



# Strong solutions and instability for the fitness gradient system in evolutionary games between two populations

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## Abstract

In this paper, we study a fitness gradient system for two populations interacting via a symmetric game. The population dynamics are governed by a conservation law, with a spatial migration flux determined by the fitness. By applying the Galerkin method, we establish the existence, regularity and uniqueness of global solutions to an approximate system, which retains most of the interesting mathematical properties of the original fitness gradient system. Furthermore, we show that a Turing instability occurs for equilibrium states of the fitness gradient system, and its approximations.

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## 1. Introduction

The ability of living things to move spatially during their struggle to survive is an inherent aspect of most biological systems, with implicit connections to evolution [8]. The fact that multiple species are moving simultaneously, some in pursuit of others, brings an additional richness to ecological dynamics. The particular mechanisms of this motion manifest themselves at the population level as dispersal or migration relations, written as a spatial flux which depends on various effects, including heterogeneous environmental conditions, spatial distribution of resources, and mutually attractive or repulsive interactions between individuals, among many other considerations [2,8]. The challenge for mathematical modeling is to realistically capture the relevant aspects of these effects, while nonetheless producing a set of equations which are both tractable and provide insight into the phenomena [10].

Partial differential equations have been developed to model populations interacting in a spatially extended region. Among such models, one of the first (called SKT model for short) determined by species fitness appeared in Shigesada et al. [29], who studied a Lotka–Volterra system of interacting species in a homogeneous environment. For the SKT model, Lou and Ni [14,15] showed the existence and nonexistence of nonconstant steady states, and obtained the limit of nonconstant steady states. The global existence of smooth solutions was proved by Kim [11] and Shim [26] in one dimension, Lou et al. [16] in two dimensions, and Lou and Winkler [18] in three dimensions. When the environment itself is spatially inhomogeneous, the case of one species moving up a resource gradient while the other disperses randomly was modeled by Cantrell et al. [4] based on an earlier single equation approach by Belgacem and Cosner [1]; Kareiva and Odell proposed a cross-diffusion model for predator–prey interaction [13]. We also refer to [3,10,19,21,22,28,20] and references therein.

Evolutionary game theory provides a specific form of the fitness for each population, based on the payoff matrix of the game which defines their mutual interactions [30]. Consider a population of individuals who are playing a game in competition. Every individual has a choice of  $m$  possible pure strategies available, and at each instant every individual is using one of these strategies. For each strategy  $i$ ,  $p_i$  denotes the proportion of individuals who are, at that moment, using strategy  $i$ . In a symmetric evolutionary game, the fitness of strategy  $i$  is the expected payoff for an individual playing strategy  $i$ , written as  $f_i$ , where the payoff matrix is defined by

$$\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{mm} \end{pmatrix}.$$

We adopt the fitness function as defined by Taylor & Jonker [32] and Vickers [34], where the fitness for an individual playing strategy  $i$  is defined as the expected per capita payoff:  $f_i = (\mathbf{A}\mathbf{p})_i$ , where  $\mathbf{p} = (p_1, \dots, p_m)$ .

In this paper we consider two populations, described by density functions  $u$  and  $v$ , who choose one from two strategies ( $m = 2$ ). The local fitness for each population defined above is written as

$$f_1(u, v) = \frac{a_{11}u + a_{12}v}{u + v}, \quad f_2(u, v) = \frac{a_{21}u + a_{22}v}{u + v}. \quad (1.1)$$

We assume that

$$a_{11} - a_{12} > 0, \quad a_{21} - a_{22} > 0, \quad (1.2)$$

an assumption which implies that the  $u$  population gains most by playing against itself, whereas the  $v$  population does better playing against the  $u$  population than against itself [7]. We consider each species to migrate along its fitness gradient, moving towards a more favorable environment. If we take the continuous limit of the discrete model [7], the population fluxes at the position  $x$  and time  $t$  are given by

$$J_1 = -\beta_1 u \nabla f_1, \quad J_2 = -\beta_2 v \nabla f_2,$$

where  $\beta_i$  are the proportionality constants determining each population's sensitivity to its fitness gradient. Then the population dynamics is described by the equation of continuity

$$\begin{cases} u_t = -\operatorname{div} J_1 = -\beta_1 \operatorname{div}(u \nabla f_1), \\ v_t = -\operatorname{div} J_2 = -\beta_2 \operatorname{div}(v \nabla f_2). \end{cases} \quad (1.3)$$

For the choice of fitness functions  $f_i$  in (1.1), one can show that  $\nabla f_2 = \kappa_{\mathbf{A}} \nabla f_1$  [7], where  $\kappa_{\mathbf{A}} = (a_{21} - a_{22})/(a_{11} - a_{12})$  is a constant. By dividing equations (1.3) by  $\beta_1$  and rescaling the time, this system becomes

$$\begin{cases} u_t = -\operatorname{div}(u \nabla f_1), \\ v_t = -\beta \kappa_{\mathbf{A}} \operatorname{div}(v \nabla f_1), \end{cases} \quad (1.4)$$

where  $\beta = \beta_2/\beta_1$ . We make the assumption  $\beta \kappa_{\mathbf{A}} > 1$  and define the positive parameter  $\gamma = \beta \kappa_{\mathbf{A}} - 1$  [7]. Using the functional form of  $f_1(u, v)$ , System (1.4) can be expressed as

$$\begin{cases} u_t = (a_{11} - a_{12}) \operatorname{div} \left( -\frac{uv}{(u+v)^2} \nabla u + \frac{u^2}{(u+v)^2} \nabla v \right), \\ v_t = (a_{11} - a_{12})(1 + \gamma) \operatorname{div} \left( -\frac{v^2}{(u+v)^2} \nabla u + \frac{uv}{(u+v)^2} \nabla v \right). \end{cases} \quad (1.5)$$

The resulting generalized diffusion system includes cross-diffusion effects, for which steady state solutions and numerical examples of solutions were investigated by [7]. In a related work [12], the authors proved existence of global non-negative weak solutions under the assumption that  $\mathbf{A}$  is a symmetric positive-definite matrix with the uniform ellipticity structure, and the fitness is determined by the difference between the available resources and the animal's consumption, represented by  $\mathbf{A}\mathbf{p}$ .

Due to the condition  $a_{11} - a_{12} > 0$ , the usual energy method is not applicable to the first equation of (1.5), so the cross-diffusion term can lead to serious problems. In order to overcome the mathematical difficulties, we regularize problem (1.5) and utilize a specific change of dependent variables [6,7]. The first aim of this paper is to prove the existence of strong solutions to the regularized problem shown in (2.1) for a bounded domain  $\Omega \subset \mathbb{R}^2$ .

When we consider the additional possibility of local increases or decreases of the two populations, the fitness of each population manifests itself as a growth rate (see [12,7]). The population dynamics can be modeled by the following system

$$\begin{cases} u_t = f_1 u - \beta_1 \operatorname{div}(u \nabla f_1), \\ v_t = f_2 v - \beta_2 \operatorname{div}(v \nabla f_2). \end{cases} \quad (1.6)$$

For the main results of this paper, we neglect the growth rates terms, as they do not cause any technical difficulty for the existence of strong solutions to (1.4); this allows us to focus on the intrinsic difficulties of the fitness gradient terms. However, these growth rates terms do play an important role in the instabilities of (1.6), which can destabilize the equilibrium solution. Later in this paper we address the effect of growth rates and cross-diffusion on the instability in any dimensions. Through a linear stability analysis, we analytically derive a set of sufficient conditions which guarantee that the system generates a Turing instability, as first indicated in [7].

The rest of the paper is arranged as follows. We shall regularize the problem and give our existence result in Section 2. In Section 3, we first adopt the Galerkin approximation scheme and then construct a sequence of approximate solutions  $(u_m, \theta_m)$ . In Section 4, we derive some *a priori* estimates for the approximate solutions. We then prove the main result theorem by the convergence of the approximate solutions in Section 5. In Section 6, the uniqueness of global strong solutions is obtained. The instability of the equilibrium for (1.6) and its approximate systems is discussed in Section 7.

## 2. Regularization the problem and existence result

Without loss of generality, we take  $a_{11} - a_{12} = 1$  throughout this paper. We regularize System (1.5) with small parameter  $\epsilon > 0$  as follows

$$\begin{cases} u_t = \operatorname{div} \left( \epsilon \nabla u - \frac{uv}{(u+v)^2} \nabla u + \frac{u^2}{(u+v)^2} \nabla v \right), \\ v_t = \operatorname{div} \left( -(1+\gamma) \frac{v^2}{(u+v)^2} \nabla u + (1+\gamma) \frac{uv}{(u+v)^2} \nabla v \right) + \epsilon \frac{v}{u} \Delta u. \end{cases} \quad (2.1)$$

**Remark 2.1.** It can be proved that  $u$  is a strictly positive function with lower bound, given the initial data  $u_0$  with positive lower bound. Then the regularizing term  $\frac{v}{u} \Delta u$  is mathematically well-defined. Further comments on the form of this term are given below.

We study the initial boundary value problem for (2.1) in a bounded domain with smooth boundary  $\Omega \subset \mathbb{R}^2$ , along with the following initial and boundary conditions

$$\begin{cases} u(x, 0) = u_0(x), & v(x, 0) = v_0(x) & x \in \Omega, \\ \nabla u \cdot \mathbf{n} = 0, & \nabla v \cdot \mathbf{n} = 0 & x \in \partial\Omega, \end{cases} \quad (2.2)$$

where the initial densities  $u_0(x)$  and  $v_0(x)$  are strictly positive functions and satisfy

$$\begin{aligned} 0 < \inf_{x \in \Omega} u_0(x) \leq \sup_{x \in \Omega} u_0(x) < \infty, \\ 0 < \inf_{x \in \Omega} \frac{u_0(x)}{u_0(x) + v_0(x)} \leq \sup_{x \in \Omega} \frac{u_0(x)}{u_0(x) + v_0(x)} < 1. \end{aligned} \quad (2.3)$$

Before stating our main result, we explain the notations and conventions that will be used throughout the paper. We set  $Q_T = [0, T] \times \Omega$ . Let us denote the usual Sobolev spaces by  $W^{m,q}(Q_T)$  with the norm  $\|\cdot\|_{W^{m,q}(Q_T)}$ . For simplicity, the norm of the Sobolev space  $W^{m,q}(\Omega)$

is written as  $\|\cdot\|_{W^{m,q}}$ . When  $q = 2$  or  $m = 0$ , we will set  $H^m(\Omega) = W^{m,2}(\Omega)$  and  $L^q(\Omega) = W^{0,q}(\Omega)$ , respectively. As usual,  $\langle \cdot, \cdot \rangle$  stands for the scalar  $L^2(\Omega)$ -inner product. For any Banach space  $B$  and any  $T > 0$ , we will denote by  $L^r(0, T; B)$  the Banach space of the  $B$ -valued (classes of) functions defined a.e. in  $[0, T]$  that are  $L^r$ -integrable in the sense of Bochner. Frequently, we will consider Banach spaces  $L^r(0, T; B)$  with  $B = W^{m,q}(\Omega)$ .

Our main result is the following:

**Theorem 2.1.** *Let  $\Omega \subset \mathbb{R}^2$  be a bounded domain with smooth boundary and  $q > 2$ , and the initial value  $u_0(x) \in H^2(\Omega)$ ,  $\frac{u_0}{u_0+v_0}(x) \in W^{2,q}(\Omega)$  satisfy (2.3). Then the initial boundary value problem (2.1)–(2.3) has a unique global in time strong solution.*

In order to prove Theorem 2.1, we utilize the proportionate variable  $\theta = u/(u + v)$  defined in [6], which transforms System (2.1) into the equivalent form

$$\begin{cases} u_t - \epsilon \Delta u + \operatorname{div}(u \nabla \theta) = 0, \\ \theta_t - \gamma \theta(1 - \theta) \Delta \theta = \gamma \theta(1 - \theta) \frac{1}{u} \nabla u \nabla \theta - (1 + \gamma) |\nabla \theta|^2. \end{cases} \quad (2.4)$$

The initial and boundary conditions can be rewritten as

$$\begin{cases} u(x, 0) = u_0(x), & \theta(x, 0) = \theta_0(x) & x \in \Omega, \\ \nabla u \cdot \mathbf{n} = 0, & \nabla \theta \cdot \mathbf{n} = 0 & x \in \partial\Omega, \end{cases} \quad (2.5)$$

and the initial values  $u_0(x)$  and  $\theta_0(x)$  are strictly positive functions satisfying

$$0 < \inf_{x \in \Omega} u_0(x) \leq \sup_{x \in \Omega} u_0(x) < \infty, \quad 0 < \inf_{x \in \Omega} \theta_0(x) \leq \sup_{x \in \Omega} \theta_0(x) < 1. \quad (2.6)$$

**Remark 2.2.** Mathematically, System (2.4) bears some similarity to the viscous homogeneous Hamilton–Jacobi equation for  $\theta$  and its relation to fluid system as [5].

