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A CLASSIFICATION OF LAGRANGIAN PLANES IN HOLOMORPHIC SYMPLECTIC VARIETIES

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Abstract Classically, an indecomposable class R in the cone of effective curves on a K3 surface X is representable by a smooth rational curve if and only if $R^2 = -2$. We prove a higher-dimensional generalization conjectured by Hassett and Tschinkel: for a holomorphic symplectic variety M deformation equivalent to a Hilbert scheme of n points on a K3 surface, an extremal curve class $R \in H_2(M, \mathbb{Z})$ in the Mori cone is the line in a Lagrangian n-plane $\mathbb{P}^n \subset M$ if and only if certain intersection-theoretic criteria are met. In particular, any such class satisfies $(R, R) = -\frac{n+3}{2}$, and the primitive such classes are all contained in a single monodromy orbit.

Keywords: holomorphic symplectic variety; extremal ray; Lagrangian subvariety

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Statement of results

Let M be an (irreducible) holomorphic symplectic variety, that is, a smooth simply connected projective variety admitting a unique (up to scalars) everywhere nondegenerate holomorphic 2-form. M comes equipped with a quadratic form $(\cdot, \cdot)$ on $H^2(M, \mathbb{Z})$ called the Beauville–Bogomolov form; it is primitive, integral, nondegenerate, and deformation invariant of signature $(3, b_2(M) - 3)$. A K3 surface X, for example, is holomorphic symplectic, and the Beauville–Bogomolov form in this case is simply the intersection pairing. The Hilbert scheme of n points on X, $M = X^{[n]}$, is holomorphic symplectic as well, and the Beauville–Bogomolov form yields an orthogonal decomposition

$$H^2(M, \mathbb{Z}) = H^2(X, \mathbb{Z}) \oplus \mathbb{Z}\delta,$$

where $H^2(X, \mathbb{Z})$ is isometrically embedded via pullback along the Hilbert–Chow map $X^{[n]} \to X^{(n)}$, and $(\delta, \delta) = 2 - 2n$, where 2δ is the divisor of nonreduced subschemes. More generally, any proper moduli space $M(\mathbf{v})$ of stable sheaves on X of Mukai vector $\mathbf{v}$ with $\mathbf{v}^2 = 2n - 2$ is a holomorphic symplectic variety deformation equivalent to a Hilbert scheme of n points on a K3 surface. We say in this case that M is 'of K3 type', or sometimes 'of $\mathrm{K3}^{[n]}$ type' if we want to specify the dimension.

Classically, much of the geometry of a projective K3 surface is encoded in the intersection pairing on the Néron–Severi group $\mathrm{NS}(X)$. If h is an ample divisor, then the closed cone $\overline{NE}_1(X)$ of effective curves (also called the Mori cone), for instance, is the closure of the cone generated by

$$\{R \in \mathrm{NS}(X) \mid h.R > 0 \text{ and } R^2 \geqslant -2\}.$$

A primitive curve class R of nonpositive self-intersection generating an extremal ray of $\overline{NE}_1(X)$ is dual to a face of the nef cone whose generic divisor induces either a contraction of a smooth rational curve R to an ordinary double point, when $R^2 = -2$, or an elliptic fibration $X \to \mathbb{P}^1$, when $R^2 = 0$.

A program to analogously understand the birational geometry of M purely in terms of the intersection theory of the Beauville–Bogomolov form was initiated by Hassett and Tschinkel in [14] and proven for fourfolds in [16]. Great strides toward fleshing out this program in higher dimensions have been made recently by using Bridgeland stability conditions to analyze moduli spaces $M(\mathbf{v})$ and then deforming to arbitrary K3 type varieties. Indeed, due to work of Bayer and Macrì [5], Bayer *et al.* [3], and Mongardi [24], there is now a complete description of the nef, movable, and Mori cones of M (see § 1.9 below). In particular, we have the following.

Theorem 1 ([3, Proposition 2] or [24, Corollary 2.4]). *Let M be a holomorphic symplectic variety of K3 type and dimension $2n$. If $R \in H_2(M, \mathbb{Z})$ is the primitive generator of an extremal ray of the Mori cone, then $(R, R) \geqslant -\frac{n+3}{2}$.*

Here we have used the embedding $H^2(M, \mathbb{Z}) \subset H_2(M, \mathbb{Z})$ induced by the Beauville–Bogomolov form, and the resulting extension of $(\cdot, \cdot)$ to a rational form on $H_2(M, \mathbb{Z})$.

The next step in the Hassett–Tschinkel program is to classify the geometry of extremal contractions in terms of the intersection theory of their contracted curves. The exceptional loci of such contractions generically look like a fibration of k-dimensional projective spaces[1] over a $(2n-2k)$-dimensional holomorphic symplectic variety, contracting via the projection (see, for example, [9, 26]). In particular, Lagrangian planes contract to points.

Our main result is to provide a numerical classification of curve classes $R \in H_2(M, \mathbb{Z})$ that sweep out a Lagrangian plane $\mathbb{P} \subset M$, thus proving a conjecture of Hassett and Tschinkel [17]. In particular, we have the following.

Theorem 2. *Let M be a holomorphic symplectic variety of K3 type and dimension $2n$, and suppose that $R \in H_2(M, \mathbb{Z})$ is the class of a line in a Lagrangian n-plane $\mathbb{P}^n \subset M$. Then R satisfies $(R, R) = -\frac{n+3}{2}$ and $2R \in H^2(M, \mathbb{Z})$.*

In Theorem 22, we classify such curve classes (in particular nonprimitive ones) in terms of Markman's extended weight-2 Hodge structure. In general, the numerical criteria of Theorem 2 are likely not sufficient (see Example 9), but for primitive extremal classes they are.

[1] If $k = 1$, an ADE configuration of rational curves can occur in the generic fiber.

Theorem 3. *With M as above, a primitive class $R \in H_2(M, \mathbb{Z})$ generating an extremal ray of the Mori cone is the line in a Lagrangian plane if and only if $(R, R) = -\frac{n+3}{2}$ and $2R \in H^2(M, \mathbb{Z})$.*

In view of Theorems 1 and 3, the extremal contractions of Lagrangian planes are singled out as being the 'most extreme', in the sense that the class of the line achieves the minimal square of the generator of an extremal ray in the Mori cone. They are the higher-dimensional analog of -2-curves on K3 surfaces. On the other side of the extremal contraction spectrum, Markman [23] (in the general case) and Bayer and Macrì [5] (in the case of Bridgeland moduli spaces) have recently resolved a long-standing conjecture asserting that a nef class D with $(D, D) = 0$ induces a fibration $M \to \mathbb{P}^n$ with Lagrangian tori fibers. As an application of Theorem 2, we have a necessary condition for the existence of sections of Lagrangian fibrations.

Corollary 4. *M admits a Lagrangian fibration with a section only if $H_2(M, \mathbb{Z}) \cap H^{n-1,n-1}(M)$ contains the sublattice*

$$\begin{pmatrix} 0 & 1 \\ 1 & -\frac{n+3}{2} \end{pmatrix}.$$

Lagrangian planes are also of interest because twisting by their structure sheaf yields an autoequivalence of the derived category $D^b(M)$ in much the same way that a smooth rational curve on a K3 surface yields a spherical twist (see [15]).

Theorem 2 was demonstrated for varieties of $\mathrm{K3}^{[2]}$ type by Hassett and Tschinkel [16], for those of $\mathrm{K3}^{[3]}$ type by Harvey *et al.* [12], and for those of $\mathrm{K3}^{[4]}$ type by the author and Jorza [4] using the representation theory of the monodromy group to exhibit possible classes of lines sweeping out a Lagrangian plane as integral points on arithmetic curves. As a product of the analysis, in all three cases a universal formula for the class $[\mathbb{P}^n] \in H_{2n}(M, \mathbb{Z})$ in terms of Hodge classes and the class of the line R is determined. It follows that there is a unique orbit of the classes of such lines under the Zariski closure of the monodromy group $\mathrm{Mon}(M)$. We conclude from Theorem 2 that the same is true without taking the Zariski closure for primitive classes.

Corollary 5. *The primitive classes $R \in H_2(M, \mathbb{Z})$ occurring as the line in a Lagrangian plane belong to a single $\mathrm{Mon}(M)$ orbit.*

In general it may not be the case that the class of the line is primitive (see Remark 28). We expect our method to also allow for an intersection-theoretic classification of Lagrangian planes in holomorphic symplectic manifolds deformation equivalent to generalized Kummer varieties, using the work of [32]. It was conjectured by Hassett and Tschinkel that an analog of Theorem 2 is true but with $(R, R) = -\frac{n+1}{2}$, and the necessity of the numerical conditions was established in the case of fourfolds [18]. Corollary 5, however, is not salvageable in this case; it is expected (and proven for $n = 2$ [18]) that there will always be multiple monodromy orbits.

