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Well-posedness on a new hydrodynamic model of the fluid with the dilute charged particles

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Abstract

We use an energetic variational approach to derive a new hydrodynamic model, which could be called a generalized Poisson–Nernst–Planck–Navier–Stokes system. Such the system could describe the dynamics of the compressible conductive fluid with the dilute charged particles and be used to analyze the interactions between the macroscopic fluid motion and the microscopic charge transportation. Then, we develop a general method to obtain the unique local classical solution, the unique global solution under small perturbations and the optimal decay rates of the solution and its derivatives of any order.

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Keywords: Poisson–Nernst–Planck–Navier–Stokes equations; Energetic variational approach; Well-posedness; Optimal decay

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1. Introduction

In real life, there are many fluids with the charged particles which transport under the influence of the (self-)electrostatic potential. For instance, the flow in semi-conductors [29,39,47,51], ion channels [8,23,48], plasmas [10,12,53], and so on. Then, some mathematical models were proposed to describe and simulate the dynamics of such the fluids in light of different considerations, such as, the Vlasov–Poisson–Boltzmann model [7,36,39], the hydrodynamic model (for example, Euler–Poisson model, Navier–Stokes–Poisson model, etc.) [5,9,29,31], the Poisson–Nernst–Planck (PNP) model (also called drift-diffusion or electro-diffusion model) [8,29,33], etc.

In this paper, we use an Energetic Variational Approach (EVA) to explore a new hydrodynamical model in $\mathbb{R}^3$:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) \\ \quad = \mu \Delta u + (\mu + \mu') \nabla \operatorname{div} u - \nabla \varphi(v) - \nabla \psi(w) + \epsilon \Delta \phi \nabla \phi, \\ v_t + \operatorname{div}(v u) = \operatorname{div}\left(D_v \nabla \varphi(v) + D_v z_v e v \nabla \phi \right), \\ w_t + \operatorname{div}(w u) = \operatorname{div}\left(D_w \nabla \psi(w) + D_w z_w e w \nabla \phi \right), \\ -\epsilon \Delta \phi = z_v e v + z_w e w, \end{cases} \tag{1.1}$$

which could be called a generalized Poisson–Nernst–Planck–Navier–Stokes (PNP-NS) system since it seems that the (microscopic) PNP system couples with the (macroscopic) compressible NS system. Here, the unknown variables ρ, u, $v(w)$, ϕ and p represent the fluid density, the fluid velocity, the distribution of the negative (positive) charge, the electrostatic potential and the pressure function, respectively. The functions φ, ψ are determined by the entropy density σ, h given in (2.4) in light of the relations (2.8) and (2.11). The physical parameters are the viscosity coefficients of fluid μ, μ', the dielectric constant ϵ, the charge of one electron e, the diffusion coefficient of negative (positive) ions $D_v(D_w)$ and the valence of negative (positive) ions $z_v(z_w)$.

Now, we review the history about the Poisson–Nernst–Planck (PNP) system. The original PNP model can be read as the following diffusion equations

$$\begin{cases} v_t = \operatorname{div}\left[D_v \left(\nabla v + \dfrac{z_v e}{k_B T} v \nabla \phi \right) \right], \\ w_t = \operatorname{div}\left[D_w \left(\nabla w + \dfrac{z_w e}{k_B T} w \nabla \phi \right) \right], \\ -\epsilon \Delta \phi = z_v e v + z_w e w, \end{cases} \tag{1.2}$$

with the corresponding energy dissipation law

$$\begin{aligned} \frac{d}{dt} E^{total} &:= \frac{d}{dt} \int k_B T \left(v \ln v + w \ln w \right) + \frac{1}{2} \phi (z_v e v + z_w e w) \\ &= -\int \frac{k_B T}{D_v} v \left| u_v \right|^2 + \frac{k_B T}{D_w} w \left| u_w \right|^2 := -\triangle. \end{aligned} \tag{1.3}$$

The above k_B, T and $u_v(u_w)$ denote the Boltzmann constant, the absolute temperature and the effective velocity of the negative (positive) charge, respectively. Then Hsieh et al. [27] derived the equations $(1.2)_{1,2}$ from the energy dissipation law (1.3) by using an energetic variational approach combined with the equations (2.12)–(2.13) and the Poisson equation $(1.2)_3$. Such a system (1.2) is usually used to describe the dynamics of the conductive fluid with the dilute charged particles [17,18,39]. We could also refer to [4,21,24,35,41,43,49,59] for existence and asymptotic behavior of solutions to (1.2). In addition, the quasi-neutral limit (i.e., $\epsilon \to 0$) of the PNP system has been extensively studied in [22,34,56,57] and the references therein. For the incompressible Poisson–Nernst–Planck–Navier–Stokes system, the readers could refer to [6, 30–32,37,50,60].

In spirit of [27], we consider a general energy dissipation law stated as (2.3)–(2.4) and (2.7). Here, we characterize the micro–macro interplay by introducing the combined total micro–macro energy and choose more general entropy densities $\sigma(v)$ and $h(w)$. Then, we derive the above system (1.1) by the EVA as in Section 2. Such a system could be used to describe the micro–macro dynamics of the compressible viscous conductive fluid with the *dilute* charged particles. Of course, there is another case that the charged particles are crowded (dense). Such a case can be also found in ion channels and the electrodes of batteries [13–15]. At this time, the derived model will be more complex, and an important phenomenon on the cross-diffusion happens, the readers could refer to our forthcoming paper [58].

In this paper, without loss of generality, we set $\mu = \epsilon = e = z_w = D_v = D_w = 1$, $\mu' = 0$ and $z_v = -1$ in (1.1). Then, the system (1.1) may be reduced to:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) - \Delta u - \nabla \operatorname{div} u = -\nabla\varphi(v) - \nabla\psi(w) + \Delta\phi\nabla\phi, \\ v_t + \operatorname{div}(vu) - \Delta\varphi(v) = -\operatorname{div}(v\nabla\phi), \\ w_t + \operatorname{div}(wu) - \Delta\psi(w) = \operatorname{div}(w\nabla\phi), \\ \Delta\phi = v - w. \end{cases} \tag{1.4}$$

Here we assume that the functions $p(\rho)$, $\varphi(v)$ and $\psi(w)$ are smooth ones such that

$$p'(\rho) > 0 \text{ for } \rho > 0,\ \varphi'(v),\ \psi'(w) > 0 \text{ for } v, w \geqslant 0. \tag{1.5}$$

Without loss of generality, we assume $p'(1) = \varphi'(1) = \psi'(1) = 1$. We look for the solutions $(\rho, u, v, w)(x, t)$ to the Cauchy problem for (1.4) with the initial data

$$(\rho, u, v, w)(x, t)\,|_{t=0} = (\rho_0, u_0, v_0, w_0)(x),\ x \in \mathbb{R}^3, \tag{1.6}$$

and the far-field behavior

$$\lim_{|x|\to+\infty} (\rho, u, v, w)(x, t) = (1, 0, 1, 1). \tag{1.7}$$

In addition, the Poisson equation $(1.4)_5$ implies the electrical neutrality of the far-field, i.e.,

$$\lim_{|x|\to+\infty} \phi(x, t) = 0. \tag{1.8}$$

We define the perturbation by

$$\varrho = \rho - 1,\ u = u,\ V = v - 1,\ W = w - 1,\ \phi = \phi.$$

Then, the Cauchy problem (1.4)–(1.8) becomes

$$\begin{cases} \varrho_t + \operatorname{div}((\varrho+1)u) = 0, \\ (\varrho+1)\,(u_t + u\cdot\nabla u) + p'(\varrho+1)\nabla\varrho - \Delta u - \nabla \operatorname{div} u \\ \quad = -\varphi'(V+1)\nabla V - \psi'(W+1)\nabla W + \Delta\phi\nabla\phi, \\ V_t + \operatorname{div}((V+1)u) - \Delta\varphi(V+1) = -\operatorname{div}\left((V+1)\nabla\phi\right), \\ W_t + \operatorname{div}((W+1)u) - \Delta\psi(W+1) = \operatorname{div}\left((W+1)\nabla\phi\right), \\ \Delta\phi = V - W, \\ (\varrho, u, V, W)\,|_{t=0} = (\varrho_0, u_0, V_0, W_0). \end{cases} \tag{1.9}$$

Since the perturbed system (1.9) is a new coupled hyperbolic–parabolic–elliptic equations, the analyses for its well-posedness will be relatively complicated. Firstly, we develop a simple approximation scheme to prove the local solution as in Section 3. Then we establish the refined a priori estimates by making some detailed energy estimates. Thus, the global solution is obtained by combing the local solution and the a priori estimates as well as a continuous argument. To prove the optimal time decay rates, we do an important observation. We find that the electrostatic potential ϕ satisfies a special equation stated as (5.12). Then some effective energy estimates could be established as Lemmas 5.1 and 5.2. With the help of Lemmas 5.1 and 5.2, we can prove the optimal decay rates of the solution and its derivatives of any order.

Notation In this paper, we use $H^s(\mathbb{R}^3)$, $s \in \mathbb{R}$ to denote the usual Sobolev spaces with norm $\|\cdot\|_{H^s}$ and $L^p(\mathbb{R}^3)$, $1 \leqslant p \leqslant \infty$ to denote the usual L^p spaces with norm $\|\cdot\|_{L^p}$. The symbol ∇^ℓ with an integer $\ell \geqslant 0$ stands for the usual any spatial derivatives of order ℓ. When $\ell < 0$ or ℓ is not a positive integer, ∇^ℓ stands for Λ^ℓ defined by $\Lambda^\ell f := \mathcal{F}^{-1}(|\xi|^\ell \mathcal{F} f)$, where $\mathcal{F}$ is the usual Fourier transform operator and $\mathcal{F}^{-1}$ is its inverse. In particular, we use $\langle\cdot,\cdot\rangle$ to denote the standard L^2 inner product in $\mathbb{R}^3$.

We then recall the homogeneous Besov spaces, cf. [3]. Let $g \in C_0^\infty(\mathbb{R}^3_\xi)$ be such that $g(\xi) = 1$ when $|\xi| \leqslant 1$ and $g(\xi) = 0$ when $|\xi| \geqslant 2$. Let $\tilde{g}(\xi) := g(\xi) - g(2\xi)$ and $\tilde{g}_j(\xi) := \tilde{g}(2^{-j}\xi)$ for $j \in \mathbb{Z}$. Then by the construction, $\sum_{j\in\mathbb{Z}} \tilde{g}_j(\xi) = 1$ if $\xi \neq 0$. We define $\dot{\Delta}_j f := \mathcal{F}^{-1}(\tilde{g}_j) * f$, then for $s \in \mathbb{R}$ and $1 \leqslant p, r \leqslant \infty$, we define the homogeneous Besov spaces $\dot{B}^s_{p,r}(\mathbb{R}^3)$ with norm $\|\cdot\|_{\dot{B}^s_{p,r}}$ defined by

$$\|f\|_{\dot{B}^s_{p,r}} := \Big(\sum_{j\in\mathbb{Z}} 2^{rsj} \|\dot{\Delta}_j f\|^r_{L^p}\Big)^{\frac{1}{r}}.$$

Particularly, if $r = \infty$, then

$$\|f\|_{\dot{B}^s_{p,\infty}} := \sup_{j\in\mathbb{Z}} 2^{sj} \left\|\dot{\Delta}_j f\right\|_{L^p}.$$

Throughout this paper we let C denote some positive universal constants. We will use $a \lesssim b$ if $a \leqslant Cb$. We use C_0 to denote the constants depending on the initial data. For simplicity, we write $\|(A, B)\|_X := \|A\|_X + \|B\|_X$ and $\int f := \int_{\mathbb{R}^3} f\,dx$.