**Remark 2.3.** It is worth mentioning that the inclusion of two regularization terms in System (2.1) aims to make the transformed system (2.4), with the new variable  $\theta$ , into a parabolic system, by which we can obtain the  $W^{1,q}$  estimate of  $u$  for some  $q > 2$ . Otherwise, System (2.4) is hyperbolic–parabolic, which can also be regarded as a variation model of the Navier–Stokes equations. Then due to the appearance of the higher order term  $\nabla u$  in the second equation of (2.4), we cannot close *a priori* estimates.

Next we have

**Proposition 2.1.** *Let  $\Omega \subset \mathbb{R}^2$  be a bounded domain with smooth boundary, assume that  $\|u_0\|_{H^2} \leq C_0 < \infty$ ,  $\|\theta_0\|_{W^{2,q}} \leq C_0 < \infty$  for some  $q > 2$  and (2.6) holds. Then, for any  $T$ , the initial boundary problem (2.4)–(2.5) has a unique strong solution  $(u, \theta)(x, t)$  defined on the time interval  $[0, T]$ , satisfying the following properties*

$$0 < \inf_{x \in \Omega} u_0(x) \leq u(x, t) \leq \sup_{x \in \Omega} u_0(x) < \infty \quad \text{for } x \in \Omega, t \in [0, T];$$

$$0 < \inf_{x \in \Omega} \theta_0 \leq \theta(x, t) \leq \sup_{x \in \Omega} \theta_0 < 1 \quad \text{for } x \in \Omega, \quad t \in [0, T],$$

and there exists a positive constant  $C$  depending on  $T, \Omega, C_0, \epsilon$  such that

$$\begin{aligned} \|u\|_{L^\infty(0, T; W^{1, q}(\Omega))} + \|\nabla u\|_{L^2(0, T; H^1(\Omega))} + \|u_t\|_{L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))} &\leq C; \\ \|\nabla \theta\|_{W^{1, q}(Q_T)} + \|\theta_t\|_{L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))} &\leq C. \end{aligned}$$

### 3. The Galerkin approximation

We start by defining the operator

$$\mathcal{A}: H^2(\Omega) \cap \{u \in H^1(\Omega) : \nabla u \cdot \mathbf{n} = 0\} \mapsto L^2(\Omega)$$

as

$$\mathcal{A}u = -\Delta u + u.$$

Let  $\lambda_j$  be the eigenvalues of  $\mathcal{A}$  and  $\varphi_j$  be the corresponding eigenfunction which is an element of  $C^\infty(\Omega) \cap H^2(\Omega)$ , that is, for  $j = 1, 2, \dots$ ,

$$\begin{cases} -\Delta \varphi_j + \varphi_j = \lambda_j \varphi_j & \text{in } \Omega, \\ \nabla \varphi_j \cdot \mathbf{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Then  $\{\varphi_j\}_{j=1}^\infty$  form an orthonormal basis of  $L^2(\Omega)$ . We set a sequence of finite dimensional spaces

$$V_m = \text{span}\{\varphi_j : j \leq m\}, \quad m = 1, 2, \dots.$$

For any fixed integer  $m > 0$ , we consider the following Galerkin type approximation of (2.4)

$$\begin{cases} \partial_t u_m - \epsilon \Delta u_m + \text{div}(u_m \nabla \theta_m) = 0, \\ \partial_t \theta_m - \gamma \theta_m (1 - \theta_m) \Delta \theta_m = \gamma \theta_m (1 - \theta_m) \frac{1}{u_m} \nabla u_m \nabla \theta_m - (1 + \gamma) |\nabla \theta_m|^2, \\ (u_m, \theta_m)(0) = (u_0, \theta_0). \end{cases} \quad (3.1)$$

#### 3.1. The Cauchy problem for the density $u_m$

In this subsection, our aim is to look for approximate solutions  $u_m$  of (3.1)<sub>1</sub> in  $V_m$  for any integer  $m > 0$ .

**Lemma 3.1.** *For any integer  $m > 0$ , there exists a  $T_m > 0$ , if  $\theta_m \in C([0, T_m]; H^1(\Omega))$ , such that the Cauchy problem (3.1)<sub>1</sub> has a unique solution  $u_m \in C([0, T_m]; V_m)$ .*

**Proof.** We may set  $u_m = \sum_{j=1}^m d_j(t) \varphi_j$ . Taking the inner product of (3.1)<sub>1</sub> and  $\varphi_k$  ( $k = 1, 2, \dots, m$ ), we find that  $d_k(t)$  satisfies

$$\begin{cases} \frac{d}{dt} d_k(t) + \sum_{j=1}^m d_j(t) \int_{\Omega} (\epsilon \nabla \varphi_j \nabla \varphi_k - \nabla \theta_m \varphi_j \nabla \varphi_k) dx = 0, \\ d_k(0) = \langle u_0, \varphi_k \rangle. \end{cases} \quad (3.2)$$

Clearly, problem (3.2) is an initial value problem for an ordinary differential equation. Thus, by standard theory and using the assumption on  $\theta_m$ , it follows that (3.2) possesses a unique local solution  $u_m \in C([0, T_m]; V_m)$ .  $\square$

Next, we derive an energy inequality for  $u_m$ . Multiplying (3.1)<sub>1</sub> by  $u_m$  and  $-\Delta u_m$  respectively, we integrate by parts to have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} u_m^2 dx + \epsilon \int_{\Omega} |\nabla u_m|^2 dx = \int_{\Omega} u_m \nabla \theta_m \nabla u_m dx \\ & \leq \frac{\epsilon}{2} \|\nabla u_m\|_{L^2}^2 + C(\epsilon) \|u_m\|_{L^4}^2 \|\nabla \theta_m\|_{L^4}^2 \end{aligned} \quad (3.3)$$

$$\begin{aligned} & \leq \frac{\epsilon}{2} \|\nabla u_m\|_{L^2}^2 + C(\epsilon, \|\nabla \theta_m\|_{L^\infty(0, T_m; H^1(\Omega))}) \|u_m\|_{H^1}^2, \\ & \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla u_m|^2 dx + \epsilon \int_{\Omega} |\Delta u_m|^2 dx \\ & = \int_{\Omega} \nabla u_m \nabla \theta_m \Delta u_m dx + \int_{\Omega} u_m \Delta \theta_m \Delta u_m dx \\ & \leq \frac{\epsilon}{2} \|\Delta u_m\|_{L^2}^2 + C(\epsilon, \|\nabla \theta_m\|_{L^\infty(0, T_m; H^2(\Omega))}) \|u_m\|_{H^1}^2. \end{aligned} \quad (3.4)$$

Summing (3.3) and (3.4), it follows that

$$\frac{d}{dt} \|u_m\|_{H^1}^2 + \epsilon \|\nabla u_m\|_{H^1}^2 \leq C(\epsilon, \|\nabla \theta_m\|_{L^\infty(0, T_m; H^2(\Omega))}) \|u_m\|_{H^1}^2. \quad (3.5)$$

By Gronwall inequality, (3.6) implies

$$\begin{aligned} \|u_m\|_{H^1}^2 + \epsilon \int_0^t \|\nabla u_m\|_{H^1}^2 dt & \leq e^{C(\epsilon, \|\nabla \theta_m\|_{L^\infty(0, T_m; H^2(\Omega))})t} \|u_0\|_{H^1}^2 \\ & \leq C(\epsilon, \|\nabla \theta_m\|_{L^\infty(0, T_m; H^2(\Omega))}, u_0) \end{aligned} \quad (3.6)$$

for  $t \in [0, T_m]$ .

Now, we assume that  $u_m^1, u_m^2$  are two solutions with the same initial value of (3.1)<sub>1</sub> corresponding to  $\theta_m = \theta_m^1, \theta_m = \theta_m^2$  respectively. Multiplying the difference of the equations by  $-\Delta(u_m^1 - u_m^2)$ , we integrate by parts to obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla(u_m^1 - u_m^2)|^2 dx + \epsilon \int_{\Omega} |\Delta(u_m^1 - u_m^2)|^2 dx \\
&= \int_{\Omega} \nabla(u_m^1 - u_m^2) \nabla \theta_m^1 \Delta(u_m^1 - u_m^2) dx + \int_{\Omega} (u_m^1 - u_m^2) \Delta \theta_m^1 \Delta(u_m^1 - u_m^2) dx \\
& \quad + \int_{\Omega} \nabla u_m^2 \nabla(\theta_m^1 - \theta_m^2) \Delta(u_m^1 - u_m^2) dx + \int_{\Omega} u_m^2 \Delta(\theta_m^1 - \theta_m^2) \Delta(u_m^1 - u_m^2) dx \\
&\leq C(\epsilon) \|u_m^1 - u_m^2\|_{H^1}^2 \|\nabla \theta_m^1\|_{H^2}^2 + C(\epsilon) \|u_m^2\|_{H^1}^2 \|\nabla(\theta_m^1 - \theta_m^2)\|_{H^2}^2 + \frac{\epsilon}{2} \|\Delta(u_m^1 - u_m^2)\|^2.
\end{aligned} \tag{3.7}$$

Note that  $\int_{\Omega} (u_m^1 - u_m^2) dx = 0$  for all  $t$ , which allows us to use Poincaré inequality. By Gronwall and Poincaré inequalities, taking (3.6) and (3.7) into account, we conclude that

$$\begin{aligned}
& \sup_{0 \leq t \leq T_m} \|u_m^1(t) - u_m^2(t)\|_{H^1} \\
&\leq T_m C(\epsilon, \|\nabla \theta_m^1\|_{L^\infty(0, T_m; H^2(\Omega))}, \|\nabla \theta_m^2\|_{L^\infty(0, T_m; H^2(\Omega))}, u_0) \sup_{0 \leq t \leq T_m} \|\nabla(\theta_m^1(t) - \theta_m^2(t))\|_{H^2}.
\end{aligned}$$

Moreover, denoting  $Lu := u_t - \epsilon \Delta u + \operatorname{div}(u \nabla \theta)$ , we calculate

$$\begin{aligned}
L(e^{\int_0^t \|\Delta \theta_m\|_{L^\infty} dt}) &= e^{\int_0^t \|\Delta \theta_m\|_{L^\infty} dt} (\|\Delta \theta_m(t)\|_{L^\infty} + \Delta \theta_m) \geq 0, \quad Lu_m = 0, \\
L(e^{-\int_0^t \|\Delta \theta_m\|_{L^\infty} dt}) &= e^{-\int_0^t \|\Delta \theta_m\|_{L^\infty} dt} (-\|\Delta \theta_m(t)\|_{L^\infty} + \Delta \theta_m) \leq 0.
\end{aligned}$$

By virtue of the comparison principle, we have

$$0 < e^{-\int_0^t \|\Delta \theta_m\|_{L^\infty} dt} \leq u_m(x, t) \leq e^{\int_0^t \|\Delta \theta_m\|_{L^\infty} dt}$$

for all  $x \in \Omega$ ,  $t \in [0, T_m]$ .

The results just obtained can be summarized in the following statement:

**Lemma 3.2.** *Let the initial value  $u_0(x) \in H^2(\Omega)$ . Then, for any integer  $m > 0$ , there exists a  $T_m > 0$  and a mapping  $\mathcal{S} = \mathcal{S}(\theta_m)$ ,*

$$\mathcal{S} : \theta_m \in C([0, T_m]; H^1(\Omega)) \mapsto C([0, T_m]; V_m)$$

*such that  $u_m = \mathcal{S}(\theta_m)$  is the unique solution to (3.1)<sub>1</sub>.*

*In addition to the above assumption, if  $\theta_m \in C([0, T_m]; W^{2,\infty}(\Omega))$ , we have*

$$e^{-\int_0^t \|\Delta \theta_m\|_{L^\infty} dt} \leq u_m(x, t) \leq e^{\int_0^t \|\Delta \theta_m\|_{L^\infty} dt} \tag{3.8}$$

*for all  $x \in \Omega$ ,  $t \in [0, T]$ , and if  $\nabla \theta_m \in C([0, T_m]; H^2(\Omega))$ , we have*

$$\|\mathcal{S}(\theta_m^1) - \mathcal{S}(\theta_m^2)\|_{C([0, T_m], H^1(\Omega))} \leq T_m C(\epsilon, k, u_0) \|\nabla(\theta_m^1 - \theta_m^2)\|_{C([0, T_m], H^1(\Omega))} \tag{3.9}$$

for any  $\theta_m^1, \theta_m^2$  belonging to set

$$M_L = \{\nabla\theta_m \in C([0, T_m]; H^2(\Omega)) \mid \|\nabla\theta_m\|_{C([0, T_m]; H^2(\Omega))} < L\}.$$

### 3.2. The approximate solutions $\theta_m$

By substituting the above solution  $u_m(x, t)$  into (3.1)<sub>2</sub>, we obtain the integral equation

$$\begin{aligned} & \int_{\Omega} u_m(t)\theta_m(t)\varphi dx - \int_{\Omega} u_0\theta_0\varphi dx \\ &= \int_0^t \int_{\Omega} \left\{ \gamma u_m\theta_m(1-\theta_m)\Delta\theta_m + \gamma\theta_m(1-\theta_m)\nabla u_m\nabla\theta_m - (1+\gamma)u_m|\nabla\theta_m|^2 \right. \\ & \quad \left. + \epsilon\Delta u_m\theta_m - \operatorname{div}(u_m\nabla\theta_m)\theta_m \right\} \varphi dx dt \end{aligned} \quad (3.10)$$

for all  $t \in [0, T_m]$  and any function  $\varphi \in V_m$ . Now, we proceed to look for approximate solutions  $\theta_m \in C([0, T_m]; V_m)$  satisfying (3.10).