Outline

In § 1, we summarize the theory of Bridgeland stability conditions on K3 surfaces, and the Bayer–Macrì description of the nef cones of Bridgeland moduli spaces. In § 2, we

define and give some examples of Lagrangian planes and Grassmannians. In § 3, we prove classify Lagrangian planes for Bridgeland moduli spaces, and discuss other contractible Lagrangian subvarieties. In § 4, we extend the classification to arbitrary K3 type varieties by deformation.

1. Bayer–Macrì description of the nef cone

We very briefly summarize the basic theory of Bridgeland stability conditions and the Bayer–Macrì description of the nef cone of Bridgeland moduli spaces on K3 surfaces, or at least as much as we will need. For background on the first, see Bridgeland's original paper [8] or Macrì's survey in [2, Appendix D]. For the second, Bayer and Macrì develop the theory for general stability conditions in [6] and apply it to the case of K3 surface in [5], and our summary is mainly taken from their treatment.

1.1.

Let X be a smooth projective variety, and let $D^b(X)$ be its bounded derived category of coherent sheaves. A stability condition $\sigma = (Z, \mathcal{P})$ consists of a (group) homomorphism $Z : K(X) \to \mathbb{C}$ and full extension-closed abelian subcategories $\mathcal{P}(\varphi) \subset D^b(X)$ for each $\varphi \in \mathbb{R}$ such that

- any $0 \neq E \in \mathcal{P}(\varphi)$ has $Z(E) \in \mathbb{R}_{>0} e^{i\pi\varphi}$;
- $\mathcal{P}(\varphi + 1) = \mathcal{P}(\varphi)[1]$ for all $\varphi \in \mathbb{R}$;
- $\mathrm{Hom}(\mathcal{P}(\varphi), \mathcal{P}(\varphi')) = 0$ if $\varphi > \varphi'$; and
- any $0 \neq E \in D^b(X)$ has a Harder–Narasimhan filtration, i.e., there is a sequence in $D^b(X)$
$$0 = E_0 \to E_1 \to \cdots \to E_{n-1} \to E_n = E$$
whose factors A_i, defined as the cones
$$E_{i-1} \to E_i \to A_i \to E_{i-1}[1],$$
satisfy $A_i \in \mathcal{P}(\varphi_i)$ with
$$\varphi_1 > \varphi_2 > \cdots > \varphi_{n-1} > \varphi_n.$$

In this case we denote $\varphi^+(E) = \varphi_1$ and $\varphi^-(E) = \varphi_n$, and for any interval $I \subset \mathbb{R}$, $\mathcal{P}(I) \subset D^b(X)$ is defined as the full subcategory of $E \in D^b(X)$ for which $[\varphi^-(E), \varphi^+(E)] \subset I$. It is easy to show that the stability condition σ induces a t-structure on $D^b(X)$ whose heart is $\mathcal{P}((0, 1])$.

1.2.

The homomorphism Z is called the central charge, and we say that $0 \neq E \in D^b(X)$ has phase φ if $Z(E) \in \mathbb{R}_{>0} e^{i\pi\varphi}$. The objects of $\mathcal{P}(\varphi)$ are called σ-semistable, while the simple objects are called σ-stable. Under mild technical assumptions (fullness; see [7]), the categories $\mathcal{P}(\varphi)$ are Artinian, and every σ-semistable object $E \in \mathcal{P}(\varphi)$ has a Jordan–Hölder filtration in the above sense with σ-stable factors of phase φ. Two σ-semistable objects $E, E' \in \mathcal{P}(\varphi)$ are S-equivalent if their Jordan–Hölder factors are the same (up to permutations).

1.3.

We further say that the stability condition σ is numerical if the central charge Z factors through the cokernel $K_{\mathrm{num}}(X)$ of the Chern character $\mathrm{ch} : K(X) \to H^*(X, \mathbb{Q})$. Let $\mathrm{Stab}(X)$ be the space of numerical stability conditions. It has a natural metric topology [7, Proposition 8.1] which makes the map $Z : \mathrm{Stab}(X) \to \mathrm{Hom}(K_{\mathrm{num}}(X), \mathbb{C})$ a local homeomorphism. There is a particularly well-behaved connected component $\mathrm{Stab}^\dagger(X) \subset \mathrm{Stab}(X)$ containing stability conditions for which the structure sheaves of points $k(x)$ are all stable of the same phase. Numerical stability conditions have been constructed on surfaces (see [8] for K3 surfaces, and [1] in general).

For the remainder of this section we restrict our attention to K3 surfaces X. We note in passing that we could just as easily work throughout with a K3 surface X twisted by a Brauer class $\alpha \in \mathrm{Br}(X)$ using the results of [13], and in fact this is extremely useful (see the use of Lemma 6.3 in the proof of Theorem 1.3 in [6], based on the idea of [25]).

1.4.

The Mukai lattice $\widetilde{H}(X, \mathbb{Z})$ is the full cohomology of X,

$$\widetilde{H}(X, \mathbb{Z}) = H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z}),$$

endowed with the (pure) weight-2 Hodge structure determined by $\widetilde{H}^{2,0} = H^{2,0}$. The Mukai pairing of two vectors $\mathbf{a} = (r, D, s), \mathbf{b} = (r', D', s') \in \widetilde{H}(X, \mathbb{Z})$ is

$$(\mathbf{a}, \mathbf{b}) = D.D' - rs' - r's.$$

For any object $E \in D^b(X)$, the Mukai vector of E is

$$\mathbf{v}(E) = \mathrm{ch}(E)\sqrt{\mathrm{Td}(X)} = (\mathrm{ch}_0(E), \mathrm{ch}_1(E), \mathrm{ch}_2(E) + \mathrm{ch}_0(E)).$$

Recall that, for a K3 surface, $\mathrm{ch} : K_{\mathrm{num}}(X) \to \widetilde{H}(X, \mathbb{Z})$ is an integral isomorphism. Thus, by Grothendieck–Riemann–Roch, for $E, F \in D^b(X)$,

$$\chi(R\,\mathrm{Hom}(E, F)) = -(\mathbf{v}(E), \mathbf{v}(F)).$$

The algebraic Mukai lattice is defined to be the integral classes in the $(1, 1)$ part of the Mukai lattice, $\widetilde{H}_{\mathrm{alg}}(X, \mathbb{Z}) = \widetilde{H}^{1,1} \cap \widetilde{H}(X, \mathbb{Z})$. Note that the Chern character endows $K_{\mathrm{num}}(X)$ with a natural (pure) weight-2 Hodge structure such that $\mathrm{ch} : K_{\mathrm{num}}(X) \xrightarrow{\cong} \widetilde{H}(X, \mathbb{Z})$, and this is perhaps a more elegant definition of the Mukai lattice.

The central charge Z of any numerical stability condition on X can be represented as $Z(\cdot) = (\Omega_Z, \cdot)$ for a unique $\Omega_Z \in \widetilde{H}_{\mathrm{alg}}(X, \mathbb{Z}) \otimes \mathbb{C}$; we denote the resulting map by the same letter: $\Omega : \mathrm{Stab}(X) \to \widetilde{H}_{\mathrm{alg}}(X, \mathbb{Z}) \otimes \mathbb{C}$.

1.5.

For any Mukai vector $\mathbf{v} \in \widetilde{H}_{\mathrm{alg}}(X, \mathbb{Z})$, the space $\mathrm{Stab}(X)$ has a wall and chamber structure with respect to $\mathbf{v}$. That is, there is a locally finite collection of codimension-1 submanifolds called walls such that, for σ off a wall, the set of σ-stable objects E (and thus also the set of σ-semistable objects) is locally constant. A connected component of the complement in $\mathrm{Stab}(X)$ of the union of all walls is called an open chamber, and its closure is called a closed chamber. We say that a stability condition σ is generic with respect to $\mathbf{v}$ if it lies in an open chamber.

1.6.

By [5, § 5], every (codimension-1) wall $\mathcal{W} \subset \mathrm{Stab}(X)$ with respect to $\mathbf{v}$ has an associated saturated signature-(1, 1) sublattice $\mathbf{v} \in \mathcal{H} \subset \widetilde{H}_{\mathrm{alg}}(X, \mathbb{Z})$, intrinsically described as the set of $\mathbf{w} \in \widetilde{H}_{\mathrm{alg}}(X, \mathbb{Z})$ for which $Z(\mathbf{w})$ and $Z(\mathbf{v})$ are $\mathbb{R}$-linearly dependent (i.e., $\Im \frac{Z(\mathbf{w})}{Z(\mathbf{v})} = 0$) for all $\sigma \in \mathcal{W}$. Not all such hyperplanes $\mathcal{H}$ arise in this way, but we can always associate to $\mathcal{H}$ the *potential wall* $\mathcal{W} \subset \mathrm{Stab}(X)$ of stability conditions σ such that $Z(\mathcal{H})$ lies on a real line. In this case we say that $\mathbf{w} \in \mathcal{H}$ is effective if there is a σ-semistable object with Mukai vector $\mathbf{w}$ for a generic $\sigma \in \mathcal{W}$, and we define the effective cone of the potential wall $\mathcal{C} \subset \mathcal{H} \otimes \mathbb{R}$ to be the real cone generated by effective classes $\mathbf{w} \in \mathcal{H}$.

