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ENERGETIC VARIATIONAL APPROACHES FOR INCOMPRESSIBLE FLUID SYSTEMS ON AN EVOLVING SURFACE

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Abstract. This paper considers the equations governing incompressible fluid-flow on an evolving surface. We employ an energetic variational approach to derive the dynamical system for the motion of incompressible fluid on such an evolving surface. The focus is to understand the coupling of an incompressible fluid-flow and the evolution of a moving surface, involving both the curvature and the motion of the surface.

1. Introduction.

There has been a lot of interest in studying motions and dynamics on moving surfaces with applications such as those in geophysics and biology. We are concerned with mathematical derivations of the governing equations for the motion of incompressible fluid on an evolving surface. Although there may be several ways to derive such equations, here we apply our energetic variational approach for their derivation.

Let us first explain our setting. Let $\Gamma(t)$ be a surface in $\mathbb{R}^3$ depending on time $t \in [0, T)$ for some $T \in (0, \infty]$. Let $w = {}^t(w_1(x, t), w_2(x, t), w_3(x, t))$ be a given velocity field at a point $x = {}^t(x_1, x_2, x_3)$ of $\Gamma(t)$ which determines the velocity of $\Gamma(t)$. This velocity w may or may not be tangential to $\Gamma(t)$, in which cases $\Gamma(t)$ can change the shape. We

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consider a fluid with zero thickness moving on $\Gamma(t)$. Let $u = {}^t(u_1(x,t), u_2(x,t), u_3(x,t))$ be a relative velocity of a fluid particle at $x = {}^t(x_1, x_2, x_3)$. The velocity

$$v = v(x,t) = {}^t(v_1(x,t), v_2(x,t), v_3(x,t)) := u + w$$

is defined as the *total velocity* of the fluid particle at x. We confine ourselves to the cases where the relative velocity u is a tangential to $\Gamma(t)$ so that there is no exchange of particles between the surface and the environment. We often call w the *motion velocity (speed)* of the evolving surface and u a *surface flow (velocity)* on the evolving surface.

One of the goals in this paper is to derive the following evolution system for viscous incompressible fluid-flow on an evolving surface:

$$\begin{cases} D_t\rho = 0 & \text{on } \mathcal{S}_T, \\ \rho D_t v + \mathrm{grad}_\Gamma \sigma + \sigma H n = 2\mu_0 \mathrm{div}_\Gamma (P_\Gamma D(v) P_\Gamma) & \text{on } \mathcal{S}_T, \\ \mathrm{div}_\Gamma v = 0 & \text{on } \mathcal{S}_T. \end{cases} \tag{1.1}$$

Here

$$\mathcal{S}_T := \bigcup_{0<t<T} \{\Gamma(t) \times \{t\}\}.$$

The symbols $\rho = \rho(x,t)$ and μ_0 represent the density and the viscosity coefficient of the fluid on $\Gamma(t)$, respectively. The quantity $\sigma = \sigma(x,t)$ is a pressure associated with the incompressibility of the total velocity v, and the notation D_t denotes the material derivative, i.e., $D_t f = \partial_t f + (v, \nabla) f$. The operators grad_Γ and div_Γ denote surface gradient and surface divergence, respectively. The symbol $n = n(x,t) = {}^t(n_1, n_2, n_3)$ denotes the unit outer normal vector of $\Gamma(t)$, H denotes the mean curvature in the direction of n, and $P_\Gamma = P_\Gamma(x,t)$ denotes an orthogonal projection to the tangent space of $\Gamma(t)$ at x, which is orthogonal to n. The symbols are defined as $D(v) = ({}^t\nabla v + \nabla v)/2$, $\nabla = {}^t(\partial_1, \partial_2, \partial_3)$, and $\partial_i = \partial/\partial x_i$. We call the system (1.1) the *incompressible full Navier-Stokes system on an evolving surface* or the *incompressible full Navier-Stokes-Scriven-Koba (NSSK) system on an evolving surface* when $\mu_0 > 0$, and we call the system the *incompressible Euler system on an evolving surface* in the case of $\mu_0 = 0$.

Such systems have attracted many researchers over the years. Our results give the system (1.1) including the continuity of the momentum with the stress determined by the Boussinesq-Scriven law (Boussinesq [5], Scriven [16]):

$$S_\Gamma(v, \sigma) = 2\mu_0 P_\Gamma D(v) P_\Gamma - P_\Gamma \sigma.$$

Indeed, we can rewrite the system (1.1) as

$$\rho D_t v = \mathrm{div}_\Gamma S_\Gamma(v, \sigma).$$

In this paper, we would like to derive the system (1.1) from a unified energetic variational approach which had been studied by Rayleigh-Strutt [15] and Onsager [13,14]. Applying our variational methods depending on the variational spaces, we can derive several different types of systems of incompressible fluid-flow on a prescribed evolving surface (see

Section 4). For example, we apply our methods to derive the following system:

$$\begin{cases} D_t\rho = 0 & \text{on } \mathcal{S}_T, \\ P_\Gamma \rho\{\partial_t v + (v,\nabla)v\} + \mathrm{grad}_\Gamma \sigma = 2\mu_0 P_\Gamma \mathrm{div}_\Gamma(P_\Gamma D(v) P_\Gamma) & \text{on } \mathcal{S}_T, \\ \mathrm{div}_\Gamma v = 0 & \text{on } \mathcal{S}_T. \end{cases} \tag{1.2}$$

We call the system (1.2) the *tangential incompressible Navier-Stokes system on an evolving surface* or the *tangential incompressible NSSK system on an evolving surface* when $\mu_0 > 0$. Note that the incompressible Navier-Stokes system on a manifold introduced by Taylor [19] is different from the system (1.2) with $w \equiv 0$. On the other hand, (1.2) with $\mu_0 = 0$ agrees with the Euler system on a manifold derived by Arnol′d [2, 3] (see also Ebin and Marsden [7]). See Appendix (I) for the comparison of our systems with previous models.

Here is our subtle issue. In general, the system (1.1) is an overdetermined system for its initial value problem if the motion of $\Gamma(t)$ is given. In fact, in (1.1) there are three unknowns: tangential velocity (having essentially two unknowns) and the pressure. In the meantime there are four equations including incompressibility. We remark that the system (1.2) is not an overdetermined system for its initial value problem if the motion of $\Gamma(t)$ is given.

1.1. *Main results.* We shall state the main results of this paper. To derive equations from the variational principle we need to calculate the variation of the action integral with respect to the flow maps, as well as the variation of the dissipation energy with respect to the velocity. Of course we also need the continuity of the density.

Let $\{\Gamma(t)\}_{0\leq t<T}$ be a smoothly evolving surface in $\mathbb{R}^3$. Assume that $\Gamma(t)$ is a 2-dimensional closed Riemannian manifold for each $t \in [0,T)$.

We say that $\Omega(t) \subset \Gamma(t)$ is deformed as transported with or without domain by the velocity field $\widehat{v} = {}^t(\widehat{v}_1(x,t), \widehat{v}_2(x,t), \widehat{v}_3(x,t))$ if there exists a smooth function $x = {}^t(x_1(\xi,t), x_2(\xi,t), x_3(\xi,t))$ such that for $\xi \in \Gamma(0)$,

$$\begin{cases} \frac{dx}{dt}(\xi,t) = \widehat{v}(x(\xi,t),t), \quad t \in (0,T), \\ x|_{t=0} = \xi, \end{cases}$$

and

$$\Omega(t) = \{x = {}^t(x_1,x_2,x_3) \in \mathbb{R}^3;\ x = x(\xi,t),\ \xi \in \Omega_0,\ \Omega_0 \subset \Gamma(0)\}.$$

The mapping $\xi \mapsto x(\xi,t)$ is called a *flow map* on $\Gamma(t)$, while the mapping $t \mapsto x(\xi,t)$ is called an *orbit* starting from ξ.

Let $\rho = \rho(x,t)$ be a smooth function defined on $\mathcal{S}_T$. If $\Omega(t)$ is flowed by the total velocity v, then

$$\frac{d}{dt}\int_{\Omega(t)} \rho(x,t)\ d\mathcal{H}^2_x = \int_{\Omega(t)} \{D_t\rho + (\mathrm{div}_\Gamma v)\rho\}(x,t)\ d\mathcal{H}^2_x, \tag{1.3}$$

where $D_t\rho = \partial_t\rho + (v, \nabla)\rho$ and $d\mathcal{H}^2_x$ denotes the 2-dimensional Hausdorff measure. This is equivalent to

$$\frac{d}{dt}\int_{\Omega(t)} \rho(x,t)\ d\mathcal{H}^2_x = \int_{\Omega(t)} \{\partial_t\rho + (w\cdot n)(n,\nabla)\rho - H(w\cdot n)\rho\}(x,t)\ d\mathcal{H}^2_x$$

if $\Omega(t)$ has no boundary. This equality is often called a *Leibniz formula.*

If we use the Leibniz formula on the surface, we immediately obtain the continuity equation.

Theorem 1.1 (Continuity equation). Assume that a smooth function ρ fulfills

$$\frac{d}{dt}\int_{\Omega(t)} \rho(x,t)\ d\mathcal{H}^2_x = 0 \tag{1.4}$$

for $t \in (0,T)$ and for all $\Omega(t) \subset \Gamma(t)$ flowed by v. Then ρ satisfies the continuity equation

$$D_t\rho + (\mathrm{div}_\Gamma v)\rho = 0 \text{ on } \mathcal{S}_T. \tag{1.5}$$

Conversely, if ρ fulfills the continuity equation (1.5), then (1.4) holds for all $\Omega(t) \subset \Gamma(t)$ flowed by v.

If ρ is a constant, we have a necessary and sufficient condition for preserving area.

Theorem 1.2 (Area preserving property). The velocity v fulfills

$$\mathrm{div}_\Gamma v = 0 \text{ on } \mathcal{S}_T$$

if and only if

$$\frac{d}{dt}\int_{\Omega(t)} 1\ d\mathcal{H}^2_x = 0$$

for any $\Omega(t)$ flowed by v.

There is a chance that there is no incompressible velocity field v for w.

Theorem 1.3 (Necessary condition for the existence of incompressible fluid-flow). For a fixed $t \in [0,T)$ assume that

$$\mathrm{div}_\Gamma v = 0 \text{ on } \Gamma(t).$$

Then

$$\int_{\Gamma(t)} H(x,t)n(x,t)\cdot\{v(x,t) - w(x,t)\}\ d\mathcal{H}^2_x = 0$$

if and only if

$$\int_{\Gamma(t)} H(x,t)\{n(x,t)\cdot w(x,t)\}\ d\mathcal{H}^2_x = 0. \tag{1.6}$$

In particular, if $u = v - w$ is tangential, i.e. $(v-w)\cdot n = 0$, then (1.6) holds.

Remark 1.4. This is easy to prove. Since $\Gamma(t)$ is a closed surface, we use $\mathrm{div}_\Gamma v = 0$ and integration by parts (Lemma 2.4) to see that

$$\begin{aligned} 0 = \int_{\Gamma(t)} \mathrm{div}_\Gamma v\ d\mathcal{H}^2_x &= \int_{\Gamma(t)} \mathrm{div}_\Gamma (v-w)\ d\mathcal{H}^2_x + \int_{\Gamma(t)} \mathrm{div}_\Gamma w\ d\mathcal{H}^2_x \\ &= -\int_{\Gamma(t)} H\{n\cdot(v-w)\}\ d\mathcal{H}^2_x - \int_{\Gamma(t)} H(n\cdot w)\ d\mathcal{H}^2_x. \end{aligned}$$

Therefore the restriction (1.6) is a necessary condition for the existence of incompressible fluid-flow u on an evolving surface. For example, if w is a constant vector and $\Gamma(t)$ is symmetric with respect to some plane orthogonal to w, then such a motion $\Gamma(t)$ satisfies (1.6).

In order to derive the momentum equation, we now discuss the variation of the flow map to the action integral. Let $x(\xi,t)$ be a flow map on $\Gamma(t)$, and let v be the total velocity determined by the flow map $x(\xi,t)$ on $\Gamma(t)$. We would like to allow variation of $\Gamma(t)$ itself. For this purpose we consider a general flow map $\widehat{x}(\xi,t)$ on another evolving surface and the velocity $\widehat{v}$ determined by the flow map $\widehat{x}$, i.e. for $\xi \in \Gamma(0)$ and $0 < t < T$,

$$\begin{cases} \widehat{v} = {}^t(\widehat{v}_1(x,t), \widehat{v}_2(x,t), \widehat{v}_3(x,t)), \\ \widehat{x} = {}^t(\widehat{x}_1(\xi,t), \widehat{x}_2(\xi,t), \widehat{x}_3(\xi,t)), \\ \frac{d\widehat{x}}{dt} = \widehat{v}(\widehat{x}(\xi,t),t), \\ \widehat{x}(\xi,0) = \xi. \end{cases}$$

For each variation $\widehat{x}$ we define the action integral as

$$A[\widehat{x}] = \int_0^T \int_{\widehat{\Gamma}(t)} \frac{1}{2}\widehat{\rho}(x,t)|\widehat{v}(x,t)|^2 \, d\mathcal{H}^2_x dt.$$

Here

$$\widehat{\Gamma}(t) := \{x = {}^t(x_1,x_2,x_3) \in \mathbb{R}^3;\ x = \widehat{x}(\xi,t),\ \xi \in \Gamma(0)\}$$

and $\widehat{\rho}$ satisfies

$$\partial_t \widehat{\rho} + (\widehat{v},\nabla)\widehat{\rho} + (\operatorname{div}_{\widehat{\Gamma}}\widehat{v})\widehat{\rho} = 0 \text{ on } \widehat{\mathcal{S}}_T.$$

Here

$$\widehat{\mathcal{S}}_T = \bigcup_{0<t<T} \{\widehat{\Gamma}(t) \times \{t\}\}.$$

For $-1 < \varepsilon < 1$, let us consider a *variation* $(x^\varepsilon(\xi,t), \mathcal{S}^\varepsilon_T)$ of $(x(\xi,t), \mathcal{S}_T)$ with $\Gamma^\varepsilon(0) = \Gamma(0)$, where

$$\mathcal{S}^\varepsilon_T := \bigcup_{0<t<T} \{\Gamma^\varepsilon(t) \times \{t\}\}.$$

Here $\Gamma^\varepsilon(t)$ is an evolving surface. We say that $(x^\varepsilon(\xi,t), \mathcal{S}^\varepsilon_T)$ is a variation of a smooth $(x(\xi,t), \mathcal{S}_T)$ if $x^\varepsilon(\xi,t)$ is smooth as a function of $(\varepsilon,\xi,t) \in (-1,1)\times\Gamma(0)\times[0,T)$ and $x^\varepsilon(\xi,t)|_{\varepsilon=0} = x(\xi,t)$.