The denotation $\mathcal{C}^k(0, T; B)(k \geqslant 0)$ denotes the space of B-valued k-times continuously differentiable functions on $[0, T]$, $\mathcal{L}^k_\infty(0, T; B)(k \geqslant 0)$ denotes the space of B-valued k-times boundedly differentiable functions on $[0,T]$, and $\mathcal{L}_2(0,T;B)$ denotes the space of B-valued L^2-functions on $(0,T)$.

Our main results about the large local solution and the small global solution and the time-decay rates are stated in the following theorems:

Theorem 1.1. *Denote $U(x,t)=(\varrho,u,V,W)(x,t)$. Assume $U_0:=(\varrho_0,u_0,V_0,W_0)\in H^3$ and $\inf_{x\in\mathbb{R}^3}\varrho(x,0)>-1$. Then there exists a positive T (suitably small) such that the Cauchy problem (1.9) has a unique solution*

$$\begin{cases}\varrho(t)\in\mathcal{C}^0(0,T;H^3)\cap\mathcal{C}^1(0,T;H^2),\\ (u,V,W)(t)\in\mathcal{C}^0(0,T;H^3)\cap\mathcal{C}^1(0,T;H^1)\cap\mathcal{L}_2(0,T;H^4),\end{cases}$$

which satisfies for any $t\in[0,T]$ and some $\nu>0$,

$$\varrho(t)>-1,\ \textit{and}\ \|U(t)\|_{H^3},\ \left(\nu\int_0^t\|\nabla(u,V,W)(\tau)\|_{H^3}^2\,d\tau\right)^{1/2}\leqslant C_1\|U_0\|_{H^3},$$

where $C_1>1$ is some fixed positive constant.

Theorem 1.2. *Assume that $U_0\in H^k$ for an integer $k\geqslant 3$. If the initial H^3 norm $\|U_0\|_{H^3}$ is sufficiently small, then the Cauchy problem (1.9) admits a unique global solution $(U,\nabla\phi)(t)$ satisfying for all $t\geqslant 0$ and $3\leqslant\ell\leqslant k$,*

$$\|U(t)\|_{H^\ell}+\left(\int_0^t\|\nabla\varrho(\tau)\|_{H^{\ell-1}}^2+\|\nabla(u,V,W)(\tau)\|_{H^\ell}^2+\|\Delta\phi(\tau)\|_{H^\ell}^2\,d\tau\right)^{1/2}\leqslant C\|U_0\|_{H^\ell}. \tag{1.10}$$

If we additionally assume that $\left\|\nabla^{-1}(V_0-W_0)\right\|_{L^2}$ is sufficiently small and $U_0\in\dot{B}^{-s}_{2,\infty}$ with $0<s\leqslant 3/2$, then for all $t\geqslant 0$,

$$\left\|\nabla^\ell U(t)\right\|_{H^{k-\ell}}\leqslant C_0(1+t)^{-\frac{\ell+s}{2}}\ \textit{for}\ 0\leqslant\ell\leqslant k-1 \tag{1.11}$$

and

$$\left\|\nabla^\ell\nabla\phi(t)\right\|_{L^2}\leqslant C_0(1+t)^{-\frac{\ell+1+s}{2}}\ \textit{for}\ 0\leqslant\ell\leqslant k-2. \tag{1.12}$$

We give some remarks about our results.

Remark 1.3. We claim that the smallness of $\left\|\nabla^{-1}(V_0-W_0)\right\|_{L^2}$ could be removed by assuming only $\nabla^{-1}(V_0-W_0)\in L^2$. At this moment, we could prove that the decay rates (1.11) hold for $\ell=0,1$. But, it is hard to obtain the higher-order decay. And such the restriction $\nabla^{-1}(V_0-W_0)\in L^2$ may be reasonable since it could be regarded as the restriction for the initial electric field $\nabla\phi_0$ by the equalities

$$\left\|\nabla^{-1}(V_0 - W_0)\right\|_{L^2} = \left\|\nabla^{-1}\Delta\phi_0\right\|_{L^2} = \|\nabla\phi_0\|_{L^2}.$$

Remark 1.4. We claim that all the decay rates (1.11) with $s = 3/2$ are optimal in sense that it is consistent with the decay of the heat kernel.

Remark 1.5. In fact, our decay rates (1.11)–(1.12) with $s = 3(1/p - 1/2)$ also follow if we replace $U_0 \in \dot{B}^{-s}_{2,\infty}$ $(0 < s \leqslant 3/2)$ by $U_0 \in L^p$ $(1 \leqslant p < 2)$ since the embedding $L^p \subset \dot{B}^{-s}_{2,\infty}$ for $1 \leqslant p < 2$ by Lemma A.5.

This paper is organized as follows. In Section 2, we use the EVA to derive the generalized PNP-NS equations (1.1). Next, the local solution and the a priori estimates are shown in Sections 3–4, respectively. In Section 5, we mainly prove the global solution and obtain the decay rates of solutions. We list some analytic tools in Appendix A, which are often used in this paper.

2. Derivation of models

In this section, we will derive a new hydrodynamic model by using an Energetic Variational Approach (EVA) together with a prescribed energy dissipation law, which could be regarded as a generalized PNP-NS system. Such the system could describe the dynamics of the compressible conductive fluid with the dilute charged particles.

We first recall that the first and second laws of thermodynamics. In mathematics, they could be read respectively as:

$$\frac{d(\mathcal{K}+\mathcal{I})}{dt} = \frac{d\mathcal{W}}{dt} + \frac{d\mathcal{Q}}{dt} \tag{2.1}$$

and

$$T\frac{d\mathcal{S}}{dt} = \frac{d\mathcal{Q}}{dt} + \triangle, \tag{2.2}$$

where $\mathcal{K}, \mathcal{I}, \mathcal{W}, \mathcal{Q}, \mathcal{S}$ and $\triangle$ is the kinetic energy, the internal energy, the work of external force, the heat, the entropy and the entropy production, respectively.

In this paper, we assume that there is being isothermal and no external forces in the thermodynamical process. Then, we combine (2.1) and (2.2) to obtain the energy dissipation law:

$$\frac{d}{dt}E^{total} = -\triangle, \tag{2.3}$$

where $E^{total} = \mathcal{K} + \mathcal{I} - T\mathcal{S}$ is the total energy. In our case, we could choose the total energy

$$\begin{aligned} E^{total} &= \underbrace{\int \sigma(v) + h(w) + \frac{\epsilon}{2}|\nabla\phi|^2\,dx}_{\text{microscopic}} + \underbrace{\int \frac{\rho}{2}|u|^2 + \omega(\rho)\,dx}_{\text{macroscopic}} \\ &= \int \sigma(v) + h(w)\,dx + \frac{e^2}{2\epsilon}\iint G(x-y)(z_v v + z_w w)(x)(z_v v + z_w w)(y)\,dydx \\ &\quad + \int \frac{\rho}{2}|u|^2 + \omega(\rho)\,dx, \end{aligned} \tag{2.4}$$

where we have used Gauss's law:

$$-\epsilon \Delta\phi = z_v e v + z_w e w. \tag{2.5}$$

By solving the Poisson equation (2.5), we obtain

$$\phi(x) = \frac{e}{\epsilon} \int G(x-y)(z_v v + z_w w)(y)\, dy, \tag{2.6}$$

where the kernel $G(\cdot) = \frac{1}{4\pi|\cdot|}$ is the fundamental solution of $-\Delta$ in $\mathbb{R}^3$. Next, we choose the dissipation functional (the entropy production)

$$\triangle = \int \frac{v}{D_v}|u_v - u|^2 + \frac{w}{D_w}|u_w - u|^2 + \mu|\nabla u|^2 + (\mu + \mu')|\operatorname{div} u|^2\, dx, \tag{2.7}$$

where μ and μ' satisfy the usual physical assumptions:

$$\mu > 0, \quad \mu' + \frac{2}{3}\mu \geqslant 0.$$

Now, we begin to use the EVA to derive the equations of motion, as in [19,27,60]. We first state the sketch of EVA. The first step of EVA is to define some action functionals in light of the given total energy E^{total} and dissipation $\triangle$. Next, we find out the conservative force and the dissipative force by computing the variations for the well-defined action functionals with the help of the Least Action Principle (LAP) (or Hamilton's principle) [1,2,19,25,28] and the Maximum Dissipation Principle (MDP) (or Onsager's principle) [19,28,44–46]. Finally, we could obtain the motion equations by the total force balance. So, we first define the action functionals as

$$\mathcal{A}_1 := -\int_0^{t^*}\int \sigma(v) + h(w) + \frac{e^2}{2\epsilon}\int G(x-y)(z_v v + z_w w)(x)(z_v v + z_w w)(y)\, dy dx dt,$$

$$\mathcal{A}_2 := \int_0^{t^*}\int \frac{\rho}{2}|u|^2 - \omega(\rho)\, dx dt.$$

Taking the variation of $\mathcal{A}_1$ (for any smooth $\tilde{v}$ with compact support) with respect to v, by the LAP, we obtain

$$0 = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \mathcal{A}_1(v + \varepsilon\tilde{v}) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\left(-\int_0^{t^*}\int \sigma(v + \varepsilon\tilde{v})\, dx dt\right)$$

$$+ \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\left[-\frac{e^2}{\epsilon}\int_0^{t^*}\iint G(x-y)(z_v v + z_v\varepsilon\tilde{v} + z_w w)(x)(z_v v + z_w w)(y) dy dx dt\right]$$

$$= \int_0^{t^*} \int \left(-\sigma'(v) - z_v e\phi\right) \tilde{v}\, dx dt,$$

which implies ($\tilde{v}$ is arbitrary),

$$\frac{\delta \mathcal{A}_1}{\delta v} = -\sigma'(v) - z_v e\phi \Rightarrow F_{\text{conservative}, v} = v\nabla \frac{\delta \mathcal{A}_1}{\delta v} = -\nabla\varphi(v) - z_v e v \nabla\phi,$$

where

$$\varphi(v) := \sigma'(v)v - \sigma(v). \tag{2.8}$$

Taking the variation of $\frac{1}{2}\triangle$ (for any smooth $\tilde{u}$ with compact support) with respect to u_v, by the MDP, we obtain

$$0 = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \frac{1}{2}\triangle(u_v + \varepsilon\tilde{u}) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \frac{1}{2}\int \frac{v}{D_v} \left|u_v + \varepsilon\tilde{u} - u\right|^2 dx = \int \frac{v}{D_v}(u_v - u)\cdot \tilde{u}\, dx,$$

which implies ($\tilde{u}$ is arbitrary),

$$\frac{\delta(\frac{1}{2}\triangle)}{\delta u_v} = \frac{v}{D_v}(u_v - u) = F_{\text{dissipative}, v}.$$

By the total force balance for the negative charge, we obtain

$$F_{\text{conservative}, v} = v\nabla\frac{\delta \mathcal{A}_1}{\delta v} = \frac{\delta(\frac{1}{2}\triangle)}{\delta u_v} = F_{\text{dissipative}, v},$$

i.e.,

$$v u_v = v u - D_v \nabla\varphi(v) - D_v z_v e v \nabla\phi. \tag{2.9}$$

For the positive charge, we similarly have

$$F_{\text{conservative}, w} = w\nabla\frac{\delta \mathcal{A}_1}{\delta w} = \frac{\delta(\frac{1}{2}\triangle)}{\delta u_w} = F_{\text{dissipative}, w},$$

i.e.,

$$w u_w = w u - D_w \nabla\psi(w) - D_w z_w e w \nabla\phi, \tag{2.10}$$

where

$$\psi(w) := h'(w)w - h(w). \tag{2.11}$$

On the other hand, we have the equations of conservation for the negative and positive charge,

$$v_t + \operatorname{div}(v u_v) = 0, \tag{2.12}$$

$$w_t + \operatorname{div}(w u_w) = 0. \tag{2.13}$$

Plugging (2.9), (2.10) into (2.12), (2.13), respectively, we obtain

$$v_t + \operatorname{div}(vu) = \operatorname{div}\left(D_v \nabla \varphi(v) + D_v z_v e v \nabla \phi\right) \tag{2.14}$$

and

$$w_t + \operatorname{div}(wu) = \operatorname{div}\left(D_w \nabla \psi(w) + D_w z_w e w \nabla \phi\right). \tag{2.15}$$