1.7.

Fixing σ, for any algebraic space S we say that an S-perfect $E \in D^b(S \times X)$ is a *flat family* if the derived restriction $E_s := i_s^* E \in D^b(X)$ is in the heart $\mathcal{P}((0, 1])$ for all closed points $s \in S$. We say that E is a flat family of σ-(semi)stable objects of Mukai vector v and phase φ if E_s is σ-(semi)stable with $\mathbf{v}(E_s) = \mathbf{v}$ for all closed points $s \in S$. By [19, 20, 28], we have the following.

- The stack $\mathfrak{M}_\sigma(\mathbf{v}, \varphi)$ (respectively, $\mathfrak{M}^s_\sigma(\mathbf{v}, \varphi)$) of flat families of σ-semistable (respectively, σ-stable) objects is an Artin stack of finite type over $\mathbb{C}$ with coarse space $M_\sigma(\mathbf{v}, \varphi)$ (respectively, $M^s_\sigma(\mathbf{v}, \varphi)$).
- $\mathfrak{M}^s_\sigma(\mathbf{v}, \varphi) \subset \mathfrak{M}_\sigma(\mathbf{v}, \varphi)$ is an open substack.
- $\mathfrak{M}^s_\sigma(\mathbf{v}, \varphi)$ is a $\mathbb{G}_m$-gerbe over a symplectic algebraic space $M^s_\sigma(\mathbf{v}, \varphi)$.
- If $\mathfrak{M}^s_\sigma(\mathbf{v}, \varphi) = \mathfrak{M}_\sigma(\mathbf{v}, \varphi)$ then $M_\sigma(\mathbf{v}, \varphi)$ is proper.

The situation is even better for a generic stability condition, by results of [31] and [28].

Theorem 6 (See [5, Theorem 2.13]). *Suppose that σ is generic with respect to $\mathbf{v}$. If $\mathbf{v} = m\mathbf{v}_0$ for $\mathbf{v}_0$ primitive with $\mathbf{v}_0^2 \geqslant -2$ and $m > 0$, then $M_\sigma(\mathbf{v}, \varphi)(\mathbb{C})$ is nonempty if and only if $\mathbf{v}_0^2 \geqslant -2$. Furthermore,*

(i) *if $\mathbf{v}_0^2 > 0$, then $M_\sigma(\mathbf{v}, \varphi)$ is of the expected dimension $\mathbf{v}^2 + 2$; and*

(ii) *if $m = 1$, then there are no strictly σ-semistable objects of Mukai vector $\mathbf{v}$, and $M_\sigma(\mathbf{v}, \varphi)$ is smooth and projective of dimension $\mathbf{v}^2 + 2$.*

Henceforth we will typically drop the phase φ from the notation (because it is determined up to shifts by $\mathbf{v}$).

1.8.

By (1.5), the spaces $M = M_\tau(\mathbf{v})$ are canonically identified as τ varies in an open chamber Δ. For any $\sigma \in \mathrm{Stab}^\dagger(X)$, Bayer and Macrì [6, Lemma 3.3] construct a divisor class ℓ_σ on M, which evaluates on any map $C \to \mathfrak{M}_\tau(\mathbf{v})$ from a projective curve C with associated flat family $E \in D^b(C \times X)$ as

$$\ell_\sigma . C = -\Im\left(\frac{Z(q_* E)}{Z(\mathbf{v})}\right),$$

where q_* is the derived pushforward along the second projection $q : C \times X \to X$. $\ell_\sigma.C = 0$ if and only if E_c and $E_{c'}$ are S-equivalent for generic $c, c' \in C$. We have the following.

Theorem 7 [5, Theorem 1.2]. *The resulting map* $\ell : \mathrm{Stab}(X) \to H^2_{\mathrm{alg}}(M, \mathbb{Z}) \otimes \mathbb{R}$ *is piecewise analytic. Further, we have the following.*

(i) *The image of* ℓ *is the intersection of the big and movable cones of* M*, and the birational model associated to* ℓ_σ *(lying in an open chamber of the movable cone decomposition) is* $M_\sigma(\mathbf{v})$.

(ii) ℓ *maps the closed chamber* $\overline{\Delta}_\sigma$ *containing a generic stability condition* σ *to the nef cone of* $M_\sigma(\mathbf{v})$.

1.9.

Suppose that $E \in D^b(X \times M)$ is a universal object (though a quasiuniversal object would suffice). We define a map $\theta : \mathbf{v}^\perp \to H^2_{\mathrm{alg}}(M, \mathbb{Z})$ via

$$\mathbf{v}^\perp \xrightarrow{-\mathbf{v}^{-1}(\cdot)^\vee} K_{\mathrm{num}}(X) \xrightarrow{\Phi_E} K_{\mathrm{num}}(M) \xrightarrow{\det} H^2_{\mathrm{alg}}(M, \mathbb{Z}),$$

where $\Phi_E(\cdot) = q_*(E \otimes p^*(\cdot))$ is the K-theoretic Fourier–Mukai transform, and $p : X \times M \to X$ is the first projection. θ is an isometry, and there is a dual map $\theta^\vee : \widetilde{H}_{\mathrm{alg}}(X, \mathbb{Z}) \to H_2^{\mathrm{alg}}(M, \mathbb{Z})$. By (1.6), every face of the nef cone of M has the form $\theta(\mathcal{H}^\perp)$, and by [5] such an $\mathcal{H}$ can be taken to contain $\mathbf{a} \in \mathcal{H}$ with $\mathbf{a}^2 \geqslant -2$ and $|(\mathbf{a}, \mathbf{v})| \leqslant \frac{\mathbf{v}^2}{2}$. Similarly, every face of the movable cone is of the form $\theta(\mathcal{H}^\perp)$ for $\mathcal{H}$ containing $\mathbf{a}$ with either $\mathbf{a}^2 = -2$ and $(\mathbf{a}, \mathbf{v}) = 0$, or $\mathbf{a}^2 = 0$ and $(\mathbf{a}, \mathbf{v}) = 1$ or 2. These three possibilities correspond to divisorial contractions of Brill–Noether, Hilbert–Chow, and Li–Gieseker–Uhlenbeck type, respectively.

1.10.

(See [5, Section 14]) Given a hyperbolic lattice $\mathcal{H} \subset \widetilde{H}_{\mathrm{alg}}(X, \mathbb{Z})$ with potential wall $\mathcal{W}$ and effective cone $\mathcal{C}$, there are only finitely many $\mathbf{a} \in \mathcal{C}$ for which $\mathbf{v} - \mathbf{a} \in \mathcal{C}$. By the results of [5], we can therefore usually assume that $\mathcal{W}$ is not totally semistable with respect to any effective class. We define a partition $P = [\mathbf{v} = \sum \mathbf{a}_i]$ of $\mathbf{v}$ to be a set of Mukai vectors $\{\mathbf{a}_i\}$ with $\mathbf{v} = \sum \mathbf{a}_i$, and a second partition $P' = [\mathbf{v} = \sum \mathbf{b}_i]$ is a refinement of P if it can be obtained by partitioning each $\mathbf{a}_i$. With the above hypothesis, there is an associated locally closed stratum $M_P \subset M_\sigma(\mathbf{v})$ of objects whose Jordan–Hölder factors with respect to a generic $\sigma_0 \in \mathcal{W}$ have Mukai vectors $\mathbf{a}_i$. Moreover, a the stratum $M_{P'}$ lies in the closure of a stratum M_P if and only if P' refines P.

2. Lagrangian planes

A Lagrangian subspace of a symplectic vector space is a maximal isotropic subspace; in particular, it is half dimensional. Let M be a holomorphic symplectic variety of dimension $2n$. A *Lagrangian subvariety* is an embedded smooth subvariety $Z \subset M$ whose tangent space at each point is a Lagrangian subspace. Note that $\Omega_Z \cong N_{Z/M}$, where $N_{Z/M}$ is

the normal bundle of Z in M. A *Lagrangian plane* $\mathbb{P} \subset M$ is a Lagrangian subvariety isomorphic to projective space $\mathbb{P} = \mathbb{P}^n$.

A Lagrangian plane $\mathbb{P} \subset M$ can always be contracted to a point in the analytic category, and in fact in the category of algebraic spaces: there is an algebraic space M' and there is a map $f : M \to M'$ such that $f(\mathbb{P}) = p$ is a point and $f : M \smallsetminus \mathbb{P} \xrightarrow{\cong} M' \smallsetminus p$. It may happen that M' is not projective. We say that $\mathbb{P}$ is *extremal* if the class $R \in H_2(M, \mathbb{Z})$ of the line in $\mathbb{P}$ generates an extremal ray of the Mori cone. Note that the exceptional locus of the associated extremal contraction may strictly contain $\mathbb{P}$.

