tag{15}$$

is ISS with respect to (δ, η), then the (x, Z) system (1) in closed loop with (12) is ISS with respect to η.

Remark 1. As in [23], we can extend Theorem 1 to cases where in addition to the artificial delay T, there is a delay in the measurements of $x(t)$ from the original system (1). However, as we noted above, [23] does not provide a bounded control for (1) even if the ν_i's are all zero, and the converging-input-converging-state assumption in [23] has a k-fold integral instead of the simpler single integral we have in (9).

We next provide sufficient conditions for our converging-input-converging-state assumption to hold, and then we prove Theorem 1 in Section 4. See also Section 5 for extensions under measurement delays.

3. Checking Assumption 1

We provide sufficient conditions for our converging-input-converging-state conditions on (9) to hold, and for the ISS property of (15) from Theorem 1 to hold, based on Lyapunov functions. We use the system

$$\dot{x}(t) = \mathcal{F}(t, x(t), \Lambda_*(T)\omega(x(t)), \eta(t)), \tag{16}$$

where $\mathcal{F}$ is from (1), $\Lambda_* : [0, \infty) \to \mathbb{R}^p$ is defined by

$$\Lambda_*(T) = \int_{t-T}^{t} (\Lambda_1(\ell,t), \dots, \Lambda_p(\ell,t))^\top \mathrm{d}\ell, \tag{17}$$

the constant $T > 0$ will be specified, the Λ_i satisfy the requirements from (5), and $p \in [1, k]$ is such that $\mathcal{F}$ is a function of $(t, x, z_1, \dots, z_p, \eta)$, where $z_1, \dots, z_p$ are the first p components of the state z of the linear subsystem of (1). The definition (17) is justified by the fact that each function $\Lambda_i(\ell, t)$ for $i = 1, 2, \dots, p$ can be written as a function $\mathcal{D}_i$ of $t-\ell-T$ and $t-\ell$, so the right side of (17) can be written as

$$\begin{aligned}&\int_{t-T}^{t}(\Lambda_1(\ell,t),\dots,\Lambda_p(\ell,t))^\top \mathrm{d}\ell\\ &=\int_{t-T}^{t}(\mathcal{D}_1(t-\ell,t-\ell-T),\dots,\mathcal{D}_p(t-\ell,t-\ell-T))^\top \mathrm{d}\ell\\ &=\int_{-T}^{0}(\mathcal{D}_1(-\ell,-\ell-T),\dots,\mathcal{D}_p(-\ell,-\ell-T))^\top \mathrm{d}\ell\end{aligned}$$

and so does not depend on t; see Section 2. In the next assumption, V_t and V_x are the partial derivative with respect to t and the gradient with respect to x, respectively, and the uniform global Lipschitzness in x means that the global Lipschitz constants can be chosen independently of the other variable t:

Assumption 2. There are functions $f:[0,+\infty)\times\mathbb{R}^n\to\mathbb{R}^n$ and $g:[0,+\infty)\times\mathbb{R}^n\to\mathbb{R}^{n\times p}$ that are uniformly globally Lipschitz in x and continuous on $[0,+\infty)\times\mathbb{R}^n$, such that $\mathcal{F}(t,x,q,\eta)=f(t,x)+g(t,x)(q+\eta)$ holds for all $t\geq 0$, $x\in\mathbb{R}^n$, $q\in\mathbb{R}^p$, and $\eta\in\mathbb{R}^p$. Also, there exist a C^1 uniformly proper and positive definite function $V:[0,+\infty)\times\mathbb{R}^n\to[0,+\infty)$; a uniformly continuous positive definite function $W:\mathbb{R}^n\to[0,+\infty)$; positive constants T, r_1, and r_3; and a constant $r_2\geq 0$ such that for all $(t,x)\in[0,+\infty)\times\mathbb{R}^n$, we have

$$V_t(t,x)+V_x(t,x)\big(f(t,x)+g(t,x)\Lambda_*(T)\omega(x)\big)\leq -W(x), \tag{18a}$$

$$|V_x(t,x)g(t,x)|\leq\sqrt{W(x)}, \tag{18b}$$

$$|\omega(x)|\leq r_1\sqrt{W(x)}, \tag{18c}$$

$$|f(t,x)|\leq r_2\sqrt{W(x)}, \tag{18d}$$

$$\text{and}\quad |g(t,x)|\leq r_3, \tag{18e}$$

where $\omega:\mathbb{R}^n\to\mathbb{R}$ is bounded, satisfies $\omega(0)=0$, and admits a global Lipschitz constant $C>0$ on $\mathbb{R}^n$.

We emphasize that the linearity of $\mathcal{F}$ in the q and η variables will play a key role in this section. See [22] for conditions under which (18) can be satisfied. Set

$$\Lambda_a(T)=\int_{t-T}^{t}|(\Lambda_1,\dots,\Lambda_p)(\ell,t)|\mathrm{d}\ell, \tag{19}$$

which is independent of t because the Λ_i's can be written as functions of $t-\ell$ and $t-\ell-T$; the proof that the right side of (19) is independent of t is the same as the argument we used to show that the right side of (17) is independent of t except with a norm on the p tuples in the earlier argument. We also set $\Lambda_+(T)=\sup_{t\geq 0}\{|(\Lambda_1(\ell,t),\dots,\Lambda_p(\ell,t))|:t-T\leq\ell\leq t\}$, which is finite because of our choice of Λ.

Proposition 1. *If Assumption 2 holds, then for all integers $k\geq 2$, and for all constants $T>0$ such that*

$$4\big(T\Lambda_a(T)C\big)^2\left[2r_2^2+\frac{5}{2}\big(r_1r_3T\Lambda_+(T)\big)^2\right]<1, \tag{20}$$

Assumption 1 is satisfied. If, in addition, W is proper, then (15) is ISS with respect to (δ,η).