For the macroscopic action functional $\mathcal{A}_2$, we refer to [19] to obtain the macroscopic conservative force

$$F_{\text{macro-conservative}} = \frac{\delta \mathcal{A}_2}{\delta x} = -\left((\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho)\right),$$

where

$$p(\rho) := \omega'(\rho)\rho - \omega(\rho).$$

And taking the variation of $\frac{1}{2}\triangle$ (for any smooth $\tilde{u}$ with compact support) with respect to u, by the MDP again, we obtain

$$\begin{aligned}
0 &= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \frac{1}{2}\triangle(u+\varepsilon\tilde{u}) \\
&= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \frac{1}{2}\int \bigg(\frac{v}{D_v}|u_v - u - \varepsilon\tilde{u}|^2 + \frac{w}{D_w}|u_w - u - \varepsilon\tilde{u}|^2 \\
&\qquad\qquad + \mu|\nabla u + \varepsilon\nabla\tilde{u}|^2 + (\mu+\mu')|\operatorname{div} u + \varepsilon \operatorname{div}\tilde{u}|^2\bigg)\, dx, \\
&= \int \frac{v}{D_v}(u_v - u)\cdot(-\tilde{u}) + \frac{w}{D_w}(u_w - u)\cdot(-\tilde{u}) + \mu\nabla u : \nabla\tilde{u} + (\mu+\mu')\operatorname{div} u \operatorname{div}\tilde{u}\, dx, \\
&= \int \left(-\mu\Delta u - (\mu+\mu')\nabla \operatorname{div} u + \frac{v}{D_v}(u-u_v) + \frac{w}{D_w}(u-u_w)\right)\cdot\tilde{u}\, dx,
\end{aligned}$$

which implies ($\tilde{u}$ is arbitrary),

$$F_{\text{macro-dissipative}} = \frac{\delta(\frac{1}{2}\triangle)}{\delta u} = -\mu\Delta u - (\mu+\mu')\nabla \operatorname{div} u + \frac{v}{D_v}(u-u_v) + \frac{w}{D_w}(u-u_w).$$

By the macroscopic force balance, we obtain

$$F_{\text{macro-conservative}} = \frac{\delta \mathcal{A}_2}{\delta x} = \frac{\delta(\frac{1}{2}\triangle)}{\delta u} = F_{\text{macro-dissipative}},$$

i.e.,

$$(\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) = \mu \Delta u + (\mu + \mu')\nabla \operatorname{div} u + \frac{v}{D_v}(u_v - u) + \frac{w}{D_w}(u_w - u).$$

Plugging (2.9)–(2.10) into the above equation, by the Poisson equation (2.5), we obtain

$$(\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) = \mu \Delta u + (\mu + \mu')\nabla \operatorname{div} u - \nabla\varphi(v) - \nabla\psi(w) + \epsilon\Delta\phi\nabla\phi. \tag{2.16}$$

On the other hand, by the law of conservation of the macroscopic mass, we obtain

$$\rho_t + \operatorname{div}(\rho u) = 0. \tag{2.17}$$

Finally, we collect the equations (2.5) and (2.14)–(2.17) to obtain the compressible PNP-NS equations

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) \\ \quad = \mu \Delta u + (\mu + \mu')\nabla \operatorname{div} u - \nabla\varphi(v) - \nabla\psi(w) + \epsilon\Delta\phi\nabla\phi, \\ v_t + \operatorname{div}(vu) = \operatorname{div}\left(D_v \nabla\varphi(v) + D_v z_v e v \nabla\phi\right), \\ w_t + \operatorname{div}(wu) = \operatorname{div}\left(D_w \nabla\psi(w) + D_w z_w e w \nabla\phi\right), \\ -\epsilon\Delta\phi = z_v e v + z_w e w. \end{cases} \tag{2.18}$$

Now, we can conclude the following proposition.

Proposition 2.1. *Assume that the Least Action Principle and the Maximum Dissipation Principle. Let the total energy be*

$$E^{total} = \int \sigma(v) + h(w) + \frac{\epsilon}{2}|\nabla\phi|^2 + \frac{\rho}{2}|u|^2 + \omega(\rho)\, dx$$

and let the dissipation functional be

$$\triangle = \int \frac{v}{D_v}|u_v - u|^2 + \frac{w}{D_w}|u_w - u|^2 + \mu|\nabla u|^2 + (\mu + \mu')|\operatorname{div} u|^2\, dx.$$

Then, we could derive the above PNP-NS equations (2.18) by using an Energetic Variational Approach. Moreover, the equations (2.18) satisfies the energy dissipation law:

$$\frac{d}{dt}E^{total} = -\triangle. \tag{2.19}$$

Proof. Multiplying the first four equations in (2.18) by $\omega'(\rho), u, \sigma'(v) + z_v e\phi, h'(w) + z_w e\phi$, respectively, summing them up and then integrating over $\mathbb{R}^3$, we obtain (2.19). Given the previous derivations in this section, we have proved that the above PNP-NS equations (2.18) follows from the Energetic Variational Approach. □

Remark 2.2. By solving the ODE: $p(\rho) = \omega'(\rho)\rho - \omega(\rho)$ directly, we easily obtain

$$\omega(\rho) = \begin{cases} \rho \ln \rho, & \text{if } p(\rho) = \rho, \\ \rho^\gamma, & \text{if } p(\rho) = (\gamma - 1)\rho^\gamma, \ \gamma > 1. \end{cases}$$

Remark 2.3. The variations $\frac{\delta E^{total}}{\delta v} = \sigma'(v) + z_v e\phi$ and $\frac{\delta E^{total}}{\delta w} = h'(w) + z_w e\phi$ are called the (electro)chemical potential [16] of negative and positive ions, respectively.

3. Local solution

throughout this section, we simply denote

$$U = (\varrho, u, V, W), \ U_n = (\varrho_n, u_n, V_n, W_n), \ n = 0, 1, 2 \ldots$$

and

$$g_\epsilon = J_\epsilon * g, \ g_{0,\epsilon} = J_\epsilon * g_0,$$

where J_ϵ is the Friedrichs' mollifier for some $\epsilon > 0$, which was introduced by Friedrichs [20].

We also introduce the definitions of some function spaces:

Definition 3.1. For $l = 2, 3$, we denote the function spaces

$$\begin{aligned} \mathcal{E}(0, T; H^l) := \{ & U : \varrho(x,t) \in \mathcal{C}^0(0, T; H^l) \cap \mathcal{C}^1(0, T; H^{l-1}), \\ & (u, V, W)(x,t) \in \mathcal{C}^0(0, T; H^l) \cap \mathcal{C}^1(0, T; H^{l-2}) \} \end{aligned}$$

and

$$\begin{aligned} \mathfrak{L}(0, T; H^l) := \{ & U : \varrho(x,t) \in \mathcal{L}^0_\infty(0, T; H^l) \cap \mathcal{L}^1_\infty(0, T; H^{l-1}), \\ & (u, V, W)(x,t) \in \mathcal{L}^0_\infty(0, T; H^l) \cap \mathcal{L}^1_\infty(0, T; H^{l-2}) \}. \end{aligned}$$

In this section, we will establish the local solution. There are three key points. First, we need to reformulate properly the original system. Thus, the reformulated system (3.1) is a coupled hyperbolic–parabolic equations with some non-local terms. Next, we construct the corresponding approximating system. To solve the approximating system, we turn to solve the linearized system, see Subsections 3.1–3.2. Here, we refer to some energy estimates for the single hyperbolic equation in [40], see Lemma 3.3 and Lemma 3.6. However, we need to carefully establish some detailed energy estimates for the parabolic equations with the non-local terms, see Lemma 3.4 and Proposition 3.11. Lastly, we can prove the solution to the approximating system converges to the solution to the Cauchy problem (1.9) ((3.1)), see Subsection 3.3. Hence, the local solution is obtained.

So, we first rewrite (1.9) as

$$\begin{cases} \varrho_t + u \cdot \nabla \varrho + (\varrho + 1) \operatorname{div} u = 0, \\ u_t - \dfrac{1}{\varrho + 1} \Delta u - \dfrac{1}{\varrho + 1} \nabla \operatorname{div} u = g^1, \\ V_t - \varphi'(V+1) \Delta V = g^2, \\ W_t - \psi'(W+1) \Delta W = g^3, \\ U \,|_{t=0} = U_0, \end{cases} \tag{3.1}$$

where

$$\begin{aligned} g^1 :=& -u \cdot \nabla u - \frac{p'(\varrho+1)}{\varrho+1} \nabla \varrho + \frac{V}{\varrho+1} \nabla \Delta^{-1}(V - W) \\ & - \frac{W}{\varrho+1} \nabla \Delta^{-1}(V - W) - \frac{\varphi'(V+1)}{\varrho+1} \nabla V - \frac{\psi'(W+1)}{\varrho+1} \nabla W, \\ g^2 :=& -\operatorname{div} u - V + W + \varphi''(V+1) |\nabla V|^2 - \operatorname{div}(Vu) \\ & - V(V - W) - \nabla V \cdot \nabla \Delta^{-1}(V - W), \\ g^3 :=& -\operatorname{div} u - W + V + \psi''(W+1) |\nabla W|^2 - \operatorname{div}(Wu) \\ & + W(V - W) + \nabla W \cdot \nabla \Delta^{-1}(V - W). \end{aligned}$$

In the following, we will solve (3.1) by constructing approximating solutions. So, we need to study the approximate system corresponding to (3.1):

$$\begin{cases} \partial_t \varrho_n + u_{n-1} \cdot \nabla \varrho_n + (\varrho_{n-1} + 1) \operatorname{div} u_n = 0, \\ \partial_t u_n - \dfrac{1}{\varrho_{n-1} + 1} \Delta u_n - \dfrac{1}{\varrho_{n-1} + 1} \nabla \operatorname{div} u_n = g^1_{n-1}, \\ \partial_t V_n - \varphi'(V_{n-1}+1) \Delta V_n = g^2_{n-1}, \\ \partial_t W_n - \psi'(W_{n-1}+1) \Delta W_n = g^3_{n-1}, \\ U_n \,|_{t=0} = U_0, \ U_1 \equiv U_0, \ n = 2, 3, 4, \dots. \end{cases} \tag{3.2}$$

Here the nonlinear functions:

$$\begin{aligned} g^1_{n-1} :=& -u_{n-1} \cdot \nabla u_{n-1} - \frac{p'(\varrho_{n-1}+1)}{\varrho_{n-1}+1} \nabla \varrho_{n-1} + \frac{V_{n-1}}{\varrho_{n-1}+1} \nabla \Delta^{-1}(V_{n-1} - W_{n-1}) \\ & - \frac{W_{n-1}}{\varrho_{n-1}+1} \nabla \Delta^{-1}(V_{n-1} - W_{n-1}) - \frac{\varphi'(V_{n-1}+1)}{\varrho_{n-1}+1} \nabla V_{n-1} - \frac{\psi'(W_{n-1}+1)}{\varrho_{n-1}+1} \nabla W_{n-1}, \\ g^2_{n-1} :=& -\operatorname{div} u_{n-1} - V_{n-1} + W_{n-1} + \varphi''(V_{n-1}+1) |\nabla V_{n-1}|^2 - \operatorname{div}(V_{n-1} u_{n-1}) \\ & - V_{n-1}(V_{n-1} - W_{n-1}) - \nabla V_{n-1} \cdot \nabla \Delta^{-1}(V_{n-1} - W_{n-1}), \\ g^3_{n-1} :=& -\operatorname{div} u_{n-1} - W_{n-1} + V_{n-1} + \psi''(W_{n-1}+1) |\nabla W_{n-1}|^2 - \operatorname{div}(W_{n-1} u_{n-1}) \\ & + W_{n-1}(V_{n-1} - W_{n-1}) + \nabla W_{n-1} \cdot \nabla \Delta^{-1}(V_{n-1} - W_{n-1}). \end{aligned}$$