Example 8. (i) The prototypical example of a Lagrangian plane is the zero section $\mathbb{P} \subset \mathbb{A}(\Omega_{\mathbb{P}})$ in the total space of the cotangent bundle $\Omega_{\mathbb{P}}$. Blowing up $X = \mathbb{A}(\Omega_{\mathbb{P}})$ at $\mathbb{P}$, the exceptional fiber is isomorphic to the universal hyperplane in $\mathbb{P} \times \mathbb{P}^\vee$, and can be blown down to the second factor yielding a smooth manifold X' containing $\mathbb{P}^\vee$. This is called the Mukai flop, and X and X' admit contractions of $\mathbb{P}$ and $\mathbb{P}^\vee$, respectively, to the same analytic space X_0. Any Lagrangian plane $\mathbb{P} \subset M$ is locally analytically isomorphic to the Mukai flop, and can similarly be flopped to a complex manifold M' (again, even an algebraic space if M is a variety), but the resulting manifold M' need not be Kähler.

(ii) Let X be a K3 surface containing a smooth rational curve $C \cong \mathbb{P}^1 \subset X$. Let $X \to X'$ be the contraction of C to a double point. There is a natural embedding $\mathrm{Sym}^n C \cong \mathbb{P}^n \subset X^{[n]}$, and the plane $\mathbb{P}^n$ is contractible via the Hilbert–Chow morphism $S^{[n]} \to \mathrm{Sym}^n X'$, though the subscheme 2δ of nonreduced subschemes is contracted as well.

(iii) Here is an example due to Namikawa of a Lagrangian plane whose flop is not projective [26, Example 1.7(ii)]. Let $X \to \mathbb{P}^1$ be a projective elliptic K3 surface with two I_3 fibers (i.e., cycles of three smooth rational curves), one of which is $E_1 + E_2 + E_3$. As in the previous example, there are three disjoint Lagrangian planes $E_i^{(2)} \subset X^{[2]}$, and flopping all three yields a nonprojective manifold.

(iv) If L is an effective line bundle on a K3 surface X such that every section of L is reduced and irreducible, then let $\mathcal{C} \subset X \times \mathbb{P}$ be the universal divisor over $\mathbb{P} = \mathbb{P}H^0(L)^\vee$. The compactified relative Jacobian $\overline{\mathrm{Pic}}_{\mathbb{P}}(\mathcal{C}) \to \mathbb{P}$ is a moduli space of stable sheaves on X, and any section is a Lagrangian plane. In particular, the structure sheaf gives a section of $\overline{\mathrm{Pic}}^0_{\mathbb{P}}(\mathcal{C})$.

The machinery of the previous section (see (1.9)) allows one to very concretely describe the nef and movable cones of moduli spaces, and we give here an in-depth look at Hilbert schemes $X^{[2]}$ of two points on a K3 surface X with Picard rank 1. In this case the classification of birational transformations reduces to two Pell's equations (see [5, § 13]).

Let $\mathrm{Pic}(X) = \mathbb{Z}h$, with h the ample generator of degree $h^2 = 2d$. As described in the introduction, we have an isomorphism

$$H^2(X^{[n]}, \mathbb{Z}) = \mathbb{Z}h \oplus \mathbb{Z}\delta,$$

and the decomposition is orthogonal with respect to the Beauville–Bogomolov form. Furthermore, we have $(h, h) = 2d$ and $(\delta, \delta) = -2$. h is represented by the divisor of

subschemes one of whose points is supported on a fixed hyperplane section of X, and 2δ is the divisor of nonreduced subschemes. Note that h always generates an extremal ray of the nef (and movable) cone as it induces the Hilbert–Chow morphism $X^{[2]} \to X^{(2)}$ contracting the diagonal.

Example 9. (i) For $d = 1$, X is a degree-2 cover $X \to \mathbb{P}^2$ branched over a sextic, and hyperplane sections of X are genus-2 curves mapping to a line in $\mathbb{P}^2$ via the unique hyperelliptic cover. The Hilbert scheme $X^{[2]}$ has a Lagrangian plane, the closure of the set of reduced fibers of the map $X \to \mathbb{P}^2$. The degree-2 compactified Jacobian $\overline{\mathrm{Pic}}^2(\mathcal{C})$ of the universal hyperplane section $\mathcal{C}$ also has a Lagrangian plane, the section $\mathbb{P} \subset \overline{\mathrm{Pic}}^2(\mathcal{C})$ given by restricting the polarization $\mathcal{O}(h)$. In fact, $\overline{\mathrm{Pic}}^2(\mathcal{C})$ is the Mukai flop of $X^{[2]}$, and the flop is resolved by the relative Hilbert scheme of two points $\mathrm{Hilb}^2(\mathcal{C})$. One can show that the movable cone of $X^{[2]}$ is $\langle h, h-\delta\rangle$, which is decomposed into the nef cone $\langle h, 3h-2\delta\rangle$ of $X^{[2]}$ and the image $\langle 3h-2\delta, h-\delta\rangle$ of the nef cone of the flop $\overline{\mathrm{Pic}}^2(\mathcal{C})$. The isotropic divisor $h-\delta$ induces the Lagrangian fibration $\overline{\mathrm{Pic}}^2(\mathcal{C}) \to \mathbb{P}^{2\vee}$, and $\overline{\mathrm{Pic}}^2(\mathcal{C})$ is the only other birational model of $X^{[2]}$. The wall between them is generated by the nef class $3h-2\delta$ contracting the Lagrangian plane, and the class of the line

$$R = h - \tfrac{3}{2}\delta$$

satisfies $R^2 = -\frac{5}{2}$ and $2R \in H^2(X^{[2]}, \mathbb{Z})$.

(ii) For $d = 11$, the movable cone of $X^{[2]}$ is $\langle h, 10h-33\delta\rangle$. There are two chambers corresponding to the two birational models: the nef cone of $X^{[2]}$ is $\langle h, 7h-22\delta\rangle$, and that of the birational model is $\langle 7h-22\delta, 10h-33\delta\rangle$. The wall between them is once again a flop, and the contracted curve

$$R_1 = h - \tfrac{7}{2}\delta$$

again has $R_1^2 = -\frac{5}{2}$ and $2R_1 \in H^2(X^{[2]}, \mathbb{Z})$. Note however that

$$R_2 = 11h - \tfrac{73}{2}\delta$$

is in the Mori cone and also has these two properties, but the movable cone does not intersect $R_2^{\perp}$.

Lagrangian Grassmannians are also of interest; we will likewise say that an embedded Lagrangian Grassmannian $\mathrm{Gr}(k, \ell) \subset M$ in a holomorphic symplectic variety is extremal if the class of the minimal rational curve in $\mathrm{Gr}(k, \ell)$ is extremal in the Mori cone of M. The following example of Hassett and Tschinkel [17, Remark 3.1] shows how Lagrangian Grassmannians naturally arise.

Example 10. Let $X \subset \mathbb{P}^3$ be a general quartic (in particular, one containing no lines), and let $M = X^{[4]}$ be the Hilbert scheme of four points on X. M contains a Lagrangian Grassmannian: intersecting with any line $\ell \subset \mathbb{P}^3$ gives a length-4 subscheme, and there is an embedding $\mathrm{Gr}(2, 4) \subset M$. Let $\mathbb{P} = |\mathcal{O}_X(1)|$, and take $\mathcal{C} \subset X \times \mathbb{P}$ to be the universal

hyperplane section. The locus of subschemes supported on a hyperplane section contains this $\mathrm{Gr}(2,4)$, and is the image of a map from the relative Hilbert scheme of $\mathcal{C}$:

$$\begin{array}{ccccc} F & \longrightarrow & \mathrm{Hilb}^4_{\mathbb{P}}(\mathcal{C}) & \longrightarrow & X^{[4]} \\ \downarrow & & \downarrow & & \\ \mathbb{P} & \longrightarrow & \overline{\mathrm{Pic}}^4_{\mathbb{P}}(\mathcal{C}) & & \end{array} \tag{1}$$

A generic line bundle of degree 4 on a hyperplane section C will have a two-dimensional space of sections, so $\mathrm{Hilb}^4_{\mathbb{P}}(\mathcal{C})$ is generically a $\mathbb{P}^1$-bundle over $\overline{\mathrm{Pic}}^4_{\mathbb{P}}(\mathcal{C})$. The locus where the fiber jumps to a $\mathbb{P}^2$ is the section $\mathbb{P} \to \overline{\mathrm{Pic}}^4_{\mathbb{P}}(\mathcal{C})$ obtained by restricting $\mathcal{O}(1)$ to a hyperplane section, and the preimage F of this in $\mathrm{Hilb}^4_{\mathbb{P}}(\mathcal{C})$ is the flag variety of $\mathbb{P}^3$ parameterizing hyperplanes and lines contained in them. The leftmost map is one of the forgetful maps, $F \to \mathbb{P}$, and the composition of the top arrows is the other one, $F \to \mathrm{Gr}(2,4) \subset M$.