To this end, we introduce a family of operators  $\mathcal{M}[u] : V_m \rightarrow V_m^*$ ,

$$\langle \mathcal{M}[u]\omega, \psi \rangle = \int_{\Omega} u\omega\psi dx.$$

Clearly, these operators are bounded if  $u \in L^1(\Omega)$ , that is

$$\|\mathcal{M}[u]\|_{\mathcal{L}(V_m, V_m^*)} \leq C(m)\|u\|_{L^1},$$

and invertible provided  $u$  is strictly positive on  $\Omega$ , so we have

$$\|\mathcal{M}^{-1}[u]\|_{\mathcal{L}(V_m^*, V_m)} \leq \left( \inf_{x \in \Omega} u(x) \right)^{-1}.$$

Notice that the identity

$$\mathcal{M}^{-1}[u^1] - \mathcal{M}^{-1}[u^2] = \mathcal{M}^{-1}[u^2](\mathcal{M}^{-1}[u^2] - \mathcal{M}^{-1}[u^1])\mathcal{M}^{-1}[u^1]$$

for any  $u^1, u^2$  from the set

$$M_\eta = \{u \in L^1(\Omega) \mid \inf_{x \in \Omega} u \geq \eta > 0\},$$

leads to

$$\begin{aligned} \|\mathcal{M}^{-1}[u^1] - \mathcal{M}^{-1}[u^2]\|_{\mathcal{L}(V_m^*, V_m)} &\leq C(m, \eta)\|\mathcal{M}[u^2] - \mathcal{M}[u^1]\|_{\mathcal{L}(V_m, V_m^*)} \\ &\leq C(m, \eta)\|u_1 - u_2\|_{L^1}. \end{aligned} \quad (3.11)$$

Now, the integral identity (3.10) can be rephrased in the mild form

$$\theta_m = \mathcal{M}^{-1}[u_m] \left( q_0 + \int_0^t \mathcal{N}[u_m(s), \theta_m(s)] ds \right) \quad (3.12)$$

where for all  $\varphi \in V_m$ ,

$$\langle q_0, \varphi \rangle = \int_{\Omega} u_0 \theta_0 \varphi dx$$

and

$$\begin{aligned} \langle \mathcal{N}[u_m(s), \theta_m(s)], \varphi \rangle = & \int_{\Omega} \left\{ \gamma u_m \theta_m (1 - \theta_m) \Delta \theta_m + \gamma \theta_m (1 - \theta_m) \nabla u_m \nabla \theta_m \right. \\ & \left. - (1 + \gamma) u_m |\nabla \theta_m|^2 + \epsilon \Delta u_m \theta_m - \operatorname{div}(u_m \nabla \theta_m) \theta_m \right\} \varphi dx. \end{aligned}$$

Taking  $u_m = \mathcal{S}(\theta_m)$ , (3.12) can be rewritten as

$$\theta_m = \mathcal{M}^{-1}[\mathcal{S}(\theta_m)] \left( q_0 + \int_0^t \mathcal{N}[\mathcal{S}(\theta_m)(s), \theta_m(s)] ds \right). \quad (3.13)$$

By means of the contraction mapping principle on the Banach space  $C([0, T_m]; V_m)$ , taking (3.9) and (3.11) into account, we obtain a local solution  $\theta_m$  of (3.13) on a short time interval  $[0, T_m]$ ,  $T_m \leq T$ .

### 3.3. Approximate solutions in a time interval $[0, T]$

Our remaining task is to show that  $T_m = T$ . Let us suppose that  $T_m < T$ . The *a priori* estimates established in Section 4 prove that  $u_m$  and  $\theta_m$  stay uniformly bounded in  $V_m$  on the whole interval  $[0, T_m]$ . We take

$$u_m(T_m) = \lim_{t \rightarrow T_m} u_m(t), \quad \theta_m(T_m) = \lim_{t \rightarrow T_m} \theta_m(t)$$

as the initial values to solve (3.1). Repeating the argument for the above two functions, after a finite number of steps, we finally get  $T_m = T$ .

From the above argument, we have the following existence result.

**Lemma 3.3.** *Let  $q > 2$ , and assume initial values  $u_0(x) \in H^2(\Omega)$ ,  $\theta_0(x) \in W^{2,q}(\Omega)$ , then for any integer  $m$ , the Cauchy problem (3.1) admits a unique solution  $(u_m, \theta_m)(x, t)$  in  $C([0, T], V_m)$ .*

#### 4. *A priori* estimates

This section is devoted to deriving some *a priori* estimates for the approximate solutions  $(u_m, \theta_m)$  by iteration. For the sake of simplicity, we drop the index  $m$ , using  $u_m = u$  and  $\theta_m = \theta$  in this section. Let us assume that

$$u^0(x, t) = u_0(x), \quad \theta^0(x, t) = \theta_0(x),$$

and  $u^n(x, t)$ ,  $\theta^n(x, t)$  for any integer  $n \geq 1$ , satisfy the problem

$$\begin{cases} u_t^n - \epsilon \Delta u^n + \operatorname{div}(u^n \nabla \theta^{n-1}) = 0, \\ \theta_t^n - \gamma \theta^{n-1} (1 - \theta^{n-1}) \Delta \theta^n = \gamma \theta^{n-1} (1 - \theta^{n-1}) \frac{1}{u^{n-1}} \nabla u^{n-1} \nabla \theta^n - (1 + \gamma) \nabla \theta^{n-1} \nabla \theta^n, \end{cases} \quad (4.1)$$

with the initial and boundary value conditions

$$\begin{cases} (u^n, \theta^n)(x, 0) = (u_0(x), \theta_0(x)) & \text{in } \Omega, \\ \nabla u^n \cdot \mathbf{n} = 0, \quad \nabla \theta^n \cdot \mathbf{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

For each  $n \geq 1$ , the linear problem (4.1) is solvable in  $C([0, T]; V_m)$  from the argument in the previous section.

We recall the well-known interpolation inequality for Sobolev functions on bounded domains (see [17]).

**Lemma 4.1** (*Gagliardo–Nirenberg’s inequality*). *Suppose that  $u \in W^{1,p}(\Omega) \cap L^r(\Omega)$ . Then there exists a constant  $C = C(\Omega, N, p, r)$  such that*

$$\|u\|_{L^q} \leq C \left( \|Du\|_{L^p}^s \|u\|_{L^r}^{1-s} + \|u\|_{L^r} \right),$$

where  $\frac{1}{q} = s(\frac{1}{p} - \frac{1}{N}) + (1-s)\frac{1}{r}$ . Then range of  $q$  is  $[r, \frac{Np}{N-p}]$  when  $p < N$ , is  $[r, \infty)$  when  $p = N$ , and is  $[r, \infty]$  when  $p > N$ .

We will often use the following version of the Gagliardo–Nirenberg’s inequality for anisotropic spaces, which is a simple consequence of Lemma 4.1. We refer to [5] for the proof.

**Lemma 4.2.** *Suppose that  $u \in L^\infty(t, t + \delta; L^2(\Omega))$  and  $Du \in L^2([t, t + \delta] \times \Omega)$  for  $t, t + \delta \in [0, T]$ . Then  $u \in L^{2r'}(t, t + \delta; L^{2q'}(\Omega))$ , for all pairs  $(r', q')$  with the conjugates  $(r, q)$  (i.e.,  $\frac{1}{q'} + \frac{1}{q} = 1$ ,  $\frac{1}{r'} + \frac{1}{r} = 1$ ) satisfying*

$$\|u\|_{L^{2r'}(t, t + \delta; L^{2q'}(\Omega))}^2 \leq C(s, r, q, \Omega) \delta^s \left( \|u\|_{L^\infty(t, t + \delta; L^2(\Omega))}^2 + \|Du\|_{L^2([t, t + \delta] \times \Omega)}^2 \right)$$

where

$$s = 1 - \frac{N}{2q} - \frac{1}{r},$$

and

$$\frac{N}{2q} + \frac{1}{r} < 1 \quad \text{if } N = 2; \quad \frac{N}{2q} + \frac{1}{r} \leq 1 \quad \text{if } N = 3.$$

Next, we will derive some estimates of  $u^n$  and  $\theta^n$  uniformly in  $m$ . In the following lemma, we first obtain the bounds of  $u^n$  and  $\theta^n$  by the maximum principle. The key point is that, following the idea of [5], we can get the  $L^\infty$  estimate of  $\nabla \theta^n$ . In [5], Chen and Su established the existence and regularity of global solutions of a viscous approximation for an unsteady Euler flow potential flow, which can be regarded as a variation model of the Navier–Stokes equations. By using Moser iteration and adopting a new transformation, they mainly derived the global *a priori* estimates with large initial data.

**Lemma 4.3.** *Under the assumption (2.6), let  $\|u_0\|_{H^2} \leq C_0 < \infty$ ,  $\|\theta_0\|_{W^{2,q}} \leq C_0 < \infty$  for some  $q > 2$ . Then, for any  $T > 0$ , there exists  $C = C(T, \Omega, C_0, \epsilon)$  such that  $(u^n, \theta^n)(x, t)$  satisfies the following estimates*

$$0 < \inf_{x \in \Omega} u_0(x) \leq u^n(x, t) \leq \sup_{x \in \Omega} u_0(x) < \infty, \quad (4.2)$$

$$0 < \inf_{x \in \Omega} \theta_0(x) \leq \theta^n(x, t) \leq \sup_{x \in \Omega} \theta_0(x) < 1, \quad (4.3)$$

$$\|u^n\|_{L^\infty(0,T;W^{1,q}(\Omega))}^2 + \|\nabla u^n\|_{W^{1,2}(Q_T)}^2 \leq C, \quad (4.4)$$

$$\|\nabla \theta^n\|_{L^\infty(Q_T)}^2 + \|\nabla \theta^n\|_{W^{1,q}(Q_T)}^2 \leq C. \quad (4.5)$$

**Proof.** Let us begin with the case  $n = 1$ . We first find the bounds of  $u^1$  and  $\theta^1$ . Set

$$\left(u^1 - \inf_{x \in \Omega} u_0(x)\right)_- = \min \left\{u^1 - \inf_{x \in \Omega} u_0(x), 0\right\}.$$

Multiplying (4.1)<sub>1</sub> for  $n = 1$  by  $\left(u^1 - \inf_{x \in \Omega} u_0(x)\right)_-$ , integrating over  $\Omega$ , we find

$$\begin{aligned} & \int_{\Omega} u_t^1 \left(u^1 - \inf_{x \in \Omega} u_0(x)\right)_- dx + \epsilon \int_{\Omega} \nabla u^1 \nabla \left(u^1 - \inf_{x \in \Omega} u_0(x)\right)_- dx \\ &= \int_{\Omega} u^1 \nabla \theta^0 \nabla \left(u^1 - \inf_{x \in \Omega} u_0(x)\right)_- dx. \end{aligned} \quad (4.6)$$

Using Sobolev's and Young's inequalities, (4.6) yields

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} \left| \left(u^1 - \inf_{x \in \Omega} u_0(x)\right)_- \right|^2 dx + \epsilon \int_{\Omega} \left| \nabla \left(u^1 - \inf_{x \in \Omega} u_0(x)\right)_- \right|^2 dx \\ & \leq C \left\| \left(u^1 - \inf_{x \in \Omega} u_0(x)\right)_- \right\|_{L^2}^2 \|\nabla \theta^0\|_{L^\infty}^2. \end{aligned} \quad (4.7)$$

By Gronwall inequality, we get

$$\left\| \left( u^1(x, t) - \inf_{x \in \Omega} u_0(x) \right)_- \right\|_{L^2}^2 \leq e^{C \|\nabla \theta^0\|_{L^\infty}^2 t} \left\| \left( u^1(x, 0) - \inf_{x \in \Omega} u_0(x) \right)_- \right\|_{L^2}^2 = 0.$$