To solve the approximate system (3.2), we linearize (3.1) at $(\eta, \varpi, \tilde{v}, \tilde{w})$ to obtain the linearized system:

$$\begin{cases} L^0_{\eta,\varpi}(\varrho, u) := \varrho_t + \varpi \cdot \nabla \varrho + (\eta+1)\operatorname{div} u = f, \\ L^1_\eta(u) := u_t - \dfrac{1}{\eta+1}\Delta u - \dfrac{1}{\eta+1}\nabla \operatorname{div} u = g^1, \\ L^2_{\tilde{v}}(V) := V_t - \varphi'(\tilde{v}+1)\Delta V = g^2, \\ L^3_{\tilde{w}}(W) := W_t - \psi'(\tilde{w}+1)\Delta W = g^3. \end{cases} \tag{3.3}$$

Here we could regard the functions η, $\varpi = (\varpi^1, \varpi^2, \varpi^3)^T$, $\tilde{v}$, $\tilde{w}$, f, $g^1 = (g^{11}, g^{12}, g^{13})^T$, g^2 and g^3 as known.

Next, we begin to establish the energy estimates for the linearized equations (3.3) so that we can easily solve the linearized equations in Subsection 3.2.

3.1. Energy estimates for linearized equations

In this subsection, we also consider the single linear equation

$$L_\varpi(\varrho) := \varrho_t + \varpi \cdot \nabla \varrho = \tilde{f}, \tag{3.4}$$

where $\tilde{f} := f - (\eta+1)\operatorname{div} u$ is looked as a known function. Throughout this subsection, we denote for some $T > 0$,

$$E = \sup_{0 \leqslant t \leqslant T} \|(\eta, \varpi, \tilde{v}, \tilde{w})(t)\|_{H^3}.$$

We first show some estimates for the commutators of the operators L_ϖ, $L^0_{\eta,\varpi}$, L^1_η, $L^2_{\tilde{v}}$, $L^3_{\tilde{w}}$, $\nabla^m (m \geqslant 1)$ and mollifiers J_ϵ, which will be used later.

Lemma 3.2. *Assume that for some $T > 0$ and some constant χ,*

$$\begin{cases} (\eta, \varpi, \tilde{v}, \tilde{w})(t) \in \mathcal{L}^0_\infty(0, T; H^3), \ \eta(t) \geqslant \chi > -1, \\ U(t) \in \mathfrak{L}(0, T; H^l) \text{ for } l = 2 \text{ or } 3. \end{cases}$$

Then we have for all $t \in [0, T]$ and $0 \leqslant m \leqslant l-1$,

$$\begin{cases} \|\nabla^m L_\varpi(\varrho) - L_\varpi(\nabla^m \varrho)\|_{L^2} \leqslant CE\,\|\nabla \varrho\|_{H^m}, \\ \left\|\nabla^m L^0_{\eta,\varpi}(\varrho, u) - L^0_{\eta,\varpi}(\nabla^m \varrho, \nabla^m u)\right\|_{L^2} \leqslant CE\,\|\nabla(\varrho, u)\|_{H^m}, \\ \left\|\nabla^m L^1_\eta(u) - L^1_\eta(\nabla^m u)\right\|_{L^2} \leqslant CE\,\|\nabla u\|_{H^m}, \\ \left\|\nabla^m L^2_{\tilde{v}}(V) - L^2_{\tilde{v}}(\nabla^m V)\right\|_{L^2} \leqslant CE\,\|\nabla V\|_{H^m}, \\ \left\|\nabla^m L^3_{\tilde{w}}(W) - L^3_{\tilde{w}}(\nabla^m W)\right\|_{L^2} \leqslant CE\,\|\nabla W\|_{H^m}, \end{cases} \tag{3.5}$$

and as $\epsilon \to 0$,

$$
\begin{cases}
\|J_\epsilon * L_\varpi(\varrho) - L_\varpi(J_\epsilon * \varrho)\|_{H^{m+1}} \to 0, \\
\left\|J_\epsilon * L^0_{\eta,\varpi}(\varrho, u) - L^0_{\eta,\varpi}(J_\epsilon * \varrho, J_\epsilon * u)\right\|_{H^{m+1}} \to 0, \\
\left\|J_\epsilon * L^1_\eta(u) - L^1_\eta(J_\epsilon * u)\right\|_{H^m} \to 0, \\
\left\|J_\epsilon * L^2_{\tilde{v}}(V) - L^2_{\tilde{v}}(J_\epsilon * V)\right\|_{H^m} \to 0, \\
\left\|J_\epsilon * L^3_{\tilde{w}}(W) - L^3_{\tilde{w}}(J_\epsilon * W)\right\|_{H^m} \to 0.
\end{cases}
\tag{3.6}
$$

If we assume that $U(t) \in \mathfrak{L}(0, T; H^3)$ *and* $(\eta', \varpi', \tilde{v}', \tilde{w}')(t) \in \mathcal{L}^0_\infty(0, T; H^3)$ *with* $\eta'(t) \geqslant \chi > -1$*, then we have*

$$
\begin{cases}
\left\|L^0_{\eta,\varpi}(\varrho, u) - L^0_{\eta',\varpi'}(\varrho, u)\right\|_{H^2} \leqslant C \sup\limits_{0\leqslant t\leqslant T} \|(\varrho, u)\|_{H^3} \left\|(\eta - \eta', \varpi - \varpi')\right\|_{H^2}, \\
\left\|L^1_\eta(u) - L^1_{\eta'}(u)\right\|_{H^1} \leqslant C \sup\limits_{0\leqslant t\leqslant T} \|u\|_{H^3} \left\|\eta - \eta'\right\|_{H^2}, \\
\left\|L^2_{\tilde{v}}(V) - L^2_{\tilde{v}'}(V)\right\|_{H^1} \leqslant C \sup\limits_{0\leqslant t\leqslant T} \|V\|_{H^3} \left\|\tilde{v} - \tilde{v}'\right\|_{H^2}, \\
\left\|L^3_{\tilde{w}}(W) - L^3_{\tilde{v}'}(W)\right\|_{H^1} \leqslant C \sup\limits_{0\leqslant t\leqslant T} \|W\|_{H^3} \left\|\tilde{w} - \tilde{w}'\right\|_{H^2}.
\end{cases}
\tag{3.7}
$$

Proof. We can refer to Lemma 3.1 in [40, p. 74] by noting that the functions φ, ψ are smooth. $\square$

Now, we show the energy estimates for the linearized equations.

Lemma 3.3. *Let* $l = 1, 2$ *or* 3*. Assume that for some* $T > 0$*,*

$$
\varpi(t) \in \mathcal{L}^0_\infty(0, T; H^3), \;\; \tilde{f}(t) \in \mathcal{L}^0_\infty(0, T; H^l).
$$

If $\varrho(t) \in \mathcal{L}^0_\infty(0, T; H^l) \cap \mathcal{L}^1_\infty(0, T; H^{l-1})$ *solves Eq. (3.4), then for any* $t \in [0, T]$*,*

$$
\|\varrho(t)\|_{H^l} \leqslant e^{CEt} \left(\|\varrho_0\|_{H^l} + \int_0^t e^{-CE\tau} \left\|\tilde{f}(\tau)\right\|_{H^l} d\tau \right).
\tag{3.8}
$$

Proof. We can refer to Lemma 3.2 in [40, p. 77]. $\square$

Lemma 3.4. *Let* $l = 2$ *or* 3*. Assume that for some* $T > 0$ *and some constant* χ*,*

$$
\begin{cases}
(\eta, \tilde{v}, \tilde{w})(t) \in \mathcal{L}^0_\infty(0, T; H^3), \\
\eta(t) \geqslant \chi > -1, \\
(g^1, g^2, g^3)(t) \in \mathcal{L}^0_\infty(0, T; H^{l-1}).
\end{cases}
$$

If $(u, V, W)(t) \in \mathcal{L}^0_\infty(0, T; H^l) \cap \mathcal{L}^1_\infty(0, T; H^{l-2})$ *solves Eqs.* $(3.3)_2$*–*$(3.3)_4$*, then* $(u, V, W)(t) \in \mathcal{L}_2(0, T; H^{l+1})$ *and there exists a constant* $\nu > 0$ *such that for any* $t \in [0, T]$*,*

$$
\begin{aligned}
&\|(u, V, W)(t)\|_{L^2}^2 + \nu \int_0^t \|\nabla(u, V, W)(\tau)\|_{L^2}^2 \, d\tau \\
&\quad \leqslant e^{C(1+E^2)t} \left(\|(u_0, V_0, W_0)\|_{L^2}^2 + C \int_0^t \left\| (g^1, g^2, g^3)(\tau) \right\|_{L^2}^2 d\tau \right)
\end{aligned} \tag{3.9}
$$

and for $1 \leqslant k \leqslant l$,

$$
\begin{aligned}
&\left\| \nabla^k (u, V, W)(t) \right\|_{L^2}^2 + \nu \int_0^t \left\| \nabla^{k+1} (u, V, W)(\tau) \right\|_{L^2}^2 d\tau \\
&\quad \leqslant e^{CE^2 t} \left(\|\nabla(u_0, V_0, W_0)\|_{H^{k-1}}^2 + C \int_0^t \left\| (g^1, g^2, g^3)(\tau) \right\|_{H^{k-1}}^2 d\tau \right).
\end{aligned} \tag{3.10}
$$

Proof. Multiplying Eqs. $(3.3)_2$–$(3.3)_4$ by u, V, W, respectively, summing them up and integrating over $\mathbb{R}^3$, by the integration by parts and Cauchy's inequality, we obtain for any $\varepsilon > 0$,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \|(u, V, W)\|_{L^2}^2 + \left\langle \frac{1}{\eta+1}, |\nabla u|^2 + |\operatorname{div} u|^2 \right\rangle + \left\langle \varphi'(\tilde{v}+1), |\nabla V|^2 \right\rangle + \left\langle \psi'(\tilde{w}+1), |\nabla W|^2 \right\rangle \\
&= \left\langle \frac{1}{(\eta+1)^2} \nabla\eta, \nabla u \cdot u + \operatorname{div} u u \right\rangle + \left\langle g^1, u \right\rangle + \left\langle g^2, V \right\rangle + \left\langle g^3, W \right\rangle \\
&\quad - \left\langle \varphi''(\tilde{v}+1)\nabla\tilde{v}, \nabla V V \right\rangle - \left\langle \psi''(\tilde{w}+1)\nabla\tilde{w}, \nabla W W \right\rangle \\
&\leqslant C \|\nabla(\eta, \tilde{v}, \tilde{w})\|_{L^\infty} \|\nabla(u, V, W)\|_{L^2} \|(u, V, W)\|_{L^2} + \left\| (g^1, g^2, g^3) \right\|_{L^2} \|(u, V, W)\|_{L^2} \\
&\leqslant \varepsilon \|\nabla(u, V, W)\|_{L^2}^2 + C_\varepsilon (1+E^2) \|(u, V, W)\|_{L^2}^2 + C \left\| (g^1, g^2, g^3) \right\|_{L^2}^2.
\end{aligned}
$$

Taking the above ε sufficiently small, by (1.5), we obtain for some $\nu > 0$,

$$
\frac{d}{dt} \|(u, V, W)\|_{L^2}^2 + \nu \|\nabla(u, V, W)\|_{L^2}^2 \leqslant C(1+E^2) \|(u, V, W)\|_{L^2}^2 + C \left\| (g^1, g^2, g^3) \right\|_{L^2}^2.
$$

By Gronwall's inequality, the estimate (3.9) follows from the above inequality.