We now assume that there are $y \in [C_0^\infty(\mathbb{R}^3 \times [0,T))]^3$ and $z \in [C^\infty(\mathcal{S}_T)]^3$ such that for $\xi \in \Gamma(0)$ and $0 \le t < T$,

$$\begin{aligned} &x^\varepsilon(\xi,t)\Big|_{\varepsilon=0} = x(\xi,t), \\ &v^\varepsilon(x^\varepsilon(\xi,t),t)\Big|_{\varepsilon=0} = v(x(\xi,t),t), \\ &\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} x^\varepsilon(\xi,t) = y(\xi,t), \\ &z(x(\xi,t),t) = y(\xi,t). \end{aligned}$$

Here z is the variation vector field described by the Eulerian coordinates. Suppose that ρ and ρ^ε satisfy

$$\begin{cases} \partial_t \rho + (v, \nabla)\rho + (\mathrm{div}_\Gamma v)\rho = 0 \text{ on } \mathcal{S}_T, \\ \rho|_{t=0} = \rho_0 \end{cases} \tag{1.7}$$

and

$$\begin{cases} \partial_t \rho^\varepsilon + (v^\varepsilon, \nabla)\rho^\varepsilon + (\mathrm{div}_{\Gamma^\varepsilon} v^\varepsilon)\rho^\varepsilon = 0 \text{ on } \mathcal{S}_T^\varepsilon, \\ \rho^\varepsilon|_{t=0} = \rho_0 \end{cases} \tag{1.8}$$

for some $\rho_0 \in C(\Gamma(0))$. Moreover, we assume that for $\xi \in \Gamma(0)$ and $0 \leq t < T$,

$$\rho^\varepsilon(x^\varepsilon(\xi, t), t)|_{\varepsilon=0} = \rho(x(\xi, t), t).$$

THEOREM 1.5 (Variation of the flow map to the action integral). Let $x(\xi, t)$ be a flow map on $\Gamma(t)$, and let v be the total velocity of the fluid-flow determined by the flow map $x(\xi, t)$. Let $(x^\varepsilon(\xi, t), \mathcal{S}_T^\varepsilon)$ be a variation of $(x(\xi, t), \mathcal{S}_T)$ with $\Gamma^\varepsilon(0) = \Gamma(0)$. Assume that ρ and ρ^ε satisfy the systems (1.7) and (1.8) for some $\rho_0 \in C(\Gamma(0))$. Then

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} A[x^\varepsilon] = -\int_0^T \int_{\Gamma(t)} \rho(x, t) D_t v(x, t) \cdot z(x, t) \; d\mathcal{H}_x^2 dt.$$

(i) For every $z \in [C_0^\infty(\mathcal{S}_T)]^3$ satisfying $\mathrm{div}_\Gamma z = 0$ on $\mathcal{S}_T$, assume that

$$-\int_0^T \int_{\Gamma(t)} \rho(x, t) D_t v(x, t) \cdot z(x, t) \; d\mathcal{H}_x^2 dt = 0.$$

Then v solves

$$\rho D_t v + \mathrm{grad}_\Gamma \sigma + \sigma H n = 0 \quad \text{on } \mathcal{S}_T$$

with some $\sigma \in C^{1,0}(\mathcal{S}_T)$.

(ii) For every $z \in [C_0^\infty(\mathcal{S}_T)]^3$ satisfying $\mathrm{div}_\Gamma z = 0$ and $z \cdot n = 0$ on $\mathcal{S}_T$, assume that

$$-\int_0^T \int_{\Gamma(t)} \rho(x, t) D_t v(x, t) \cdot z(x, t) \; d\mathcal{H}_x^2 dt = 0.$$

Then v solves

$$P_\Gamma \rho D_t v + \mathrm{grad}_\Gamma \sigma = 0 \quad \text{on } \mathcal{S}_T$$

with some $\sigma \in C^{1,0}(\mathcal{S}_T)$.

In (i) variation is with respect to the total flow map, including the motion of $\Gamma(t)$. In (ii) variation is with respect to only the tangential part of the total velocity on $\Gamma(t)$. See Lemma 2.7 for the pressure term of the incompressible fluid on an evolving surface.

We now define the dissipation energy $E[\widehat{v}]$ for the velocity field

$$\widehat{v} = {}^t(\widehat{v}_1(x, t), \widehat{v}_2(x, t), \widehat{v}_3(x, t))$$

at each fixed time t. For fixed t, let μ_0 be a positive constant and

$$E[\widehat{v}] := -\int_{\Gamma(t)} \mu_0 |P_\Gamma(x, t) D(\widehat{v}(x, t)) P_\Gamma(x, t)|^2 \; d\mathcal{H}_x^2.$$

Here

$$[D(\widehat{v})]_{ij} := \frac{1}{2}\left(\frac{\partial \widehat{v}_i}{\partial x_j} + \frac{\partial \widehat{v}_j}{\partial x_i}\right).$$

See Subsection 2.2 for the notation $[\ \cdot\ ]_{ij}$. We shall study its variation.

THEOREM 1.6 (Variation of dissipation energy). Fix $t \in (0,T)$. For every vector field $\varphi \in [C_0^\infty(\Gamma(t))]^3$ the direction derivation of E at v is of the form

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} E[v+\varepsilon\varphi] = \int_{\Gamma(t)} 2\mu_0 \mathrm{div}_\Gamma (P_\Gamma(x,t) D(v(x,t)) P_\Gamma(x,t)) \cdot \varphi(x)\ d\mathcal{H}_x^2.$$

(i) If

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} E[v+\varepsilon\varphi] = 0$$

for all $\varphi \in [C_0^\infty(\Gamma(t))]^3$ satisfying $\mathrm{div}_\Gamma \varphi = 0$ on $\Gamma(t)$, then v fulfills

$$-2\mu_0 \mathrm{div}_\Gamma(P_\Gamma D(v) P_\Gamma) + \mathrm{grad}_\Gamma \sigma + \sigma H n = 0 \text{ on } \Gamma(t)$$

for some $\sigma \in C^1(\Gamma(t))$.

(ii) If

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} E[v+\varepsilon\varphi] = 0$$

for all $\varphi \in C_0^\infty(\Gamma(t))$ satisfying $\mathrm{div}_\Gamma \varphi = 0$ and $\varphi \cdot n = 0$ on $\Gamma(t)$, then v fulfills

$$-2\mu_0 P_\Gamma \mathrm{div}_\Gamma(P_\Gamma D(v) P_\Gamma) + \mathrm{grad}_\Gamma \sigma = 0 \text{ on } \Gamma(t)$$

for some $\sigma \in C^1(\Gamma(t))$.

Applying Theorems 1.1–1.6, we obtain several incompressible fluid systems on an evolving surface. See Section 4 for details.

There are three subtle issues (difficulties), sometimes confusing, in the derivation of incompressible fluid systems on an evolving surface.

• The first one is to characterize the incompressibility of the fluid on the prescribed evolving surface which follows from a continuity equation for the evolving surface.

• The second one is to calculate the variation of the action integral. This is a domain variation and there are two ways of variation: one is variation with respect to all directions while the other is variation only in the tangential direction. Since the surface is moving, one needs to use a Riemannian metric expression for all the computations.

• The third difficulty is to derive a viscous term which is obtained as the variation of the dissipation energy with respect to the total velocity. Here again we use Riemannian metric interpretation to proceed with the calculation. The resulting equation follows from the identity that the variation of the action integral with respect to the flow map agrees with a constant multiple of velocity variation of the dissipation energy. Actually, it is easy to say that the real calculation is quite involved.

We next explain some mathematical derivations of the incompressible fluid system on a manifold and surface. Arnol′d [2,3] applied the Lie group of diffeomorphisms to derive the Euler system on a manifold. Taylor [19] introduced the incompressible Navier-Stokes system on a manifold from their physical sense (see also Taylor's book [20]). Mitsumatsu and Yano [12] used their energetic approach to derive the incompressible Navier-Stokes system on a manifold. Arnaudon and Cruzeiro [1] applied the stochastic variational approach to derive the incompressible Navier-Stokes system on a manifold. We remark that the system derived by [12] agrees with one introduced by [19].

Let us state some related papers and books to fluid-flow on an evolving surface. Dziuk and Elliott [6] derived several fluid systems on an evolving surface by applying the Leibniz formula on an evolving surface and their diffusive flux. Bothe and Prüss [4] used the Boussinesq-Scriven law to make a model for the two-phase fluid flow with surface tension and surface viscosity. Koba [11] derived compressible fluid systems on an evolving surface by his energetic variational approach and thermodynamical theory and gave a mathematical justification of the Boussinesq-Scriven law. For Boussinesq-Scriven surface fluid, we refer the reader to Slattery's book [18]. Remark that the systems in [6] are different from our systems.

Finally we state the outline of the paper. In Section 2 we introduce evolving surfaces and function spaces, and study calculus on an evolving surface. In Section 3 we study incompressible fluid-flow on an evolving surface. We first consider the continuity equation for fluid on the evolving surface. Secondly we investigate the existence for incompressible fluid-flow on the evolving surface. Thirdly we use our action integral to derive the Euler system on the evolving surface. Finally, we study the dissipation energy and viscous terms of the system (1.1). In Section 4 we present various incompressible fluid systems on an evolving surface. In Appendix (I) we compare our systems with previous models such as the Euler system on a manifold and the Navier-Stokes system on a manifold. In Appendix (II) we discuss the energy law and work of the fluid in a moving domain.

2. Preliminaries.

In this section, we first introduce evolving surfaces and functions on an evolving surface. Secondly, we state convention and notation used in this paper. Especially, we define notation such as surface gradient grad_Γ, surface divergence div_Γ, mean curvature H, and an orthogonal projection to a tangent space P_Γ. Thirdly, we study integration by parts on a surface, and we give an important tool to derive a pressure of the incompressible fluid on a surface. Fourthly, we describe flow maps on an evolving surface and a variation of the flow map. Finally we use Riemannian metrics on a surface to characterize an orthogonal projection P_Γ and differential operators ∂_i^{tan}.

2.1. *Evolving surfaces and function spaces.* We first introduce 2-dimensional C^2-surfaces in $\mathbb{R}^3$ and evolving 2-dimensional $C^{2,1}$-surfaces in $\mathbb{R}^3$.

Definition 2.1 (2-dimensional C^2-surfaces in $\mathbb{R}^3$). A set Γ_0 in $\mathbb{R}^3$ is called a C^2-surface in $\mathbb{R}^3$ if for each point $x_0 \in \Gamma_0$ there are $r > 0$ and $\phi \in C^2(B_r(x_0))$ such that

$$\Gamma_0 \cap B_r(x_0) = \{x = {}^t(x_1, x_2, x_3) \in B_r(x_0);\ \phi(x) = 0\}$$

and that

$$\nabla_x \phi = {}^t\left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, \frac{\partial \phi}{\partial x_3}\right) \neq (0,0,0) \text{ on } B_r(x_0).$$

Here

$$B_r(x_0) := \{x \in \mathbb{R}^3;\ |x - x_0| < r\}.$$

In this paper we call a 2-dimensional C^2-surface in $\mathbb{R}^3$ *a 2-dimensional surface in* $\mathbb{R}^3$. Note that Γ_0 may not be a closed surface. It may have a geometric boundary $\partial\Gamma_0$. It may not be bounded. We also recall a definition of an evolving surface [8].

DEFINITION 2.2 (Evolving 2-dimensional $C^{2,1}$-surfaces in $\mathbb{R}^3$). Let $T \in (0,\infty]$. Set $I = [0,T)$. Suppose that $\Gamma(t)$ is a set in $\mathbb{R}^3$ for each $t \in I$. A family $\{\Gamma(t)\}_{t\in I}$ is called an evolving 2-dimensional $C^{2,1}$-surface in $\mathbb{R}^3$ on I if the following two properties hold:

(i) $\Gamma(0)$ is a 2-dimensional surface in $\mathbb{R}^3$.

(ii) For each $t_0 \in (0,T)$ and $x_0 \in \Gamma(t_0)$ there are $r_1, r_2 > 0$ and $\psi \in C^{2,1}(B_{r_1}(x_0) \times B_{r_2}(t_0))$ such that

$$\Gamma(t_0) \cap B_{r_1}(x_0) = \{x = {}^t(x_1,x_2,x_3) \in B_{r_1}(x_0);\ \psi(x,t_0) = 0\}$$

and that

$$\nabla_x \psi = {}^t\left(\frac{\partial\psi}{\partial x_1}, \frac{\partial\psi}{\partial x_2}, \frac{\partial\psi}{\partial x_3}\right) \neq (0,0,0) \ \text{ on } B_{r_1}(x_0) \times B_{r_2}(t_0).$$

Here

$$B_{r_1}(x_0) := \{x \in \mathbb{R}^3;\ |x - x_0| < r_1\},$$
$$B_{r_2}(t_0) := \{t \in \mathbb{R}_+;\ |t - t_0| < r_2\},$$

$$C^{2,1}(B_{r_1}(x_0) \times B_{r_2}(t_0)) := \{f \in C(B_{r_1}(x_0) \times B_{r_2}(t_0));$$
$$\partial_i f, \partial_j\partial_i f, \partial_t f, \partial_i\partial_t f, \partial_j\partial_i\partial_t f \in C(B_{r_1}(x_0) \times B_{r_2}(t_0)) \text{ for each } i,j = 1,2,3\}.$$

Throughout this paper we write $\Gamma(t)$ instead of $\{\Gamma(t)\}_{t\in I}$. As in Definition 2.1, we often suppress the word $C^{2,1}$.