Proof. We first prove the first assertion of the proposition (where $\eta=0$), and then we indicate the additional arguments needed to prove the second assertion. Fix any continuous function $\delta:[0,+\infty)\to\mathbb{R}^p$ that exponentially converges to 0. Along all solutions $x(t)$ of (9), the control affine structure of $\mathcal{F}$ gives

$$\begin{aligned}\dot{x}(t)&=\{f(t,x(t))+g(t,x(t))\Lambda_*(T)\omega(x(t))\}\\ &\quad+g(t,x(t))\left[\delta(t)+\int_{t-T}^{t}\Lambda^\flat(\ell,t)(\omega(x(\ell))-\omega(x(t)))\mathrm{d}\ell\right]\end{aligned} \tag{21}$$

where $\Lambda^\flat=(\Lambda_1,\dots,\Lambda_p)^\top$. Combining (21) with (18a)-(18b) now gives

$$\begin{aligned}\dot{V}(t)&\leq -W(x(t))+|V_x(t,x(t))g(t,x(t))|\\ &\quad\times\left[\left|\int_{t-T}^{t}\Lambda^\flat(\ell,t)(\omega(x(\ell))-\omega(x(t)))\mathrm{d}\ell\right|+|\delta(t)|\right]\\ &\leq -W(x(t))+\sqrt{W(x(t))}(\{\Lambda_a(T)\sup_{\ell\in[t-T,t]}|\omega(x(\ell))\\ &\quad-\omega(x(t))|\}+|\delta(t)|)\end{aligned} \tag{22}$$

for all $t\geq 0$, where the first inequality used the fact that the portion of the dynamics (21) contained in the curly braces agrees with the dynamics from (18a), combined with the triangle inequality and the fact that (18a) holds for all $t\geq 0$ and $x\in\mathbb{R}^n$, and where the second inequality in (22) used our bound on the function $|V_x(t,x)g(t,x)|$ from (18b) and our formula (19) for $\Lambda_a(T)$ after moving the norm inside the integral.

We next use the global Lipschitz constant C on ω and apply the Fundamental Theorem of Calculus to find a useful upper bound on the supremum that is contained in (22). To this end, we first use inequalities (18d)–(18e) from Assumption 2 to obtain the upper bound

$$|\dot{x}(t)|\leq r_2\sqrt{W(x(t))}+r_3\left\{|\Lambda_+(T)|\int_{t-T}^{t}|\omega(x(\ell))|\mathrm{d}\ell+|\delta(t)|\right\} \tag{23}$$

along all solutions of (9). Applying $(a+b)^2\leq 2(a^2+b^2)$ with $a=r_2\sqrt{W(x(t))}$ and b being the rest of the right side of (23), and then applying $(a+b)^2\leq(5/4)a^2+5b^2$ where a and b are the terms being added together in curly braces in (23) and then Jensen's inequality, it follows that along all solutions of (21):

$$\begin{aligned}|\dot{x}(t)|^2&\leq 2r_2^2W(x(t))\\ &\quad+2r_3^2\left(5|\delta(t)|^2+\frac{5}{4}Tr_1^2\Lambda_+^2(T)\int_{t-T}^{t}W(x(\ell))\mathrm{d}\ell\right),\end{aligned} \tag{24}$$

where $W(x(\ell))$ in the integrand is present because of our condition (18c) relating ω to W.

We can now combine (22) and (24) and then use Jensen's and Young's inequalities $\sqrt{W(x(t))}b\leq\frac{1}{4}W(x(t))+b^2$, with $b=|\delta(t)|$ and then b being the quantity in curly braces in (22), to get

$$\begin{aligned}\dot{V}(t)&\leq -\frac{1}{2}W(x(t))+|\delta(t)|^2+\Lambda_a^2(T)C^2\sup_{\ell\in[t-T,t]}|x(\ell)-x(t)|^2\\ &\leq -\frac{1}{2}W(x(t))+|\delta(t)|^2+\Lambda_a^2(T)C^2T\int_{t-T}^{t}|\dot{x}(q)|^2\mathrm{d}q\\ &\leq -\frac{1}{2}W(x(t))+|\delta(t)|^2\\ &\quad+\Lambda_a^2(T)C^2T\Big(2r_2^2\int_{t-T}^{t}W(x(\ell))\mathrm{d}\ell+10r_3^2T|\delta|^2_{[\max\{0,t-T\},t]}\\ &\quad+\frac{5}{2}r_3^2T^2\Lambda_+^2(T)r_1^2\int_{t-2T}^{t}W(x(\ell))\mathrm{d}\ell\Big)\\ &\leq -\frac{1}{2}W(x(t))+\mathcal{N}_1\int_{t-2T}^{t}W(x(\ell))\mathrm{d}\ell\\ &\quad+\mathcal{N}_2|\delta|^2_{[\max\{0,t-T\},t]}\end{aligned} \tag{25}$$

along all solutions of (9) for all $t\geq 0$, where

$$\mathcal{N}_1 = T(\Lambda_a(T)C)^2\left(2r_2^2 + \frac{5}{2}(r_1 r_3 \Lambda_+(T)T)^2\right) \text{ and}$$
$$\mathcal{N}_2 = 10(T\Lambda_a(T)r_3C)^2(a+1), \tag{26}$$

by using the Fundamental Theorem of Calculus and Jensen's inequality to obtain the bounds

$$\sup_{\ell\in[t-T,t]} |x(\ell)-x(t)|^2 \le \left(\int_{t-T}^t |\dot{x}(q)|\mathrm{d}q\right)^2 \le T\int_{t-T}^t |\dot{x}(q)|^2\mathrm{d}q$$

and $\int_{s-T}^s W(x(\ell))\mathrm{d}\ell \le \int_{t-2T}^t W(x(\ell))\mathrm{d}\ell$ for all $s\in[t-T,t]$. Then our condition (20) implies that $4T\mathcal{N}_1 < 1$, so we can find a constant $\lambda > 1$ that is close enough to 1 so that $2T\mathcal{N}_1\lambda < 1/2$. Then

$$V_1(t,x_t) = V(t,x(t)) + \lambda\mathcal{N}_1\int_{t-2T}^t\int_s^t W(x(\ell))\mathrm{d}\ell\mathrm{d}s \tag{27}$$

satisfies

$$\dot{V}_1 \le -c_* W(x(t)) - (\lambda-1)\mathcal{N}_1\int_{t-2T}^t W(x(\ell))\mathrm{d}\ell + \mathcal{N}_2|\delta|^2_{[\max\{0,t-T\},t]} \tag{28}$$

for all $t\ge 0$ along all solutions of (9) where $c_* = \frac{1}{2} - 2T\mathcal{N}_1\lambda$, since for all $t\ge 0$, we have

$$\frac{d}{dt}\int_{t-2T}^t\int_s^t W(x(\ell))\mathrm{d}\ell\mathrm{d}s = 2TW(x(t)) - \int_{t-2T}^t W(x(\ell))\mathrm{d}\ell. \tag{29}$$