This implies

$$u^1(x, t) \geq \inf_{x \in \Omega} u_0(x) > 0. \quad (4.8)$$

On the other hand, we set

$$\left( u^1 - \sup_{x \in \Omega} u_0(x) \right)_+ = \max \left\{ u^1 - \sup_{x \in \Omega} u_0(x), 0 \right\}.$$

In the same way, we have

$$\left\| \left( u^1(x, t) - \sup_{x \in \Omega} u_0(x) \right)_+ \right\|_{L^2}^2 \leq e^{C \|\nabla \theta^0\|_{L^\infty}^2 t} \left\| \left( u^1(x, 0) - \sup_{x \in \Omega} u_0(x) \right)_+ \right\|_{L^2}^2 = 0,$$

which gives

$$u^1(x, t) \leq \sup_{x \in \Omega} u_0(x). \quad (4.9)$$

Having proved (4.2), we in turn prove that  $\theta^1$  has a lower and upper bound. Multiplying (4.1)<sub>2</sub> for  $n = 1$  by  $\left( \theta^1 - \inf_{x \in \Omega} \theta_0(x) \right)_-$ , integrating over  $\Omega$ , we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} \left| \left( \theta^1 - \inf_{x \in \Omega} \theta_0(x) \right)_- \right|^2 dx + \gamma \int_{\Omega} \theta^0 (1 - \theta^0) \left| \nabla \left( \theta^1 - \inf_{x \in \Omega} \theta_0(x) \right)_- \right|^2 dx \\ &= -\gamma \int_{\Omega} (1 - 2\theta^0) \nabla \theta^0 \nabla \theta^1 \left( \theta^1 - \inf_{x \in \Omega} \theta_0(x) \right)_- dx \\ & \quad + \gamma \int_{\Omega} \theta^0 (1 - \theta^0) \frac{1}{u^0} \nabla u^0 \nabla \theta^1 \left( \theta^1 - \inf_{x \in \Omega} \theta_0(x) \right)_- dx \\ & \quad - (1 + \gamma) \int_{\Omega} \nabla \theta^0 \nabla \theta^1 \left( \theta^1 - \inf_{x \in \Omega} \theta_0(x) \right)_- dx. \end{aligned}$$

Since  $0 < \inf_{x \in \Omega} \theta_0(x)$ ,  $\sup_{x \in \Omega} \theta_0(x) < 1$ , there exists  $\alpha > 0$  such that

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \left| \left( \theta^1 - \inf_{x \in \Omega} \theta_0(x) \right)_- \right|^2 dx + \alpha \int_{\Omega} \left| \nabla \left( \theta^1 - \inf_{x \in \Omega} \theta_0(x) \right)_- \right|^2 dx \\ & \leq C \|\nabla \theta^0\|_{L^\infty}^2 \left\| \left( \theta^1 - \inf_{x \in \Omega} \theta_0(x) \right)_- \right\|_{L^2}^2 + C \|\nabla u^0\|_{L^4}^2 \left\| \left( \theta^1 - \inf_{x \in \Omega} \theta_0(x) \right)_- \right\|_{L^4}^2 \end{aligned}$$

$$\begin{aligned}
&\leq C \|\nabla \theta^0\|_{L^\infty}^2 \left\| \left( \theta^1 - \inf_{x \in \Omega} \theta_0(x) \right)_- \right\|_{L^2}^2 + C \|\nabla u^0\|_{H^1}^2 \times \\
&\quad \left( \left\| \left( \theta^1 - \inf_{x \in \Omega} \theta_0(x) \right)_- \right\|_{L^2} \left\| \nabla \left( \theta^1 - \inf_{x \in \Omega} \theta_0(x) \right)_- \right\|_{L^2} + \left\| \left( \theta^1 - \inf_{x \in \Omega} \theta_0(x) \right)_- \right\|_{L^2}^2 \right) \\
&\leq C \left( \|\nabla \theta^0\|_{L^\infty}^2 + \|\nabla u^0\|_{H^1}^2 + \|\nabla u^0\|_{H^1}^4 \right) \left\| \left( \theta^1 - \inf_{x \in \Omega} \theta_0(x) \right)_- \right\|_{L^2}^2.
\end{aligned}$$

Here, we have used [Lemma 4.1](#) and Young's inequality. Then Gronwall inequality gives

$$\theta^1(x, t) \geq \inf_{x \in \Omega} \theta_0(x) > 0. \quad (4.10)$$

Similarly, we have

$$\theta^1(x, t) \leq \sup_{x \in \Omega} \theta_0(x) < 1. \quad (4.11)$$

Next, we proceed to derive the  $L^\infty$  estimate for  $\nabla \theta^1$ . Multiplying (4.1)<sub>2</sub> for  $n = 1$  by  $\theta^1$  and  $-\Delta \theta^1$ , respectively, we integrate by parts to obtain

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\theta^1|^2 dx + \gamma \int_{\Omega} \theta^0 (1 - \theta^0) |\nabla \theta^1|^2 dx \\
&= -\gamma \int_{\Omega} (1 - 2\theta^0) \nabla \theta^0 \nabla \theta^1 \theta^1 dx + \gamma \int_{\Omega} \theta^0 (1 - \theta^0) \frac{1}{u^0} \nabla u^0 \nabla \theta^1 \theta^1 dx \\
&\quad - (1 + \gamma) \int_{\Omega} \nabla \theta^0 \nabla \theta^1 \theta^1 dx
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla \theta^1|^2 dx + \gamma \int_{\Omega} \theta^0 (1 - \theta^0) |\Delta \theta^1|^2 dx \\
&= -\gamma \int_{\Omega} \theta^0 (1 - \theta^0) \frac{1}{u^0} \nabla u^0 \nabla \theta^1 \Delta \theta^1 dx + (1 + \gamma) \int_{\Omega} \nabla \theta^0 \nabla \theta^1 \Delta \theta^1 dx.
\end{aligned}$$

By Sobolev's and Young's inequalities, we have

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\theta^1|^2 dx + \alpha \int_{\Omega} |\nabla \theta^1|^2 dx &\leq C \|\nabla u^0\|_{L^4}^2 \|\theta^1\|_{L^4}^2 + C \|\nabla \theta^0\|_{L^\infty}^2 \|\theta^1\|_{L^2}^2 \\
&\leq C \left( \|\nabla u^0\|_{H^1}^2 + \|\nabla \theta^0\|_{L^\infty}^2 \right) \|\theta^1\|_{H^1}^2
\end{aligned} \quad (4.12)$$

and

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla \theta^1|^2 dx + \alpha \int_{\Omega} |\Delta \theta^1|^2 dx &\leq C \|\nabla u^0\|_{L^4}^2 \|\nabla \theta^1\|_{L^4}^2 + C \|\nabla \theta^0\|_{L^\infty}^2 \|\nabla \theta^1\|_{L^2}^2 \\
&\leq C \|\nabla u^0\|_{L^4}^2 \left( \|\nabla \theta^1\|_{L^2} \|\nabla^2 \theta^1\|_{L^2} + \|\nabla \theta^1\|_{L^2}^2 \right) \\
&\quad + C \|\nabla \theta^0\|_{L^\infty}^2 \|\nabla \theta^1\|_{L^2}^2 \\
&\leq C \left( \|\nabla u^0\|_{H^1}^2 + \|\nabla u^0\|_{H^1}^4 + \|\nabla \theta^0\|_{L^\infty}^2 \right) \|\nabla \theta^1\|_{L^2}^2.
\end{aligned} \tag{4.13}$$

Adding up (4.12) and (4.13) gives

$$\frac{1}{2} \frac{d}{dt} \|\theta^1\|_{H^1}^2 + \alpha \|\nabla \theta^1\|_{H^1}^2 \leq C \|\theta^1\|_{H^1}^2,$$

by Gronwall inequality, which yields

$$\|\theta^1\|_{H^1}^2 + \alpha \int_0^t \|\nabla \theta^1\|_{H^1}^2 d\tau \leq e^{Ct} \|\theta_0\|_{H^1}^2 \leq C \tag{4.14}$$

for  $t \in [0, T]$ .

Now, differentiating (4.1)<sub>2</sub>, we get

$$\begin{aligned}
\nabla \theta_t^1 - \gamma \theta^0 (1 - \theta^0) \nabla \Delta \theta^1 &= \gamma (1 - 2\theta^0) \nabla \theta^0 \Delta \theta^1 \\
&\quad + \gamma \nabla \left( \theta^0 (1 - \theta^0) \frac{1}{u^0} \nabla u^0 \nabla \theta^1 \right) - (1 + \gamma) \nabla (\nabla \theta^0 \nabla \theta^1).
\end{aligned} \tag{4.15}$$

Setting  $\omega = \nabla \theta^1 - \|\nabla \theta_0\|_{L^\infty} \nabla \theta^1 = \nabla \theta^1 - M$ , (4.15) becomes

$$\begin{aligned}
\omega_t - \gamma \theta^0 (1 - \theta^0) \Delta \omega &= \gamma (1 - 2\theta^0) \nabla \theta^0 \operatorname{div} \omega \\
&\quad + \gamma \nabla \left( \theta^0 (1 - \theta^0) \frac{1}{u^0} \nabla u^0 (\omega + M) \right) - (1 + \gamma) \nabla (\nabla \theta^0 (\omega + M)).
\end{aligned} \tag{4.16}$$

Define  $\bar{\omega} = \max\{0, \omega\} + k$  with  $k > 1$ . Multiplying (4.16) by  $\bar{\omega}^p - k^p$  with  $p \geq 1$  and then integrating over  $[0, t] \times \Omega$  for  $t \in (0, T)$ , we have

$$\begin{aligned}
&\frac{1}{p+1} \int_{\Omega} (\bar{\omega}^{p+1} - (p+1)\bar{\omega}k^p + pk^{p+1}) dx \Big|_0^t + \frac{4p\gamma}{(p+1)^2} \int_0^t \int_{\Omega} \theta^0 (1 - \theta^0) |\nabla \bar{\omega}^{\frac{p+1}{2}}|^2 dx d\tau \\
&= -\gamma \int_0^t \int_{\Omega} (1 - 2\theta^0) \nabla \theta^0 \nabla \bar{\omega} (\bar{\omega}^p - k^p) dx d\tau + \gamma \int_0^t \int_{\Omega} (1 - 2\theta^0) \nabla \theta^0 \operatorname{div} \bar{\omega} (\bar{\omega}^p - k^p) dx d\tau \\
&\quad - p\gamma \int_0^t \int_{\Omega} \theta^0 (1 - \theta^0) \frac{1}{u^0} \nabla u^0 (\omega + M) \nabla \bar{\omega} \bar{\omega}^{p-1} dx d\tau
\end{aligned} \tag{4.17}$$

$$+ p(1 + \gamma) \int_0^t \int_{\Omega} \nabla \theta^0 (\omega + M) \nabla \bar{\omega} \bar{\omega}^{p-1} dx d\tau.$$

Notice that the fact  $\bar{\omega}^{p+1} - (p+1)\bar{\omega}k^p + pk^{p+1} \geq 0$  and  $\bar{\omega}(x, 0) = k$  as well as the inequality

$$\bar{\omega}^{p+1} - (p+1)\bar{\omega}k^p + pk^{p+1} \geq (1 - 2^{-p})\bar{\omega}^{p+1} - pk^{p+1} \geq \frac{1}{2}\bar{\omega}^{p+1} - pk^{p+1}$$

by Young's inequality. Then, we deduce from (4.17) that, for any  $t \in (0, T]$ ,

$$\begin{aligned} & \frac{1}{2(p+1)} \int_{\Omega} \bar{\omega}^{p+1} dx - \frac{p}{p+1} \int_{\Omega} k^{p+1} dx + \frac{4p\gamma}{(p+1)^2} \int_0^t \int_{\Omega} \theta^0 (1 - \theta^0) |\nabla \bar{\omega}^{\frac{p+1}{2}}|^2 dx d\tau \\ & \leq C \frac{1}{p+1} \int_0^t \int_{\Omega} \left| \nabla \theta^0 \nabla \bar{\omega}^{\frac{p+1}{2}} \bar{\omega}^{\frac{p+1}{2}} \right| dx d\tau \\ & \quad + C \int_0^t \int_{\Omega} \left( |\nabla u^0| + |\nabla \theta^0| \right) \left( 1 + \frac{M}{k} \right) \nabla \bar{\omega}^{\frac{p+1}{2}} \bar{\omega}^{\frac{p+1}{2}} dx d\tau. \end{aligned} \quad (4.18)$$