Next we define functions on an evolving surface. Let Γ_0 be a 2-dimensional surface in $\mathbb{R}^3$, and let $\Gamma(t)$ be an evolving 2-dimensional $C^{2,1}$-surface in $\mathbb{R}^3$ on $[0,T)$ for some $T \in (0,\infty]$. Set

$$\mathcal{S}_T \equiv \mathcal{S}_{T,\Gamma(t)} := \left\{(x,t) = {}^t(x_1,x_2,x_3,t) \in \mathbb{R}^4;\ (x,t) \in \bigcup_{0<t<T} \{\Gamma(t) \times \{t\}\}\right\}.$$

For each $m \in \mathbb{N} \cup \{0,\infty\}$ we define

$$C^m(\Gamma_0) := \{f : \Gamma_0 \to \mathbb{R};\ g|_{\Gamma_0} = f \text{ for some } g \in C^m(\mathbb{R}^3)\},$$
$$C_0^m(\Gamma_0) := \{f \in C^m(\Gamma_0);\ \text{supp} f \text{ does not intersect the geometric boundary of } \Gamma_0\},$$
$$C(\mathcal{S}_T) := \{f : \mathcal{S}_T \to \mathbb{R};\ g|_{\mathcal{S}_T} = f \text{ for some } g \in C(\mathbb{R}^3 \times \mathbb{R})\},$$
$$C_0(\mathcal{S}_T) := \{f \in C(\mathcal{S}_T);\ \text{supp} f \text{ is included in } \mathcal{S}_T \text{ and}$$
$$\text{supp} f(\cdot,t) \text{ does not intersect the geometric boundary of } \Gamma(t)\}.$$

Moreover, we write

$$C^{1,0}(\mathcal{S}_T) := \{f \in C(\mathcal{S}_T);\ \partial_i f \in C(\mathcal{S}_T) \text{ for each } i = 1,2,3\},$$
$$C^{2,1}(\mathcal{S}_T) := \{f \in C^{1,0}(\mathcal{S}_T);\ \partial_j\partial_i f, \partial_t f, \partial_i\partial_t f, \partial_j\partial_i\partial_t f \in C(\mathcal{S}_T) \text{ for each } i,j = 1,2,3\},$$
$$C_0^{2,1}(\mathcal{S}_T) := C^{2,1}(\mathcal{S}_T) \cap C_0(\mathcal{S}_T),\ C^\infty(\mathcal{S}_T) := C^\infty(\mathbb{R}^4) \cap C(\mathcal{S}_T),$$
$$\text{and } C_0^\infty(\mathcal{S}_T) := C(\mathbb{R}^4) \cap C_0(\mathcal{S}_T).$$

2.2. *Convention and notation.* Let us explain some conventions used in this paper. We use italic characters $i, j, k, \ell, i'j'$ as $1, 2, 3$, and use Greek characters $\alpha, \beta, \zeta, \eta, \alpha', \beta'$ as $1, 2$. Moreover, we use the following Einstein summation convention:

$$a_{ij}b_j = \sum_{j=1}^{3} a_{ij}b_j,\ a_{ij}b_{ij\ell} = \sum_{i,j=1}^{3} a_{ij}b_{ij\ell},\ a_{ij}b_{i\alpha}c_{\alpha\beta} = \sum_{i=1}^{3}\sum_{\alpha=1}^{2} a_{ij}b_{i\alpha}c_{\alpha\beta}.$$

Let $\mathcal{X}$ be a set. The symbol $M_{p\times q}(\mathcal{X})$ denotes the set of all $p \times q$ matrices whose component belongs to $\mathcal{X}$; that is, $M \in M_{p\times q}(\mathcal{X})$ if and only if

$$M = \begin{pmatrix} [M]_{11} & [M]_{12} & \cdots & [M]_{1q} \\ [M]_{21} & [M]_{22} & \cdots & [M]_{2q} \\ \vdots & \vdots & & \vdots \\ [M]_{p1} & [M]_{p2} & \cdots & [M]_{pq} \end{pmatrix},$$

and $[M]_{\mathfrak{i}\mathfrak{j}} \in \mathcal{X}$ ($\mathfrak{i} = 1, 2, \dots, p,\ \mathfrak{j} = 1, 2, \dots, q$), where $[M]_{\mathfrak{i}\mathfrak{j}}$ denotes the $(\mathfrak{i}, \mathfrak{j})$-th component of the matrix M.

Let $\Gamma(t)$ be an evolving 2-dimensional $C^{2,1}$-surface in $\mathbb{R}^3$ on $[0, T)$ for some $T \in (0, \infty]$. By $n = n(x_0, t_0) = {}^t(n_1, n_2, n_3)$ we mean the unit outer normal vector of $\Gamma(t_0)$ at $x_0 \in \Gamma(t_0)$ for each fixed $t_0 \in [0, T)$. In this paper, we use the following notation:

$$\begin{aligned} \partial_i^{tan} &:= (\delta_{ij} - n_i n_j)\partial_j \quad \left(= \sum_{j=1}^{3} (\delta_{ij} - n_i n_j)\partial_j \right), \\ \nabla^{tan} &:= {}^t(\partial_1^{tan}, \partial_2^{tan}, \partial_3^{tan}), \\ \Delta^{tan} &:= (\partial_1^{tan})^2 + (\partial_2^{tan})^2 + (\partial_3^{tan})^2. \end{aligned}$$

Here δ_{ij} is Kronecker's delta. Moreover, for $f = {}^t(f_1, f_2, f_3) \in [C^{1,0}(\mathcal{S}_T)]^3$ and $g \in C^{2,0}(\mathcal{S}_T)$,

$$\begin{aligned} \mathrm{div}_\Gamma f &:= \partial_1^{tan} f_1 + \partial_2^{tan} f_2 + \partial_3^{tan} f_3, \\ \mathrm{grad}_\Gamma g &:= \nabla^{tan} g, \\ \Delta_\Gamma g &:= \Delta^{tan} g. \end{aligned}$$

Let H and P_Γ be the mean curvature and the orthogonal projection to a tangent space defined by

$$H = H(x, t) := -\mathrm{div}_\Gamma n, \tag{2.1}$$

$$[P_\Gamma]_{ij} = [P_\Gamma(x, t)]_{ij} := \delta_{ij} - n_i n_j \ (i, j = 1, 2, 3), \tag{2.2}$$

respectively. Note that $P_\Gamma = I - n \otimes n$ and that $n_1^2 + n_2^2 + n_3^2 = 1$.

2.3. *Calculus on surfaces.* Let Γ_0 be a 2-dimensional surface in $\mathbb{R}^3$, and let $n = n(x) = {}^t(n_1, n_2, n_3)$ be its unit outer normal vector at $x \in \Gamma_0$. Let H and P_Γ be the mean curvature and orthogonal projection to tangent defined by (2.1) and (2.2), respectively.

Let us first study the relation between H and P_Γ.

LEMMA 2.3. Assume that $\sigma \in C^1(\Gamma_0)$. Then

$$\mathrm{div}_\Gamma(\sigma P_\Gamma) = \mathrm{grad}_\Gamma \sigma + \sigma H n. \tag{2.3}$$

Proof of Lemma 2.3. Fix $j \in \{1,2,3\}$. A direct calculation gives

$$\begin{aligned}\partial_i^{tan}(\sigma[P_\Gamma]_{ij}) =&(\partial_i^{tan}\sigma)(\delta_{ij} - n_i n_j) + \sigma(\partial_i^{tan}[P_\Gamma]_{ij})\\ =&\partial_j^{tan}\sigma + \sigma(\partial_i^{tan}[P_\Gamma]_{ij}).\end{aligned} \tag{2.4}$$

It is easy to check that

$$\begin{aligned}\partial_i^{tan}[P_\Gamma]_{ij} =&(\delta_{ik} - n_i n_k)\partial_k(\delta_{ij} - n_i n_j)\\ =&-(\delta_{ik} - n_i n_k)(\partial_k n_i)n_j - (\delta_{ik} - n_i n_k)n_i(\partial_k n_j)\\ =&-(\delta_{ik} - n_i n_k)(\partial_k n_i)n_j.\end{aligned} \tag{2.5}$$

By definition, we observe that

$$\begin{aligned}H =& -\partial_i^{tan} n_i\\ =& -(\delta_{ik} - n_i n_k)\partial_k n_i.\end{aligned} \tag{2.6}$$

Combining (2.4)-(2.6), we obtain

$$\partial_i^{tan}(\sigma[P_\Gamma]_{ij}) = \partial_j^{tan}\sigma + \sigma H n_j.$$

Therefore we see (2.3). □

Next we state one useful lemma to deal with integration by parts on a surface.

LEMMA 2.4 (Integration by parts). Let $f = {}^t(f_1, f_2, f_3) \in [C^1(\Gamma_0)]^3$. Then

$$-\int_{\partial\Gamma_0} \nu \cdot f \; d\mathcal{H}_x^1 = \int_{\Gamma_0} \mathrm{div}_\Gamma f \; d\mathcal{H}_x^2 + \int_{\Gamma_0} Hn \cdot f \; d\mathcal{H}_x^2.$$

In particular,

$$\begin{aligned}\int_{\Gamma_0} \mathrm{div}_\Gamma f \; d\mathcal{H}_x^2 &= -\int_{\partial\Gamma_0} \nu \cdot f \; d\mathcal{H}_x^1 \text{ when } f \cdot n = 0,\\ \int_{\Gamma_0} \mathrm{div}_\Gamma f \; d\mathcal{H}_x^2 &= -\int_{\Gamma_0} Hn \cdot f \; d\mathcal{H}_x^2 \text{ when } f \in [C_0^1(\Gamma_0)]^3.\end{aligned} \tag{2.7}$$

Here ν is the inward pointing unit co-normal of $\partial\Gamma_0$; that is, $|\nu| = 1$, where ν is normal to $\partial\Gamma_0$ and tangent to Γ_0.

The proof of Lemma 2.4 is found for example in Simon's book [17].

LEMMA 2.5. Assume that Γ_0 is a closed manifold. Then for every $f = {}^t(f_1, f_2, f_3) \in [C^1(\Gamma_0)]^3$,

$$0 = \int_{\Gamma_0} \mathrm{div}_\Gamma f \; d\mathcal{H}_x^2 + \int_{\Gamma_0} Hn \cdot f \; d\mathcal{H}_x^2.$$

Proof of Lemma 2.5. Fix $x_0 \in \Gamma_0$. We choose $r > 0$ sufficiently small such that $\Gamma_0 \setminus B_r(x_0) \subset \Gamma_0$. Using Lemma 2.4, we have

$$-\int_{\partial(\Gamma_0 \setminus B_r(x_0))} \nu \cdot f \; d\mathcal{H}_x^1 = \int_{\Gamma_0 \setminus B_r(x_0)} \mathrm{div}_\Gamma f \; d\mathcal{H}_x^2 + \int_{\Gamma_0 \setminus B_r(x_0)} Hn \cdot f \; d\mathcal{H}_x^2.$$

Since $|\nu| = 1$ and $f \in [C^1(\Gamma_0)]^3$, we let $r \to +0$ to get the desired result. □

LEMMA 2.6 (Integration by parts). For each $f \in C^1(\Gamma_0)$, $g \in C_0^1(\Gamma_0)$, and $m \in \{1,2,3\}$,

$$\int_{\Gamma_0} (\partial_m^{tan} f) g \; d\mathcal{H}_x^2 = - \int_{\Gamma_0} f(\partial_m^{tan} g) \; d\mathcal{H}_x^2 - \int_{\Gamma_0} H n_m f g \; d\mathcal{H}_x^2. \tag{2.8}$$

Proof of Lemma 2.6. We give the proof only for the case when $m = 1$ since other cases are similar. Set $h = {}^t(h_1, h_2, h_3) := {}^t(fg, 0, 0)$. It is easy to check that

$$\begin{aligned} \mathrm{div}_\Gamma h =& \partial_1^{tan} h_1 \\ =& (\partial_1^{tan} f) g + f(\partial_1^{tan} g). \end{aligned}$$

By (2.7), we observe that

$$\begin{aligned} \int_{\Gamma_0} (g \partial_1^{tan} f + f \partial_1^{tan} g) \; d\mathcal{H}_x^2 =& \int_{\Gamma_0} \mathrm{div}_\Gamma h \; d\mathcal{H}_x^2 \\ =& - \int_{\Gamma_0} H n \cdot h \; d\mathcal{H}_x^2 \\ =& - \int_{\Gamma_0} H n_1 f g \; d\mathcal{H}_x^2. \end{aligned}$$

Therefore the lemma follows. □

The following lemma is important to derive the pressure of the incompressible fluid on an evolving surface.

LEMMA 2.7. Set

$$E := \left\{ f \in [L^2(\Gamma_0)]^3 ; \int_{\Gamma_0} f \cdot \varphi \; d\mathcal{H}_x^2 = 0 \text{ for all } \varphi \in [C_0^\infty(\Gamma_0)]^3 \text{ with } \mathrm{div}_\Gamma \varphi = 0 \right\}.$$

Then $f \in E$ if and only if there is $\mathfrak{p} \in W^{1,2}(\Gamma_0)$ such that

$$f = \nabla^{tan} \mathfrak{p} + \mathfrak{p} H n.$$

Moreover, if f is continuous, then $\mathfrak{p} \in C^1(\Gamma_0)$.