Next, we will prove the estimates (3.10) for $1 \leqslant k \leqslant l$. Applying $J_\epsilon *$ to Eqs. $(3.3)_2$–$(3.3)_4$, we have

$$
\begin{cases}
L_\eta^1(u_\epsilon) = g_\epsilon^1 + H_1^\epsilon, & H_1^\epsilon := L_\eta^1(u_\epsilon) - J_\epsilon * L_\eta^1(u), \\
L_{\tilde{v}}^2(V_\epsilon) = g_\epsilon^2 + M_1^\epsilon, & M_1^\epsilon := L_{\tilde{v}}^2(V_\epsilon) - J_\epsilon * L_{\tilde{v}}^2(V), \\
L_{\tilde{w}}^3(W_\epsilon) = g_\epsilon^3 + N_1^\epsilon, & N_1^\epsilon := L_{\tilde{w}}^3(W_\epsilon) - J_\epsilon * L_{\tilde{w}}^3(W).
\end{cases} \tag{3.11}
$$

Multiplying Eqs. $(3.11)_1$–$(3.11)_3$ by $-\Delta u_\epsilon$, $-\Delta V_\epsilon$, $-\Delta W_\epsilon$, respectively, summing them up and integrating over $\mathbb{R}^3$, by Cauchy's inequality, we obtain for any $\varepsilon > 0$,

$$
\begin{aligned}
&-\left\langle L^1_\eta(u_\epsilon), \Delta u_\epsilon\right\rangle - \left\langle L^2_{\tilde v}(V_\epsilon), \Delta V_\epsilon\right\rangle - \left\langle L^3_{\tilde w}(W_\epsilon), \Delta W_\epsilon\right\rangle \\
&= -\left\langle g^1_\epsilon + H^\epsilon_1, \Delta u_\epsilon\right\rangle - \left\langle g^2_\epsilon + M^\epsilon_1, \Delta V_\epsilon\right\rangle - \left\langle g^3_\epsilon + N^\epsilon_1, \Delta W_\epsilon\right\rangle \\
&\leqslant \varepsilon \left\|\Delta(u_\epsilon, V_\epsilon, W_\epsilon)\right\|^2_{L^2} + C_\varepsilon \left(\left\|(g^1_\epsilon, g^2_\epsilon, g^3_\epsilon)\right\|^2_{L^2} + \left\|(H^\epsilon_1, M^\epsilon_1, N^\epsilon_1)\right\|^2_{L^2}\right).
\end{aligned}
\tag{3.12}
$$

For the left-hand side of the above inequality, we have for some $\nu_0 > 0$,

$$
\begin{aligned}
&-\left\langle L^1_\eta(u_\epsilon), \Delta u_\epsilon\right\rangle - \left\langle L^2_{\tilde v}(V_\epsilon), \Delta V_\epsilon\right\rangle - \left\langle L^3_{\tilde w}(W_\epsilon), \Delta W_\epsilon\right\rangle \\
&= -\left\langle \partial_t u_\epsilon - \frac{1}{\eta+1}\Delta u_\epsilon - \frac{1}{\eta+1}\nabla \operatorname{div} u_\epsilon, \Delta u_\epsilon\right\rangle \\
&\quad - \left\langle \partial_t V_\epsilon - \varphi'(\tilde v+1)\Delta V_\epsilon, \Delta V_\epsilon\right\rangle - \left\langle \partial_t W_\epsilon - \psi'(\tilde w+1)\Delta W_\epsilon, \Delta W_\epsilon\right\rangle \\
&= \frac{1}{2}\frac{d}{dt}\left\|\nabla(u_\epsilon, V_\epsilon, W_\epsilon)\right\|^2_{L^2} + \left\langle \frac{1}{\eta+1}, |\Delta u_\epsilon|^2\right\rangle + \left\langle \varphi'(\tilde v+1), |\Delta V_\epsilon|^2\right\rangle \\
&\quad + \left\langle \psi'(\tilde w+1), |\Delta W_\epsilon|^2\right\rangle + \left\langle \frac{1}{\eta+1}\nabla \operatorname{div} u_\epsilon, \Delta u_\epsilon\right\rangle \\
&\geqslant \frac{1}{2}\frac{d}{dt}\left\|\nabla(u_\epsilon, V_\epsilon, W_\epsilon)\right\|^2_{L^2} + \nu_0 \left\|\Delta(u_\epsilon, V_\epsilon, W_\epsilon)\right\|^2_{L^2} + \left\langle \frac{1}{\eta+1}\nabla \operatorname{div} u_\epsilon, \Delta u_\epsilon\right\rangle.
\end{aligned}
\tag{3.13}
$$

By the integration by parts and Cauchy's inequality, the last term can be estimated by

$$
\begin{aligned}
\left\langle \frac{1}{\eta+1}\nabla \operatorname{div} u_\epsilon, \Delta u_\epsilon\right\rangle &\geqslant \left\langle \frac{1}{(\eta+1)^2}\operatorname{div} u_\epsilon, \nabla\eta\cdot\Delta u_\epsilon\right\rangle - \left\langle \frac{1}{(\eta+1)^2}\operatorname{div} u_\epsilon, \nabla\eta\cdot\nabla \operatorname{div} u_\epsilon\right\rangle \\
&\geqslant -\varepsilon\left\|\Delta u_\epsilon\right\|^2_{L^2} - C_\varepsilon E^2\left\|\nabla u_\epsilon\right\|^2_{L^2}
\end{aligned}
\tag{3.14}
$$

for any $\varepsilon > 0$. Plugging (3.13)–(3.14) into (3.12), we obtain for some $\nu > 0$,

$$
\begin{aligned}
&\frac{d}{dt}\left\|\nabla(u_\epsilon, V_\epsilon, W_\epsilon)\right\|^2_{L^2} + \nu\left\|\nabla^2(u_\epsilon, V_\epsilon, W_\epsilon)\right\|^2_{L^2} \\
&\quad\leqslant CE^2\left\|\nabla(u_\epsilon, V_\epsilon, W_\epsilon)\right\|^2_{L^2} + C\left(\left\|(g^1_\epsilon, g^2_\epsilon, g^3_\epsilon)\right\|^2_{L^2} + \left\|(H^\epsilon_1, M^\epsilon_1, N^\epsilon_1)\right\|^2_{L^2}\right).
\end{aligned}
\tag{3.15}
$$

By Gronwall's inequality, we obtain

$$
\begin{aligned}
&\left\|\nabla(u_\epsilon, V_\epsilon, W_\epsilon)(t)\right\|^2_{L^2} + \nu\int_0^t \left\|\nabla^2(u_\epsilon, V_\epsilon, W_\epsilon)(\tau)\right\|^2_{L^2} d\tau \\
&\quad\leqslant e^{CE^2 t}\Bigg(\left\|\nabla(u_{0,\epsilon}, V_{0,\epsilon}, W_{0,\epsilon})\right\|^2_{L^2} \\
&\qquad + C\int_0^t \left(\left\|(g^1_\epsilon, g^2_\epsilon, g^3_\epsilon)(\tau)\right\|^2_{L^2} + \left\|(H^\epsilon_1, M^\epsilon_1, N^\epsilon_1)(\tau)\right\|^2_{L^2}\right) d\tau\Bigg).
\end{aligned}
\tag{3.16}
$$

As $\epsilon \to 0$, by (3.6) of Lemma 3.2, we obtain

$$\|\nabla(u,V,W)(t)\|_{L^2}^2 + \nu \int_0^t \left\|\nabla^2(u,V,W)(\tau)\right\|_{L^2}^2 d\tau \leqslant e^{CE^2 t}\left(\|\nabla(u_0,V_0,W_0)\|_{L^2}^2 + C\int_0^t \left\|(g^1,g^2,g^3)(\tau)\right\|_{L^2}^2 d\tau\right). \tag{3.17}$$

Thus, we prove (3.10) for $k=1$. Applying ∇ to Eqs. (3.11), we obtain

$$\begin{cases} L_\eta^1(\nabla u_\epsilon) = \nabla g_\epsilon^1 + \nabla H_1^\epsilon + H_2^\epsilon, & H_2^\epsilon := L_\eta^1(\nabla u_\epsilon) - \nabla L_\eta^1(u_\epsilon), \\ L_{\tilde v}^2(\nabla V_\epsilon) = \nabla g_\epsilon^2 + \nabla M_1^\epsilon + M_2^\epsilon, & M_2^\epsilon := L_{\tilde v}^2(\nabla V_\epsilon) - \nabla L_{\tilde v}^2(V_\epsilon), \\ L_{\tilde w}^3(\nabla W_\epsilon) = \nabla g_\epsilon^3 + \nabla N_1^\epsilon + N_2^\epsilon, & N_2^\epsilon := L_{\tilde w}^3(\nabla W_\epsilon) - \nabla L_{\tilde w}^3(W_\epsilon). \end{cases} \tag{3.18}$$

Here we can apply the inequality (3.15) to $\nabla(u_\epsilon, V_\epsilon, W_\epsilon)$ in (3.18) to prove (3.10) for $k=2$. In fact, for $k=2$, we deduce from (3.15) and (3.18) that for some $\nu > 0$,

$$\begin{aligned} &\frac{d}{dt}\left\|\nabla^2(u_\epsilon,V_\epsilon,W_\epsilon)\right\|_{L^2}^2 + \nu\left\|\nabla^3(u_\epsilon,V_\epsilon,W_\epsilon)\right\|_{L^2}^2 \\ &\quad\leqslant CE^2\left\|\nabla^2(u_\epsilon,V_\epsilon,W_\epsilon)\right\|_{L^2}^2 \\ &\qquad + C\left(\left\|\nabla(g_\epsilon^1,g_\epsilon^2,g_\epsilon^3)\right\|_{L^2}^2 + \|\nabla(H_1^\epsilon,M_1^\epsilon,N_1^\epsilon)\|_{L^2}^2 + \|(H_2^\epsilon,M_2^\epsilon,N_2^\epsilon)\|_{L^2}^2\right) \\ &\quad\leqslant CE^2\left\|\nabla^2(u_\epsilon,V_\epsilon,W_\epsilon)\right\|_{L^2}^2 + C\left(\left\|\nabla(g_\epsilon^1,g_\epsilon^2,g_\epsilon^3)\right\|_{L^2}^2 + \|\nabla(H_1^\epsilon,M_1^\epsilon,N_1^\epsilon)\|_{L^2}^2\right) \\ &\qquad + CE^2\|\nabla(u_\epsilon,V_\epsilon,W_\epsilon)\|_{L^2}^2, \end{aligned}$$

where we have used the estimate

$$\left\|(H_2^\epsilon,M_2^\epsilon,N_2^\epsilon)\right\|_{L^2}^2 \leqslant CE^2\|\nabla(u_\epsilon,V_\epsilon,W_\epsilon)\|_{H^1}^2$$

by (3.5) of Lemma 3.2. By Gronwall's inequality, we obtain

$$\begin{aligned} &\left\|\nabla^2(u_\epsilon,V_\epsilon,W_\epsilon)(t)\right\|_{L^2}^2 + \nu\int_0^t \left\|\nabla^3(u_\epsilon,V_\epsilon,W_\epsilon)(\tau)\right\|_{L^2}^2 d\tau \\ &\quad\leqslant e^{CE^2 t}\Bigg(\left\|\nabla^2(u_{0,\epsilon},V_{0,\epsilon},W_{0,\epsilon})\right\|_{L^2}^2 + CE^2\int_0^t \|\nabla(u_\epsilon,V_\epsilon,W_\epsilon)(\tau)\|_{L^2}^2 d\tau \\ &\qquad + C\int_0^t\left(\left\|\nabla(g_\epsilon^1,g_\epsilon^2,g_\epsilon^3)(\tau)\right\|_{L^2}^2 + \|\nabla(H_1^\epsilon,M_1^\epsilon,N_1^\epsilon)(\tau)\|_{L^2}^2\right)d\tau\Bigg). \end{aligned}$$