By our assumption on δ, we can find positive constants $\bar{\delta}_1$ and $\bar{\delta}_2$ such that $|\delta(t)| \le \bar{\delta}_1 e^{-\bar{\delta}_2 t}$ for all $t\ge 0$. Since $c_* > 0$ and $\lambda > 1$, we can integrate (28) on $[0, M]$ for any constant $M > 0$ to get

$$\begin{aligned}\sup_{t\ge 0} V_1(t,x_t) &\le V_1(0,x_0) + \mathcal{N}_2\int_0^{+\infty} |\delta|^2_{[\max\{\ell-T,0\},\ell]}\mathrm{d}\ell \\ &\le V_1(0,x_0) + \mathcal{N}_2\bar{\delta}_1^2 T + \bar{\delta}_1^2\mathcal{N}_2\int_T^{+\infty} e^{-2\bar{\delta}_2(\ell-T)}\mathrm{d}\ell \\ &\le V_1(0,x_0) + \mathcal{N}_2\bar{\delta}_1^2 T + \mathcal{N}_2\bar{\delta}_1^2/(2\bar{\delta}_2) < +\infty.\end{aligned}$$

Since V is uniformly proper and positive definite, we conclude that $|x(t)|$ is bounded, so $x(t)$ is uniformly continuous, by the structure of the dynamics (9) when $\eta = 0$. Since W is uniformly continuous, it follows that $W(x(t))$ is a uniformly continuous function of t, and integrating (28) gives

$$\int_0^{+\infty} W(x(\ell))\mathrm{d}\ell < +\infty. \tag{30}$$

Therefore, Barbalat's Lemma implies that $\lim_{t\to+\infty} W(x(t)) = 0$, so since W is positive definite, we conclude that $\lim_{t\to+\infty} x(t) = 0$. This proves the first assertion of the proposition.

To prove the second assertion of the proposition, fix a choice of the measurable essentially bounded function η. Then the preceding analysis applies to the corresponding system (15), save for the fact that we must add the additional term $V_x(t,x(t))g(t,x(t))\eta(t)$ to the right sides of the decay estimates on V. We can use (18b) and Jensen's inequality to check that this additional term is bounded above by

$$\sqrt{W(x(t))}|\eta(t)| \le \frac{c_*}{2}W(x(t)) + \frac{1}{2c_*}|\eta(t)|^2. \tag{31}$$

Adding the right sides of (31) and (28) and using the fact that

$$\int_{t-2T}^t\int_s^t W(x(\ell))\mathrm{d}\ell\mathrm{d}s \le 2T\int_{t-2T}^t W(x(\ell))\mathrm{d}\ell \tag{32}$$

for all $t\ge 0$, we can find a function $\gamma_0\in\mathcal{K}_\infty$ and a constant $k_* > 0$ such that

$$\dot{V}_1 \le -\gamma_0(V_1(t,x_t)) + k_*|(\delta,\eta)|^2_{[0,t]} \tag{33}$$

along all solutions of (9), using the properness of V and W to find a $\gamma_1\in\mathcal{K}_\infty$ such that $\gamma_1(V(t,x)) \le (c_*/2)W(x)$ for all $t\ge 0$ and $x\in\mathbb{R}^n$, then choosing $\gamma_0(\ell) = \min\{\gamma_1(\ell/2), r_*\ell\}$ with $r_* = (\lambda-1)/(4T\lambda)$ (by the relation $\gamma_0(a+b) \le \gamma_0(2a) + \gamma_0(2b)$ where a and b are the terms being added in the formula (27)). Hence, V_1 is an ISS Lyapunov function for (9), so the ISS properties follow by standard arguments [13]. □

Remark 2. Proposition 2 requires $T > 0$ to be small enough, but due to the structure of our control (12), we cannot pick $T = 0$. In Section 5, we will see how picking T small enough can ensure that ISS is maintained under measurement delays. Conditions (18) agree with the sufficient conditions in [23], except that instead of (18b), [23] required a constant $r_0 > 0$ such that $|V_x(t,x)g(t,x)| \le r_0\sqrt{W(x)}$ for all t and x. However, one may assume that $r_0 = 1$, by replacing g, ω, r_1, and r_3 by g/r_0, $r_0\omega$, $r_1 r_0$, and r_3/r_0, respectively. □

4. Proof of Theorem 1

The forward completeness of the closed loop systems defined in the statement of the theorem will follow from the Lyapunov analysis to follow, which will ensure that finite time blowups cannot occur. Theorem 1 will now follow from three more lemmas, which we state next. The first of these lemmas follows from [29, Lemma A.3.2] (applied to the entire function $\mathcal{E}(x) = e^{xt}$ for any $t\in\mathbb{R}$ to compute $\mathcal{E}(J_{2k-1})$):

Lemma 3. *For the Jordan matrix J_{2k-1} defined in (10), the equality*

$$e^{J_{2k-1}t} = e^{-t}\begin{bmatrix} 1 & t & \frac{t^2}{2} & \dots & \frac{t^{2(k-1)}}{(2(k-1))!} \\ 0 & 1 & t & \dots & \frac{t^{2k-3}}{(2k-3)!} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & t \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix} \tag{34}$$

holds for all $t\in\mathbb{R}$ and integers $k\ge 2$.

Later in the proof of Theorem 1, we specialize the following lemma to the case where $\mu_0(t) = \omega(x(t))$:

Lemma 4. *Let $\mu_0 : [-T,+\infty) \to [-\bar{\mu},\bar{\mu}]$ be any continuous function having a bound $\bar{\mu}$. Then the functions μ_i from (2) in Lemma 1, and the functions $\Psi(Y_t) = Y(t) - e^{TJ_{2k-1}}Y(t-T)$ for all solutions Y of*

$$\dot{Y}(t) = J_{2k-1}Y(t) + \frac{e_{2k-1}}{T}\mu_0(t), \tag{35}$$

are such that for all $t\ge 0$, we have

$$\nu_{2k-1}(t) = \Psi(Y_t) \text{ and } |\Psi(Y_t)| \le e^{|J_{2k-1}|T}\bar{\mu}, \tag{36}$$

where $\nu_{2k-1}(t) = (\mu_{2k-1}(t),\dots,\mu_1(t))^\top$ for all $t \ge 0$. □

Proof. By integrating (35) over $[t-T,t]$ for any $t\ge 0$, we deduce that

$$\Psi(Y_t) = Y(t) - e^{TJ_{2k-1}}Y(t-T) = \varrho(t), \text{ where}$$
$$\varrho(t) = \int_{t-T}^t e^{J_{2k-1}(t-\ell)}\frac{e_{2k-1}}{T}\mu_0(\ell)\mathrm{d}\ell \tag{37}$$