Setting  $\Lambda = \bar{\omega}^{\frac{p+1}{2}}$ , since

$$\int_{\Omega} k^{p+1} dx \leq \frac{1}{T} \int_0^T \int_{\Omega} \bar{\omega}^{p+1} dx d\tau,$$

it follows from (4.18) that

$$\begin{aligned} & \frac{1}{2(p+1)} \int_{\Omega} \Lambda^2 dx + \frac{p\alpha}{(p+1)^2} \int_0^t \int_{\Omega} |\nabla \Lambda|^2 dx d\tau \\ & \leq C \|\Lambda\|_{L^2(Q_T)}^2 + C \frac{(p+1)^2}{p} \|\nabla \theta^0\|_{L^\infty(Q_T)}^2 \|\Lambda\|_{L^2(Q_T)}^2 + C \frac{(p+1)^2}{p} \int_0^T \|\nabla u^0\|_{L^4}^2 \|\Lambda\|_{L^4}^2 d\tau \\ & \leq C(p+1) \|\Lambda\|_{L^2(0,T;L^4(\Omega))}^2. \end{aligned}$$

This implies that

$$\|\Lambda\|_{L^\infty(0,T;L^2(\Omega))}^2 \leq C(p+1)^2 \|\Lambda\|_{L^2(0,T;L^4(\Omega))}^2, \quad (4.19)$$

$$\|\nabla \Lambda\|_{L^2(0,T;L^2(\Omega))}^2 \leq C(p+1)^2 \|\Lambda\|_{L^2(0,T;L^4(\Omega))}^2. \quad (4.20)$$

Making use of Lemma 4.2 and (4.19)–(4.20), we have, for  $1 < \kappa < \frac{3}{2}$ ,

$$\|\Lambda\|_{L^{2\kappa}(0,T;L^{4\kappa}(\Omega))}^2 = \|\Lambda^\kappa\|_{L^2(0,T;L^4(\Omega))}^{\frac{2}{\kappa}} \leq C(p+1)^2 \|\Lambda\|_{L^2(0,T;L^4(\Omega))}^2. \quad (4.21)$$

For  $i = 0, 1, 2, \dots$ , let

$$\frac{p+1}{2} = \kappa^i, \quad \sigma_i = \|\bar{\omega}^{\kappa^i}\|_{L^2(0,T;L^4(\Omega))}^{\frac{2}{\kappa^i}}$$

so that  $i \rightarrow \infty$  when  $p \rightarrow \infty$ . Then (4.21) can be rewritten as

$$\sigma_{i+1} \leq C^{\frac{1}{\kappa^i}} \kappa^{\frac{2}{\kappa^i}} \sigma_i.$$

By iteration, this gives

$$\sigma_{i+1} \leq C^{\sum_0^i \frac{1}{\kappa^j}} \kappa^{2 \sum_0^i \frac{j}{\kappa^j}} \sigma_0.$$

Since

$$\sigma_i = \|\bar{\omega}^{\frac{p+1}{2}}\|_{L^2(0,T;L^4(\Omega))}^{\frac{4}{p+1}} = \|\bar{\omega}\|_{L^{p+1}(0,T;L^{2(p+1)}(\Omega))}^2,$$

let  $i \rightarrow \infty$ , we obtain

$$\|\bar{\omega}\|_{L^\infty(Q_T)}^2 \leq C \sigma_0 \leq C.$$

Here we have used the fact that  $\sigma_0 = \|\bar{\omega}\|_{L^2(0,T;L^4(\Omega))}^2 \leq C$  from (4.14). Thus, we get

$$\|\nabla \theta^1\|_{L^\infty(Q_T)}^2 \leq C. \quad (4.22)$$

By the  $L^q$  estimate of (4.1), we have

$$\|\nabla \theta^1\|_{W^{1,q}(Q_T)} \leq C \left( \|\theta^0\|_{W^{2,q}} + \|\nabla u^0\|_{L^q} \|\nabla \theta^1\|_{L^\infty(Q_T)} \right) \leq C. \quad (4.23)$$

$$\|\nabla u^1\|_{W^{1,2}(Q_T)} \leq C(\|u^0\|_{H^2} + \|u^1\|_{L^\infty(Q_T)} \|\Delta \theta^0\|_{L^2}) \leq C. \quad (4.24)$$

Applying  $\nabla$  to (4.1)<sub>1</sub>, and then multiplying the resulting equation by  $|\nabla u^1|^{q-2} \nabla u^1$  for  $q > 2$ , we integrate over  $\Omega$  and obtain

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} |\nabla u^1|^q dx + \epsilon(q-1) \int_{\Omega} |\Delta u^1|^2 |\nabla u^1|^{q-2} dx \\ &= (q-1) \int_{\Omega} (\nabla u^1 \nabla \theta^0 + u^1 \Delta \theta^0) \Delta u^1 |\nabla u^1|^{q-2} dx \\ &\leq \frac{\epsilon}{2} (q-1) \int_{\Omega} |\Delta u^1|^2 |\nabla u^1|^{q-2} dx + C(q-1) \|\nabla \theta^0\|_{L^\infty}^2 \|\nabla u^1\|_{L^q}^q \end{aligned}$$

$$\begin{aligned}
& + C(q-1)\|u^1\|_{L^\infty(Q_T)}^2 \|\Delta\theta^0\|_{L^q}^2 \|\nabla u^1\|_{L^q}^{q-2} \\
& \leq \frac{\epsilon}{2}(q-1) \int_{\Omega} |\Delta u^1|^2 |\nabla u^1|^{q-2} dx + C(q-1) \|\nabla\theta^0\|_{L^\infty}^2 \|\nabla u^1\|_{L^q}^q \\
& \quad + C(q-1)\|u^1\|_{L^\infty(Q_T)}^2 \left( \|\Delta\theta^0\|_{L^q}^q + \|\nabla u^1\|_{L^q}^q \right),
\end{aligned}$$

together with Gronwall inequality and (4.8)–(4.9) again, which implies that

$$\|\nabla u^1\|_{L^\infty(0,T;L^q(\Omega))} \leq C. \quad (4.25)$$

Then, in view of (4.8)–(4.9), (4.10)–(4.11), (4.14) and (4.22)–(4.25), we finish the proof of (4.2)–(4.5) for  $n = 1$ .

Suppose  $(u^{n-1}, \theta^{n-1})(x, t)$  for  $n = 2, 3, \dots$  satisfies (4.2)–(4.5). The proof is similar to the above argument that dealt with the case  $n = 1$ . Due to  $\|\nabla^2 u^{n-1}\|_{L^2} \in L^2([0, T])$ , we control the term  $\|\nabla u^{n-1}\|_{L^4}$  by inequality

$$\|\nabla u^{n-1}\|_{L^4}^2 \leq C \left( \|\nabla u^{n-1}\|_{L^2} \|\nabla^2 u^{n-1}\|_{L^2} + \|\nabla u^{n-1}\|_{L^2}^2 \right).$$

It is also worth noting that, since  $\nabla u^{n-1} \in L^\infty(0, T; L^q(\Omega))$ ,

$$\begin{aligned}
\|\Lambda\|_{L^\infty(0,T;L^2(\Omega))}^2 & \leq C(p+1)^2 \|\Lambda\|_{L^2(0,T;L^{\frac{2q}{q-2}}(\Omega))}^2, \\
\|\nabla \Lambda\|_{L^2(0,T;L^2(\Omega))}^2 & \leq C(p+1)^2 \|\Lambda\|_{L^2(0,T;L^{\frac{2q}{q-2}}(\Omega))}^2.
\end{aligned}$$

Then, for  $1 < \kappa < 2 - \frac{2}{q}$ ,

$$\|\Lambda\|_{L^{2\kappa}(0,T;L^{\frac{2q\kappa}{q-2}}(\Omega))}^2 = \|\Lambda^\kappa\|_{L^2(0,T;L^{\frac{2q}{q-2}}(\Omega))}^{\frac{2}{\kappa}} \leq C(p+1)^2 \|\Lambda\|_{L^2(0,T;L^{\frac{2q}{q-2}}(\Omega))}^2.$$

Therefore, we can conclude that (4.2)–(4.5) hold for any  $T$  by repeating our procedure.  $\square$

**Lemma 4.4.** Under the assumptions in Lemma 4.3, it holds that

$$\|u_t^n\|_{L^\infty(0,T;L^2(\Omega))}^2 + \epsilon \|\nabla u_t^n\|_{L^2(Q_T)}^2 \leq C, \quad (4.26)$$

$$\|\theta_t^n\|_{L^\infty(0,T;L^2(\Omega))}^2 + \alpha \|\nabla \theta_t^n\|_{L^2(Q_T)}^2 \leq C \quad (4.27)$$

for any  $T$ . Here  $C$  is a positive constant dependent on  $T$ ,  $\Omega$ ,  $C_0$  and  $\epsilon$ .

**Proof.** For  $n = 1$ , applying  $\partial_t$  to (4.1), multiplying  $u_t^1$  and  $\theta_t^1$  by the first and second resulting equation, we integrate them over  $\Omega$  to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} |u_t^1|^2 dx + \epsilon \int_{\Omega} |\nabla u_t^1|^2 dx &= \int_{\Omega} u_t^1 \nabla \theta^0 \nabla u_t^1 dx \\ &\leq C \|\nabla \theta^0\|_{L^\infty}^2 \|u_t^1\|_{L^2}^2 + \frac{\epsilon}{2} \|\nabla u_t^1\|_{L^2}^2, \end{aligned} \quad (4.28)$$

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\theta_t^1|^2 dx + \gamma \int_{\Omega} \theta^0 (1 - \theta^0) |\nabla \theta_t^1|^2 dx \\ &= -\gamma \int_{\Omega} (1 - 2\theta^0) \nabla \theta^0 \nabla \theta_t^1 dx + \gamma \int_{\Omega} \theta^0 (1 - \theta^0) \frac{1}{u^0} \nabla u^0 \nabla \theta_t^1 \theta_t^1 dx \\ &\quad - (1 + \gamma) \int_{\Omega} \nabla \theta^0 \nabla \theta_t^1 \theta_t^1 dx \\ &\leq C \|\nabla \theta^1\|_{L^\infty}^2 + C \|\nabla u^0\|_{H^1}^2 (\|\theta_t^1\|_{L^2} \|\nabla \theta_t^1\|_{L^2} + \|\theta_t^1\|_{L^2}^2) + C \|\nabla \theta^0\|_{L^\infty}^2 \|\theta_t^1\|_{L^2}^2 + \frac{\alpha}{4} \|\nabla \theta_t^1\|_{L^2}^2 \\ &\leq C \|\nabla \theta^1\|_{L^\infty}^2 + C (\|\nabla u^0\|_{H^1}^2 + \|\nabla u^0\|_{H^1}^4 + \|\nabla \theta^0\|_{L^\infty}^2) \|\theta_t^1\|_{L^2}^2 + \frac{\alpha}{2} \|\nabla \theta_t^1\|_{L^2}^2. \end{aligned} \quad (4.29)$$

By using (4.5) in Lemma 4.3, it follows from Gronwall inequality that

$$\|u_t^1\|_{L^2}^2 + \epsilon \int_0^t \|\nabla u_\tau^1\|_{L^2}^2 d\tau \leq e^{Ct} \|u_t^1(0)\|_{L^2}^2, \quad (4.30)$$

$$\|\theta_t^1\|_{L^2}^2 + \alpha \int_0^t \|\nabla \theta_\tau^1\|_{L^2}^2 d\tau \leq e^{Ct} \|\theta_t^1(0)\|_{L^2}^2. \quad (4.31)$$

Taking  $t = 0$  in (4.1), we find that

$$\begin{aligned} \|u_t^n(0)\|_{L^2} &\leq C(\epsilon \|\Delta u^n(0)\|_{L^2} + \|\nabla u^n(0)\|_{L^4} \|\nabla \theta^{n-1}(0)\|_{L^4} + \|u^n(0)\|_{L^\infty} \|\Delta \theta^{n-1}(0)\|_{L^2}) \\ &\leq C \end{aligned} \quad (4.32)$$

and

$$\begin{aligned} \|\theta_t^n(0)\|_{L^2} &\leq C(\alpha \|\Delta \theta^n(0)\|_{L^2} + \|\nabla u^{n-1}(0)\|_{L^4} \|\nabla \theta^n(0)\|_{L^4} + \|\nabla \theta^{n-1}(0)\|_{L^4} \|\nabla \theta^n(0)\|_{L^4}) \\ &\leq C. \end{aligned} \quad (4.33)$$

In view of (4.30)–(4.33), we conclude that (4.26) and (4.27) hold when  $n = 1$ .