Lagrangian subvarieties are rigid in the following sense.

Lemma 11. *Let $G \subset M$ be a Lagrangian Grassmannian in a holomorphic symplectic variety. Then G does not deform as a subscheme. If $G \cong \mathbb{P}$ is a Lagrangian plane, then no curve $C \subset \mathbb{P}$ deforms out of $\mathbb{P}$.*

Proof. For the first statement, since G is Lagrangian, we have $N_{G/M} \cong \Omega_G$, and therefore $H^0(N_{G/M}) = 0$. For the second, it follows from the Euler sequence

$$0 \to \Omega_{\mathbb{P}} \to \mathcal{O}_P(-1)^{n+1} \to \mathcal{O}_{\mathbb{P}} \to 0$$

together with the sequence

$$0 \to I_{\mathbb{P}/M} \to I_{C/M} \to I_{C/\mathbb{P}} \to 0$$

that $\mathrm{Hom}(I_{\mathbb{P}/M}, \mathcal{O}_C) = 0$, and therefore that the map $\mathrm{Hom}(I_{C/\mathbb{P}}, \mathcal{O}_C) \to \mathrm{Hom}(I_{C/M}, \mathcal{O}_C)$ is an isomorphism. □

3. Bridgeland Moduli Spaces

Let X be a K3 surface, let $\mathbf{v} \in \widetilde{H}_{\mathrm{alg}}(X, \mathbb{Z})$ be a primitive Mukai vector with $\mathbf{v}^2 > 0$, and let σ be a generic stability condition. We first prove Theorem 2 for $M = M_\sigma(\mathbf{v})$. Note that by Verbitsky's Torelli theorem [29] and Markman's results on the monodromy group for the K3 deformation type (see § 4), the weight-2 Hodge structure $\widetilde{H}(X, \mathbb{Z})$ from (1.4) together with the class $\mathbf{v} \in \widetilde{H}_{\mathrm{alg}}(X, \mathbb{Z})$ determines $M_\sigma(\mathbf{v})$ up to birational equivalence.

Definition 12. A *pointed period* $(\widetilde{\Lambda}, \mathbf{v})$ is a (pure) weight-2 polarized Hodge structure on the Mukai lattice $\widetilde{\Lambda}$ with Hodge number $h^{2,0} = 1$ together with a primitive algebraic class $\mathbf{v} \in \widetilde{\Lambda}_{\mathrm{alg}}$. A *pointed sublattice* is a saturated sublattice $\mathcal{H} \subset \widetilde{\Lambda}_{\mathrm{alg}}$ containing $\mathbf{v}$. We will adopt the convention that, when a pointed sublattice is specified by its intersection form, the distinguished class will be the first basis vector.

Roughly speaking, if there is a partition $\mathbf{v} = \mathbf{a} + \mathbf{b}$ with $\mathbf{a}, \mathbf{b} \in \widetilde{H}_{\mathrm{alg}}(X, \mathbb{Z})$, then σ-stable objects with Mukai vector $\mathbf{v}$ can be built as extensions of objects A, B with Mukai vectors $\mathbf{a}$ and $\mathbf{b}$, and the projectivized extension group $\mathbb{P} = \mathbb{P}\operatorname{Ext}^1(A, B)^\vee$ will map into $M_\sigma(\mathbf{v})$. The geometry of $\mathbb{P}$ depends on the particulars of the pointed sublattice $\mathcal{H}$ generated by $\mathbf{a}$ and $\mathbf{b}$. Recall that $\mathbf{a} \in \widetilde{H}_{\mathrm{alg}}(X, \mathbb{Z})$ is *spherical* if $\mathbf{a}^2 = -2$. An object $A \in D^b(X)$ is *rigid* if $\operatorname{Ext}^1(A, A) = 0$, and *spherical* if it is rigid and $\operatorname{Hom}(A, A) = \mathbb{C}\,\mathrm{id}$.

Definition 13. A pointed sublattice $\mathcal{H} \subset \widetilde{H}_{\mathrm{alg}}(X, \mathbb{Z})$ is a $\mathbb{P}$ *type* sublattice if the following hold.

(i) There is a spherical class $\mathbf{s} \in \mathcal{H}$ such that $|(\mathbf{s}, \mathbf{v})| = \frac{\mathbf{v}^2}{2}$.

(ii) There is no spherical class $\mathbf{s}' \in \mathcal{H}$ with $|(\mathbf{s}', \mathbf{v})| < \frac{\mathbf{v}^2}{2}$.

Further, we say that a $\mathbb{P}$ type sublattice $\mathcal{H}$ is *extremal* with respect to a (generic) stability condition σ if $\theta(\mathcal{H}^\perp)$ is a wall of the nef cone of $M_\sigma(\mathbf{v})$.

Note that $\mathcal{H}$ being extremal with respect to some stability condition is equivalent to $\theta(\mathcal{H}^\perp)$ intersecting the movable cone of each $M_\sigma(\mathbf{v})$. A pointed sublattice of the form

$$\begin{pmatrix} \mathbf{v}^2 & \frac{\mathbf{v}^2}{2} \\ \frac{\mathbf{v}^2}{2} & -2 \end{pmatrix}$$

is automatically of $\mathbb{P}$ type, though not every one is of this form. For the following lemma, we say that $\mathbf{v}$ is minimal in $\mathcal{H}$ if there is no effective spherical class $\mathbf{s} \in \mathcal{H}$ with $(\mathbf{s}, \mathbf{v}) < 0$.

Lemma 14. *Let $\mathcal{H}$ be a $\mathbb{P}$ type sublattice, and let $\mathcal{W}$ be the associated potential wall with generic $\sigma_0 \in \mathcal{W}$. If $\mathbf{v}$ is minimal in $\mathcal{H}$, then there are two σ_0-stable spherical objects S, T with Mukai vectors $\mathbf{s}, \mathbf{t}$ such that $\mathbf{v} = \mathbf{s} + \mathbf{t}$.*

Proof. By definition, there is a spherical class $\mathbf{s} \in \mathcal{H}$ with $(\mathbf{s}, \mathbf{v}) = \frac{\mathbf{v}^2}{2}$, and it is effective. Note that $\mathbf{t} = \mathbf{v} - \mathbf{s}$ satisfies

$$\mathbf{t}^2 = \mathbf{v}^2 - 2(\mathbf{s}, \mathbf{v}) + (-2) = -2,$$

so $\mathbf{t}$ is spherical, and moreover $(\mathbf{t}, \mathbf{v}) = \frac{\mathbf{v}^2}{2}$ as well. By [5, Proposition 6.3], there are exactly two σ_0-stable objects S, T with Mukai vectors $\mathbf{s}_0, \mathbf{t}_0$, and the Jordan–Hölder factors of an object representing $\mathbf{s}$ consist entirely of $\mathbf{s}_0$'s and $\mathbf{t}_0$'s. Thus, $\mathbf{s} = x\mathbf{s}_0 + y\mathbf{t}_0$ with $x, y \geqslant 0$, but since

$$(\mathbf{s}, \mathbf{v}) = x(\mathbf{s}_0, \mathbf{v}) + y(\mathbf{t}_0, \mathbf{v}) = \frac{\mathbf{v}^2}{2}$$

we must have either $\mathbf{s} = \mathbf{s}_0$ or $\mathbf{s} = \mathbf{t}_0$ by condition (ii). Similarly, $\mathbf{t} = \mathbf{t}_0$ or $\mathbf{t} = \mathbf{s}_0$. □

The utility of Definition 13 is hinted at by the following.