for all $t\ge 0$. On the other hand, using (34), we obtain

$$\varrho(t) = \frac{1}{T}\int_{t-T}^t e^{-(t-\ell)}\begin{bmatrix}\frac{(t-\ell)^{2(k-1)}}{(2(k-1))!} \\ \vdots \\ t-\ell \\ 1\end{bmatrix}\mu_0(\ell)d\ell = \nu_{2k-1}(t), \tag{38}$$

which proves the first conclusion of the lemma. The second conclusion of the lemma follows since (37) gives

$$\left|Y(t) - e^{TJ_{2k-1}}Y(t-T)\right| = |\varrho(t)| \le \int_{t-T}^{t} \left|e^{J_{2k-1}(t-\ell)}\frac{e_{2k-1}}{T}\right| |\mu_0(\ell)|\mathrm{d}\ell \le \int_{t-T}^{t} \left|e^{J_{2k-1}(t-\ell)}\frac{e_{2k-1}}{T}\right| \bar{\mu}\mathrm{d}\ell \tag{39}$$

for all $t \geq 0$, because of the bound $\bar{\mu}$ on μ_0, which proves the lemma.

Lemma 5. *Let $\mu_0 : [-T, +\infty) \to [-\bar{\mu}, \bar{\mu}]$ be any continuous function having a bound $\bar{\mu}$, and let the constants ν_i and the function ϑ satisfy the requirements from Lemma 2. Consider the linear system*

$$\begin{cases} \dot{z}_i(t) = z_{i+1}(t), \quad i \in \{1, \dots, k-1\} \\ \dot{z}_k(t) = u(t) + \sum_{j=1}^{k} \nu_j z_j(t) \end{cases} \tag{40}$$

in closed loop with the control

$$u(Z(t), Y_t, x_t) = \sigma_{\bar{c}}(\mathcal{M}(Y_t)) + g_0(T)\mu_0(t) + g_{-1}(T)\mu_0(t-T) + \vartheta(Z_\star(t)) \tag{41}$$

with the saturation level $\bar{c}$ for $\sigma_{\bar{c}}$ defined by

$$\bar{c} = \left|\sum_{j=1}^{k} \nu_j \mathcal{C}_j(T) - \mathcal{G}(T)\right| e^{|J_{2k-1}|T}\bar{\mu} \tag{42}$$

and where Y satisfies (35) and $\mathcal{M}$, $Z_\star$, $\mathcal{G}$, and the $\mathcal{C}_j$'s and g_j's are defined as in Theorem 1. Then the dynamics for the vector $\tilde{Z}(t) = (\tilde{z}_1(t), \dots, \tilde{z}_k(t))$ are globally asymptotically and locally exponentially stable to the origin, where $\tilde{z}_i(t) = z_i(t) - \Omega_i(t)$ for $i = 1, 2, \dots, k$ and the Ω_i's are defined in (2) in Lemma 1.

Proof. The fact that $\dot{\Omega}_i = \Omega_{i+1}$ for all $i \in \{1, 2, \dots, k\}$ and the structure of the dynamics (40) allow us to conclude that the dynamics for the functions $\tilde{z}_i(t) = z_i(t) - \Omega_i(t)$ are

$$\begin{cases} \dot{\tilde{z}}_i(t) = \tilde{z}_{i+1}(t), \quad i \in \{1, \dots, k-1\} \\ \dot{\tilde{z}}_k(t) = u(t) - \Omega_{k+1}(t) + \sum_{j=1}^{k} \nu_j[\tilde{z}_j(t) + \Omega_j(t)] \end{cases}. \tag{43}$$

Using our conclusion from Lemma 4 that

$$\nu_{2k-1}(t) = \Psi(Y_t) \tag{44}$$

where $\nu_{2k-1}(t) = (\mu_{2k-1}(t), \dots, \mu_1(t))^\top$ as before, it follows from (3) that:

$$\dot{\tilde{z}}_k(t) = u(t) - \mathcal{G}(T)\nu_{2k-1}(t) - g_0(T)\mu_0(t) - g_{-1}(T)\mu_0(t-T) + \sum_{j=1}^{k} \nu_j \tilde{z}_j(t) + \sum_{j=1}^{k} \nu_j \mathcal{C}_j(T)\nu_{2k-1}. \tag{45}$$

Hence, (44) gives

$$\begin{cases} \dot{\tilde{z}}_i(t) = \tilde{z}_{i+1}(t), \quad i \in \{1, \dots, k-1\} \\ \dot{\tilde{z}}_k(t) = u(t) + \sum_{j=1}^{k} \nu_j \tilde{z}_j(t) - g_0(T)\mu_0(t) \\ \qquad - g_{-1}(T)\mu_0(t-T) + \bar{g}\Psi(Y_t) \end{cases} \tag{46}$$

where

$$\bar{g} = \sum_{j=1}^{k} \nu_j \mathcal{C}_j(T) - \mathcal{G}(T). \tag{47}$$

Next note that since Lemma 1 gives $\Omega_j = \mathcal{C}_j(T)\nu_{2k-1}$ for $j = 1, \dots, k$, it follows that:

$$\tilde{z}_i(t) = z_i(t) - \Omega_i(t) = z_i(t) - \mathcal{C}_i(T)\nu_{2k-1}(t) \text{ for } i \in \{1, \dots, k\}. \tag{48}$$

Thus, (44) gives $\tilde{z}_i(t) = z_i(t) - \mathcal{C}_i(T)\Psi(Y_t)$ for all $t \geq 0$ and all $i \in \{1, \dots, k\}$, so $Z_\star(t) = \tilde{Z}(t) = (\tilde{z}_1(t), \dots, \tilde{z}_k(t))$ for all $t \geq 0$. Also, $\mathcal{M}(Y_t) = -\bar{g}\Psi(Y_t)$. Therefore, our choice (41) of the control gives

$$\begin{cases} \dot{\tilde{z}}_i(t) = \tilde{z}_{i+1}(t), \quad i \in \{1, \dots, k-1\} \\ \dot{\tilde{z}}_k(t) = \sum_{j=1}^{k} \nu_j \tilde{z}_j(t) + \sigma_{\bar{c}}(-\bar{g}\Psi(Y_t)) + \bar{g}\Psi(Y_t) + \vartheta(Z_\star(t)). \end{cases} \tag{49}$$