Thus, by (3.6) of Lemma 3.2 and (3.17), as $\epsilon \to 0$, we obtain

$$\begin{aligned}
&\left\|\nabla^2(u, V, W)(t)\right\|_{L^2}^2 + \nu \int_0^t \left\|\nabla^3(u, V, W)(\tau)\right\|_{L^2}^2 d\tau \\
&\leqslant CE^2 e^{CE^2 t} \int_0^t \|\nabla(u, V, W)(\tau)\|_{L^2}^2 \, d\tau \\
&\quad + e^{CE^2 t}\left(\left\|\nabla^2(u_0, V_0, W_0)\right\|_{L^2}^2 + C\int_0^t \left\|\nabla(g^1, g^2, g^3)(\tau)\right\|_{L^2}^2 d\tau\right) \\
&\leqslant e^{CE^2 t}\left(\|\nabla(u_0, V_0, W_0)\|_{H^1}^2 + C\int_0^t \left\|(g^1, g^2, g^3)(\tau)\right\|_{H^1}^2 d\tau\right). \qquad (3.19)
\end{aligned}$$

Thus, we prove (3.10) with $k = 2$. Similar to the case for $k = 2$, we can prove (3.10) with $k = 3$. Hence, the proof of Lemma 3.4 is completed. □

Now, we can use Lemmas 3.3 and 3.4 to obtain the energy estimates for the solution $U(t)$.

Lemma 3.5. *Let $l = 2$ or 3. Assume that for some $T > 0$ and some constant χ,*

$$\begin{cases}
(\eta, \varpi, \tilde{v}, \tilde{w})(t) \in \mathcal{L}_\infty^0(0, T; H^3), \\
\eta(t) \geqslant \chi > -1, \\
f(t) \in \mathcal{L}_\infty^0(0, T; H^l), \\
(g^1, g^2, g^3)(t) \in \mathcal{L}_\infty^0(0, T; H^{l-1}).
\end{cases}$$

If $U(t) \in \mathfrak{L}(0, T; H^l)$ solves the system (3.3), then $(u, V, W)(t) \in \mathcal{L}_2(0, T; H^{l+1})$ and there exists a constant $\nu > 0$ such that for any $t \in [0, T]$,

$$\begin{aligned}
&\|U(t)\|_{H^l}, \left(\nu \int_0^t \|\nabla(u, V, W)(\tau)\|_{H^l}^2 d\tau\right)^{1/2} \\
&\leqslant e^{C(1+E)^2 t}\left(\|U_0\|_{H^l} + \int_0^t \|f(\tau)\|_{H^l} d\tau + \left(C\int_0^t \left\|(g^1, g^2, g^3)(\tau)\right\|_{H^{l-1}}^2 d\tau\right)^{1/2}\right).
\end{aligned}$$

Proof. By the estimates (3.8)–(3.10) in Lemmas 3.3 and 3.4, we obtain

$$\|U(t)\|_{H^l} \leqslant e^{C(1+E)^2 t}\left(\|\varrho_0\|_{H^l} + \int_0^t \left\|\tilde{f}(\tau)\right\|_{H^l} d\tau\right.$$

$$
\begin{aligned}
&+\|(u_0,V_0,W_0)\|_{H^l}+\left(C\int_0^t\left\|(g^1,g^2,g^3)(\tau)\right\|_{H^{l-1}}^2\,d\tau\right)^{1/2}\Bigg)\\
&\leqslant e^{C(1+E)^2t}\Bigg(\|\varrho_0\|_{H^l}+\int_0^t\|f(\tau)\|_{H^l}\,d\tau\\
&+C(1+E)t^{1/2}e^{C(1+E^2)t}\left(\|(u_0,V_0,W_0)\|_{H^l}^2+\int_0^t\left\|(g^1,g^2,g^3)(\tau)\right\|_{H^{l-1}}^2\,d\tau\right)^{1/2}\\
&+\|(u_0,V_0,W_0)\|_{H^l}+\left(C\int_0^t\left\|(g^1,g^2,g^3)(\tau)\right\|_{H^{l-1}}^2\,d\tau\right)^{1/2}\Bigg)\\
&\leqslant e^{C(1+E)^2t}\left(\|U_0\|_{H^l}+\int_0^t\|f(\tau)\|_{H^l}\,d\tau+\left(C\int_0^t\left\|(g^1,g^2,g^3)(\tau)\right\|_{H^{l-1}}^2\,d\tau\right)^{1/2}\right),
\end{aligned}
$$

where we have used that

$$
\int_0^t\left\|\tilde f(\tau)\right\|_{H^l}\,d\tau\leqslant\int_0^t\|f(\tau)\|_{H^l}\,d\tau+C(1+E)t^{1/2}\left(\int_0^t\|\nabla u\|_{H^l}^2\,d\tau\right)^{1/2}
$$

and

$$
C(1+E)t^{1/2}e^{C(1+E^2)t}\leqslant C(1+E)^2t+e^{C(1+E^2)t}\leqslant e^{C(1+E)^2t}.
$$

By (3.9)–(3.10), we obtain

$$
\nu\int_0^t\|\nabla(u,V,W)(\tau)\|_{H^l}^2\,d\tau\leqslant e^{C(1+E)^2t}\left(\|U_0\|_{H^l}^2+C\int_0^t\left\|(g^1,g^2,g^3)(\tau)\right\|_{H^{l-1}}^2\,d\tau\right).
$$

The proof of Lemma 3.5 is completed. □

3.2. Solving the linearized equations

We assume that for some $T>0$ and some constant χ,

$$
\begin{cases}
(\eta,\varpi,\tilde v,\tilde w)(t)\in\mathcal{C}^0(0,T;H^3),\\
\eta(t)\geqslant\chi>-1,\\
f(t)\in\mathcal{C}^0(0,T;H^3),\\
(g^1,g^2,g^3)(t)\in\mathcal{C}^0(0,T;H^2).
\end{cases}
$$

Next, we will solve the Cauchy problem for the linearized equations (3.3):

$$\begin{cases} L^0(\varrho, u) := \varrho_t + \varpi \cdot \nabla\varrho + (\eta+1)\operatorname{div} u = f, \\ L^1(u) := u_t - \dfrac{1}{\eta+1}\Delta u - \dfrac{1}{\eta+1}\nabla \operatorname{div} u = g^1, \\ L^2(V) := V_t - \varphi'(\tilde{v}+1)\Delta V = g^2, \\ L^3(W) := W_t - \psi'(\tilde{w}+1)\Delta W = g^3, \\ U\mid_{t=0} = U_0. \end{cases} \tag{3.20}$$

We first solve the following Cauchy problem

$$\begin{cases} L(\varrho) := \varrho_t + \varpi \cdot \nabla\varrho = \tilde{f}, \\ \varrho\mid_{t=0} = \varrho_0, \end{cases} \tag{3.21}$$

where ϖ and $\tilde{f} = f - (\eta+1)\operatorname{div} u$ are regarded as known.

Lemma 3.6. *Let $l = 1$ or 2. Assume that for some $T > 0$,*

$$\begin{cases} \varpi(t) \in \mathcal{C}^0(0, T; H^3), \\ \tilde{f}(t) \in \mathcal{C}^0(0, T; H^l). \end{cases}$$

If $\varrho_0 \in H^l$, then the Cauchy problem (3.21) has a unique solution

$$\varrho(t) \in \mathcal{C}^0(0, T; H^l) \cap \mathcal{C}^1(0, T; H^{l-1})$$

such that for any $t \in [0, T]$,

$$\|\varrho(t)\|_{H^l} \leqslant e^{CEt}\left(\|\varrho_0\|_{H^l} + \int_0^t e^{-CE\tau}\left\|\tilde{f}(\tau)\right\|_{H^l} d\tau\right). \tag{3.22}$$

Proof. By referring to Proposition 4.1 in [40, p. 83], we omit the details. □

Next, we solve the Cauchy problem of $(3.20)_2$–$(3.20)_4$.

Lemma 3.7. *Let $l = 2$ or 3. Assume that for some $T > 0$ and some constant χ,*

$$\begin{cases} (\eta, \tilde{v}, \tilde{w})(t) \in \mathcal{C}^0(0, T; H^3), \\ \eta(t) \geqslant \chi > -1, \\ (g^1, g^2, g^3)(t) \in \mathcal{C}^0(0, T; H^{l-1}). \end{cases}$$

If $(u_0, V_0, W_0) \in H^l$, then the Cauchy problem of $(3.20)_2$–$(3.20)_4$ has a unique solution

$$(u, V, W)(t) \in \mathcal{C}^0(0, T; H^l) \cap \mathcal{C}^1(0, T; H^{l-2})$$

such that for any $t \in [0, T]$,

$$\|(u,V,W)(t)\|_{H^l}^2+\nu\int_0^t\|\nabla(u,V,W)(\tau)\|_{H^l}^2\,d\tau$$
$$\leqslant e^{C(1+E^2)t}\left(\|(u_0,V_0,W_0)\|_{H^l}^2+C\int_0^t\left\|(g^1,g^2,g^3)(\tau)\right\|_{H^{l-1}}^2\,d\tau\right), \tag{3.23}$$

where $\nu>0$ is some constant.

Proof. For the existence and uniqueness of solutions, we could refer to Theorem 2.5.1 in [11, p. 108]. Then we can use Lemma 3.4 to obtain the energy estimates (3.23). $\square$

Now, we can obtain the solution to the Cauchy problem (3.20) by using Lemmas 3.6 and 3.7.

Proposition 3.8. *Assume that for some $T>0$ and some constant χ,*

$$\begin{cases}(\eta,\varpi,\tilde{v},\tilde{w})(t)\in\mathcal{C}^0(0,T;H^3),\\ \eta(t)\geqslant\chi>-1,\\ f(t)\in\mathcal{C}^0(0,T;H^2),\\ (g^1,g^2,g^3)(t)\in\mathcal{C}^0(0,T;H^1).\end{cases}$$

If $U_0\in H^2$, then the Cauchy problem (3.20) has a unique solution

$$U(t)\in\mathcal{E}(0,T;H^2) \tag{3.24}$$

such that

$$(u,V,W)(t)\in\mathcal{L}_2(0,T;H^3) \tag{3.25}$$

and for any $t\in[0,T]$,

$$\|U(t)\|_{H^2},\ \left(\nu\int_0^t\|\nabla(u,V,W)(\tau)\|_{H^2}^2\,d\tau\right)^{1/2}$$
$$\leqslant e^{C(1+E)^2t}\left(\|U_0\|_{H^2}+\left(C\int_0^t\left\|(g^1,g^2,g^3)(\tau)\right\|_{H^1}^2\,d\tau\right)^{1/2}\right), \tag{3.26}$$

where $\nu>0$ is some constant.