Lemma 15. *If $\mathcal{H} \subset \widetilde{H}_{\mathrm{alg}}(X, \mathbb{Z})$ is an extremal $\mathbb{P}$ type sublattice, then $M_\sigma(\mathbf{v})$ contains an extremal Lagrangian plane $\mathbb{P} \subset M_\sigma(\mathbf{v})$ for σ generic on either side of the wall associated to $\mathcal{H}$.*

Proof. First assuming that $\mathbf{v}$ is minimal in $\mathcal{H}$, by Lemma 14 there exist σ_0-stable objects $S, T \in \mathcal{P}_0(1)$ of classes $\mathbf{s}, \mathbf{t}$ (respectively) for a generic $\sigma_0 \in \mathcal{W}$, and $\mathbf{v} = \mathbf{s} + \mathbf{t}$. We therefore have that $\mathrm{Ext}^k(T, S) = 0$ for $k < 0$ and $k > 2$ as S, T are both in the heart of σ_0, and further $\mathrm{Hom}(T, S) = \mathrm{Ext}^2(T, S) = 0$ by stability, so $\mathrm{ext}^1(T, S) = (\mathbf{s}, \mathbf{t}) = n + 1$. If σ is a generic stability condition on one side of $\mathcal{W}$, assume that $\varphi(S) < \varphi(T)$, and let $\mathbb{P} = \mathbb{P}\,\mathrm{Ext}^1(T, S)^\vee$. Denoting by $p : \mathbb{P} \times X \to \mathbb{P}$ the first projection,

$$p^*S(1) \to E \to p^*T \to p^*S(1)[1] \tag{2}$$

defines a flat family $E \in D^b(\mathbb{P} \times X)$ with $\mathbf{v}(E_x) = \mathbf{v}$ for all $x \in \mathbb{P}$. Restricting to $x \in \mathbb{P}$ and applying $\mathrm{Hom}(T, \cdot)$ to the above sequence, we see that $\mathrm{Hom}(T, E_x) = 0$, and therefore, by the following simple lemma, E_x is σ-stable.

Lemma 16 (See [5, Lemma 6.9]). *Let A, B be simple objects in an abelian category, and let*

$$0 \to A^x \to E \to B^y \to 0$$

be any extension with the property that either (i) $x = 1$ *and* $\mathrm{Hom}(B, E) = 0$, *or* (ii) $y = 1$ *and* $\mathrm{Hom}(E, A) = 0$. *Then, in case* (i), *every proper quotient of E is isomorphic to B^z for some z; in case* (ii), *every proper subobject of E is isomorphic to A^z for some z.*

Further, it is easy to see that any σ-stable object E with Jordan–Hölder partition $P = [\mathbf{v} = \mathbf{s} + \mathbf{t}]$ with respect to σ_0 is of the form (2), so we have an isomorphism between the stratum $M_P \subset M_\sigma(\mathbf{v})$ and $\mathbb{P}$. The case when $\varphi(S) > \varphi(T)$ likewise produces a Lagrangian plane on the other side of the wall.

Finally, if $\mathbf{v}$ is not minimal in $\mathcal{H}$, then there is a minimal $\mathbf{v}_0 \in \mathcal{H}$ such that $\mathbf{v}$ is obtained from $\mathbf{v}_0$ by successive spherical reflections. If we let $\mathrm{ST} : D^b(X) \to D^b(X)$ be the composition of the corresponding sequence of spherical twists by σ-stable spherical objects, for σ on one side of the wall, then, by the same argument as [5, Proposition 6.8], ST applied to the family in (2) will be stable on that side of the wall. □

We can compute the class of the line in the Lagrangian plane of Lemma 15. Recall from § 1.9 that, for any curve $C \subset M_\sigma(\mathbf{v})$,

$$\theta(\mathbf{w}).C = (\mathbf{w}, \mathbf{v}(\Phi_E(\mathcal{O}_C))).$$

Let $\mathbb{P}^1 \subset \mathbb{P}$ be a line, and let $q : \mathbb{P}^1 \times X \to X$ be the projection onto the second factor. The class $R = [\mathbb{P}^1] \in H_2(M_\sigma(\mathbf{v}))$ is determined by intersecting with all divisors: for any $\mathbf{w} \in \mathbf{v}^\perp$,

$$\theta(\mathbf{w}).R = (\mathbf{w}, \mathbf{v}(q_*E|_{\mathbb{P}^1 \times X})) = (\mathbf{w}, 2\mathbf{s} + \mathbf{t}) = (\mathbf{w}, \mathbf{s}),$$

and thus $R = \theta^\vee(\mathbf{s})$ in the case $\varphi(S) < \varphi(T)$. If $\varphi(S) > \varphi(T)$ we obtain $R = -\theta^\vee(\mathbf{s})$.

The Lagrangians planes constructed as in Lemma 15 are clearly extremal. Indeed, by the classification in [5, Theorem 5.7], passing through the potential wall associated to the hyperbolic lattice $\mathcal{H}$ will flop the projective space and change the sign of the class of the line.

The rest of this section will be devoted to showing that this is in fact the only way such planes arise.

Proposition 17. *Let* $\mathbf{v} \in \widetilde{H}_{\mathrm{alg}}(X, \mathbb{Z})$ *be primitive with* $\mathbf{v}^2 > 0$*, and let* σ *be a generic stability condition with respect to* $\mathbf{v}$*.* $M_\sigma(\mathbf{v})$ *contains an extremal Lagrangian plane* $\mathbb{P} \subset M_\sigma(\mathbf{v})$ *if and only if* $\widetilde{H}_{\mathrm{alg}}(X, \mathbb{Z})$ *admits an extremal* $\mathbb{P}$ *type sublattice* $\mathcal{H}$ *with respect to* σ*. Further, in this case the class of the line in* $\mathbb{P}$ *is* $\pm\theta^\vee(\mathbf{s})$*, for* $\mathbf{s} \in \mathcal{H}$ *a spherical class with* $(\mathbf{s}, \mathbf{v}) = \frac{\mathbf{v}^2}{2}$*.*

Proof. The reverse direction and the computation of the curve class is Lemma 15 and the ensuing discussion, so we just need to demonstrate the necessity of the lattice condition.

Suppose that for some generic stability condition σ there is an extremal Lagrangian plane $\mathbb{P} \subset M_\sigma(\mathbf{v})$, and let $\pi : M_\sigma(\mathbf{v}) \to M$ be the contraction. π is realized by crossing some wall $\mathcal{W} \subset \mathrm{Stab}(M)$; let $\mathcal{H} \subset \widetilde{H}_{\mathrm{alg}}(X, \mathbb{Z})$ be the associated hyperbolic lattice, with effective cone $\mathcal{C} \subset \mathcal{H} \otimes \mathbb{R}$. Then π contracts curves parameterizing objects that are S-equivalent with respect to σ_0, and, since $\mathbb{P}$ is contracted to a point, a generic point $x \in \mathbb{P}$ has fixed Jordan–Hölder factors A_i with respect to σ_0. Let $P = [\mathbf{v} = \sum \mathbf{a}_i]$ be the corresponding partition, where $\mathbf{a}_i = \mathbf{v}(A_i)$, and let $M_P \subset M_\sigma(\mathbf{v})$ be the locally closed subvariety of points with the same Jordan–Hölder decomposition.

Lemma 18. *The* A_i *are all rigid. In particular, they are spherical.*

Proof. Obviously, if the A_i deformed, then, because the dimensions of the extension groups between the A_i locally remain constant (as the A_i locally remain stable with respect to σ_0), a curve C contracted by π would deform outside of $\mathbb{P}$, contradicting Lemma 11. Thus, the A_i are simple in $\mathcal{P}_0(1)$ and rigid, and therefore spherical. □

As in the proof of Lemma 15, there are exactly two spherical objects $S, T \in \mathcal{P}_0(1)$, so in fact the partition P must be of the form $\mathbf{v} = x\mathbf{s} + y\mathbf{t}$, where $\mathbf{s} = \mathbf{v}(S)$ and $\mathbf{t} = \mathbf{v}(T)$. Suppose that $\varphi(S) < \varphi(T)$ for σ. It follows that an object E associated to a point of M_P is an extension of the form

$$S \otimes U^\vee \to E \to T \otimes V \to S \otimes U^\vee[1], \tag{3}$$

where U, V are vector spaces of dimensions x, y, respectively. Indeed, since neither S nor T admits self-extensions, (3) is just the Harder–Narasimhan filtration on the other side of the wall[2]. Such an extension is stable only if the rightmost map $U \otimes V \to \mathrm{Ext}^1(T, S)$ from (3) satisfies

(i) $U \to \mathrm{Hom}(V, \mathrm{Ext}^1(T, S))$ is injective, and

(ii) $V \to \mathrm{Hom}(U, \mathrm{Ext}^1(T, S))$ is injective,

and therefore we can identify M_P with a subscheme of the space of bidegree $(1, 1)$ maps $\mathbb{P}U \times \mathbb{P}V \to \mathbb{P}\,\mathrm{Ext}^1(B, A)$, up to the action of $\mathrm{PGL}(U) \times \mathrm{PGL}(V)$. But $\mathbb{P} \subset M_P$ and

$$\dim M_P \leqslant xy((\mathbf{s}, \mathbf{t}) - xy) = \frac{\mathbf{v}^2}{2} + x^2 + y^2 - x^2y^2 = n + (x^2 - 1)(y^2 - 1).$$

In order for this to be half dimensional, we need either $x = 1$ or $y = 1$. In this case, by Lemma 16 we have $M_P = \mathrm{Gr}(x, (\mathbf{s}, \mathbf{t}))$ (for $y = 1$, and likewise if $x = 1$), and, in order for $\mathbb{P} = M_P$, we need $x = y = 1$. □

[2]Therefore, we can retrospectively realize that $U^\vee = \mathrm{Hom}(S, E)$ and $V^\vee = \mathrm{Hom}(E, T)$.