Proof of Lemma 2.7. We first show the necessary condition $\Leftarrow$). Fix $\varphi \in [C_0^\infty(\Gamma_0)]^3$ with $\mathrm{div}_\Gamma \varphi = 0$. From (2.7), we see that

$$\int_{\Gamma_0} \mathrm{div}_\Gamma (\mathfrak{p} \varphi) \; d\mathcal{H}_x^2 = - \int_{\Gamma_0} H n \cdot (\mathfrak{p} \varphi) \; d\mathcal{H}_x^2.$$

Since

$$\mathrm{div}_\Gamma (\mathfrak{p} \varphi) = (\nabla^{tan} \mathfrak{p}) \cdot \varphi + \mathfrak{p} \mathrm{div}_\Gamma \varphi,$$

we have

$$\int_{\Gamma_0} (\nabla^{tan} \mathfrak{p}) \cdot \varphi \; d\mathcal{H}_x^2 = - \int_{\Gamma_0} H n \cdot (\mathfrak{p} \varphi) \; d\mathcal{H}_x^2.$$

Next we prove the sufficient condition $\Rightarrow$). We assume that

$$\int_{\Gamma_0} f_{tan} \cdot \varphi_{tan} \; d\mathcal{H}_x^2 = 0 \text{ for all } \varphi \in [C_0^\infty(\Gamma_0)]^3 \text{ with } \mathrm{div}_\Gamma \varphi_{tan} = 0.$$

Here $f_{tan} := P_\Gamma f$ and $\varphi_{tan} := P_\Gamma \varphi$. Note that $f = f_{tan} + (f \cdot n) n$. We easily check that for every circle $\mathcal{C}$ in Γ_0,

$$\int_{\mathcal{C}} f_{tan} \; d\mathcal{H}_x^2 = 0.$$

From Weyl's Theorem, there is a $\mathfrak{p} \in W^{1,2}(\Gamma_0)$ such that $f_{tan} = \nabla^{tan}\mathfrak{p}$. Therefore, we have

$$\int_{\Gamma_0} f \cdot \varphi \, d\mathcal{H}_x^2 = \int_{\Gamma_0} \nabla^{tan}\mathfrak{p} \cdot \varphi \, d\mathcal{H}_x^2 + \int_{\Gamma_0} (f \cdot n)n \cdot \varphi \, d\mathcal{H}_x^2.$$

On the other hand,

$$\begin{aligned} \int_{\Gamma_0} (\nabla^{tan}\mathfrak{p})\varphi \, d\mathcal{H}_x^2 &= \int_{\Gamma_0} \operatorname{div}_\Gamma(\mathfrak{p}\varphi) \, d\mathcal{H}_x^2 \\ &= -\int_{\Gamma_0} Hn \cdot (\mathfrak{p}\varphi) \, d\mathcal{H}_x^2. \end{aligned}$$

Thus, we check that that

$$\begin{aligned} 0 &= \int_{\Gamma_0} f \cdot \varphi \, d\mathcal{H}_x^2 \\ &= -\int_{\Gamma_0} Hn \cdot (\mathfrak{p}\varphi) \, d\mathcal{H}_x^2 + \int_{\Gamma_0} (f \cdot n)n \cdot \varphi \, d\mathcal{H}_x^2. \end{aligned}$$

This implies that $f \cdot n = \mathfrak{p}H$. Therefore, we conclude that $f = \nabla^{tan}\mathfrak{p} + \mathfrak{p}Hn$. Moreover, we see that $\mathfrak{p} \in C^1(\Gamma_0)$ when f is continuous since Γ_0 is a smooth surface. □

2.4. *Flow maps on evolving surfaces.* In this section we consider flow maps on evolving surfaces. In what follows we only consider closed surfaces for simplicity. We shall introduce an evolving surface $\Gamma(t)$ and a flow map on $\Gamma(t)$. Then we study surface area by applying such flow maps.

DEFINITION 2.8 (Evolving surface). Let $\{\Gamma(t)\}_{0 \leq t < T}$ be a given evolving 2-dimensional $C^{2,1}$-surface in $\mathbb{R}^3$ on $[0, T)$ for some $T \in (0, \infty]$. We simply call $\Gamma(t)$ a 2-*dimensional evolving surface in* $\mathbb{R}^3$ if for each fixed $t \in [0, T)$, $\Gamma(t)$ is a Riemannian 2-dimensional manifold.

DEFINITION 2.9 (Flow map on an evolving surface). Let $\Gamma(t)$ be a given evolving 2-dimensional surface in $\mathbb{R}^3$ on $[0, T)$ for some $T \in (0, \infty]$. Let $x \in [C^\infty(\mathbb{R}^4)]^3$. We call x a *flow map* on $\Gamma(t)$ if the three properties hold:

(i) for $\xi \in \Gamma_0 := \Gamma(0)$

$$x(\xi, 0) = \xi,$$

(ii) for $\xi \in \Gamma_0$ and $0 \leq t < T$

$$x(\xi, t) \in \Gamma(t),$$

(iii) for $0 \leq t < T$

$$x(\cdot, t) : \Gamma_0 \to \Gamma(t) \text{ is bijective.}$$

Let $\Gamma(t)$ be a given evolving 2-dimensional surface in $\mathbb{R}^3$ on $[0, T)$ for some $T \in (0, \infty]$. Let x be a flow map on $\Gamma(t)$. Suppose that there is a smooth function $v(\cdot, \cdot)$ such that

$$\frac{dx}{dt} = x_t(\xi, t) = v(x(\xi, t), t) = v = {}^t(v_1, v_2, v_3).$$

We call the vector-valued function v a *velocity* determined by the flow map x. We assume that v is the total velocity.

Since $\Gamma_0 = \Gamma(0)$ is a 2-dimensional closed Riemannian manifold, there is a partition of unity, i.e. there are $\Gamma_m \subset \Gamma_0$, $\Phi_m \in C^\infty(\mathbb{R}^2)$, $U_m \subset \mathbb{R}^2$, $\Psi_m \in C^\infty(\mathbb{R}^3)$ $(m = 1, 2, \cdots, N)$ such that

$$\bigcup_{m=1}^{N} \Gamma_m = \Gamma_0,$$
$$\Gamma_m = \Phi_m(U_m),$$
$$\mathrm{supp}\Psi_m \subset \Gamma_m,$$
$$\|\Psi_m\|_{L^\infty} = 1,$$
$$\sum_{m=1}^{N} \Psi_m = 1 \text{ on } \Gamma_0.$$

Fix $\xi \in \Gamma_0$. Assume that $\xi \in \Gamma_m$ for some $m \in \{1, 2, \cdots, N\}$. Since we can write $\xi = \Phi_m(X)$ for some $X = {}^t(X_1, X_2) \in U_m \subset \mathbb{R}^2$, we set

$$\widetilde{x} = \widetilde{x}(X, t) = x(\Phi_m(X), t)(= x(\xi, t)).$$

Then

$$\begin{cases} \frac{d\widetilde{x}}{dt} = \widetilde{x}_t(X, t) = v(\widetilde{x}(X, t), t), \\ \widetilde{x}|_{t=0} = \Phi_m(X)(= \xi). \end{cases}$$

Now we write

$$\Phi := \Phi_m \text{ if } \xi \in \Gamma_m.$$

Then for each $\xi \in \Gamma_0$,

$$\begin{cases} \frac{d\widetilde{x}}{dt} = \widetilde{x}_t(X, t) = v(\widetilde{x}(X, t), t), \\ \widetilde{x}|_{t=0} = \Phi(X)(= \xi). \end{cases}$$

We also call $\widetilde{x}(X, t)$ a flow map on $\Gamma(t)$. For the map $\widetilde{x} = \widetilde{x}(X, t)$, we define

$$F = F(X, t) := \nabla_X \widetilde{x} = \begin{pmatrix} \frac{\partial \widetilde{x}_1}{\partial X_1} & \frac{\partial \widetilde{x}_1}{\partial X_2} \\ \frac{\partial \widetilde{x}_2}{\partial X_1} & \frac{\partial \widetilde{x}_2}{\partial X_2} \\ \frac{\partial \widetilde{x}_3}{\partial X_1} & \frac{\partial \widetilde{x}_3}{\partial X_2} \end{pmatrix}.$$

Next we study the surface area integral by applying flow maps $x(\xi, t)$ on $\Gamma(t)$. For each $f(\cdot, \cdot) \in C(\mathbb{R}^3 \times \mathbb{R})$, we see that

$$\int_{\Gamma(t)} f(x, t)\, d\mathcal{H}^2_x = \int_U \widetilde{\Psi}(X) f(\widetilde{x}(X, t), t) \sqrt{\det({}^tFF)}\, dX.$$

Here, the right-hand side is a shorthand notation of the form

$$\int_U \widetilde{\Psi}(X) f(\widetilde{x}(X, t), t) \sqrt{\det({}^tFF)}\, dX$$
$$:= \sum_{m=1}^{N} \int_{U_m} \Psi_m(\Phi_m(X)) f(\widetilde{x}(X, t), t) \sqrt{\det({}^tFF)}\, dX.$$

By the bijective of the flow map $x(\cdot, t)$, one can write

$$\Gamma(t) = \{x \in \mathbb{R}^3;\ x = x(\xi, t),\ \xi \in \Gamma_0\}.$$

Using the change of variables and the usual surface area integral, we observe that for $0 < t < T$,

$$\int_{\Gamma(t)} f(x,t)\, d\mathcal{H}^2_x = \int_{\Gamma_0} f(x(\xi,t),t)(\det(\nabla_\xi x))\, d\mathcal{H}^2_\xi$$
$$= \sum_{m=1}^N \int_{\Gamma_m} \Psi_m(\xi) f(x(\xi,t),t)(\det(\nabla_\xi x))\, d\mathcal{H}^2_\xi$$
$$= \sum_{m=1}^N \int_{U_m} \Psi_m(\Phi_m(X)) f(x(\Phi_m(X),t),t)(\det(\nabla_\xi x))\sqrt{\det({}^t\nabla_X \Phi_m \nabla_X \Phi_m)}\, dX$$
$$= \sum_{m=1}^N \int_{U_m} \Psi_m(\Phi_m(X)) f(\widetilde{x}(X,t),t)\sqrt{\det({}^tFF)}\, dX.$$

2.5. *Riemannian metrics on evolving surfaces.* Let us prepare key tools to analyze fluid-flow on an evolving surface. Let $\Gamma(t)$ be a given evolving surface. Let $x = x(\xi,t)$ or $\widetilde{x} = \widetilde{x}(X,t)$ be a flow map on $\Gamma(t)$. Assume that $\Gamma_0 := \Gamma(0)$ satisfies the same conditions in Subsection 2.4. See Subsection 2.4 for details. For the flow map $\widetilde{x} = \widetilde{x}(X,t)$,

$$g_\alpha := \left(\frac{\partial \widetilde{x}_i}{\partial X_\alpha}\right)_{i=1,2,3} = {}^t\left(\frac{\partial \widetilde{x}_1}{\partial X_\alpha}, \frac{\partial \widetilde{x}_2}{\partial X_\alpha}, \frac{\partial \widetilde{x}_3}{\partial X_\alpha}\right).$$

Write

$$g_{\alpha\beta} := g_\alpha \cdot g_\beta = \frac{\partial \widetilde{x}_i}{\partial X_\alpha}\frac{\partial \widetilde{x}_i}{\partial X_\beta} = \sum_i \frac{\partial \widetilde{x}_i}{\partial X_\alpha}\frac{\partial \widetilde{x}_i}{\partial X_\beta}.$$

Set

$$g^{\alpha\beta} := (g_{\alpha\beta})^{-1},$$
$$g^\alpha := g^{\alpha\beta} g_\beta,$$
$$\acute{g}_\alpha := \frac{d}{dt} g_\alpha = \frac{\partial v}{\partial X_\alpha}.$$

Note that

$$g^{\alpha\beta} = g^\alpha \cdot g^\beta,$$
$$g_\alpha = g_{\alpha\beta} g^\beta$$

and

$$\acute{g}_{\alpha\beta} = \acute{g}_\alpha \cdot g_\beta + g_\alpha \cdot \acute{g}_\beta,$$
$$\acute{g}_\alpha = \frac{\partial v}{\partial X_\alpha} = \frac{\partial \widetilde{x}_i}{\partial X_\alpha}\frac{\partial v}{\partial \widetilde{x}_i} = \sum_i \frac{\partial \widetilde{x}_i}{\partial X_\alpha}\frac{\partial v}{\partial \widetilde{x}_i},$$
$$\acute{g}_\alpha \cdot g_\beta = \frac{\partial \widetilde{x}_i}{\partial X_\alpha}\frac{\partial v_j}{\partial \widetilde{x}_i}\frac{\partial \widetilde{x}_j}{\partial X_\beta} = \sum_{i,j} \frac{\partial \widetilde{x}_i}{\partial X_\alpha}\frac{\partial v_j}{\partial \widetilde{x}_i}\frac{\partial \widetilde{x}_j}{\partial X_\beta},$$
$$\acute{g}_{\alpha\beta} = \frac{\partial \widetilde{x}_i}{\partial X_\alpha}\left(\frac{\partial v_j}{\partial \widetilde{x}_i} + \frac{\partial v_i}{\partial \widetilde{x}_j}\right)\frac{\partial \widetilde{x}_j}{\partial X_\beta} = 2\sum_{i,j} \frac{\partial \widetilde{x}_i}{\partial X_\alpha}[D(v)]_{ij}\frac{\partial \widetilde{x}_j}{\partial X_\beta}.$$

Using the above symbols we obtain

LEMMA 2.10. The projection in (2.2) can be expressed in the Lagrange coordinate X as

$$[P_\Gamma]_{ij} = \frac{\partial \widetilde{x}_i}{\partial X_\alpha}\frac{\partial \widetilde{x}_j}{\partial X_\beta} g^{\alpha\beta}.$$

Proof of Lemma 2.10. Using rotation, we may assume that at $x = a$ such that

$$\begin{pmatrix} \frac{\partial \widetilde{x}_1}{\partial X_1} & \frac{\partial \widetilde{x}_2}{\partial X_1} & \frac{\partial \widetilde{x}_3}{\partial X_1} \\ \frac{\partial \widetilde{x}_1}{\partial X_2} & \frac{\partial \widetilde{x}_2}{\partial X_2} & \frac{\partial \widetilde{x}_3}{\partial X_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Therefore we see that

$$\left(\frac{\partial \widetilde{x}_i}{\partial X_\alpha}\frac{\partial \widetilde{x}_j}{\partial X_\beta} g^{\alpha\beta}\right)_{i,j=1,2,3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = I - n \otimes n.$$