Proof. We consider the solution $(\varrho_\epsilon,u_\epsilon,V_\epsilon,W_\epsilon)(t)$ with any $\epsilon>0$ to the system

$$\begin{cases} L^0(\varrho_\epsilon, u_\epsilon) = f, \\ L^1(u_\epsilon) = g_\epsilon^1, \\ L^2(V_\epsilon) = g_\epsilon^2, \\ L^3(W_\epsilon) = g_\epsilon^3, \end{cases} \tag{3.27}$$

with the initial data

$$\varrho_\epsilon \mid_{t=0} = \varrho_0, \ (u_\epsilon, V_\epsilon, W_\epsilon) \mid_{t=0} = (u_{0,\epsilon}, V_{0,\epsilon}, W_{0,\epsilon}).$$

Since $(g_\epsilon^1, g_\epsilon^2, g_\epsilon^3)(t) \in \mathcal{C}^0(0,T;H^2)$ and $(u_{0,\epsilon}, V_{0,\epsilon}, W_{0,\epsilon}) \in H^3$, by Lemma 3.7, we obtain

$$(u_\epsilon, V_\epsilon, W_\epsilon)(t) \in \mathcal{C}^0(0,T;H^3) \cap \mathcal{C}^1(0,T;H^1)$$

and

$$-(\eta+1)\operatorname{div} u_\epsilon(t) \in \mathcal{C}^0(0,T;H^2).$$

Thus, by Lemma 3.6, we obtain

$$\varrho_\epsilon(t) \in \mathcal{C}^0(0,T;H^2) \cap \mathcal{C}^1(0,T;H^1).$$

Consider the difference of solutions for any $\epsilon, \epsilon' > 0$. By Lemma 3.5, we obtain for any $t \in [0,T]$,

$$\begin{aligned} &\|(\varrho_\epsilon - \varrho_{\epsilon'}, u_\epsilon - u_{\epsilon'}, V_\epsilon - V_{\epsilon'}, W_\epsilon - W_{\epsilon'})(t)\|_{H^2}, \\ &\left(\nu \int_0^t \|\nabla(u_\epsilon - u_{\epsilon'}, V_\epsilon - V_{\epsilon'}, W_\epsilon - W_{\epsilon'})(\tau)\|_{H^2}^2 \, d\tau\right)^{1/2} \\ &\quad \leqslant e^{C(1+E)^2 t}\Bigg(\left\|(u_{0,\epsilon} - u_{0,\epsilon'}, V_{0,\epsilon} - V_{0,\epsilon'}, W_{0,\epsilon} - W_{0,\epsilon'})\right\|_{H^2} \\ &\qquad + \left(C \int_0^t \left\|(g_\epsilon^1 - g_{\epsilon'}^1, g_\epsilon^2 - g_{\epsilon'}^2, g_\epsilon^3 - g_{\epsilon'}^3)(\tau)\right\|_{H^1}^2 d\tau \right)^{1/2} \Bigg) \to 0, \end{aligned} \tag{3.28}$$

as $\epsilon, \epsilon' \to 0$. Thus, we obtain the solution $U(t)$ to the Cauchy problem (3.20) as the limit $\epsilon \to 0$, which satisfies (3.24). The estimate (3.26) follows from Lemma 3.5. Then the estimate (3.26) gives the uniqueness and (3.25). The proof of Proposition 3.8 is completed. □

Lastly, we prove the existence of the solution in $\mathcal{E}(0,T;H^3)$ for some $T > 0$ of the Cauchy problem (3.20).

Proposition 3.9. *Assume that for some $T > 0$ and some constant χ,*

$$\begin{cases} (\eta, \varpi, \tilde{v}, \tilde{w})(t) \in \mathcal{C}^0(0, T; H^3), \\ \eta(t) \geqslant \chi > -1, \\ f \equiv 0, \ (g^1, g^2, g^3)(t) \in \mathcal{C}^0(0, T; H^2). \end{cases}$$

If $U_0 \in H^3$, then the Cauchy problem (3.20) has a unique solution

$$U(t) \in \mathcal{E}(0, T; H^3) \tag{3.29}$$

such that

$$(u, V, W)(t) \in \mathcal{L}_2(0, T; H^4) \tag{3.30}$$

and for any $t \in [0, T]$,

$$\begin{aligned} &\|U(t)\|_{H^3}, \ \left(\nu \int_0^t \|\nabla(u, V, W)(\tau)\|_{H^3}^2 \, d\tau\right)^{1/2} \\ &\leqslant e^{C(1+E)^2 t} \left(\|U_0\|_{H^3} + \left(C \int_0^t \left\|(g^1, g^2, g^3)(\tau)\right\|_{H^2}^2 d\tau\right)^{1/2}\right), \end{aligned} \tag{3.31}$$

where $\nu > 0$ is some constant.

Proof. Applying ∇ to (3.20), we obtain

$$\begin{cases} L^0(\nabla\varrho, \nabla u) = -(\nabla\varpi)^T \cdot \nabla\varrho - \nabla\eta \operatorname{div} u := f', \\ L^1(\nabla u) = -\dfrac{1}{(\eta+1)^2} \Delta u \cdot (\nabla\eta)^T - \dfrac{1}{(\eta+1)^2} \nabla \operatorname{div} u \cdot (\nabla\eta)^T + \nabla g^1 := g^{1'}, \\ L^2(\nabla V) = \varphi''(\tilde{v}+1) \Delta V \nabla\tilde{v} + \nabla g^2 := g^{2'}, \\ L^3(\nabla W) = \psi''(\tilde{w}+1) \Delta W \nabla\tilde{w} + \nabla g^3 := g^{3'}, \\ \nabla U \mid_{t=0} = \nabla U_0 \in H^2. \end{cases} \tag{3.32}$$

Then, we could combine an iteration method and Proposition 3.8 to solve the Cauchy problem (3.32) in $\mathcal{E}(0, T; H^2)$. Denote

$$\mathcal{U}^m(t) := ([\nabla\varrho]^{(m)}, [\nabla u]^{(m)}, [\nabla V]^{(m)}, [\nabla W]^{(m)})(t) \text{ for } m = 0, 1, 2, \ldots.$$

Noting

$$\begin{aligned} &\left\|f'\right\|_{H^2} \leqslant CE \left\|\nabla(\varrho, u)\right\|_{H^2}, \\ &\left\|(g^{1'}, g^{2'}, g^{3'})\right\|_{H^1} \leqslant \left\|\nabla(g^1, g^2, g^3)\right\|_{H^1} + CE \left\|\nabla^2(u, V, W)\right\|_{H^1}, \end{aligned}$$

we can solve the Cauchy problem (3.32) by the iteration process:

$$\mathcal{U}^0(t) := \nabla U_0,$$

and $\mathcal{U}^m(t)$, $m = 1, 2, 3, \ldots$, is the solution, belonging to $\mathcal{C}^0(0, T; H^2)$, of the problem

$$\begin{cases} L^0([\nabla\varrho]^{(m)}, [\nabla u]^{(m)}) = -(\nabla\varpi)^T \cdot [\nabla\varrho]^{(m-1)} - \nabla\eta \sum\limits_{i=1}^{3}([u_{x_i}]^{(m-1)}), \\ L^1([\nabla u]^{(m)}) = -\dfrac{1}{(\eta+1)^2}\left(\operatorname{div}([\nabla u]^{(m-1)}) + \nabla \sum\limits_{i=1}^{3}([u_{x_i}]^{(m-1)})\right) \cdot (\nabla\eta)^T + \nabla g^1, \\ L^2([\nabla V]^{(m)}) = \varphi''(\tilde{v}+1)\nabla\tilde{v} \operatorname{div}[\nabla V]^{(m-1)} + \nabla g^2, \\ L^3([\nabla W]^{(m)}) = \psi''(\tilde{w}+1)\nabla\tilde{w} \operatorname{div}[\nabla W]^{(m-1)} + \nabla g^3, \\ \mathcal{U}^m |_{t=0} = \nabla U_0. \end{cases} \tag{3.33}$$

Next, we estimate the approximation $\{\mathcal{U}^m(t)\}$. First, we have

$$\left\|\mathcal{U}^0(t)\right\|_{H^2}^2 \leqslant \|\nabla U_0\|_{H^2}^2$$

and

$$\begin{aligned} \left\|\mathcal{U}^1(t)\right\|_{H^2}^2 &\leqslant e^{C(1+E^2)t}\Bigg(\|\nabla U_0\|_{H^2}^2 + \int_0^t CE^2 \|\nabla(\varrho_0, u_0)\|_{H^2}^2 \, d\tau \\ &\quad + C\int_0^t \left(\left\|\nabla(g^1, g^2, g^3)(\tau)\right\|_{H^1}^2 + CE^2 \left\|\nabla^2(u_0, V_0, W_0)\right\|_{H^1}^2 \right) d\tau \Bigg) \\ &\leqslant e^{C(1+E^2)t}\left(\|\nabla U_0\|_{H^2}^2 + C\int_0^t \left\|\nabla(g^1, g^2, g^3)(\tau)\right\|_{H^1}^2 \, d\tau \right). \end{aligned}$$

For $m = 1, 2, 3, \ldots$, we consider the difference system

$$\begin{cases} L^0([\nabla\varrho]^{(m+1)} - [\nabla\varrho]^{(m)}, [\nabla u]^{(m+1)} - [\nabla u]^{(m)}) = G_0^m, \\ L^1([\nabla u]^{(m+1)} - [\nabla u]^{(m)}) = G_1^m, \\ L^2([\nabla V]^{(m+1)} - [\nabla V]^{(m)}, [\nabla W]^{(m+1)} - [\nabla W]^{(m)}) = G_2^m, \\ L^3([\nabla V]^{(m+1)} - [\nabla V]^{(m)}, [\nabla W]^{(m+1)} - [\nabla W]^{(m)}) = G_3^m, \end{cases}$$

where

$$G_0^m := -(\nabla\varpi)^T \cdot \left([\nabla\varrho]^{(m)} - [\nabla\varrho]^{(m-1)}\right) - \nabla\eta \sum_{i=1}^{3}\left([u_{x_i}]^{(m)} - [u_{x_i}]^{(m-1)}\right),$$

$$G_1^m := -\frac{1}{(\eta+1)^2}\left(\operatorname{div}\left([\nabla u]^{(m)} - [\nabla u]^{(m-1)}\right) + \nabla\sum_{i=1}^{3}\left([u_{x_i}]^{(m)} - [u_{x_i}]^{(m-1)}\right)\right)\cdot(\nabla\eta)^T,$$

$$G_2^m := \varphi''(\tilde v+1)\nabla\tilde v \operatorname{div}\left([\nabla V]^{(m)} - [\nabla V]^{(m-1)}\right),$$

$$G_3^m := \psi''(\tilde w+1)\nabla\tilde w \operatorname{div}\left([\nabla W]^{(m)} - [\nabla W]^{(m-1)}\right).$$

By Lemma 3.5, we obtain

$$\begin{aligned}
\left\|(\mathcal{U}^{m+1}-\mathcal{U}^m)(t)\right\|_{H^2}^2 &\leqslant e^{C(1+E^2)t}\int_0^t CE^2\left\|(\mathcal{U}^m-\mathcal{U}^{m-1})(\tau_1)\right\|_{H^2}^2 d\tau_1,\\
&\leqslant e^{C(1+E^2)t}\int_0^t CE^2 e^{C(1+E^2)\tau_1}\int_0^{\tau_1} CE^2\left\|(\mathcal{U}^{m-1}-\mathcal{U}^{m-2})(\tau_2)\right\|_{H^2}^2 d\tau_2\, d\tau_1,\\
&\leqslant \cdots \leqslant \left(\frac{E^2}{1+E^2}\right)^m e^{C(1+E^2)T}\left(4\|\nabla U_0\|_{H^2}^2 + C\int_0^T \left\|\nabla(g^1,g^2,g^3)(\tau)\right\|_{H^1}^2 d\tau\right)\\
&\to 0,\ m\to\infty,
\end{aligned}$$

which implies that there exists

$$\nabla U(t) = \lim_{m\to\infty}\mathcal{U}^m(t)\in\mathcal{C}^0(0,T;H^2),$$

which is the unique solution to the Cauchy problem (3.32). Thus, we obtain the solution $U(t)\in\mathcal{E}(0,T;H^3)$ to the Cauchy problem (3.20). Then, by Lemma 3.5, we can obtain (3.30), the energy estimate (3.31) and the uniqueness of the solution. This completes the proof of Proposition 3.9. □