According to (36), we have

$$|\bar{g}\Psi(Y_t)| \le |\bar{g}|e^{|J_{2k-1}|T}\overline{\mu} = \bar{c} \tag{50}$$

for all $t \geq 0$. From the definition of the saturation level $\bar{c}$ of $\sigma_{\bar{c}}$, it follows that for all $t \geq 0$, we have

$$\begin{cases} \dot{\tilde{z}}_i(t) = \tilde{z}_{i+1}(t), \quad i \in \{1, \dots, k-1\} \\ \dot{\tilde{z}}_k(t) = \vartheta(\tilde{Z}(t)) + \sum_{i=1}^{k} \nu_i \tilde{z}_i(t) \end{cases} \tag{51}$$

so the lemma follows from our choice of ϑ in Lemma 2. □

We now combine the preceding lemmas to prove Theorem 1. We begin by proving the first conclusion of the theorem, in which $\eta = 0$. In this case, the closed loop system defined in our theorem is

$$\begin{cases} \dot{x}(t) = \mathcal{F}(t, x(t), z(t), 0) \\ \dot{z}_i(t) = z_{i+1}(t), \quad i \in \{1, \dots, k-1\} \\ \dot{z}_k(t) = u(Z(t), Y_t, x_t) + \sum_{j=1}^{k} \nu_j z_j(t) \\ \dot{Y}(t) = J_{2k-1}Y(t) + \frac{e_{2k-1}}{T}\omega(x(t)). \end{cases} \tag{52}$$

Using the fact that the control (41) from Lemma 5 agrees with our control (12) from Theorem 1 when we select $\mu_0(t) = \omega(x(t))$, it follows from using Lemma 5 with the choice $\mu_0(t) = \omega(x(t))$ that:

$$\lim_{t \to +\infty} |z_i(t) - \Omega_i(t)| = 0 \tag{53}$$

for all $i = 1$ to k, and $\tilde{z}_i = z_i - \Omega_i$ exponentially converges to 0 for all i.

Next notice that the x subsystem of (52) can be written as

$$\dot{x}(t) = \mathcal{F}(t, x(t), \Omega(t) + \tilde{z}(t), 0) \quad \text{where } \Omega = (\Omega_1, \dots, \Omega_k)^\top \tag{54}$$

when we choose the bounded function $\mu_0(t) = \omega(x(t))$. Hence, we can use the converging-input-converging-state portion of our Assumption 1 (with the choices $\delta = \tilde{z}$ and $\eta = 0$) to conclude that $\lim_{t \to +\infty} |x(t)| = 0$ and therefore that for all $i \in \{1, 2, \dots, k\}$, we have $\lim_{t \to +\infty} \Omega_i(t) = 0$, since $\omega(0) = 0$ and ω is continuous at 0. It follows that:

$$\lim_{t \to +\infty} z(t) = \lim_{t \to +\infty} (z(t) - \Omega(t)) + \lim_{t \to +\infty} \Omega(t) = 0. \tag{55}$$

On the other hand,

$$\dot{Y} = J_{2k-1}Y + \varepsilon \tag{56}$$

is ISS with respect to ε, by the Hurwitzness of J_{2k-1} as we defined this matrix in (10), which makes it possible to use a Riccati equation to find a quadratic Lyapunov function for $\dot{Y} = J_{2k-1}Y$ of the form $Y^\top PY$ for some positive definite matrix P which is then an ISS Lyapunov function for (56) with ε playing the role of the uncertainty. This provides positive constants c_a and c_b such that $|Y(t)| \le c_a(|Y(t/2)|e^{-c_b t} + \sup\{|\varepsilon(\ell)| : t/2 \le \ell \le t\})$ along all solutions of (56) for all $t \geq 0$. Specializing the preceding argument to the function $\varepsilon(t)=e_{2k-1}\omega(x(t))/T$ which converges to 0 as $t\to+\infty$ now gives the first conclusion of Theorem 1. This follows because all solutions of (56) for bounded choices of ε are bounded, so for each constant $\delta_0>0$, we can find a constant $T_0>0$ that is large enough so that $\max\{|Y(t/2)|e^{-c_b t}, \sup\{|\varepsilon(\ell)|: t/2\le\ell\le t\}\}<\delta_0/(2c_a)$ for all $t>T_0$, which gives $|Y(t)|\le\delta_0$ for all $t\ge T_0$.

It remains to prove the second conclusion of the theorem. To this end, first note that with the notation from our proof of the first conclusion of the theorem, the dynamics for $\tilde{z}$ are globally asymptotically stable to 0, so the interconnection of the perturbed dynamics $\dot{x}(t)=\mathcal{F}(t,x(t),\Omega(t)+\tilde{z}(t),\eta(t))$ with $\mu_0(t)=\omega(x(t))$ and the $\tilde{z}$ dynamics will be ISS with respect to η, by standard small gain arguments. Then the structure of the function Ω implies that the $(x,z)=(x,\tilde{z}+\Omega)$ dynamics are ISS with respect to η. This completes the proof of our theorem.

5. Extension to systems with measurement delays

This section is connected with, and provides a nontrivial extension of, Section 2, by explaining how the framework of Theorem 1 is general enough to allow cases where current values $x(t)$ are not available for measurement or for use in the control. Such cases occur in engineering applications where the control must be computed on a computer that is far from the actual plant, which was the case for instance in the work [26] which used small marine robots to search for oil pollution. Our strategy in this section is to find values of T that ensure that the required converging-input-converging-state assumption is satisfied for cases where current values $x(t)$ are not available for use in the control. See Remark 3 for a detailed description of how our work in this section adds value relative to the existing delay compensation literature.

Although [23] did not provide a bounded backstepping controller for the original system (1), it allowed cases where current values of the x components of the state of the original system were not available for use in the control, leading to feedback controls in which $x(t)$ must be replaced by time delayed values $x(t-D)$ of x for a constant delay $D>0$. In the same way, we can extend Theorem 1 above to allow cases where one must use time lagged values of x instead of current ones. This is done by replacing $\omega(x(\ell))$ in the preceding analysis by $\omega(x(\ell-D))$ for constant values of the delay D, so instead of placing a converging-input-converging-state assumption on (9) in Assumption 1, we must replace (9) by the delayed version

$$\dot{\xi}(t)=\mathcal{F}\left(t,\xi(t),\int_{t-T}^{t}\Lambda(\ell,t)\omega(\xi(\ell-D))\mathrm{d}\ell+\delta(t),0\right),\tag{57}$$

and then the conclusions of the theorem remain true with $x(\ell)$ replaced by $x(\ell-D)$ in the feedback control. However, our sufficient conditions from Proposition 1 do not apply in cases such as (57) with measurement delays. This motivates the following analog of Proposition 1 that provides sufficient conditions for our delayed version of the converging-input-converging-state condition to hold, and which can therefore facilitate checking the requirements of our theorem when constant measurement delays D are introduced in the x measurements. In what follows, we use the same choices of $\Lambda_a(T)$ from (19) and $\Lambda_+(T)=\sup_{t\ge0}\{|(\Lambda_1(\ell,t),\ldots,\Lambda_p(\ell,t))|: t-T\le\ell\le t\}$ as in Section 3, which are still independent of t (by the argument we gave in Section 3), and which also do not depend on D.