□

LEMMA 2.11. For each fixed $\Omega_0 \subset \Gamma_0$ and $0 < t < T$,

$$\Omega(t) := \{x \in \mathbb{R}^3;\ x = x(\xi, t),\ \xi \in \Omega_0\},$$

where $x(\xi, t)$ is a flow map on $\Gamma(t)$. Then the following two assertions hold:

(i) For $f = {}^t(f_1, f_2, f_3) \in [C^1(\mathbb{R}^3 \times \mathbb{R})]^3$,

$$\int_{\Omega(t)} \mathrm{div}_\Gamma f(x,t)\ d\mathcal{H}^2_x = \int_U 1_{\Omega_0}(\Phi(X))\widetilde{\Psi}(X) g^\alpha \cdot \frac{\partial f}{\partial X_\alpha}\sqrt{\det({}^tFF)}\ dX. \tag{2.9}$$

(ii) For $f \in C^1(\mathbb{R}^3 \times \mathbb{R})$,

$$\int_{\Omega(t)} f(x,t)\mathrm{div}_\Gamma v(x,t)\ d\mathcal{H}^2_x$$
$$= \int_U 1_{\Omega_0}(\Phi(X))\widetilde{\Psi}(X) f(\widetilde{x}(X,t),t)\left(\frac{\partial}{\partial t}\sqrt{\det({}^tFF)}\right)\ dX. \tag{2.10}$$

Here

$$1_{\Omega_0}(\xi) := \begin{cases} 1, & \xi \in \Omega_0, \\ 0, & \xi \in \mathbb{R}^3 \setminus \Omega_0. \end{cases}$$

Proof of Lemma 2.11. We first show (i). Fix $\Omega_0 \subset \Gamma_0$ and $0 < t < T$. By definition, we check that

$$\begin{aligned}\int_{\Omega(t)} \mathrm{div}_\Gamma f(x,t)\ d\mathcal{H}^2_x &= \int_{\Omega_0} \mathrm{div}_\Gamma f(x(\xi,t),t)(\det \nabla_\xi x)\ d\mathcal{H}^2_\xi \\ &= \int_{\Gamma_0} 1_{\Omega_0}(\xi)\mathrm{div}_\Gamma f(x(\xi,t),t)(\det \nabla_\xi x)\ d\mathcal{H}^2_\xi \\ &= \int_U 1_{\Omega_0}(\Phi(X))\widetilde{\Psi}(X)\mathrm{div}_\Gamma f(\widetilde{x}(X,t),t)\sqrt{\det({}^tFF)}\ dX.\end{aligned}$$

A direct calculation shows that

$$\begin{aligned}
g^\alpha \cdot \frac{\partial f}{\partial X_\alpha} &= g^{\alpha\beta} g_\beta \cdot \frac{\partial f}{\partial X_\alpha} = g^{1\beta} g_\beta \cdot \frac{\partial f}{\partial X_1} + g^{2\beta} g_\beta \cdot \frac{\partial f}{\partial X_2} \\
&= \left(g^{11} g_1 \cdot \frac{\partial f}{\partial X_1} + g^{12} g_2 \cdot \frac{\partial f}{\partial X_1} \right) + \left(g^{21} g_1 \cdot \frac{\partial f}{\partial X_2} + g^{22} g_2 \cdot \frac{\partial f}{\partial X_2} \right) \\
&= \left(g^{11} \frac{\partial \widetilde{x}_1}{\partial X_1} \frac{\partial f_1}{\partial X_1} + g^{12} \frac{\partial \widetilde{x}_1}{\partial X_2} \frac{\partial f_1}{\partial X_1} + g^{21} \frac{\partial \widetilde{x}_1}{\partial X_1} \frac{\partial f_1}{\partial X_2} + g^{22} \frac{\partial \widetilde{x}_1}{\partial X_2} \frac{\partial f_1}{\partial X_2} \right) \\
&+ \left(g^{11} \frac{\partial \widetilde{x}_2}{\partial X_1} \frac{\partial f_2}{\partial X_1} + g^{12} \frac{\partial \widetilde{x}_2}{\partial X_2} \frac{\partial f_2}{\partial X_1} + g^{21} \frac{\partial \widetilde{x}_2}{\partial X_1} \frac{\partial f_2}{\partial X_2} + g^{22} \frac{\partial \widetilde{x}_2}{\partial X_2} \frac{\partial f_2}{\partial X_2} \right) \\
&+ \left(g^{11} \frac{\partial \widetilde{x}_3}{\partial X_1} \frac{\partial f_3}{\partial X_1} + g^{12} \frac{\partial \widetilde{x}_3}{\partial X_2} \frac{\partial f_3}{\partial X_1} + g^{21} \frac{\partial \widetilde{x}_3}{\partial X_1} \frac{\partial f_3}{\partial X_2} + g^{22} \frac{\partial \widetilde{x}_3}{\partial X_2} \frac{\partial f_3}{\partial X_2} \right) \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

It is easy to check that

$$\begin{aligned}
I_1 &= \left(g^{11} \frac{\partial \widetilde{x}_1}{\partial X_1} \frac{\partial \widetilde{x}_1}{\partial X_1} + g^{12} \frac{\partial \widetilde{x}_1}{\partial X_1} \frac{\partial \widetilde{x}_1}{\partial X_2} + g^{21} \frac{\partial \widetilde{x}_1}{\partial X_1} \frac{\partial \widetilde{x}_1}{\partial X_2} + g^{22} \frac{\partial \widetilde{x}_1}{\partial X_2} \frac{\partial \widetilde{x}_1}{\partial X_2} \right) \frac{\partial f_1}{\partial \widetilde{x}_1} \\
&+ \left(g^{11} \frac{\partial \widetilde{x}_1}{\partial X_1} \frac{\partial \widetilde{x}_2}{\partial X_1} + g^{12} \frac{\partial \widetilde{x}_1}{\partial X_1} \frac{\partial \widetilde{x}_2}{\partial X_2} + g^{21} \frac{\partial \widetilde{x}_1}{\partial X_1} \frac{\partial \widetilde{x}_2}{\partial X_2} + g^{22} \frac{\partial \widetilde{x}_1}{\partial X_2} \frac{\partial \widetilde{x}_2}{\partial X_2} \right) \frac{\partial f_1}{\partial \widetilde{x}_2} \\
&+ \left(g^{11} \frac{\partial \widetilde{x}_1}{\partial X_1} \frac{\partial \widetilde{x}_3}{\partial X_1} + g^{12} \frac{\partial \widetilde{x}_1}{\partial X_1} \frac{\partial \widetilde{x}_3}{\partial X_2} + g^{21} \frac{\partial \widetilde{x}_1}{\partial X_1} \frac{\partial \widetilde{x}_3}{\partial X_2} + g^{22} \frac{\partial \widetilde{x}_1}{\partial X_2} \frac{\partial \widetilde{x}_3}{\partial X_2} \right) \frac{\partial f_1}{\partial \widetilde{x}_3} \\
&= \left(\frac{\partial \widetilde{x}_1}{\partial X_\alpha} \frac{\partial \widetilde{x}_1}{\partial X_\beta} g^{\alpha\beta} \frac{\partial}{\partial \widetilde{x}_1} + \frac{\partial \widetilde{x}_1}{\partial X_\alpha} \frac{\partial \widetilde{x}_2}{\partial X_\beta} g^{\alpha\beta} \frac{\partial}{\partial \widetilde{x}_2} + \frac{\partial \widetilde{x}_1}{\partial X_\alpha} \frac{\partial \widetilde{x}_3}{\partial X_\beta} g^{\alpha\beta} \frac{\partial}{\partial \widetilde{x}_3} \right) f_1 = \partial_1^{tan} f_1.
\end{aligned}$$

Similarly, we see that

$$\begin{aligned}
I_2 &= \partial_2^{tan} f_2, \\
I_3 &= \partial_3^{tan} f_3.
\end{aligned}$$

Therefore we obtain (2.9).

Next we prove (ii). A direct calculation shows that

$$\begin{aligned}
\frac{\partial}{\partial t} \sqrt{\det({}^t F F)} &= \frac{1}{2\sqrt{\det({}^t F F)}} \frac{\partial}{\partial t} (\det({}^t F F)) \\
&= \frac{1}{2\sqrt{\det({}^t F F)}} \det({}^t F F) \cdot \mathrm{tr} \left(({}^t F F)^{-1} \cdot \frac{\partial ({}^t F F)}{\partial t} \right) \\
&= \frac{1}{2} \sqrt{\det({}^t F F)} \cdot \mathrm{tr} \left(({}^t F F)^{-1} \cdot \left({}^t \left(\frac{\partial F}{\partial t} \right) F + {}^t F \frac{\partial F}{\partial t} \right) \right).
\end{aligned}$$

It is easy to check that

$$\begin{aligned}
\mathrm{tr} \left(({}^t F F)^{-1} \cdot \left({}^t \left(\frac{\partial F}{\partial t} \right) F + {}^t F \frac{\partial F}{\partial t} \right) \right) &= \mathrm{tr}\ g^{\alpha\beta} \left(\frac{\partial \widetilde{x}}{\partial X_\beta} \cdot \frac{\partial v}{\partial X_\alpha} + \frac{\partial \widetilde{x}}{\partial X_\alpha} \cdot \frac{\partial v}{\partial X_\beta} \right) \\
&= g^\alpha \cdot \frac{\partial v}{\partial X_\alpha} + g^\beta \cdot \frac{\partial v}{\partial X_\beta} (= 2 \mathrm{div}_\Gamma v).
\end{aligned}$$

Therefore the lemma follows. □

3. On incompressible fluid-flow on evolving surfaces. In this section we study incompressible fluid-flow on an evolving surface. Let $\Gamma(t)$ be an evolving surface on $[0,T)$ for some $T \in (0,\infty]$. Assume that $\Gamma_0 := \Gamma(0)$ satisfies the same conditions as in Subsection 2.4. Let $x = x(\xi,t)$ or $\widetilde{x} = \widetilde{x}(X,t)$ be a flow map on $\Gamma(t)$.

3.1. *Continuity equation and local surface area preserving.* Now we study the continuity equation of fluids on the evolving surface $\Gamma(t)$. A proof of Theorem 1.1 is found in [6, Appendix]. We give here a complete proof for our later purpose and completeness.

LEMMA 3.1. For each fixed $\Omega_0 \subset \Gamma_0$ and $0 < t < T$,

$$\Omega(t) := \{x \in \mathbb{R}^3;\ x = x(\xi,t),\ \xi \in \Omega_0\}.$$

Then for each $\Omega_0 \subset \Gamma_0$ and $0 < t < T$,

$$\frac{d}{dt}\int_{\Omega(t)} \rho(x,t)\ d\mathcal{H}^2_x = \int_{\Omega(t)} \{D_t\rho(x,t) + (\mathrm{div}_\Gamma v(x,t))\rho(x,t)\}\ d\mathcal{H}^2_x. \tag{3.1}$$

Proof of Lemma 3.1. Fix $\Omega_0 \subset \Gamma_0$ and $0 < t < T$. By definition, we observe that

$$\begin{aligned}\int_{\Omega(t)} \rho(x,t)\ d\mathcal{H}^2_x &= \int_{\Omega_0} \rho(x(\xi,t),t)(\det \nabla_\xi x)\ d\mathcal{H}^2_\xi \\ &= \int_{\Gamma_0} 1_{\Omega_0}(\xi)\rho(x(\xi,t),t)(\det \nabla_\xi x)\ d\mathcal{H}^2_\xi \\ &= \int_U 1_{\Omega_0}(\Phi(X))\widetilde{\Psi}(X)\rho(\widetilde{x}(X,t),t)\sqrt{\det({}^tFF)}\ dX.\end{aligned}$$

Here

$$1_{\Omega_0}(\xi) := \begin{cases} 1, & \xi \in \Omega_0, \\ 0, & \xi \in \mathbb{R}^3 \setminus \Omega_0. \end{cases}$$

Since

$$\begin{aligned}\frac{d}{dt}\int_{\Omega(t)} \rho(x,t)\ d\mathcal{H}^2_x = &\int_U 1_{\Omega_0}(\Phi(X))\widetilde{\Psi}(X)\left(\frac{d}{dt}\rho(\widetilde{x}(X,t),t)\right)\sqrt{\det({}^tFF)}\ dX \\ &+ \int_U 1_{\Omega_0}(\Phi(X))\widetilde{\Psi}(X)\rho(\widetilde{x}(X,t),t)\left(\frac{\partial}{\partial t}\sqrt{\det({}^tFF)}\right)\ dX,\end{aligned}$$

it follows from Lemma 2.11 to derive (3.1). ☐

Proof of Theorem 1.1. Fix $t \in (0,T)$ and $\Omega(t) \subset \Gamma(t)$. Since the flow map $x(\xi,t)$ is bijective, there is $\Omega_0 \subset \Gamma_0$ such that

$$\Omega(t) = \{x \in \mathbb{R}^3;\ x = x(\xi,t),\ \xi \in \Omega_0\}.$$

From Lemma 3.1, we observe that

$$\frac{d}{dt}\int_{\Omega(t)} \rho(x,t)\ d\mathcal{H}^2_x = \int_{\Omega(t)} \{D_t\rho(x,t) + (\mathrm{div}_\Gamma v(x,t))\rho(x,t)\}\ d\mathcal{H}^2_x.$$

Since $\Omega(t)$ can be taken arbitrary, this implies that

$$D_t\rho + (\mathrm{div}_\Gamma v)\rho = 0 \text{ on } \mathcal{S}_T.$$

Therefore we see the continuity equation of the fluid on $\Gamma(t)$. ☐

Proof of Theorem 1.2. Assume that $\operatorname{div}_\Gamma v = 0$. Fix $\Omega(t) \subset \Gamma(t)$. By an argument similar to that in the proof of Lemma 3.1, we check that

$$\frac{d}{dt}\int_{\Omega(t)} 1 \; d\mathcal{H}^2_x = \int_{\Omega(t)} \operatorname{div}_\Gamma v(x,t) \; d\mathcal{H}^2_x = 0.$$

Therefore Theorem 1.2 is proved. □

3.2. *Necessary condition for the existence of an incompressible fluid-flow.* This is stated in Theorem 1.3 whose proof is given in Remark 1.4 and is justified in Subsections 2.2 and 2.3.