Moreover, we assume that (4.26) and (4.27) hold for the case  $n - 1$ . Similar to (4.28)–(4.29), we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} |u_t^n|^2 dx + \epsilon \int_{\Omega} |\nabla u_t^n|^2 dx = \int_{\Omega} u_t^n \nabla \theta^{n-1} \nabla u_t^1 dx + \int_{\Omega} u^n \nabla \theta_t^{n-1} \nabla u_t^1 dx \\ & \leq C \|\nabla \theta^{n-1}\|_{L^\infty}^2 \|u_t^n\|_{L^2}^2 + C \|\nabla \theta_t^{n-1}\|_{L^2}^2 \|u^n\|_{L^\infty}^2 + \frac{\epsilon}{2} \|\nabla u_t^n\|_{L^2}^2, \end{aligned} \quad (4.34)$$

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\theta_t^n|^2 dx + \gamma \int_{\Omega} \theta^{n-1} (1 - \theta^{n-1}) |\nabla \theta_t^n|^2 dx \\ & = \gamma \int_{\Omega} (1 - 2\theta^{n-1}) \theta_t^{n-1} \Delta \theta^n \theta_t^n dx - \gamma \int_{\Omega} (1 - 2\theta^{n-1}) \nabla \theta^{n-1} \nabla \theta_t^n \theta_t^n dx \\ & \quad + \gamma \int_{\Omega} \partial_t \left( \theta^{n-1} (1 - \theta^{n-1}) \frac{1}{u^{n-1}} \nabla u^{n-1} \nabla \theta^n \right) \theta_t^n dx - (1 + \gamma) \int_{\Omega} \partial_t (\nabla \theta^{n-1} \nabla \theta^n) \theta_t^n dx \\ & \leq C \left( \|\nabla u^{n-1}\|_{L^4}^2 + \|\Delta \theta^n\|_{L^2}^2 \right) \|\theta_t^n\|_{L^4}^2 + C \|\nabla u_t^{n-1}\|_{L^2}^2 \\ & \quad + C \|\nabla u^{n-1}\|_{L^4}^2 \left( \|u_t^{n-1}\|_{L^4}^2 + \|\theta_t^{n-1}\|_{L^4}^2 \right) + C \|\theta_t^{n-1}\|_{H^1}^2 \\ & \quad + C \left( \|\nabla \theta^{n-1}\|_{L^\infty}^2 + \|\nabla \theta^n\|_{L^\infty}^2 \right) \|\theta_t^n\|_{L^2}^2 + \frac{\alpha}{4} \|\nabla \theta_t^n\|_{L^2}^2 \\ & \leq C \left( \|\nabla u^{n-1}\|_{H^1}^2 + \|\nabla u^{n-1}\|_{L^2}^4 + \|\nabla u^{n-1}\|_{L^2}^2 \|\nabla^2 u^{n-1}\|_{L^2}^2 + \|\Delta \theta^n\|_{L^2}^2 \right) \|\theta_t^n\|_{L^2}^2 \\ & \quad + C \|\nabla u_t^{n-1}\|_{L^2}^2 + C \|\nabla u^{n-1}\|_{L^2}^2 \|\nabla^2 u^{n-1}\|_{L^2}^2 + C \|u_t^{n-1}\|_{L^2}^2 \left( \|\nabla u_t^{n-1}\|_{L^2}^2 + \|\nabla u^{n-1}\|_{H^1}^2 \right) \\ & \quad + C \|\nabla u^{n-1}\|_{L^2}^2 \|u_t^{n-1}\|_{H^1}^2 + C \|\theta_t^{n-1}\|_{L^2}^2 \left( \|\nabla \theta_t^{n-1}\|_{L^2}^2 + \|\nabla u^{n-1}\|_{H^1}^2 \right) \\ & \quad + C \|\nabla u^{n-1}\|_{L^2}^2 \|\theta_t^{n-1}\|_{H^1}^2 + C \|\theta_t^{n-1}\|_{H^1}^2 + C \left( \|\nabla \theta^{n-1}\|_{L^\infty}^2 + \|\nabla \theta^n\|_{L^\infty}^2 \right) \|\theta_t^n\|_{L^2}^2 \\ & \quad + \frac{\alpha}{2} \|\nabla \theta_t^n\|_{L^2}^2, \end{aligned} \quad (4.35)$$

where we have used Lemma 4.1 in the last inequality. The Gronwall inequality then implies

$$\begin{aligned} & \|u_t^n\|_{L^2}^2 + \epsilon \int_0^t \|\nabla u_t^n\|_{L^2}^2 d\tau \\ & \leq e^{C \|\nabla \theta^{n-1}\|_{L^\infty(Q_T)}^2 t} \left( \|u_t^n(0)\|_{L^2}^2 + C \int_0^t \|\nabla \theta_t^{n-1}\|_{L^2}^2 \|u^n\|_{L^\infty}^2 d\tau \right), \end{aligned} \quad (4.36)$$

$$\begin{aligned}
& \|\theta_t^1\|_{L^2}^2 + \alpha \int_0^t \|\nabla \theta_t^1\|_{L^2}^2 d\tau \\
& \leq e^{C \int_0^t (\|\nabla u^{n-1}\|_{H^1}^2 + \|\nabla u^{n-1}\|_{L^2}^4 + \|\nabla u^{n-1}\|_{L^2}^2 \|\nabla^2 u^{n-1}\|_{L^2}^2 + \|\Delta \theta^n\|_{L^2}^2 + \|\nabla \theta^{n-1}\|_{L^\infty}^2 + \|\nabla \theta^n\|_{L^\infty}^2) d\tau} \\
& \quad \times \left( \|\theta_t^n(0)\|_{L^2}^2 + C \int_0^\tau ((\|\nabla u^{n-1}\|_{L^2}^2 + 1) \|\theta_t^{n-1}\|_{H^1}^2 + \|\nabla u_t^{n-1}\|_{L^2}^2 \right. \\
& \quad \left. + \|\nabla u^{n-1}\|_{L^2}^2 \|u_t^{n-1}\|_{H^1}^2) d\tau \right).
\end{aligned} \tag{4.37}$$

In view of (4.36)–(4.37) and Lemma 4.3, we get (4.26)–(4.27).  $\square$

## 5. Existence of strong solutions

In this section, we shall establish the existence of the global strong solution by applying the convergence method and *a priori* estimates obtained in the previous section.

### 5.1. The limit passage $n \rightarrow \infty$

The task of this section is to employ the estimates obtained in Lemmas 4.3–4.4 to get the limit of the sequence  $(u_m^n, \theta_m^n)$  as  $n \rightarrow \infty$ .

By virtue of the estimates (4.4)–(4.5) and (4.26)–(4.27), and applying the compactness results of Sell and You (see Lemma 63.2 in [27]), we can find a subsequence of  $(u_m^n, \theta_m^n)$ , relabeled as  $(u_m^n, \theta_m^n)$ , so that

$$\begin{aligned}
& u_m^n(x, t) \rightarrow u_m \text{ weakly-* in } L^\infty(0, T; W^{1,q}(\Omega)), \\
& \nabla u_m^n(x, t) \rightarrow \nabla u_m \text{ weakly in } L^2(0, T; H^1(\Omega)), \\
& \nabla \theta_m^n(x, t) \rightarrow \nabla \theta_m \text{ weakly in } W^{1,q}(Q_T), \\
& (\partial_t u_m^n, \partial_t \theta_m^n)(x, t) \rightarrow (\tilde{u}_m, \tilde{\theta}_m) \text{ weakly in } L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \\
& (u_m^n, \theta_m^n)(x, t) \rightarrow (u_m, \theta_m) \text{ strongly in } L^2(0, T; H^1(\Omega)) \times W^{1,q}(Q_T),
\end{aligned} \tag{5.1}$$

and for almost every  $t \in [0, T]$ ,

$$(u^n, \theta^n)(x, t) \rightarrow (u, \theta) \text{ strongly in } H^1(\Omega). \tag{5.2}$$

Standard arguments can then be used to show that  $\tilde{u}_m = \partial_t u_m$  and  $\tilde{\theta}_m = \partial_t \theta_m$ . Then, by the compactness property of  $L^2$ -space, we have

$$(u_t^n, \theta_t^n)(x, t) \rightarrow (u_t, \theta_t) \text{ strongly in } L^2(0, T; L^2(\Omega)). \tag{5.3}$$

**Lemma 5.1.** *If the initial conditions of Lemma 4.3 hold, then the couple  $(u_m, \theta_m)(x, t)$  is the solution of (3.1), and has the following properties: for any  $T$ ,*

$$0 < \sup_{x \in \Omega} u_0(x) \leq u_m(x, t) \leq \sup_{x \in \Omega} u_0(x) < \infty, \quad t \in [0, T], \quad (5.4)$$

$$0 < \inf_{x \in \Omega} \theta_0(x) \leq \theta_m(x, t) \leq \sup_{x \in \Omega} \theta_0(x) < 1, \quad t \in [0, T], \quad (5.5)$$

$$\|u^n\|_{L^\infty(0, T; W^{1, q}(\Omega))}^2 + \|\nabla u^n\|_{W^{1, 2}(Q_T)}^2 \leq C, \quad (5.6)$$

$$\|\nabla \theta^n\|_{L^\infty(Q_T)}^2 + \|\nabla \theta^n\|_{W^{1, q}(Q_T)}^2 \leq C, \quad (5.7)$$

$$\|\partial_t u_m\|_{L^\infty(0, T; L^2(\Omega))}^2 + \epsilon \|\nabla \partial_t u_m\|_{L^2(Q_T)}^2 \leq C, \quad (5.8)$$

$$\|\partial_t \theta_m\|_{L^\infty(0, T; L^2(\Omega))}^2 + \alpha \|\nabla \partial_t \theta_m\|_{L^2(Q_T)}^2 \leq C. \quad (5.9)$$

Here,  $\alpha, C$  are positive constants given in [Lemmas 4.3–4.4](#).

**Proof.** Due to the above convergence, one can see easily that the couple  $(u_m, \theta_m)(x, t)$  is the solution of (3.1). Applying the lower semicontinuity of the norms, and using the estimates in [Lemmas 4.3–4.4](#), we deduce the estimates (5.4)–(5.9).  $\square$

## 5.2. Proof of existence

By using [Lemma 5.1](#) and repeating the same procedure in Subsection 3.2 to pass to the limit for  $m \rightarrow \infty$ , we prove that the limit function  $(u, \theta)$  of the approximate solutions  $(u_m, \theta_m)$  is a global strong solution of (2.4)–(2.5).

## 6. Uniqueness of the strong solution

In this section, we consider the uniqueness of the strong solution. Let  $(u_1, \theta_1)$  and  $(u_2, \theta_2)$  be two strong solutions of (2.4)–(2.5) with the same initial value and satisfy the regularities furnished by [Theorem 2.1](#). We introduce  $(\eta, \xi)$  with

$$\eta = u_1 - u_2, \quad \xi = \theta_1 - \theta_2.$$

Then these functions satisfy the following equations:

$$\begin{cases} \eta_t - \epsilon \Delta \eta = -\operatorname{div}(\eta \nabla \theta_1) - \operatorname{div}(u_2 \nabla \xi), \\ \xi_t - \gamma \theta_1(1 - \theta_1) \Delta \xi = \gamma \xi \Delta \theta_2 + \gamma \xi (\theta_1 + \theta_2) \Delta \theta_2 + \gamma \theta_1(1 - \theta_1) \frac{1}{u_1} \nabla u_1 \nabla \xi \\ \quad + \gamma \theta_1(1 - \theta_1) \frac{1}{u_1} \nabla \eta \nabla \theta_2 - \gamma \theta_1(1 - \theta_1) \eta \frac{1}{u_1 + u_2} \nabla u_2 \nabla \theta_2 + \gamma \eta \frac{1}{u_2} \nabla u_2 \nabla \theta_2 \\ \quad - \gamma \eta (\theta_1 + \theta_2) \frac{1}{u_2} \nabla u_2 \nabla \theta_2 - (1 + \gamma) \nabla \theta_1 \nabla \xi - (1 + \gamma) \nabla \theta_2 \nabla \xi. \end{cases} \quad (6.1)$$

**Proof of uniqueness.** Multiplying the first equation in (6.1) by  $\eta$  and  $-\Delta \eta$ , and then integrating over  $\Omega$ , we obtain

$$\frac{1}{2} \frac{d}{dt} \|\eta\|_{L^2}^2 + \epsilon \|\nabla \eta\|_{L^2}^2 \leq C \|\eta\|_{L^2}^2 \|\nabla \theta_1\|_{L^\infty}^2 + C \|u_2\|_{L^\infty}^2 \|\nabla \xi\|_{L^2}^2, \quad (6.2)$$

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\nabla \eta\|_{L^2}^2 + \epsilon \|\Delta \eta\|_{L^2}^2 \\
& \leq C \|\nabla \eta\|_{L^2}^2 \|\nabla \theta_1\|_{L^\infty}^2 + C \|\eta\|_{H^1}^2 \|\Delta \theta_1\|_{L^p}^2 + \tilde{C} \|u_2\|_{L^\infty}^2 \|\Delta \xi\|_{L^2}^2 \\
& \quad + C \left( \|\nabla u_2\|_{L^2} \|\nabla^2 u_2\|_{L^2} + \|\nabla u_2\|_{L^2}^2 \right) \left( \|\nabla \xi\|_{L^2} \|\nabla^2 \xi\|_{L^2} + \|\nabla \xi\|_{L^2}^2 \right) \\
& \leq C \|\nabla \eta\|_{L^2}^2 \|\nabla \theta_1\|_{L^\infty}^2 + C \|\eta\|_{H^1}^2 \|\Delta \theta_1\|_{L^p}^2 \\
& \quad + C \|\nabla u_2\|_{H^1}^2 \|\nabla \xi\|_{L^2}^2 + \tilde{C} \left( \|u_2\|_{L^\infty}^2 + \|\nabla u_2\|_{L^2}^2 \right) \|\Delta \xi\|_{L^2}^2.
\end{aligned} \tag{6.3}$$