Proposition 2. *If Assumption 2 holds, and if the constants $T>0$ and $D>0$ are such that*

$$\mathcal{R}(T)<1\ \text{and}\tag{58a}$$

$$C\Lambda_a(T)(r_2+r_3\Lambda_+(T)r_1(D+T))(D+T)<\frac{1-\mathcal{R}(T)}{8\sqrt{2}},\ \text{where}\tag{58b}$$

$$\mathcal{R}(T)=4(T\Lambda_a(T)C)^2\left[2r_2^2+\frac{5}{2}(r_1r_3T\Lambda_+(T))^2\right]\tag{58c}$$

then the following is true: For each continuous function $\delta:[0,+\infty)\to\mathbb{R}^p$ that exponentially converges to zero, all solutions of (57) converge to 0 as $t\to+\infty$. If, in addition, the function W from Assumption 2 is proper, then the system

$$\dot{\xi}(t)=\mathcal{F}\left(t,\xi(t),\int_{t-T}^{t}\Lambda(\ell,t)\omega(\xi(\ell-D))\mathrm{d}\ell+\delta(t),\eta(t)\right)\tag{59}$$

is ISS with respect to (δ,η).

Proof. We indicate the changes needed in the proof of Proposition 1. We let $c_*>0$ be the constant from (28) as before, where λ is chosen as in the proof of Proposition 1. We may assume that $\lambda>1$ is close enough to 1 so that the requirements from (58) are still true if we replace $\mathcal{R}(T)$ by $\mathcal{R}(T)=4\lambda(T\Lambda_a(T)C)^2[2r_2^2+2.5(r_1r_3T\Lambda_+(T))^2]$ (by the strictness of the inequalities in (58)), and we make this replacement in the rest of the proof. Then, using our notation from the proof of Proposition 1, we have $c_*=0.5(1-\mathcal{R}(T))=0.5(1-4T\mathcal{N}_1\lambda)$. In what follows, we use Λ_a and Λ_+ to mean $\Lambda_a(T)$ and $\Lambda_+(T)$, respectively, to keep our notation simple. Using the function V from Assumption 2 and Young's Inequality, the additional term that must be added to the decay estimate on V can be bounded above as follows:

$$\begin{aligned}
&V_x(t,x(t))g(t,x)\int_{t-T}^{t}\Lambda^\flat(\ell,t)[\omega(x(\ell-D))-\omega(x(\ell))]\mathrm{d}\ell\\
&\le C\Lambda_a\sqrt{W(x(t))}\int_{t-T-D}^{t}|\dot{x}(s)|\mathrm{d}s\\
&\le C\Lambda_a\sqrt{W(x(t))}\int_{t-T-D}^{t}\Big\{|f(s,x(s))|\\
&\quad+|g(s,x(s))|\Big[\Lambda_+\int_{s-T}^{s}|\omega(x(\ell-D))|\mathrm{d}\ell+|\eta(s)|\Big]\Big\}\mathrm{d}s\\
&\le C\Lambda_a\sqrt{W(x(t))}\int_{t-T-D}^{t}\Big(r_2\sqrt{W(x(s))}\\
&\quad+r_3\Lambda_+\int_{t-D-2T}^{t}|\omega(x(\ell-D))|\mathrm{d}\ell+r_3|\eta(s)|\Big)\mathrm{d}s\\
&\le C\Lambda_a\sqrt{W(x(t))}\Big[(r_2+r_3\Lambda_+r_1(D+T))\int_{t-2D-2T}^{t}\\
&\quad\times\sqrt{W(x(\ell))}\mathrm{d}\ell+r_3(D+T)|\eta|_{[0,t]}\Big]\\
&\le\frac{c_*}{4}W(x(t))+\frac{1}{c_*}\Big(C\Lambda_a\Big[(r_2+r_3\Lambda_+r_1(D+T))\int_{t-2(D+T)}^{t}\\
&\times\sqrt{W(x(\ell))}\mathrm{d}\ell+r_3(D+T)|\eta|_{[0,t]}\Big]\Big)^2\\
&\le\frac{c_*}{4}W(x(t))+\frac{4}{c_*}\{C\Lambda_a(r_2+r_3\Lambda_+r_1(D+T))\}^2(D+T)\\
&\quad\times\int_{t-2D-2T}^{t}W(x(\ell))\mathrm{d}\ell\\
&+\frac{2}{c_*}(Cr_3\Lambda_a(D+T))^2|\eta|_{[0,t]}^2,
\end{aligned}$$

where the last inequality also used Young's inequality, the relations $ab\le a^2/4+b^2$ and $(a+b)^2\le 2a^2+2b^2$ for suitable nonnegative values of a and b, and then Jensen's inequality. Using the inequality (58b) and choosing $\lambda>1$ close enough to 1, it follows that we can find a constant $\lambda_*>1$ that is close enough to 1 and which is such that

$$\tfrac{8\lambda_*}{c_*}\{C\Lambda_a(D+T)(r_2+r_3\Lambda_+r_1(D+T))\}^2<\tfrac{c_*}{4}, \tag{60}$$

since $c_*=0.5(1-\mathcal{R}(T))$. Then reasoning analogously to the argument that produced (28) shows that the time derivative of

$$V_2(t,x_t)=V_1(t,x_t)+\frac{4\lambda_*}{c_*}\{C\Lambda_a(r_2+r_3\Lambda_+r_1(D+T))\}^2 \times (D+T)\int_{t-2(D+T)}^{t}\int_{\ell}^{t}W(x(s))\mathrm{d}s\mathrm{d}\ell \tag{61}$$

along all solutions of (59) admits positive constants c_{**} and c_{***} such that

$$\dot V_2\le -c_{**}W(x(t))+c_{***}|(\delta,\eta)|^2_{[0,t]}. \tag{62}$$

If, in addition, W is proper, then we can argue as in the proof of Proposition 1 to find a function $\gamma_a\in\mathcal{K}_\infty$ and a positive constant k_a such that

$$\dot V_2\le -\gamma_a(V_2(t,x_t))+k_a|(\delta,\eta)|^2_{[0,t]} \tag{63}$$

(by using the bound (32) except with T in (32) replaced by $D+T$). Then the rest of the proof is the same as in the last part of the proof of Proposition 1 except with V_1 replaced by V_2. □