3.3. *Action integral.* Let us recall our action integral. Let $\Gamma^\varepsilon(t)$ be a variation of $\Gamma(t)$ with $\Gamma^\varepsilon(0) = \Gamma_0$. Let $(x^\varepsilon(\xi,t), \mathcal{S}^\varepsilon_T)$ be a variation of $(x, \mathcal{S}_T)$, and let v^ε be the velocity determined by x^ε; that is,

$$\begin{cases} \frac{dx^\varepsilon}{dt}(\xi,t) = v^\varepsilon(x^\varepsilon(\xi,t),t), \\ x^\varepsilon(\xi,0) = \xi. \end{cases}$$

Suppose there are $y \in [C^\infty(\mathbb{R}^3 \times \mathbb{R})]^3$ and $z \in [C^\infty(\mathcal{S}_T)]^3$ such that for $\xi \in \Gamma_0$ and $0 \leq t < T$,

$$\begin{aligned} &x^\varepsilon(\xi,t)\Big|_{\varepsilon=0} = x(\xi,t), \\ &v^\varepsilon(x^\varepsilon(\xi,t),t)\Big|_{\varepsilon=0} = v(x(\xi,t),t), \\ &\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} x^\varepsilon(\xi,t) = y(\xi,t), \\ &z(x(\xi,t),t) = y(\xi,t). \end{aligned}$$

For such a flow map $x^\varepsilon = x^\varepsilon(\xi,t)$ of the flow map, we define $A[x^\varepsilon]$ by

$$A[x^\varepsilon] = \int_0^T \int_{\Gamma^\varepsilon(t)} \frac{1}{2}\rho^\varepsilon(x,t)|v^\varepsilon(x,t)|^2 \; d\mathcal{H}^2_x dt.$$

Here

$$\Gamma^\varepsilon(t) = \{x \in \mathbb{R}^3; \; x = x^\varepsilon(\xi,t), \; \xi \in \Gamma_0\}$$

and ρ^ε satisfies

$$\begin{cases} \partial_t \rho^\varepsilon + (v^\varepsilon, \nabla)\rho^\varepsilon + (\operatorname{div}_{\Gamma^\varepsilon} v^\varepsilon)\rho^\varepsilon = 0 \text{ on } \mathcal{S}^\varepsilon_T, \\ \rho^\varepsilon|_{t=0} = \rho_0 \end{cases} \tag{3.2}$$

for some $\rho_0 \in C(\Gamma(0))$. Suppose that ρ satisfies

$$\begin{cases} \partial_t \rho + (v, \nabla)\rho + (\operatorname{div}_\Gamma v)\rho = 0 \text{ on } \mathcal{S}_T, \\ \rho|_{t=0} = \rho_0. \end{cases}$$

Moreover, we assume that for $\xi \in \Gamma(0)$ and $0 \leq t < T$,

$$\rho^\varepsilon(x^\varepsilon(\xi,t),t)|_{\varepsilon=0} = \rho(x(\xi,t),t).$$

By the same argument as in Subsection 2.4, we write

$$\widetilde{x}^\varepsilon(X,t) := x^\varepsilon(\Phi_m(X),t)$$

and

$$F^\varepsilon = F^\varepsilon(X,t) := \nabla_X \widetilde{x}^\varepsilon = \begin{pmatrix} \frac{\partial \widetilde{x}_1^\varepsilon}{\partial X_1} & \frac{\partial \widetilde{x}_1^\varepsilon}{\partial X_2} \\ \frac{\partial \widetilde{x}_2^\varepsilon}{\partial X_1} & \frac{\partial \widetilde{x}_2^\varepsilon}{\partial X_2} \\ \frac{\partial \widetilde{x}_3^\varepsilon}{\partial X_1} & \frac{\partial \widetilde{x}_3^\varepsilon}{\partial X_2} \end{pmatrix}.$$

We also see that for $f \in C(\mathbb{R}^4)$,

$$\int_{\Gamma^\varepsilon(t)} f(x,t)\ d\mathcal{H}_x^2 = \int_U \widetilde{\Psi}(X) f(\widetilde{x}^\varepsilon(X,t),t)\sqrt{\det({}^tF_\varepsilon F_\varepsilon)}\ dX.$$

Here

$$\int_U \widetilde{\Psi}(X) f(\widetilde{x}^\varepsilon(X,t),t)\sqrt{\det({}^tF_\varepsilon F_\varepsilon)}\ dX$$
$$:= \sum_{m=1}^N \int_{U_m} \Psi_m(\Phi_m(X)) f(\widetilde{x}^\varepsilon(X,t),t)\sqrt{\det({}^tF_\varepsilon F_\varepsilon)}\ dX.$$

LEMMA 3.2. The variation of the action integral with respect to the flow map is of the form

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} A[x^\varepsilon] = -\int_0^T \int_{\Gamma(t)} \rho(x,t) D_t v(x,t)\cdot z(x,t)\ d\mathcal{H}_x^2 dt. \tag{3.3}$$

Proof of Lemma 3.2. We first show that

$$y(\xi,0) = 0. \tag{3.4}$$

Set

$$\widetilde{y}(X,t) := y(\Phi(X),t).$$

Since

$$x^\varepsilon(\xi,0) - x(\xi,0) = \xi - \xi = 0,$$

we see that

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} x^\varepsilon(\xi,0) = 0 = y(\xi,0).$$

Thus, we have (3.4).

Next we show the key lemma to study the variation of our action integral.

LEMMA 3.3. We have

$$\int_0^T \int_{\Gamma(t)} \frac{1}{2}\rho(x,t)|v(x,t)|^2\ d\mathcal{H}_x^2 dt = \int_0^T \int_U \frac{1}{2}\widetilde{\rho}_0(X)\widetilde{\Psi}(X)|\widetilde{x}_t(X,t)|^2\ dXdt, \tag{3.5}$$

$$\int_0^T \int_{\Gamma^\varepsilon(t)} \frac{1}{2}\rho^\varepsilon(x,t)|v^\varepsilon(x,t)|^2\ d\mathcal{H}_x^2 dt = \int_0^T \int_U \frac{1}{2}\widetilde{\rho}_0(X)\widetilde{\Psi}(X)|\widetilde{x}_t^\varepsilon(X,t)|^2\ dXdt. \tag{3.6}$$

Here

$$\widetilde{\rho}_0(X) = \rho_0(\widetilde{x}(X,0))\sqrt{\{\det({}^tFF)\}(X,0)}.$$

Proof of Lemma 3.3. We only prove (3.5). By the definition, we check that

$$\int_0^T \int_{\Gamma(t)} \frac{1}{2}\rho(x,t)|v(x,t)|^2 \, d\mathcal{H}^2_x dt$$
$$= \int_0^T \int_U \frac{1}{2}\rho(\widetilde{x}(X,t),t)\widetilde{\Psi}(X)|\widetilde{x}_t(X,t)|^2 \sqrt{\det({}^tFF)} \, dX dt.$$

Set

$$Q(X,t) = \rho(\widetilde{x}(X,t),t)\sqrt{\{\det\,({}^tFF)\}(X,t)}.$$

From Lemma 2.11 and the assumption of ρ, we observe that

$$\frac{d}{dt}Q(X,t) = \{(v,\nabla)\rho + \partial_t\rho + \rho(\mathrm{div}_\Gamma v)\}\sqrt{\det({}^tFF)} = 0.$$

Since $Q(X,0) = \rho_0(\widetilde{x}(X,0))\sqrt{\{\det({}^tFF)\}(X,0)}$, we have

$$\rho(\widetilde{x}(X,t),t) = \frac{\widetilde{\rho}_0(X)}{\sqrt{\det\,({}^tFF)}}.$$

Therefore we obtain (3.5). Similarly, we see (3.6) and that

$$\rho^\varepsilon(\widetilde{x}^\varepsilon(X,t),t) = \frac{\widetilde{\rho}_0(X)}{\sqrt{\det\,({}^tF_\varepsilon F_\varepsilon)}}.$$

Note that $\widetilde{x}^\varepsilon(X,0) = \widetilde{x}(X,0) = \Phi(X) = \xi$. □

From (3.6), we have

$$\int_0^T \int_{\Gamma^\varepsilon(t)} \frac{1}{2}\rho^\varepsilon(x,t)|v^\varepsilon(x,t)|^2 \, d\mathcal{H}^2_x dt = \int_0^T \int_U \frac{1}{2}\widetilde{\rho}_0(X)\widetilde{\Psi}(X)|\widetilde{x}^\varepsilon_t(X,t)|^2 \, dX dt.$$

A direct calculation gives

$$\frac{d}{d\varepsilon}A[x^\varepsilon] = \int_0^T \int_U \widetilde{\rho}_0(X)\widetilde{\Psi}(X)\left(\frac{d}{d\varepsilon}\widetilde{x}^\varepsilon_t(X,t)\right)\cdot \widetilde{x}^\varepsilon_t(X,t) \, dX dt.$$

Hence, we have

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} A[x^\varepsilon] = \int_0^T \int_U \widetilde{\rho}_0(X)\widetilde{\Psi}(X)\widetilde{y}_t(X,t)\cdot \widetilde{x}_t(X,t) \, dX dt.$$

Next we prove that

$$\int_0^T \int_U \widetilde{\rho}_0(X)\widetilde{\Psi}(X)\widetilde{y}_t(X,t)\cdot \widetilde{x}_t(X,t) \, dX dt$$
$$= -\int_0^T \int_{\Gamma(t)} \rho(x,t)D_t v(x,y)\cdot z(x,t) \, d\mathcal{H}^2_x dt.$$

By integration by parts and (3.4), we see that

$$\begin{aligned}
&\int_0^T \int_U \widetilde{\rho}_0(X)\widetilde{\Psi}(X)\widetilde{y}_t(X,t)\cdot\widetilde{x}_t(X,t)\ dXdt \\
&\qquad\qquad = \int_0^T \int_U \widetilde{\rho}_0(X)\widetilde{\Psi}(X)\widetilde{y}_t(X,t)\cdot v(\widetilde{x}(X,t),t)\ dXdt \\
&\qquad\qquad = -\int_0^T \int_U \widetilde{\rho}_0(X)\widetilde{\Psi}(X)\widetilde{y}(X,t)\cdot\left(\frac{d}{dt}v(\widetilde{x}(X,t),t)\right)\ dXdt \\
&\qquad = -\int_0^T \int_U \frac{\widetilde{\rho}_0(X)}{\sqrt{\det({}^tFF)}}\widetilde{\Psi}(X)\widetilde{y}(X,t)\cdot\left(\frac{d}{dt}v(\widetilde{x}(X,t),t)\right)\sqrt{\det({}^tFF)}\ dXdt.
\end{aligned}$$

Therefore we see that

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} A[x^\varepsilon] = -\int_0^T \int_{\Gamma(t)} \rho(x,t)D_tv(x,t)\cdot z(x,t)\ d\mathcal{H}_x^2dt,$$

which is (3.3). □

Proof of Theorem 1.5. We first show (i). Assume that for each $z \in [C_0^\infty(\mathcal{S}_T)]^3$ satisfying $\mathrm{div}_\Gamma z = 0$,

$$-\int_0^T \int_{\Gamma(t)} \rho(x,t)D_tv(x,t)\cdot z(x,t)\ d\mathcal{H}_x^2dt = 0.$$

From Lemma 2.7, there is $\sigma \in C^{1,0}(\mathcal{S}_T)$ such that

$$\rho\{v_t + (v,\nabla)v\} + \nabla^{tan}\sigma + \sigma Hn = 0.$$

Next we attack (ii). Assume that for each $z \in [C_0^\infty(\mathcal{S}_T)]^3$ satisfying $\mathrm{div}_\Gamma z = 0$, and $z\cdot n = 0$,

$$-\int_0^T \int_{\Gamma(t)} \rho(x,t)D_tv(x,t)\cdot z(x,t)\ d\mathcal{H}_x^2dt = 0.$$

Since

$$\int_0^T \int_{\Gamma(t)} \rho(x,t)D_tv(x,t)\cdot z(x,t)\ d\mathcal{H}_x^2dt = \int_0^T \int_{\Gamma(t)} P_\Gamma\rho(x,t)D_tv(x,t)\cdot z(x,t)\ d\mathcal{H}_x^2dt,$$

it follows from Lemma 2.7 to see that there is $\sigma \in C^{1,0}(\mathcal{S}_T)$ such that

$$P_\Gamma\rho(x,t)\{v_t + (v,\nabla)v\} + \nabla^{tan}\sigma = 0.$$

□

3.4. *Dissipation energy.* In this subsection we consider the dissipation energy.

LEMMA 3.4. Fix $t \in (0,T)$. Define $E[v](t)$ by

$$E[v](t) = -\int_{\Gamma(t)} \mu_0|P_\Gamma(x,t)D(v(x,t))P_\Gamma(x,t)|^2\ d\mathcal{H}_x^2.$$

Then for all $\varphi = {}^t(\varphi_1,\varphi_2,\varphi_3) \in [C_0^\infty(\Gamma(t))]^3$

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} E[v+\varepsilon\varphi] = 2\mu_0\int_{\Gamma_0} \mathrm{div}_\Gamma(P_\Gamma(x,t)D(v(x,t))P_\Gamma(x,t))\cdot\varphi\ d\mathcal{H}_x^2.$$

We first prepare the following lemma.