In a similar way, from the second equation in (6.1), we find that

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\xi\|_{L^2}^2 + \alpha \|\nabla \xi\|_{L^2}^2 & \leq C (\|\nabla \theta_1\|_{L^\infty}^2 + \|\nabla \theta_2\|_{L^\infty}^2 + \|\Delta \theta_2\|_{L^2}^2) \|\xi\|_{L^2}^2 + C \|\nabla u_1\|_{H^1}^2 \|\xi\|_{H^1}^2 \\
& \quad + C \|\eta\|_{H^1}^2 \|\nabla u_2\|_{H^1}^2 + C \|\nabla \eta\|_{L^2}^2,
\end{aligned} \tag{6.4}$$

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\nabla \xi\|_{L^2}^2 + \alpha \|\Delta \xi\|_{L^2}^2 & \leq C (\|\nabla \theta_1\|_{L^\infty}^2 + \|\Delta \theta_2\|_{L^p}^2 + \|\nabla \theta_2\|_{L^\infty}^2) \|\xi\|_{H^1}^2 + C \|\nabla u_1\|_{L^4}^2 \|\nabla \xi\|_{L^4}^2 \\
& \quad + C \|\nabla \eta\|_{L^4}^2 \|\nabla \theta_2\|_{L^4}^2 + \|\eta\|_{H^1}^2 \|\nabla u_2\|_{H^1}^2 \|\nabla \theta_2\|_{L^\infty}^2 \\
& \leq C (\|\nabla \theta_1\|_{L^\infty}^2 + \|\Delta \theta_2\|_{L^p}^2 + \|\nabla \theta_2\|_{L^\infty}^2 + \|\nabla u_1\|_{L^2}^2 \\
& \quad + \|\nabla u_1\|_{L^2}^2 \|\nabla^2 u_1\|_{L^2}^2) \|\xi\|_{H^1}^2 \\
& \quad + C \|\nabla \eta\|_{L^2}^2 \|\nabla \theta_2\|_{L^\infty}^2 + \|\eta\|_{H^1}^2 \|\nabla u_2\|_{H^1}^2 \|\nabla \theta_2\|_{L^\infty}^2.
\end{aligned} \tag{6.5}$$

We can choose a small constant  $d$  such that

$$d \tilde{C} (\|u_2\|_{L^\infty}^2 + \|\nabla u_2\|_{L^2}^2) \leq \frac{1}{2} \alpha.$$

Multiplying (6.3) by  $d$ , combining it with (6.2) and (6.4)–(6.5), we get

$$\frac{d}{dt} \left( \|\eta\|_{H^1}^2 + \|\xi\|_{H^1}^2 \right) + \epsilon \|\nabla \eta\|_{H^1}^2 + \alpha \|\nabla \xi\|_{H^1}^2 \leq Ch(t) \left( \|\eta\|_{H^1}^2 + \|\xi\|_{H^1}^2 \right),$$

where

$$\begin{aligned}
h(t) & = C \left( 1 + \|\nabla \theta_1\|_{L^\infty}^2 + \|\Delta \theta_1\|_{L^p}^2 + \|\nabla \theta_2\|_{L^\infty}^2 + \|\Delta \theta_2\|_{L^p}^2 + \|\nabla u_1\|_{H^1}^2 \right. \\
& \quad \left. + \|\nabla u_2\|_{H^1}^2 + \|\nabla u_1\|_{L^2}^2 \|\nabla^2 u_1\|_{L^2}^2 + \|\nabla u_2\|_{H^1}^2 \|\nabla \theta_2\|_{L^\infty}^2 \right).
\end{aligned}$$

Observe that  $h$  is an integrable function in  $[0, T]$ , in view of the regularity of  $(u_1, \theta_1)$  and  $(u_2, \theta_2)$ . Consequently, we can apply Gronwall inequality, which gives

$$\|\eta\|_{H^1}^2 + \|\xi\|_{H^1}^2 = 0,$$

that is,  $u_1 = u_2$  and  $\theta_1 = \theta_2$  a.e. in  $Q_T$ . This ends the proof of uniqueness.  $\square$

## 7. Turing instability in a bounded domain

The Turing instability was proposed by A.M. Turing in 1952 [31] as an explanation for pattern formation in reaction–diffusion systems. It is the phenomenon that an initially stable steady state of a dynamical system can become unstable if diffusion is additionally taken into account. This is both surprising and unexpected, because diffusion usually makes things more smooth and uniform. This loss of stability due to diffusion is what is known as the Turing instability [24]. For this issue, there are many works made on population dynamics of biological systems [9,25,23,33]. In this section, we present a stability analysis of the nonlinear cross-diffusion system (1.6) for two interacting populations in a bounded domain  $\Omega \subset \mathbb{R}^N$ .

Suppose that  $(u^*, v^*)$  is a constant equilibrium solution, i.e.

$$f_1(u^*, v^*)u^* = 0, \quad \text{and} \quad f_2(u^*, v^*)v^* = 0, \quad (7.1)$$

where the fitness  $f_i(u, v)$ ,  $i = 1, 2$ , are defined by (1.1). Clearly,  $(u^*, v^*)$  is also constant equilibrium solution of a system of ordinary differential equations:

$$\begin{cases} \dot{u} = f_1(u, v)u, & t > 0, \\ \dot{v} = f_2(u, v)v, & t > 0. \end{cases} \quad (7.2)$$

If a positive equilibrium solution exists, a sufficient condition is that

$$a_{11}a_{22} = a_{12}a_{21}, \quad a_{11}a_{12} < 0.$$

Then we have the positive equilibrium solutions

$$(u^*, v^*) = (c, -\frac{a_{11}}{a_{12}}c)$$

for any positive constant  $c$ .

We linearize the ODE system (7.2) about the constant equilibrium  $(u^*, v^*)$ . Let  $U = u - u^*$ ,  $V = v - v^*$  be a spatial perturbation for which, we have

$$\begin{cases} \dot{U} = \partial_u f_1(u^*, v^*)u^*U + \partial_v f_1(u^*, v^*)u^*V, \\ \dot{V} = \partial_u f_2(u^*, v^*)v^*U + \partial_v f_2(u^*, v^*)v^*V. \end{cases} \quad (7.3)$$

We define the matrix  $J$  by the following

$$J = \begin{pmatrix} \partial_u f_1(u^*, v^*)u^* & \partial_v f_1(u^*, v^*)u^* \\ \partial_u f_2(u^*, v^*)v^* & \partial_v f_2(u^*, v^*)v^* \end{pmatrix}. \quad (7.4)$$

In fact the stability of  $(0, 0)$  in (7.3) is equivalent to the stability of matrix  $J$ , which depends on the signs of the eigenvalues  $\lambda$  of  $J$ , i.e., all which must have negative real parts. The characteristic polynomial of  $J$  is given by

$$P_J(\lambda) = \lambda^2 - \text{Trace}(J)\lambda + \text{Det}(J).$$

A direct calculation yields

$$\text{Det}(J) = \text{Det} \begin{pmatrix} (a_{11} - a_{12}) \frac{u^* v^*}{(u^* + v^*)^2} & (a_{12} - a_{11}) \frac{u^* u^*}{(u^* + v^*)^2} \\ (a_{21} - a_{22}) \frac{v^* v^*}{(u^* + v^*)^2} & (a_{22} - a_{21}) \frac{u^* v^*}{(u^* + v^*)^2} \end{pmatrix} = 0. \quad (7.5)$$

It is easy to see that at least one eigenvalue of  $J$  is zero. Thus,  $(u^*, v^*)$  is not a stable equilibrium with respect to (7.3). We summarize the result in the theorem below.

**Theorem 7.1.** *Let  $a_{11}a_{22} = a_{12}a_{21}$ ,  $a_{11}a_{12} < 0$ . Suppose that  $(u^*, v^*)$  is a positive equilibrium solution of (1.6). Then  $(u^*, v^*)$  is not a stable solution with respect to the ODE system (7.2) and the cross-diffusion system (1.6).*

Next, we add two diffusion terms to the ODE system (7.2), that is, for the parameters  $v_1, v_2 > 0$ ,

$$\begin{cases} u_t = f_1(u, v)u + v_1 \Delta u, \\ v_t = f_2(u, v)v + v_2 \Delta v. \end{cases} \quad (7.6)$$

We consider the situation in which the stability of the equilibrium changes from unstable for the ODE system (7.2) to stable for the diffusion system (7.6). Clearly, if  $a_{11}a_{22} = a_{12}a_{21}$  and  $a_{11}a_{12} < 0$ , then for any positive constant  $c$ ,

$$(u^*, v^*) = (c, -\frac{a_{11}}{a_{12}}c)$$

are also the positive equilibrium solutions of (7.6). Linearizing the diffusion system (7.6) about the positive equilibrium  $(u^*, v^*)$ , we have

$$\Psi_t = J\Psi + D\Delta\Psi, \quad (7.7)$$

where  $\Psi = (U, V)^T$  and  $D = \text{diag}(v_1, v_2)$ . Let  $0 = \lambda_1 < \lambda_2 < \dots$  be the eigenvalues of operator  $-\Delta$  on  $\Omega$  with the homogeneous Neumann boundary condition, and  $E(\lambda_i)$  be the eigenspace corresponding to  $\lambda_i$  in  $C^2(\Omega)$ . Let  $\mathbf{X} = \{\mathbf{u} \in [C^1(\Omega)]^2 \mid \frac{\partial \mathbf{u}}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega\}$ ,  $\{\phi_{ij}\}_{j=1,2,\dots,\dim E(\lambda_i)}$  be an orthonormal basis of  $E(\lambda_i)$ , and  $\mathbf{X}_{ij} = \{\mathbf{c}\phi_{ij} \mid \mathbf{c} \in \mathbb{R}^2\}$ . Then

$$\mathbf{X} = \bigotimes_{i=1}^{\infty} \mathbf{X}_i \quad \text{and} \quad \mathbf{X}_i = \bigotimes_{j=1}^{\dim E(\lambda_i)} \mathbf{X}_{ij}.$$

For each  $i \geq 1$ ,  $\mathbf{X}_i$  is invariant under the operator  $J + D\Delta$ . Then problem (7.5) has a non-trivial solution of the form  $\Psi = \mathbf{c}\phi \exp(\mu t)$  if and only if  $(\mu, \mathbf{c})$  is an eigenpair for the matrix  $J - \lambda_i D$ , where  $\mathbf{c}$  is a constant vector. The equilibrium  $(u^*, v^*)$  is stable if all the eigenvalues have negative real parts for each  $\lambda_i > 0$ .

The characteristic polynomial of  $J - \lambda_i D$  is given by

$$P_i(\mu) = \mu^2 - \text{Trace}(J - \lambda_i D)\mu + \text{Det}(J - \lambda_i D),$$

where

$$\begin{aligned}\text{Trace}(J - \lambda_i D) &= (a_{11} - a_{12} + a_{22} - a_{21}) \frac{u^* v^*}{(u^* + v^*)^2} - v_1 \lambda_i - v_2 \lambda_i, \\ \text{Det}(J - \lambda_i D) &= v_1 v_2 \lambda_i^2 - [v_1(a_{22} - a_{21}) + v_2(a_{11} - a_{12})] \lambda_i \frac{u^* v^*}{(u^* + v^*)^2}.\end{aligned}$$

We denote  $\mu_1(\lambda_i)$  and  $\mu_2(\lambda_i)$  as the roots of  $P_i(\mu) = 0$ , and then we have

$$\mu_1(\lambda_i) + \mu_2(\lambda_i) = \text{Trace}(J - \lambda_i D) \quad \text{and} \quad \mu_1(\lambda_i) \mu_2(\lambda_i) = \text{Det}(J - \lambda_i D).$$

In order to get  $\text{Re} \mu_1(\lambda_i) < 0$  and  $\text{Re} \mu_2(\lambda_i) < 0$ , a sufficient condition is that

$$\text{Trace}(J - \lambda_i D) < 0 \quad \text{and} \quad \text{Det}(J - \lambda_i D) > 0$$

for each  $\lambda_i > 0$ . Thus,

$$a_{11} - a_{12} + a_{22} - a_{21} < 0 \quad \text{and} \quad v_2(a_{11} - a_{12}) + v_1(a_{22} - a_{21}) < 0. \quad (7.8)$$

As a consequence, we give sufficient conditions on diffusion which leads to stability.