Remark 3. There is a large recent literature on delay compensating control design for nonlinear systems, largely involving prediction, which replaces time lagged state values in controls by predicted state values [2,3,9–12,25,27,28,35]. While prediction is useful for eliminating delays from control variables, it generally leads to dynamic controls that contain distributed terms (i.e., terms that use all values of the control or the state along certain time intervals), which can be difficult to implement in practice [12]. See also the reduction model controls [14] which are expressed implicitly as solutions of integral equations that do not admit explicit solutions. Hence, potential advantages of the controls that can be obtained using our approach from this section include (a) the lack of distributed terms in our controls, (b) our ability to satisfy control bounds, (c) our ability to prove global asymptotic stability of the closed loop system from Theorem 1 under any measurement delay $D>0$ for which (57) satisfies the required converging-input-converging-state condition (with no other restriction on the size of D), and (d) the robust performance of our controls in terms of ISS.

6. Illustrations

Our Lyapunov function based sufficient conditions are convenient for checking our assumptions from Theorem 1. We illustrate this point in this section, in two examples. In our first example, we apply our Lyapunov sufficient conditions directly. In our second example, our Lyapunov sufficient conditions do not apply directly, but we use a mixture of our Lyapunov and trajectory based methods to check our converging-input-converging-state conditions. Our second example illustrates the point that it may only be necessary to check our sufficient conditions locally in a neighborhood of the equilibrium, instead of globally, which eliminates the need to find a global Lyapunov function as required in Assumption 2. For simplicity, this section only considers cases where there are no measurement delays D, but we can apply the methods from the preceding section to cover measurement delays as well.

6.1. First Illustration

Consider the three-dimensional system

$$\begin{cases}\dot x(t)=\dfrac{|x(t)|}{1+|x(t)|}+z_1(t)\\ \dot z_1(t)=z_2(t)\\ \dot z_2(t)=u(t)\end{cases} \tag{64}$$

which is not amenable to classical backstepping, because the right side of $\dot x(t)$ in the dynamics is not differentiable. In terms of our notation from Section 3, we choose $k=2,\ n=1,\ p=1$, and

$$\mathcal{F}(t,x,z_1)=\frac{|x|}{1+|x|}+z_1 \text{ and } \omega(x)=-\frac{1}{\Lambda_*(T)}\left(\frac{|x|}{1+|x|}+2\frac{x}{1+|x|}\right), \tag{65}$$

where

$$\Lambda_*(T)=\int_{-T}^{0}\Lambda_1(\ell+t,t)\mathrm{d}\ell=\int_{-T}^{0}e^{\ell}\ell^{k-1}(\ell+T)^{k-1}\mathrm{d}\ell =2-T-e^{-T}(2+T). \tag{66}$$

We compute a constant $T>0$ such that Assumption 1 is satisfied. First note that since $p=1$, and since $\Lambda_1(\ell,t)\le 0$ for all $t\ge 0$ and $\ell\in[t-T,t]$, we have $\Lambda_a(T)=-\Lambda_*(T)=|\Lambda_*(T)|$. Since (65) are globally Lipschitz functions and $\mathcal{F}$ is an affine function of z_1 and ω is bounded, it suffices to find constants r_i for $i=1,2,3$ and functions V and W such that Assumption 2 is satisfied with

$$f(t,x)=\frac{|x|}{1+|x|} \text{ and } g(t,x)=1 \tag{67}$$

and then to choose T such that our condition (20) holds.

To this end, we check that Assumption 2 is satisfied using the functions

$$V(t,x)=\int_0^x\sigma_1(\ell)\mathrm{d}\ell \ \text{ and } \ W(x)=\frac{2\sigma_1(x)x}{1+|x|}. \tag{68}$$

Since (67) give

$$f(t,x)+g(t,x)\Lambda_*(T)\omega(x)=-\frac{2x}{1+|x|} \tag{69}$$

our conditions (18) on the r_i's from Assumption 2 for the preceding choices of f, g, V, and W will be satisfied if

$$|\sigma_1(x)|\le\sqrt{\frac{2\sigma_1(x)x}{1+|x|}},\ \frac{1}{|\Lambda_*(T)|}\frac{3|x|}{1+|x|}\le r_1\sqrt{\frac{2\sigma_1(x)x}{1+|x|}}, \frac{|x|}{1+|x|}\le r_2\sqrt{\frac{2\sigma_1(x)x}{1+|x|}},\ \text{ and } 1\le r_3. \tag{70}$$

By separately considering points $x\in[-1,1]$ and points $x\notin[-1,1]$, it follows easily that Assumption 2 is satisfied with the choices

$$C=\frac{3}{|\Lambda_*(T)|},\ r_1=\frac{3}{\sqrt{2}|\Lambda_*(T)|},\ r_2=1,\ \text{and } r_3=1. \tag{71}$$

Hence, our requirement (20) on $T>0$ from Proposition 1 holds if

$$1>4(3T)^2\left[2+\frac{5}{2}\left(\frac{3T^3}{\sqrt{2}|2-T-e^{-T}(2+T)|}\right)^2\right] \tag{72}$$

and we can use Mathematica [34] to check that the right side of (72) takes the value 0.912536 at $T=0.11$. Hence, Assumption 1 is satisfied with $T=0.11$, and then the desired controller is provided by Theorem 1.

6.2. Second Illustration

We can sometimes apply Theorem 1 by checking Assumption 1 through a mixture of Lyapunov and direct trajectory analyses. For instance, consider the three dimensional system

$$\begin{cases} \dot{x} = x^2 - x^3 + z_1 \\ \dot{z}_1 = z_2 \\ \dot{z}_2 = u. \end{cases} \tag{73}$$