LEMMA 3.5 (Projected strain rate). Set

$$D_\Gamma(v) := P_\Gamma D(v) P_\Gamma.$$

Then for each $i, j = 1, 2, 3$,

$$[D_\Gamma(v)]_{ij} = \frac{1}{2}\pi_i \cdot (\partial_j^{tan} v) + \frac{1}{2}\pi_j \cdot (\partial_i^{tan} v). \tag{3.7}$$

We call $D_\Gamma(v)$ a *projected strain rate.*

Proof of Lemma 3.5. By definition, we see that

$$\begin{aligned}[P_\Gamma D(v) P_\Gamma]_{ij} =& \frac{1}{2}\sum_{k,\ell}(\delta_{ik} - n_i n_k)\left(\frac{\partial v_k}{\partial x_\ell} + \frac{\partial v_\ell}{\partial x_k}\right)(\delta_{j\ell} - n_j n_\ell) \\ =& \frac{1}{2}\sum_k (\delta_{ik} - n_i n_k)\partial_j^{tan} v_k + \frac{1}{2}\sum_\ell (\delta_{j\ell} - n_j n_\ell)\partial_i^{tan} v_\ell \\ =& \frac{1}{2}\sum_k [P_\Gamma]_{ik}\partial_j^{tan} v_k + \frac{1}{2}\sum_\ell [P_\Gamma]_{j\ell}\partial_i^{tan} v_\ell \\ =& \frac{1}{2}\pi_i \cdot (\partial_j^{tan} v) + \frac{1}{2}\pi_j \cdot (\partial_i^{tan} v).\end{aligned}$$

Therefore we obtain (3.7). □

Proof of Lemma 3.4. Fix $\varphi = {}^t(\varphi_1, \varphi_2, \varphi_3) \in [C_0^\infty(\Gamma_0)]^3$. A direct calculation gives

$$\begin{aligned}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\int_{\Gamma_0} E(v + \varepsilon\varphi)\, d\mathcal{H}_x^2 =& -\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\int_{\Gamma_0} 2\mu_0 |P_\Gamma D(v + \varepsilon\varphi) P_\Gamma|^2\, d\mathcal{H}_x^2 \\ =& -2\mu_0 \int_{\Gamma_0} \mathrm{Tr}(P_\Gamma D(v) P_\Gamma P_\Gamma D(\varphi) P_\Gamma)\, d\mathcal{H}_x^2.\end{aligned} \tag{3.8}$$

Note that

$$\begin{aligned}&4[P_\Gamma]_{ik}[D(v + \varepsilon\varphi)]_{ij}[P_\Gamma]_{j\ell}[D(v + \varepsilon\varphi)]_{k\ell} \\ &\quad = [P_\Gamma]_{ik}\left(\frac{\partial(v_i + \varepsilon\varphi_i)}{\partial x_j} + \frac{\partial(v_j + \varepsilon\varphi_j)}{\partial x_i}\right)[P_\Gamma]_{j\ell}\left(\frac{\partial(v_k + \varepsilon\varphi_k)}{\partial x_\ell} + \frac{\partial(v_\ell + \varepsilon\varphi_\ell)}{\partial x_k}\right).\end{aligned}$$

Now we prove that

$$\int_{\Gamma_0} \mathrm{Tr}(D_\Gamma(v) D_\Gamma(\varphi))\, d\mathcal{H}_x^2 = -\int_{\Gamma_0} \mathrm{div}_\Gamma D_\Gamma(v) \cdot \varphi\, d\mathcal{H}_x^2. \tag{3.9}$$

Since

$$\begin{aligned}&(\pi_i \partial_j^{tan} v + \pi_j \partial_i^{tan} v)(\pi_i \partial_j^{tan}\varphi + \pi_j \partial_i^{tan}\varphi) \\ &\qquad = 2(\pi_i \partial_j^{tan} v)(\pi_i \partial_j^{tan}\varphi) + 2(\pi_j \partial_i^{tan} v)(\pi_i \partial_j^{tan}\varphi),\end{aligned}$$

we have

$$\begin{aligned}&\int_{\Gamma_0} \mathrm{Tr}(D_\Gamma(v) D_\Gamma(\varphi))\, d\mathcal{H}_x^2 \\ &\qquad = -\int_{\Gamma_0} (\pi_i \partial_j^{tan} v)(\pi_i \partial_j^{tan}\varphi)\, d\mathcal{H}_x^2 - \int_{\Gamma_0} (\pi_j \partial_i^{tan} v)(\pi_i \partial_j^{tan}\varphi)\, d\mathcal{H}_x^2.\end{aligned}$$

Applying Lemma 2.6, we check that

$$\begin{aligned}\int_{\Gamma_0} (\pi_i \partial_j^{tan} v)(\pi_i \partial_j^{tan} \varphi) \, d\mathcal{H}_x^2 \\ = - \int_{\Gamma_0} \{\partial_j^{tan}((\pi_i \partial_j^{tan} v)\pi_i)\}\varphi \, d\mathcal{H}_x^2 - \int_{\Gamma_0} H n_j (\pi_i \partial_j^{tan} v)\pi_i \, d\mathcal{H}_x^2 \\ = - \int_{\Gamma_0} \{\partial_j^{tan}((\pi_i \partial_j^{tan} v)\pi_i)\}\varphi \, d\mathcal{H}_x^2.\end{aligned}$$

Here we used the fact that $n_j \partial_j^{tan} = 0$. By definition, we see that

$$\begin{aligned}[P_\Gamma]_{ik}\pi_i v =& (\delta_{ik} - n_i n_k)(\delta_{i\ell} - n_i n_\ell) v_\ell \\ =& (\delta_{k\ell} - n_k n_\ell) v_\ell = \pi_k v \end{aligned} \tag{3.10}$$

and that

$$\begin{aligned}[P_\Gamma]_{ik}\partial_i^{tan} v =& (\delta_{ik} - n_i n_k)(\delta_{i\ell} - n_i n_\ell)\partial_\ell v \\ =& (\delta_{k\ell} - n_k n_\ell)\partial_\ell v \\ =& \partial_k^{tan} v. \end{aligned} \tag{3.11}$$

Using (3.10) and (3.11), we observe that

$$\begin{aligned}\{\partial_j^{tan}((\pi_i \partial_j^{tan} v)\pi_i)\}\varphi =& \{\partial_j^{tan}[(\delta_{ik} - n_i n_k)\partial_j^{tan} v_k (\delta_{i\ell} - n_i n_\ell)]\}\varphi_\ell \\ =& \{\partial_j^{tan}(\delta_{k\ell} - n_k n_\ell)\partial_j^{tan} v_k\}\varphi_\ell \\ =& \{\partial_j^{tan}(\pi_\ell \partial_j^{tan} v)\}\varphi_\ell.\end{aligned}$$

Thus, we have

$$\int_{\Gamma_0} (\pi_i \partial_j^{tan} v)(\pi_i \partial_j^{tan} \varphi) \, d\mathcal{H}_x^2 = - \int_{\Gamma_0} \partial_j^{tan}(\pi_\ell \partial_j^{tan} v)\varphi_\ell \, d\mathcal{H}_x^2.$$

Similarly, we see that

$$\begin{aligned}\int_{\Gamma_0} (\pi_j \partial_i^{tan} v)(\pi_i \partial_j^{tan} \varphi) \, d\mathcal{H}_x^2 \\ = - \int_{\Gamma_0} \{\partial_j^{tan}((\pi_j \partial_i^{tan} v)\pi_i)\}\varphi \, d\mathcal{H}_x^2 - \int_{\Gamma_0} H n_j (\pi_j \partial_i^{tan} v)\pi_i \varphi \, d\mathcal{H}_x^2 \\ = - \int_{\Gamma_0} \{\partial_j^{tan}((\pi_j \partial_i^{tan} v)\pi_i)\}\varphi \, d\mathcal{H}_x^2\end{aligned}$$

and that

$$\begin{aligned}\partial_j^{tan}(\pi_j(\partial_i^{tan} v)\pi_i)\varphi =& \varphi_{i'}\{\partial_j^{tan}[(\delta_{j\ell} - n_j n_\ell)(\delta_{ik} - n_i n_k)(\partial_k v_\ell)(\delta_{ii'} - n_i n_{i'})]\} \\ =& \varphi_{i'}\{\partial_j^{tan}[(\delta_{j\ell} - n_j n_\ell)(\delta_{ki'} - n_k n_{i'})\partial_k v_\ell]\} \\ =& \varphi_{i'}(\partial_j^{tan}(\pi_j \partial_{i'}^{tan} v)).\end{aligned}$$

Consequently, we obtain

$$\int_{\Gamma_0} (\pi_i \partial_j^{tan} v)(\pi_j \partial_i^{tan} \varphi) \, d\mathcal{H}_x^2 = - \int_{\Gamma_0} \partial_j^{tan} (\pi_j \partial_\ell^{tan} v) \varphi_\ell \, d\mathcal{H}_x^2,$$

which is (3.9). Combining (3.8), (3.9), and Lemma 3.5, we finish the proof of the lemma. □

Proof of Theorem 1.6. We first show (i). Fix $t \in (0,T)$. From Lemma 3.4 and the assumption, we see that for all $\varphi = {}^t(\varphi_1, \varphi_2, \varphi_3) \in [C_0^\infty(\Gamma(t))]^3$ satisfying $\mathrm{div}_\Gamma \varphi = 0$,

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} E[v+\varepsilon\varphi](t) = 2\mu_0 \int_{\Gamma_0} \mathrm{div}_\Gamma (P_\Gamma(x,t) D(v(x,t)) P_\Gamma(x,t)) \cdot \varphi(x) \, d\mathcal{H}_x^2 = 0.$$

From Lemma 2.7, there is $\sigma \in C^{1,0}(\Gamma(t))$ such that

$$-2\mu_0 \mathrm{div}_\Gamma (P_\Gamma D(v) P_\Gamma) + \nabla^{tan} \sigma + \sigma H n = 0.$$

Next we prove (ii). Fix $t \in (0,T)$. From Lemma 3.4 and the assumption, we check that for all $\varphi = {}^t(\varphi_1, \varphi_2, \varphi_3) \in [C_0^\infty(\Gamma(t))]^3$ satisfying $\mathrm{div}_\Gamma \varphi = 0$ and $\varphi \cdot n = 0$,

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} E[v+\varepsilon\varphi] = 2\mu_0 \int_{\Gamma_0} P_\Gamma \mathrm{div}_\Gamma (P_\Gamma(x,t) D(v(x,t)) P_\Gamma(x,t)) \cdot \varphi(x) \, d\mathcal{H}_x^2 = 0.$$

From Lemma 2.7, there is $\sigma \in C^{1,0}(\mathcal{S}_T)$ such that

$$-2\mu_0 P_\Gamma \mathrm{div}_\Gamma (P_\Gamma D(v) P_\Gamma) + \nabla^{tan} \sigma = 0.$$

□

Finally, we state the reason why $E_\Gamma[v]$ is a candidate of an energy of fluid-flow systems on an evolving surface.

LEMMA 3.6 (Surface energy density). Set

$$e_\Gamma(v) := \frac{1}{4} \acute{g}_{\alpha\beta} \acute{g}_{\zeta\eta} g^{\alpha\zeta} g^{\beta\eta} = \frac{1}{4} \sum_{\alpha,\beta,\zeta,\eta} \acute{g}_{\alpha\beta} \acute{g}_{\zeta\eta} g^{\alpha\zeta} g^{\beta\eta}.$$

Then

$$e_\Gamma(v) = |D_\Gamma(v)|^2 = |P_\Gamma D(v) P_\Gamma|^2.$$

Proof. It is easy to check that

$$e_\Gamma(v) = \frac{\partial x_i}{\partial X_\alpha} [D(v)]_{ij} \frac{\partial x_j}{\partial X_\beta} g^{\alpha\zeta} g^{\beta\eta} \frac{\partial x_k}{\partial X_\zeta} [D(v)]_{k\ell} \frac{\partial x_\ell}{\partial X_\eta}.$$

Since

$$[P_\Gamma]_{ij} = [I - n \otimes n]_{ij} = \frac{\partial x_i}{\partial X_\alpha} \frac{\partial x_j}{\partial X_\beta} g^{\alpha\beta}$$

by Lemma 2.10, we see that

$$\begin{aligned} e_\Gamma(v) =& [D(v)]_{ij} [D(v)]_{k\ell} [I - n\otimes n]_{ik} [I - n \otimes n]_{j\ell} \\ =& \mathrm{Tr}(D(v) P_\Gamma D(v) P_\Gamma) \\ =& \mathrm{Tr}(P_\Gamma D(v) P_\Gamma P_\Gamma D(v) P_\Gamma) \\ =& |P_\Gamma D(v) P_\Gamma|^2. \end{aligned}$$

Therefore the lemma follows. □

4. Incompressible fluid systems on an evolving surface. Let us derive the incompressible fluid flow system (1.1). Let $\Gamma(t)$ be a given evolving surface and $x = x(\xi, t)$ a flow map on $\Gamma(t)$. From Theorems 1.1 and 1.2, we see the following incompressible condition:

$$\mathrm{div}_\Gamma v = 0 \text{ on } \mathcal{S}_T.$$

Assume that $D_t\rho = 0$ on $\mathcal{S}_T$. From Theorems 1.5 and 1.6, we apply our energetic variational approach (Least Action Principle and Minimum Dissipation Principle) to obtain

$$\rho D_t v + \mathrm{grad}_\Gamma \sigma + \sigma H n = 2\mu_0 \mathrm{div}_\Gamma (P_\Gamma D(v) P_\Gamma) \text{ on } \mathcal{S}_T.$$

Therefore we have the system (1.1).

Applying Theorems 1.1-1.6 and the above argument, we can derive several incompressible fluid systems on a given evolving surface.