**Theorem 7.2.** *Let  $a_{11}a_{22} = a_{12}a_{21}$ ,  $a_{11}a_{12} < 0$ ,  $v_1 > 0$ ,  $v_2 > 0$  and the condition (7.8) hold. Suppose that  $(u^*, v^*)$  is a positive equilibrium solution of (7.6). Then  $(u^*, v^*)$  is a stable equilibrium solution with respect to the diffusion system (7.6).*

Furthermore, we study the diffusion system (7.6) with cross-diffusion effects, which takes the form

$$\begin{cases} u_t = f_1(u, v)u + v_1 \Delta u + \beta_1(a_{11} - a_{12}) \text{div} \left( -\frac{uv}{(u+v)^2} \nabla u + \frac{u^2}{(u+v)^2} \nabla v \right), \\ v_t = f_2(u, v)v + v_2 \Delta v + \beta_2(a_{21} - a_{22}) \text{div} \left( -\frac{v^2}{(u+v)^2} \nabla u + \frac{uv}{(u+v)^2} \nabla v \right). \end{cases} \quad (7.9)$$

We linearize the cross-diffusion system (7.9) about the positive equilibrium  $(u^*, v^*)$  to obtain

$$\Psi_t = J\Psi + (D + H)\Delta\Psi, \quad (7.10)$$

where

$$H = \begin{pmatrix} -\beta_1(a_{11} - a_{12}) \frac{u^* v^*}{(u^* + v^*)^2} & \beta_1(a_{11} - a_{12}) \frac{u^* u^*}{(u^* + v^*)^2} \\ -\beta_2(a_{21} - a_{22}) \frac{v^* v^*}{(u^* + v^*)^2} & \beta_2(a_{21} - a_{22}) \frac{u^* v^*}{(u^* + v^*)^2} \end{pmatrix}.$$

Then the characteristic polynomial of  $J - \lambda_i(D + H)$  is

$$P_i(\mu) = \mu^2 - \text{Trace}(J - \lambda_i(D + H))\mu + \text{Det}(J - \lambda_i(D + H)),$$

where

$$\begin{aligned} \text{Trace}(J - \lambda_i(D + H)) &= (a_{11} - a_{12} + a_{22} - a_{21}) \frac{u^* v^*}{(u^* + v^*)^2} - \lambda_i(v_1 + v_2) \\ &\quad + \lambda_i[(a_{11} - a_{12})\beta_1 + (a_{22} - a_{21})\beta_2] \frac{u^* v^*}{(u^* + v^*)^2}, \end{aligned}$$

$$\begin{aligned} \text{Det}(J - \lambda_i(D + H)) &= \text{Det}(D + H) \lambda_i^2 - [v_2(a_{11} - a_{12}) + v_1(a_{22} - a_{21})] \frac{u^* v^*}{(u^* + v^*)^2} \lambda_i + \text{Det} J \\ &= \left\{ v_1 v_2 - [v_2 \beta_1(a_{11} - a_{12}) + v_1 \beta_2(a_{22} - a_{21})] \frac{u^* v^*}{(u^* + v^*)^2} \right\} \lambda_i^2 \\ &\quad - [v_2(a_{11} - a_{12}) + v_1(a_{22} - a_{21})] \frac{u^* v^*}{(u^* + v^*)^2} \lambda_i. \end{aligned}$$

Then the equilibrium  $(u^*, v^*)$  is stable with respect to (7.9) if all eigenvalues have negative real parts, that is, for all  $\lambda_i > 0$ ,

$$\text{Trace}(J - \lambda_i(D + H)) < 0, \quad \text{Det}(J - \lambda_i(D + H)) > 0.$$

Under the assumption (7.8), the sufficient conditions are that

$$\begin{aligned} [(a_{11} - a_{12})\beta_1 + (a_{22} - a_{21})\beta_2] \frac{u^* v^*}{(u^* + v^*)^2} &< v_1 + v_2, \\ [v_2 \beta_1(a_{11} - a_{12}) + v_1 \beta_2(a_{22} - a_{21})] \frac{u^* v^*}{(u^* + v^*)^2} &\leq v_1 v_2. \end{aligned} \quad (7.11)$$

In order to have a Turing instability, the polynomial  $P_i$  must at least have one eigenvalue with positive real part for some  $\lambda_i$ . Thus, we need one of the two conditions

$$[(a_{11} - a_{12})\beta_1 + (a_{22} - a_{21})\beta_2] \frac{u^* v^*}{(u^* + v^*)^2} \geq v_1 + v_2, \quad (7.12)$$

$$[v_2 \beta_1(a_{11} - a_{12}) + v_1 \beta_2(a_{22} - a_{21})] \frac{u^* v^*}{(u^* + v^*)^2} < v_1 v_2. \quad (7.13)$$

**Theorem 7.3.** Let  $a_{11}a_{22} = a_{12}a_{21}$ ,  $a_{11}a_{12} < 0$ ,  $v_1 > 0$ ,  $v_2 > 0$  and the condition (7.8) hold. Suppose that  $(u^*, v^*)$  is a positive equilibrium solution of (7.9). Then,  $(u^*, v^*)$  is a stable equilibrium solution with respect to the diffusion system (7.9) if (7.11) is true,  $(u^*, v^*)$  is an unstable equilibrium solution with respect to the diffusion system (7.9) if (7.12) (or (7.13)) is true.

Moreover, we make a modification to the ODE system (7.2), that is,

$$\begin{cases} \dot{u} = \frac{a_{11}u + a_{12}v + \varepsilon}{u + v} u, & t > 0, \\ \dot{v} = \frac{a_{21}u + a_{22}v + \varepsilon}{u + v} v, & t > 0. \end{cases} \quad (7.14)$$

It is easy to see that if

$$\frac{a_{12} - a_{22}}{a_{11}a_{22} - a_{12}a_{21}}\varepsilon > 0, \quad \frac{a_{21} - a_{11}}{a_{11}a_{22} - a_{12}a_{21}}\varepsilon > 0,$$

the ODE system (7.14) has a unique positive equilibrium  $(\bar{u}, \bar{v})$  which is given by

$$\bar{u} = \frac{a_{12} - a_{22}}{a_{11}a_{22} - a_{12}a_{21}}\varepsilon, \quad \bar{v} = \frac{a_{21} - a_{11}}{a_{11}a_{22} - a_{12}a_{21}}\varepsilon.$$

Using the similar argument above, we find that  $(\bar{u}, \bar{v})$  is a stable equilibrium with respect to the ODE system (7.14) if  $a_{11} - a_{12} + a_{22} - a_{21} < 0$ ,  $\varepsilon > 0$  and

$$\frac{a_{12} - a_{22}}{a_{11}a_{22} - a_{12}a_{21}} > 0, \quad \frac{a_{21} - a_{11}}{a_{11}a_{22} - a_{12}a_{21}} > 0. \quad (7.15)$$

We proceed to examine Turing instability for the following system with cross-diffusion

$$\begin{cases} u_t = \frac{a_{11}u + a_{12}v + \varepsilon}{u + v}u + \beta_1(a_{11} - a_{12})\operatorname{div}\left(-\frac{uv}{(u+v)^2}\nabla u + \frac{u^2}{(u+v)^2}\nabla v\right), \\ v_t = \frac{a_{21}u + a_{22}v + \varepsilon}{u + v}v + \beta_2(a_{21} - a_{22})\operatorname{div}\left(-\frac{v^2}{(u+v)^2}\nabla u + \frac{uv}{(u+v)^2}\nabla v\right). \end{cases} \quad (7.16)$$

Linearizing the cross-diffusion system (7.16) about the positive equilibrium  $(\bar{u}, \bar{v})$  gives

$$\Phi_t = G\Phi + K\Delta\Phi,$$

where  $\Phi = (u - \bar{u}, v - \bar{v})^T$  and

$$G = \begin{pmatrix} \frac{(a_{11} - a_{12})\bar{v} - \varepsilon}{(\bar{u} + \bar{v})^2}\bar{u} & \frac{(a_{12} - a_{11})\bar{u} - \varepsilon}{(\bar{u} + \bar{v})^2}\bar{u} \\ \frac{(a_{21} - a_{22})\bar{v} - \varepsilon}{(\bar{u} + \bar{v})^2}\bar{v} & \frac{(a_{22} - a_{21})\bar{u} - \varepsilon}{(\bar{u} + \bar{v})^2}\bar{v} \end{pmatrix},$$

$$K = \begin{pmatrix} -\beta_1(a_{11} - a_{12})\frac{\bar{u}\bar{v}}{(\bar{u} + \bar{v})^2} & \beta_1(a_{11} - a_{12})\frac{\bar{u}^2}{(\bar{u} + \bar{v})^2} \\ -\beta_2(a_{21} - a_{22})\frac{\bar{v}^2}{(\bar{u} + \bar{v})^2} & \beta_2(a_{21} - a_{22})\frac{\bar{u}\bar{v}}{(\bar{u} + \bar{v})^2} \end{pmatrix}.$$

We obtain that the characteristic polynomial of  $G - \lambda_i K$  is

$$P_i(\mu) = \mu^2 - \operatorname{Trace}(G - \lambda_i K)\mu + \operatorname{Det}(G - \lambda_i K),$$

where

$$\begin{aligned}\text{Trace}(G - \lambda_i K) &= (a_{11} - a_{12} + a_{22} - a_{21}) \frac{\bar{u}\bar{v}}{(\bar{u} + \bar{v})^2} - \frac{\varepsilon}{\bar{u} + \bar{v}} \\ &\quad + \lambda_i [(a_{11} - a_{12})\beta_1 + (a_{22} - a_{21})\beta_2] \frac{\bar{u}\bar{v}}{(\bar{u} + \bar{v})^2}, \\ \text{Det}(G - \lambda_i K) &= \text{Det}K \lambda_i^2 - (a_{11} - a_{12} + a_{22} - a_{21}) \frac{\varepsilon \bar{u}\bar{v}}{(\bar{u} + \bar{v})^3} \lambda_i + \text{Det}G \\ &= -(a_{11} - a_{12} + a_{22} - a_{21}) \frac{\varepsilon \bar{u}\bar{v}}{(\bar{u} + \bar{v})^3} \lambda_i - (a_{11} - a_{12} + a_{22} - a_{21}) \frac{\varepsilon \bar{u}\bar{v}}{(\bar{u} + \bar{v})^3}.\end{aligned}$$

Notice that  $\text{Det}(G - \lambda_i K) > 0$ . If  $\text{Trace}(G - \lambda_i K) < 0$  for all  $\lambda_i > 0$ , then the two roots of  $P_i(\mu) = 0$  have negative real parts. To this end, we need the sufficient condition  $(a_{11} - a_{12})\beta_1 + (a_{22} - a_{21})\beta_2 \leq 0$ . In this case, a Turing instability does not occur, and the equilibrium  $(\bar{u}, \bar{v})$  is stable for (7.16).

In the other case,  $\text{Trace}(G - \lambda_i K) > 0$  for some  $\lambda_i$  if  $(a_{11} - a_{12})\beta_1 + (a_{22} - a_{21})\beta_2 > 0$ . Then both roots have positive real parts, the equilibrium  $(\bar{u}, \bar{v})$  is Turing unstable for (7.16). It is observed that the cross-diffusion effect is able to destabilize the positive equilibrium.

**Theorem 7.4.** *Let  $a_{11} - a_{12} + a_{22} - a_{21} < 0$ ,  $\varepsilon > 0$  and the condition (7.15) hold. Then if  $(a_{11} - a_{12})\beta_1 + (a_{22} - a_{21})\beta_2 \leq 0$ , the unique positive equilibrium  $(\bar{u}, \bar{v})$  is stable for (7.14) and (7.16). If  $(a_{11} - a_{12})\beta_1 + (a_{22} - a_{21})\beta_2 > 0$ , the unique positive equilibrium  $(\bar{u}, \bar{v})$  is unstable for (7.16), but is stable for (7.14).*

## 8. Conclusions

In this paper we have considered the dynamics of two populations, interacting via a symmetric game, who drive their migration by spatial gradients in the fitness function determined by the game payoffs. We have established existence and uniqueness results for strong solutions to the regularized fitness gradient system. However, it is still a very challenging problem to study the solutions of the original fitness gradient system, namely, the convergence of approximate solutions  $(u_\epsilon, v_\epsilon)$  as  $\epsilon$  tends to 0. Further, we have shown the occurrence of a Turing instability when growth rate terms are included. These equations represent in only an average, population-level way the variety of competitive interactions between organisms which comprise these populations [2,8], yet our mathematical analysis puts into evidence some of the intrinsic difficulties of this system. Whether these aspects, some of which have been seen in numerical simulations [6,7], represent possible occurrences in corresponding ecological systems, or whether they point directly to the ways in which the mathematical representation should be refined, remains unclear and is a subject for future studies.

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