As noted in [13, pages 593–594], the system (73) is globally asymptotically stabilized to 0 by the control

$$\begin{aligned} u(x,z) = &-\frac{\partial V_0}{\partial z_1}(x,z_1) + \frac{\partial \phi}{\partial z_1}(x,z_1)z_2 - z_2 \\ &+ \frac{\partial \phi}{\partial x}(x,z_1)(x^2 - x^3 + z_1) + \phi(x,z_1), \\ &\text{where } V_0(x,z_1) = \frac{1}{2}x^2 + \frac{1}{2}(z_1 + x + x^2)^2 \\ &\text{and } \phi(x,z_1) = -2x - (1+2x)(x^2 - x^3 + z_1) - z_1 - x^2, \end{aligned}$$

which is unbounded since it satisfies $\lim_{x\to\infty} u(x,0) = -\infty$. Our work [23] provided the unbounded control

$$\begin{aligned} u(t) = &\frac{1}{(1-e^{-\tau})^2} \\ &\times \{J(x(t)) - 2e^{-\tau}J(x(t-\tau)) + e^{-2\tau}J(x(t-2\tau))\} \\ &- 2z_2(t) - z_1(t), \\ &\text{where } J(x) = -\sin\left(\frac{\pi x}{2}\right)\mathbf{1}_{[-2,2]}(x) \end{aligned} \tag{74}$$

that rendered (73) globally asymptotically stable to 0, where the indicator function $\mathbf{1}_{[-2,2]}$ is defined to be 1 on $[-2,2]$, and 0 on $\mathbb{R}\setminus[-2,2]$. Here we show how our new Theorem 1 provides a globally bounded globally asymptotically stabilizing controller for (73), using the choice of $\omega = J/\Lambda_*(T)$ with J as defined in (74), and with $p=1$, and $k=2$ and with the artificial $T>0$ to be specified.

To verify Assumption 1 with the preceding choices, first note that for each continuous function $\delta : \mathbb{R} \to \mathbb{R}$ that exponentially converges to 0 and each initial state $x_0 \in \mathbb{R}$, we can find a value $T_*(x_0, \delta) \in [0, \infty)$ such that the corresponding solution of

$$\dot{x}(t) = x^2(t) - x^3(t) + \int_{t-T}^{t} \Lambda_1(\ell,t)\omega(x(\ell))\mathrm{d}\ell + \delta(t) \tag{75}$$

satisfies $x(t) \in [-0.8, 3/2]$ for all $t \geq T_*(x_0, \delta)$. This can be done by noting that the integral in (75) is bounded by 1 (since $\Lambda_a(T) = |\Lambda_*(T)|$), that $x^2 - x^3 \leq -1.125$ for all $x \geq 3/2$, and that $x^2 - x^3 \geq 1.152$ for all $x \leq -0.8$, so the right side terms $x^2(t) - x^3(t)$ in (75) dominate the other right side terms, since we may assume that t is large enough so that $|\delta(t)| \leq 0.12$. Hence, it suffices to check the inequalities (18) from Assumption 2 for all $x \in [-0.8, 3/2]$, by only considering time values $t \geq T_*(x_0, \delta)$.

We now check the estimates from (18) for all $x \in [-0.8, 3/2]$ using $V(x) = \frac{1}{2}x^2$, $W(x) = x^2$, $f(x) = x^2 - x^3$, and $g(x) = 1$. First note that simple calculations (e.g., using Mathematica [34]) give $x^2 - x^3 - \sin(\pi x/2) \leq -x$ (resp., $\geq -x$) for all $x \in [0, 3/2]$ (resp., $x \in [-0.8, 0]$) which gives $\nabla V(x)(f(x) + \Lambda_*(T)\omega(x)) \leq -W(x)$, $|x^2 - x^3| \leq 1.44|x|$, and $|\sin(\pi x/2)| \leq (\pi/2)|x|$ when $x \in [-0.8, 3/2]$, so we can choose $r_1 = \pi/(2|\Lambda_*(T)|)$, $r_2 = 1.44$, $r_3 = 1$, and $C = \pi/(2|\Lambda_*(T)|)$. Hence, we can use our formula (66) for $\Lambda_*(T)$ to check that the sufficient condition (20) from Proposition 1 (for $\lim_{t\to\infty} x(t) = 0$ to hold) is satisfied if

$$1 > (T\pi)^2\left(2(1.44)^2 + \frac{5\pi^2}{8}\left(\frac{T^3}{2 - T - e^{-T}(2+T)}\right)^2\right) \tag{76}$$

which is satisfied for all $T \in (0, 0.0209]$. Therefore, we can satisfy our requirements with $T = 0.0209$, and then the desired bounded control is provided by Theorem 1.

7. Conclusions

We provided a new bounded backstepping technique for a large class of cascaded partially linear systems with arbitrarily large numbers of integrators, under a converging-input-converging-state assumption involving the nonlinear subsystems. For many cases where the nonlinear part of the system is control affine, we used Lyapunov functions to provide sufficient conditions for our converging-input-converging-state assumption to be satisfied. Although our controller involves a dynamic extension, it has an advantage that it provides bounded controllers for the original system, which would not have been possible under our assumptions if we had instead relied on previous results. We plan to combine our new methods with the time delay methods in [1] and [14] to also allow arbitrarily long measurement delays.

Appendix A. Proof of Lemma 1

For each $j \in \{1, 2, \ldots, k\}$, Ω_j will be a linear combination of integrals, each of which having an integrand of the form $e^{\ell-t}(t-\ell)^\alpha(t-\ell-T)^\beta$ with integers $\alpha \in \{0, 1, \ldots, k-1\}$ and $\beta \in \{0, 1, \ldots, k-1\}$, so the required constants $c_{i,j}$ can be obtained by applying the binomial formula

$$(a+b)^\beta = \sum_{j=0}^{\beta} \frac{\beta!}{j!(\beta-j)!} a^j b^{\beta-j} \tag{A.1}$$

with the choices $a = t - \ell$ and $b = -T$ for those integrals in the sums having positive β values. If $j < k$, then all of the α's and β's in the sums will be positive integers. On the other hand, if $j = k$, then the linear combination of integrals in the formula for Ω_k will include multiples of the integrals

$$\int_{t-T}^{t} e^{\ell-t}(t-\ell)^{k-1}\mu_0(\ell)\mathrm{d}\ell \text{ and } \int_{t-T}^{t} e^{\ell-t}(t-\ell-T)^{k-1}\mu_0(\ell)\mathrm{d}\ell \tag{A.2}$$

and the derivatives of (A.2) in the formula for $\Omega_{k+1} = \Omega_k'$ will be linear combinations of terms that include $-e^{-T}T^{k-1}\mu_0(t-T)$ and $(-T)^{k-1}\mu_0(t)$, which will provide the constants g_{-1} and g_0 in the lemma. The remaining terms $T_i(t)$ in the linear combination in the formula for Ω_k will only have positive powers α and β, and computing their derivatives $T_i'(t)$ will produce the g_i's in the formula for Ω_{k+1} for $i = 1, 2, \ldots, 2k-1$, by again applying the binomial formula (A.1) to the integrand factors $(t-\ell-T)^\beta$ with positive integers β.

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