This provides an easy verification that Lagrangian planes contract to isolated singularities, since the decomposition $\mathbf{v} = \mathbf{s} + \mathbf{t}$ cannot be refined.

Corollary 19. *If $\mathbb{P} \subset M_\sigma(\mathbf{v})$ is an extremal Lagrangian plane, then $\mathbb{P}$ is a connected component of the exceptional locus of the associated extremal contraction $\pi : M_\sigma(\mathbf{v}) \to M$.*

The proof of Proposition 17 begs the same classification question for Lagrangian Grassmannians, and a similar argument shows that they arise as the strata corresponding to partitions of the form $P = [\mathbf{v} = \mathbf{s} + k\mathbf{t}]$, for spherical $\mathbf{s}, \mathbf{t}$. As remarked above, however, it is not the case that an extremal Grassmannian G is contracted to an isolated singularity, because the partition

$$\mathbf{v} = \mathbf{s} + \mathbf{t} + \cdots + \mathbf{t} \tag{4}$$

is a common refinement of the partitions $\mathbf{v} = (\mathbf{s} + m\mathbf{t}) + (k - m)\mathbf{t}$ for all $0 \leqslant m < k$, and, since $(\mathbf{s} + m\mathbf{t})^2 \geqslant 0$, these Jordan–Hölder factors deform. In fact, G will lie in the closure of each of these strata, since spherical objects have no self-extensions and therefore (4) and $\mathbf{v} = \mathbf{s} + k\mathbf{t}$ have the same associated strata. There will thus always be a rational curve in G which sweeps out a larger exceptional locus (as in Example 10).

Example 20. Here, we revisit Example 10 in the above language. $X^{[4]}$ is the moduli space $M_\sigma(\mathbf{v})$ for $\mathbf{v} = (1, 0, -3)$, parameterizing ideal sheaves I_Z of length-4 subschemes $Z \subset X$, for σ in some chamber $\mathcal{C}$ of the stability manifold. The Grassmannian $\mathrm{Gr}(2, 4) \subset X^{[4]}$ arises as the ideal sheaves that are complete intersections of hyperplane sections of X,

$$0 \to \mathcal{O}_X(-2) \to \mathcal{O}_X(-1)^2 \to I_Z \to 0,$$

and can therefore be thought of as extensions:

$$\mathcal{O}_X(-1)^2 \to I_Z \to \mathcal{O}_X(-2)[1] \to \mathcal{O}_X(-1)^2[1].$$

Thus, the corresponding partition is $\mathbf{v} = 2\mathbf{s} + \mathbf{t}$, for $\mathbf{s} = (1, -H, 3)$ and $\mathbf{t} = (-1, 2H, -9)$, and $\mathbf{v} = \mathbf{s} + \mathbf{s} + \mathbf{t}$ is a refinement of it. As in the above, both of these partitions have the same associated stratum, but the second also refines $\mathbf{v} = \mathbf{s} + \mathbf{a}$, where $\mathbf{a} = \mathbf{s} + \mathbf{t} = (0, H, -6)$. The stratum M_P of $\mathbf{v} = \mathbf{s} + \mathbf{a}$ parameterizes ideal sheaves of the form

$$0 \to \mathcal{O}_X(-1) \to I_Z \to F \to 0$$

with $\mathbf{v}(F) = \mathbf{a}$, that is, with Z lying entirely on a hyperplane section. F moves in a $2 + \mathbf{a}^2 = 6$ dimensional family isomorphic to $\overline{\mathrm{Pic}}^4_{\mathbb{P}}(\mathcal{H})$, and M_P is the image of the complement of the Lagrangian section $\mathbb{P}$ in Example 10. Note that $\mathbb{P}$ is the Lagrangian plane corresponding to the partition $\mathbf{a} = \mathbf{s} + \mathbf{t}$, by Lemma 15.

The phenomenon in Example 20 is generally true: a Lagrangian Grassmannian $\mathrm{Gr}(k, \ell)$ in a moduli space $M_\sigma(\mathbf{v})$ always 'comes from' a Lagrangian Grassmannian in a smaller-dimensional moduli space $M_\sigma(\mathbf{w})$ with respect to the same stability condition σ, and the process terminates at a Lagrangian plane. A general notion of 'stratified Mukai flops' such as these were first studied by Markman [21].

4. Holomorphic symplectic varieties of K3 type

We now turn to the general case. Much of the Hodge-theoretic structure of Bridgeland moduli spaces on K3 surfaces is echoed by arbitrary holomorphic symplectic manifolds of K3 type. For M a K3 type manifold of dimension $2n$, Markman [22, Corollary 9.5] constructs a monodromy invariant extension of pure weight-2 Hodge structures (we will blur the notational distinction between the Hodge structure and the underlying lattice)

$$0 \to H^2(M,\mathbb{Z}) \to \widetilde{\Lambda}(M) \to Q(M) \to 0,$$

where $\widetilde{\Lambda}(M)$ is a (pure) weight-2 Hodge structure on the Mukai lattice $\widetilde{\Lambda}$ polarized by the intersection form, and $Q(M)$ is rank 1 of type $(1,1)$. In the language introduced in the previous section, this yields a pointed period $(\widetilde{\Lambda}(M), \mathbf{v}(M))$ which determines M up to birational equivalence, again by Verbitsky's Torelli theorem [29]. The subgroup of the oriented isometry group $O^+(\widetilde{\Lambda})$ preserving the embedding $H^2(M,\mathbb{Z}) \to \widetilde{\Lambda}(M)$ is equal to $\mathrm{Mon}^2(M)$, the image of the restriction map $\mathrm{Mon}(X) \to O(H^2(M,\mathbb{Z}))$, and there is a natural lift of the monodromy action to $\widetilde{\Lambda}(M)$.

In the case of a Bridgeland moduli space $M = M_\sigma(\mathbf{v})$ of objects on a K3 surface X, $\widetilde{\Lambda}(M) = \widetilde{H}(X,\mathbb{Z})$ is the pointed period described above, and the embedding $H^2(M,\mathbb{Z}) \to \widetilde{\Lambda}(M)$ is the inverse of the Mukai map $\theta : \mathbf{v}^\perp \to H^2(M,\mathbb{Z})$. In general we will still denote by $\mathbf{v}(M)$ a primitive generator of $H^2(M,\mathbb{Z})^\perp \subset \widetilde{\Lambda}(M)$; we always have $\mathbf{v}(M)^2 = 2n-2$. We will also denote by $\theta^\vee : \widetilde{\Lambda}(M) \to H_2(M,\mathbb{Z})$ the dual of the embedding.

From [3], the description of the nef cone of moduli spaces in terms of their pointed periods in § 1.9 deforms to all holomorphic symplectic varieties of K3 type, and, in particular, we have the following.

Theorem 21 [3, Theorem 1]. *Let (M,h) be a polarized holomorphic symplectic variety of K3 type. The Mori cone of M is generated by the positive cone and classes of the form*

$$\left\{\theta^\vee(\mathbf{a}) \mid \mathbf{a} \in \widetilde{\Lambda}(M)_{\mathrm{alg}} \text{ with } \mathbf{a}^2 \geqslant -2, |(\mathbf{a},\mathbf{v})| \leqslant \frac{\mathbf{v}^2}{2}, h.\theta^\vee(\mathbf{a}) > 0\right\}.$$

Before proving the general case of Theorem 2, recall that a parallel transport operator is an isometry $\varphi : H^2(M,\mathbb{Z}) \to H^2(M',\mathbb{Z})$ that arises from parallel transport of the local system $R^2 f_* \mathbb{Z}$ for some smooth proper family $f : \mathcal{M} \to B$ along a path with endpoint fibers M and M'. Recall also that, for an embedding $i : Y \hookrightarrow M$ of a Lagrangian submanifold into a holomorphic symplectic manifold, the deformations of the pair (M,i) are those of M that preserve the sub-Hodge structure $\ker i^* \subset H^*(M,\mathbb{Z})$, and they are unobstructed (see [30] and [27]). In our case, for $Y = \mathbb{P}$ and $R \in H_2(M_\sigma(\mathbf{v}),\mathbb{Z})$ the class of the line R, as long as R remains algebraic in a family the plane will deform as well.

We now prove the first main theorem.