For example, under area conservation we get $D_t\rho = 0$ from Theorems 1.1 and 1.2. If it is a critical point of the action integral $A[x] = \int_0^T \int_{\Gamma(t)} \frac{1}{2}\rho|v|^2 \, d\mathcal{H}_x^2 dt$ among all possible perturbations including $\Gamma(t)$, one get from Theorem 1.5 an overdetermined system:

Incompressible Euler system (I)

$$\begin{cases} D_t\rho = 0 & \text{on } \mathcal{S}_T, \\ \rho D_t v + \mathrm{grad}_\Gamma \sigma + \sigma H n = 0 & \text{on } \mathcal{S}_T, \\ \mathrm{div}_\Gamma v = 0 & \text{on } \mathcal{S}_T. \end{cases} \tag{4.1}$$

In this case the system (4.1) satisfies the energy law

$$\frac{d}{dt}\int_{\Gamma(t)} \frac{1}{2}\rho|v|^2 \, d\mathcal{H}_x^2 = 0.$$

On the other hand, with the prescribed variation of the motion z is tangent to $\Gamma(t)$. Then Theorem 1.5 (ii) gives the incompressible Euler system (II) as follows:

Incompressible Euler system (II)

$$\begin{cases} D_t\rho = 0 & \text{on } \mathcal{S}_T, \\ P_\Gamma \rho\{\partial_t v + (v, \nabla) v\} + \mathrm{grad}_\Gamma \sigma = 0 & \text{on } \mathcal{S}_T, \\ \mathrm{div}_\Gamma v = 0 & \text{on } \mathcal{S}_T. \end{cases} \tag{4.2}$$

The equation (4.2) avoids the problem of being overdetermined since there is a function $g = g(x, t)$ such that $\rho D_t + \mathrm{grad}_\Gamma \sigma + g n = 0$. Note that $g(x, t)$ is not prescribed.

In this case when $v \cdot n = 0$ on $\Gamma(t)$, the system still satisfies the energy law

$$\frac{d}{dt}\int_{\Gamma(t)} \frac{1}{2}\rho|v|^2 \, d\mathcal{H}_x^2 = 0.$$

However, in general, the energy law becomes

$$\frac{d}{dt}\int_{\Gamma(t)} \frac{1}{2}\rho|v|^2 \, d\mathcal{H}_x^2 = \int_{\Gamma(t)} (v \cdot n)(\sigma H - g) \, d\mathcal{H}_x^2.$$

In this case the right-hand side corresponds to the work done by the moving surface to the fluid.

Now we consider the effect of viscosity. The equation is formally of the form

$$\frac{\delta A}{\delta x} = -\frac{\delta E}{\delta v}.$$

If both variations are general, not necessarily being in a tangential direction including the variation of the surface, then we have an overdetermined incompressible NSSK system (I) as follows:

Incompressible NSSK system (I)

$$\begin{cases} D_t\rho = 0 & \text{on } \mathcal{S}_T, \\ \rho D_t v + \mathrm{grad}_\Gamma \sigma + \sigma H n = 2\mu_0 \mathrm{div}_\Gamma (P_\Gamma D(v) P_\Gamma) & \text{on } \mathcal{S}_T, \\ \mathrm{div}_\Gamma v = 0 & \text{on } \mathcal{S}_T. \end{cases} \tag{4.3}$$

If both variations are tangent to $\Gamma(t)$, then we set the incompressible NSSK system (II):

Incompressible NSSK system (II)/Tangential incompressible NSSK system

$$\begin{cases} D_t\rho = 0 & \text{on } \mathcal{S}_T, \\ P_\Gamma \rho\{\partial_t v + (v, \nabla) v\} + \mathrm{grad}_\Gamma \sigma = 2\mu_0 P_\Gamma \mathrm{div}_\Gamma (P_\Gamma D(v) P_\Gamma) & \text{on } \mathcal{S}_T, \\ \mathrm{div}_\Gamma v = 0 & \text{on } \mathcal{S}_T. \end{cases} \tag{4.4}$$

We notice the ambiguity of the formal variations above. In fact, if we choose the variation of v generally, while the variation of E with respect to u is tangential to $\Gamma(t)$, then we have the noncanonical incompressible NSSK system (III) which is again an overdetermined system:

Noncanonical incompressible NSSK system (III)

$$\begin{cases} D_t\rho = 0 & \text{on } \mathcal{S}_T, \\ \rho D_t v + \mathrm{grad}_\Gamma \sigma + \sigma H n = 2\mu_0 P_\Gamma \mathrm{div}_\Gamma (P_\Gamma D(u) P_\Gamma) & \text{on } \mathcal{S}_T, \\ \mathrm{div}_\Gamma v = 0 & \text{on } \mathcal{S}_T. \end{cases} \tag{4.5}$$

In general, for the system (4.3), we can compute the energy law

$$\frac{d}{dt}\int_{\Gamma(t)} \frac{1}{2}\rho|v|^2 \, d\mathcal{H}^2_x = -\int_{\Gamma(t)} 2\mu_0 |P_\Gamma D(v) P_\Gamma|^2 \, d\mathcal{H}^2_x$$

by Lemma 3.4. However, the system will include the work done by the evolving surface, hence, may not be necessarily dissipate. In (4.4) again we compute the energy law:

$$\frac{d}{dt}\int_{\Gamma(t)} \frac{1}{2}\rho|v|^2 \, d\mathcal{H}^2_x = -\int_{\Gamma(t)} 2\mu_0 D_\Gamma(v) : D_\Gamma(P_\Gamma v) \, d\mathcal{H}^2_x.$$

However, for (4.5) the identity becomes more complicated. It is of the form

$$\frac{d}{dt}\int_{\Gamma(t)} \frac{1}{2}\rho|v|^2 \, d\mathcal{H}^2_x = \int_{\Gamma(t)} (v\cdot n)(\sigma H - g) \, d\mathcal{H}^2_x - \int_{\Gamma(t)} 2\mu_0 D_\Gamma(u) : D_\Gamma(P_\Gamma v) \, d\mathcal{H}^2_x,$$

where the first term on the right-hand side corresponds to the work done by the evolving surface.

5. Appendix (I): Comparison to the Euler and the Navier-Stokes systems on a manifold. In this section we first compare the incompressible Euler system (II) on a fixed surface with the Euler system on a manifold derived by Arnol′d [2,3]. Next we compare the incompressible NSSK system (II) on a fixed surface with the Navier-Stokes system on a manifold introduced by Taylor [19]. More precisely, we prove that our incompressible Euler system (4.2) on a fixed surface is the same as the Euler system on a manifold derived by Arnol′d, and show that our incompressible NSSK system (4.4) on a fixed surface is different from the Navier-Stokes system on a manifold obtained by Taylor. The difference between Taylor's model and our model is the dissipative functional.

Let us introduce the Euler system on a manifold derived by Arnol′d [2, 3] and the Navier-Stokes system on a manifold obtained by Taylor [19]. Arnol′d derived the following Euler system on a manifold $\mathcal{M}$:

$$\begin{cases} u_t + \nabla_u u + \mathrm{grad}_{\mathcal{M}}\sigma = 0, \\ \mathrm{div}_{\mathcal{M}} u = 0. \end{cases} \tag{5.1}$$

See [10, Chapters 8 and 9] for a mathematical derivation of the system (5.1). Taylor [19] introduced the following Navier-Stokes system on a manifold $\mathcal{M}$:

$$\begin{cases} u_t + \nabla_u u - \Delta_B u + Ku + \mathrm{grad}_{\mathcal{M}}\sigma = 0, \\ \mathrm{div}_{\mathcal{M}} u = 0. \end{cases} \tag{5.2}$$

Here $\mathcal{M}$ is a closed 2-dimensional Riemannian manifold, u is a 1-form on $\mathcal{M}$, Δ_B is the Borhner-Laplacian, K is Gaussian curvature, $\mathrm{grad}_{\mathcal{M}}$ is a gradient operator on $\mathcal{M}$, and $\mathrm{div}_{\mathcal{M}}$ is a divergence operator on $\mathcal{M}$.

Let us compare our systems with the previous models. Let $\Gamma(t)$ be a fixed surface, that is, $\Gamma(t) = \Gamma_0$ for $t \in [0, T)$. Suppose that $\Gamma_0 = \mathcal{M}$. Let v be a total velocity on Γ_0. Assume that $v \cdot n = 0$ and $\mathrm{div}_\Gamma v = 0$. Now we consider Γ_0 as a manifold and v a 1-form on Γ_0.

Let us compare the incompressible Euler system (4.2) on a fixed surface Γ_0 with the Euler system (5.1) on a manifold Γ_0. Using local coordinate representation and $v \cdot n = 0$, we easily check that

$$P_\Gamma v_t + P_\Gamma (v, \nabla) v = v_t + \nabla_v v,$$
$$\mathrm{div}_\Gamma v = \mathrm{div}_{\mathcal{M}} v = 0.$$

Therefore we conclude that the Euler system (4.2) on a fixed surface Γ_0 is the same as system (5.1) on a manifold Γ_0.

Next we use an energetic variational approach to derive the system (5.2). For fixed $t \in (0,T)$ let

$$E_1[v](t) = -\int_{\Gamma_0} \mu_0 |D^{tan}(v(x,t))|^2 \, d\mathcal{H}^2_x.$$

Here $D^{tan}(v) := 1/2(\nabla^{tan} v + {}^t(\nabla^{tan} v))$. We call $D^{tan}(v)$ a *tangential strain rate*, comparing it to the definition of the projected strain rate in Lemma 3.5. Note that in our case we use the projected strain rate $D_\Gamma(v) = P_\Gamma D(v) P_\Gamma$.

By the same argument as in the proof of Theorem 1.6, we can obtain the following theorem:

THEOREM 5.1 (Variation of dissipation energy). For every vector field $\varphi \in [C_0^\infty(\Gamma(t))]^3$ satisfying $\mathrm{div}_\Gamma \varphi = 0$ and $\varphi \cdot n = 0$, the direction derivation of E_1 at v is of the form

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} E_1[v + \varepsilon\varphi] = \int_{\Gamma(t)} 2\mu_0 \mathrm{div}_\Gamma (P_\Gamma(x,t) D^{tan}(v(x,t))) \cdot \varphi(x) \, d\mathcal{H}^2_x.$$

The proof of Theorem 5.1 is left to the reader.

Next we prove that $\mathrm{div}_\Gamma(P_\Gamma D^{tan}(v)) = \Delta_\Gamma v + Kv$. Here $\Delta_\Gamma = (\partial_1^{tan})^2 + (\partial_2^{tan})^2 + (\partial_3^{tan})^2$. We now use the following principal coordinates at the origin (Gibarg and Trudinger [9, the Appendix in Chapter 14]):

$$\begin{aligned} \partial_1 n_1 &= -\kappa_1, \\ \partial_1 n_2 &= -\kappa_2, \\ n &= {}^t(0,0,1), \\ \kappa_1 + \kappa_2 &= H, \\ \kappa_1 \kappa_2 &= K. \end{aligned}$$

Using the above principal coordinates, $\mathrm{div}_\Gamma v = 0$, $v \cdot n = 0$, and $v_3 = 0$ at the origin, we are able to conclude that

$$P_\Gamma \mathrm{div}_\Gamma (P_\Gamma D^{tan}(v)) = \Delta_B v + Kv.$$

Here we note that $\Delta_B = \Delta_\Gamma$. Therefore our incompressible NSSK system (4.4) on a fixed surface Γ_0 is different from system (5.2) on a manifold Γ_0. Here we notice that, in general,

$$P_\Gamma \mathrm{div}_\Gamma (P_\Gamma D(v) P_\Gamma) \neq P_\Gamma \mathrm{div}_\Gamma (P_\Gamma D^{tan}(v))$$

even if $\mathrm{div}_\Gamma v = 0$ and $v \cdot n = 0$. In fact, we see at once that

$$P_\Gamma \mathrm{div}_\Gamma (P_\Gamma D(v) P_\Gamma) - P_\Gamma \mathrm{div}_\Gamma (P_\Gamma D^{tan}(v)) = \begin{pmatrix} H(n \cdot \partial_1^{tan} v) \\ H(n \cdot \partial_2^{tan} v) \\ H(n \cdot \partial_3^{tan} v) \end{pmatrix}.$$

Note also that it is easy to check that

$$P_\Gamma D(v) P_\Gamma = P_\Gamma D^{tan}(v) P_\Gamma.$$

6. Appendix (II): Energy law and work. In general, if the fluid is contained in a moving domain, the kinetic energy may not be conserved. In other words, the moving boundary is doing work to the fluid. To illustrate this, we consider the simple example of an incompressible Euler equation in a prescribed smooth moving domain $Q_T = \bigcup_{0<t<T} \{\Omega(t) \times \{t\}\}$, where $\Omega(t)$ is a smooth bounded domain. The Euler equation is written by

$$\begin{cases} u_t + (u, \nabla)u + \nabla p = 0 \text{ in } Q_T, \\ \nabla \cdot u = 0 \text{ in } Q_T, \\ u \cdot n = V \text{ on } \partial\Omega(t) \times (0, T), \end{cases}$$

where $u = u(t, x) = {}^t(u_1, u_2, u_3)$ denotes the velocity of the fluid in Q_T, p the pressure of the fluid, V is a given normal velocity of $\partial\Omega(t)$, and n is the unit outer normal. The rate of change of the kinetic energy is

$$\frac{d}{dt}\int_{\Omega(t)} \frac{1}{2}|u|^2 \, dx = \int_{\Omega(t)} \partial_t \left\{\frac{1}{2}|u|^2\right\} \, dx + \int_{\partial\Omega(t)} \frac{1}{2}|u|^2 V \, d\mathcal{H}^2_x.$$

Meanwhile,

$$\int_{\Omega(t)} \partial_t \left\{\frac{1}{2}|u|^2\right\} \, dx = \int_{\Omega(t)} u \cdot u_t \, dx = -\int_{\Omega(t)} u \cdot \{(u, \nabla)u\} \, dx - \int_{\Omega(t)} u \nabla p \, dx.$$

Integrating by parts, we get

$$\frac{d}{dt}\int_{\Omega(t)} \frac{1}{2}|u|^2 \, dx = -\int_{\partial\Omega(t)} (u \cdot n)\frac{1}{2}|u|^2 \, d\mathcal{H}^2_x + \int_{\partial\Omega(t)} V\frac{1}{2}|u|^2 \, d\mathcal{H}^2_x - \int_{\partial\Omega(t)} (u \cdot n)p \, d\mathcal{H}^2_x.$$

Thus, we conclude that

$$\frac{d}{dt}\int_{\Omega(t)} \frac{1}{2}|u|^2 \, dx = -\int_{\partial\Omega(t)} Vp \, d\mathcal{H}^2_x$$

since $u \cdot n = V$. The right-hand side is the work through pressure caused by the motion of $\partial\Omega(t)$. From the observation it is clear that the energy law for our Euler system (II) includes the work caused by the motion of $\Gamma(t)$.

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