Theorem 22. *Let (M,h) be a holomorphic symplectic variety of K3 type and dimension $2n$ with a Lagrangian plane $\mathbb{P} \subset M$, and let $R \in H_2(M,\mathbb{Z})$ be the class of the line. Then $\widetilde{\Lambda}(M)$ admits a $\mathbb{P}$ type sublattice $\mathcal{H}$, and $R = \theta^\vee(\mathbf{s})$ for a spherical class $\mathbf{s} \in \mathcal{H}$ with $|(\mathbf{s},\mathbf{v})| = \frac{\mathbf{v}^2}{2}$.*

Proof. We know that $R^2 < 0$, and so, by the argument of [3, Proposition 3], there is a smooth proper family $f : \mathcal{M} \to B$ over an irreducible analytic base specializing to M over some point $0 \in B$ such that there is an algebraic section ρ of $R^{2n-2} f_* \mathbb{Z}$ specializing to R over 0 (the argument of Proposition 3 in [3] only uses that $R^2 < 0$ and the line deforms sideways). By the above discussion, the Lagrangian plane $\mathbb{P}$ also deforms to the general fiber. As the periods of moduli spaces are dense in the base of the Kuranishi family of the pair $(M, \mathbb{P})$, we can find a specialization to a moduli space for which the plane $\mathbb{P}$ does not degenerate and such that $\mathbb{P}$ is extremal. Transporting the $\mathbb{P}$ type lattice guaranteed by 17 then yields the claim. □

The existence of a $\mathbb{P}$ type lattice does not only depend on $H^2(M, \mathbb{Z})$, but we always have a simple necessary criterion for a curve class to be the class of a line in a Lagrangian plane.

Corollary 23. *Let (M, h) be as above, and let R be the class of a line in a Lagrangian plane. Then $(R, R) = -\frac{n+3}{2}$ and $2R \in H^2(M, \mathbb{Z})$.*

Proof. The statement follows from the following observation.

Lemma 24. *If $\widetilde{\Lambda}(M)$ admits a spherical $\mathbf{a} \in \widetilde{\Lambda}(M)$ such that $(\mathbf{a}, \mathbf{v}) = \frac{\mathbf{v}^2}{2}$, then $R = \theta^\vee(\mathbf{a})$ has $(R, R) = -\frac{n+3}{2}$ and order 2 in the discriminant group of $H^2(M, \mathbb{Z})$.*

Proof. Since $\theta^\vee$ is the composition of the orthogonal projection onto $\mathbf{v}^\perp$ and the inclusion $H^2(M, \mathbb{Z}) \to H_2(M, \mathbb{Z})$ given by the quadratic form, we have

$$(R, R) = \left(\mathbf{a} - \frac{\mathbf{v}}{2}\right)^2 = \mathbf{a}^2 - (\mathbf{a}, \mathbf{v}) + \frac{\mathbf{v}^2}{4} = -\frac{n+3}{2}.$$

We have $2\mathbf{a} - \mathbf{v} \in \mathbf{v}^\perp$, so R is 2-torsion in the discriminant group D of $H^2(M, \mathbb{Z})$, but clearly $R \neq 0$ in D, so R has order 2. □

□

Running the argument of Theorem 22 backwards yields a partial converse.

Theorem 25. *Let (M, h) be as above, and suppose that $R \in H_2(M, \mathbb{Z})$ is a primitive generator of an extremal ray of the Mori cone. Then R is the class of a line in a Lagrangian plane if and only if $(R, R) = -\frac{n+3}{2}$ and $2R \in H^2(M, \mathbb{Z})$.*

Proof. The 'only if' part follows from the previous theorem, so we only need to prove the sufficiency of the numerical conditions in this setting. As R is extremal, we know by Theorem 21 that it is a multiple of a class of the form $\theta^\vee(\mathbf{a})$ for $\mathbf{a} \in \mathcal{H} \subset \widetilde{\Lambda}(M)$ as in the Theorem, and it is not hard to see that in fact $\mathcal{H}$ must be $\mathbb{P}$ type and further that $R = \theta^\vee(\mathbf{s})$ for a spherical class $\mathbf{s}$ with $|(\mathbf{s}, \mathbf{v})| = \frac{\mathbf{v}^2}{2}$. Again by [3, Proposition 3 and Corollary 6], there is a smooth proper family along which R remains algebraic specializing to a moduli space M' for which the image $R' \in H_2(M', \mathbb{Z})$ of R is extremal, and therefore by Theorem 17 R' is the class of a line in a Lagrangian plane $\mathbb{P}$ which then deforms to the general fiber of the family, as above. The primitivity and extremity assumptions on R ensure that $\mathbb{P}$ does not degenerate in M. □

A full converse to Corollary 23 is not expected without some indecomposability constraint on the curve class R(indeed, such a hypothesis is also needed in the case of smooth rational curves on K3 surfaces), but the exact condition is at the moment unclear. If we drop the extremity and primitivity condition in Theorem 25 and only insist that R comes from a $\mathbb{P}$ type lattice, then the argument carries through except at the last step, where we must show that the plane $\mathbb{P}$ does not degenerate in M. For example, the class R_2 in Example 9(ii) is such a class, and there is even a family keeping R_2 algebraic whose associated parallel transport operator sends R_2 to R_1, but the Lagrangian plane on the generic fiber of this family could easily degenerate.

With Corollary 23 in mind, Corollary 5 follows by lattice theory. Let L be the lattice $H^2(M,\mathbb{Z})$, and let $D(L) = L^\vee/L$ be its discriminant group. Denote by $O(L)$ the isometry group of L and by $\widetilde{O}(L)$ the group of isometries acting trivially on $D(L)$. By a result of Eichler [10, § 10] (see also [11, Lemma 3.5]), the orbit of a primitive class $\mathbf{a} \in L$ under the group $\widetilde{O}(L)$ is determined by its square $(\mathbf{a},\mathbf{a})$ and the class of its dual $\mathbf{a}^\vee = \frac{1}{\mathrm{div}(\mathbf{a})}(\mathbf{a},\cdot) \in D(L)$ in the discriminant group. Recall that $\mathrm{div}(\mathbf{a})$ is defined by $(\mathbf{a}, L) = \mathrm{div}(\mathbf{a})\mathbb{Z}$. By [22, Lemma 9.2], $\widetilde{O}(H^2(M,\mathbb{Z}))$ is an index-2 subgroup of $\mathrm{Mon}^2(M)$, and therefore we deduce the following.

Corollary 26. *There is a single monodromy orbit containing all primitive classes arising as lines in Lagrangian planes embedded in holomorphic symplectic varieties of K3 type.*

Remark 27. In fact, the same proof shows that the number of monodromy orbits containing the classes of lines in Lagrangian planes is at most equal to the number of square divisors of $(n+3)(n-1)$.

Remark 28. It is not in general true that the class of a line in an extremal Lagrangian plane is primitive. Indeed, if $\mathbf{v}$ is minimal in a $\mathbb{P}$ type sublattice $\mathcal{H}$, so that $\mathbf{v} = \mathbf{s} + \mathbf{t}$ in the notation of § 4, this will be the case if and only if the parallelogram with vertexes $0, \mathbf{s}, \mathbf{t}, \mathbf{v}$ contains no interior lattice point; i.e., if $\mathbf{s}$ and $\mathbf{t}$ generate $\mathcal{H}$. Since the effective cone is generated by $\mathbf{s}, \mathbf{t}$, any other contracted stratum M_P corresponding to a partition $P = [\mathbf{v} = \sum \mathbf{a}_i]$ must have each $\mathbf{a}_i$ in the interior of the parallelogram with vertexes $0, \mathbf{s}, \mathbf{t}, \mathbf{v}$, so this is in turn equivalent to $\mathbb{P}$ being the only exceptional locus.

In dimensions $\leqslant 8$, the method of proving the sufficiency of the numerical criteria in Corollary 23 in [4, 12, 16] also provides universal expressions for the class $[\mathbb{P}] \in H^{2n}(M,\mathbb{Z})$ of a Lagrangian plane in terms of Hodge classes and the dual to the class of the line $\rho = 2R \in H^2(M,\mathbb{Z})$:

$$\begin{aligned}
[\mathbb{P}^2] &= \tfrac{1}{24}(3\rho^2 + \mathrm{c}_2(M)) \\
[\mathbb{P}^3] &= \tfrac{1}{48}(\rho^3 + \rho\mathrm{c}_2(M)) \\
[\mathbb{P}^4] &= \tfrac{1}{337920}(880\rho^4 + 1760\rho^2\mathrm{c}_2(M) - 3520\theta^2 + 4928\theta\mathrm{c}_2(M) - 1408\mathrm{c}_2(M)^2).
\end{aligned}$$

Here $\theta \in \mathrm{Sym}^2 H^2(M,\mathbb{Z})^* \subset H^4(M,\mathbb{Q})$ is the class of the Beauville–Bogomolov form. Given the monodromy invariance in Corollary 26, such universal expressions must exist.

Question 29. *What are the universal polynomials for the class of a Lagrangian plane (with primitive line class) in a holomorphic symplectic variety of K3 type in terms of the dual to the line and Hodge classes?*

As is clear from the $n = 4$ case, the class of a Lagrangian plane cannot always be expressed purely in terms of chern classes and ρ.

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