3.3. Local solution for the nonlinear equations

In this subsection, we will prove that the Cauchy problem (3.1) ((1.9)) admits a unique solution in $\mathcal{E}(0,T;H^3)$ for some $T>0$ such that $\varrho>-1$. By (3.3), we can simply denote the approximate equations (3.2) as

$$\begin{cases}
L^0_{\varrho_{n-1},u_{n-1}}(\varrho_n,u_n)=0,\\
L^1_{\varrho_{n-1}}(u_n)=g^1_{n-1},\\
L^2_{V_{n-1}}(V_n)=g^2_{n-1},\\
L^3_{W_{n-1}}(W_n)=g^3_{n-1},\\
U_1\equiv U_0,\ U_n\,|_{t=0}=U_0,\ n=2,3,4,\dots,
\end{cases}\tag{3.34}$$

which satisfies

$$\inf_{x\in\mathbb{R}^3} \varrho_0(x) > -1. \tag{3.35}$$

Define

$$\begin{cases} E_0 := \|U_0\|_{H^3} < \infty, \\ \chi := (\inf \varrho_0(x) - 1)/2 > -1. \end{cases} \tag{3.36}$$

Next, we will estimate the approximate sequence $\{U_n(t)\}_{n=1}^{\infty}$.

Lemma 3.10. *If T is suitably small, then we have for all $n \geqslant 1$*

$$U_n(t) \in \mathcal{E}(0, T; H^3), \tag{3.37}$$

such that for any $t \in [0, T]$,

$$\|U_n(t)\|_{H^3}, \left(\nu \int_0^t \|\nabla(u_n, V_n, W_n)(\tau)\|_{H^3}^2 \, d\tau\right)^{1/2} \leqslant C_1 E_0 \tag{3.38}$$

and

$$\varrho_n(t) \geqslant \chi > -1, \tag{3.39}$$

where $\nu > 0$ and $C_1 > 1$ are two fixed constants.

Proof. We will prove this lemma by using an induction. First, it is trivial for $n = 1$ by (3.36). Assume (3.37)–(3.39) follow for $U_k(t)$, $k = 2, 3, \ldots, n-1$. Then by Proposition 3.9, we have

$$U_n(t) \in \mathcal{E}(0, T; H^3)$$

and

$$\begin{aligned}
&\|U_n(t)\|_{H^3}, \left(\nu \int_0^t \|\nabla(u_n, V_n, W_n)(\tau)\|_{H^3}^2 \, d\tau\right)^{1/2} \\
&\leqslant e^{C(1+E_0)^2 t} \left(\|U_0\|_{H^3} + \left(\int_0^t \left\| (g_{n-1}^1, g_{n-1}^2, g_{n-1}^3)(\tau) \right\|_{H^2}^2 d\tau \right)^{1/2} \right) \\
&\leqslant e^{C(1+E_0)^2 t} \left(\|U_0\|_{H^3} + \left(\int_0^t C(E_0) \|U_{n-1}(\tau)\|_{H^3}^2 \, d\tau \right)^{1/2} \right)
\end{aligned}$$

$$\leqslant e^{C(1+E_0)^2 t}\left(CE_0 + \left(\int_0^t C(E_0)E_0^2\, d\tau\right)^{1/2}\right) \leqslant C_1 E_0 \ (C_1 > 1), \tag{3.40}$$

provided that $t \in [0, T_1]$ for some small $T_1 > 0$. By Eq. (3.34)$_1$ and (3.40), we have that the sequence $\{\partial_t \varrho_n(t)\}$ is uniformly bounded, which implies that the sequence $\{\varrho_n(t)\}$ is equicontinuous with respect to t. Then, there exists a small $T_2 > 0$ such that for all $n \geqslant 1$,

$$\varrho_n(t) \geqslant \chi > -1 \text{ for any } t \in [0, T_2].$$

Fix $T = \min\{T_1, T_2\}$. Hence, (3.37)–(3.39) follow for any $n = 1, 2, 3, \ldots$. The proof of Lemma 3.10 is completed. □

Now, we could show the local solution to the Cauchy problem (1.9).

Proposition 3.11. *Assume $U_0 \in H^3$ and $\inf \varrho(x, 0) > -1$. Then there exists a positive T (suitably small) such that the Cauchy problem (1.9) has a unique solution*

$$\begin{cases} U(t) \in \mathcal{E}(0, T; H^3), \\ (u, V, W)(t) \in \mathcal{L}_2(0, T; H^4), \end{cases} \tag{3.41}$$

which satisfies for any $t \in [0, T]$ and some $\nu > 0$,

$$\begin{cases} \varrho(t) \geqslant (\inf \varrho_0(x) - 1)/2 > -1, \\ \|U(t)\|_{H^3}, \left(\nu \int_0^t \|\nabla(u, V, W)(\tau)\|_{H^3}^2\, d\tau\right)^{1/2} \leqslant C_1 \|U_0\|_{H^3}, \end{cases} \tag{3.42}$$

where $C_1 > 1$ is some fixed constant.

Proof. First, we could show a convergence of the approximate sequence $\{U_n(t)\}$ constructed in Lemma 3.10. Subtracting the equations (3.34) with $n = m \geqslant 2$ from that for $n = m + 1$, we obtain

$$\begin{cases} L^0_{\varrho_m, u_m}(\varrho_{m+1} - \varrho_m, u_{m+1} - u_m) = L^0_{\varrho_{m-1}, u_{m-1}}(\varrho_m, u_m) - L^0_{\varrho_m, u_m}(\varrho_m, u_m), \\ L^1_{\varrho_m}(u_{m+1} - u_m) = L^1_{\varrho_{m-1}}(u_m) - L^1_{\varrho_m}(u_m) + \left(g^1_m - g^1_{m-1}\right), \\ L^2_{V_m}(V_{m+1} - V_m) = L^2_{V_{m-1}}(V_m) - L^2_{V_m}(V_m) + \left(g^2_m - g^2_{m-1}\right), \\ L^3_{W_m}(W_{m+1} - W_m) = L^3_{W_{m-1}}(W_m) - L^3_{W_m}(W_m) + \left(g^3_m - g^3_{m-1}\right), \end{cases} \tag{3.43}$$

with the initial data

$$(U_{m+1} - U_m)\,|_{t=0} = 0, \quad m = 2, 3, 4, \ldots. \tag{3.44}$$

Next, we will use Lemma 3.5 to estimate the solution $(U_{m+1}-U_m)(t)$ to (3.43)–(3.44). For some fixed $m \geqslant 2$, by (3.7) of Lemma 3.2 and Lemma 3.10, we obtain for some $T>0$,

$$\begin{cases} U_m(t) \in \mathcal{L}^0_\infty(0,T;H^3),\ \varrho_m(t) \geqslant \chi > -1, \\ L^0_{\varrho_m,u_m}(\varrho_m,u_m) - L^0_{\varrho_{m-1},u_{m-1}}(\varrho_m,u_m) \in \mathcal{L}^0_\infty(0,T;H^2), \\ L^1_{\varrho_m}(u_m) - L^1_{\varrho_{m-1}}(u_m), L^2_{V_m}(V_m) - L^2_{V_{m-1}}(V_m), L^3_{W_m}(W_m) - L^3_{W_{m-1}}(W_m) \in \mathcal{L}^0_\infty(0,T;H^1), \\ g^1_m - g^1_{m-1}, g^2_m - g^2_{m-1}, g^3_m - g^3_{m-1} \in \mathcal{L}^0_\infty(0,T;H^1), \\ (U_{m+1}-U_m)(t) \in \mathfrak{L}(0,T;H^2). \end{cases}$$

In addition, we easily estimate

$$\left\| (g^1_m - g^1_{m-1}, g^2_m - g^2_{m-1}, g^3_m - g^3_{m-1}) \right\|_{H^1} \leqslant C(E_0) \left\| (U_m - U_{m-1}) \right\|_{H^2}. \tag{3.45}$$

So, by Lemma 3.5, (3.7) of Lemma 3.2 and (3.45), we have

$$\|(U_{m+1}-U_m)(t)\|^2_{H^2} \leqslant e^{C(1+E_0)^2T} C(E_0) \int_0^t \|(U_m - U_{m-1})(\tau_1)\|^2_{H^2}\, d\tau_1, \tag{3.46}$$

which implies

$$\begin{aligned} \|(U_{m+1}-U_m)(t)\|^2_{H^2} &\leqslant e^{C(1+E_0)^2T} C(E_0) \int_0^t \|(U_m - U_{m-1})(\tau_1)\|^2_{H^2}\, d\tau_1 \\ &\leqslant (e^{C(1+E_0)^2T} C(E_0))^{m-2} \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{m-3}} \|(U_3 - U_2)(\tau_{m-2})\|^2_{H^2}\, d\tau_{m-2} \\ &\leqslant (e^{C(1+E_0)^2T} C(E_0))^{m-1} \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{m-3}} \tau_{m-2}\, d\tau_{m-2} \\ &\leqslant \frac{(e^{C(1+E_0)^2T} C(E_0) T)^{m-1}}{(m-1)!} \to 0, \quad m \to \infty. \end{aligned}$$

Hence, $\{U_n(t)\}$ with $n \geqslant 1$ is a Cauchy sequence in $\mathcal{C}^0(0,T;H^2)$ and then there exists $U(t) \in \mathcal{C}^0(0,T;H^2)$ such that

$$U_n(t) \to U(t) \text{ strongly in } \mathcal{C}^0(0,T;H^2),\ n \to \infty. \tag{3.47}$$

By Lemma 3.10, there exists a subsequence $\{n'\} \subset \{n\}$ such that

$$\nabla(u,V,W)_{n'}(t) \to \nabla(u,V,W)(t) \text{ weakly in } \mathcal{L}_2(0,T;H^3),\ n' \to \infty.$$

By Lemma 3.10 again, we have that for any fixed $t \in [0, T]$, there exists a subsequence $\{n''\} \subset \{n'\}$ such that

$$U_{n''}(t) \to U(t) \text{ weakly in } H^3,\ n'' \to \infty.$$

So far, we have proved that the Cauchy problem (3.1) has a solution $U(t) \in \mathfrak{L}(0, T; H^3)$ such that

$$\varrho(t) \geqslant \chi > -1.$$

Next, we will show that such a solution $U(t)$ belongs to $\mathcal{E}(0, T; H^3)$. Since $U(t) \in \mathcal{C}^0(0, T; H^2)$ by (3.47), we have $U_\epsilon(t) \in \mathcal{C}^0(0, T; H^\infty)$. Applying $J_\epsilon *$ to Eqs. $(3.1)_1$–$(3.1)_4$, we obtain

$$\begin{cases} L^0_{\varrho,u}(\varrho_\epsilon, u_\epsilon) = R^0_\epsilon, \quad R^0_\epsilon := L^0_{\varrho,u}(\varrho_\epsilon, u_\epsilon) - J_\epsilon * L^0_{\varrho,u}(\varrho, u), \\ L^1_\varrho(u_\epsilon) = g^1_\epsilon + R^1_\epsilon, \quad R^1_\epsilon := L^1_\varrho(u_\epsilon) - J_\epsilon * L^1_\varrho(u), \\ L^2_V(V_\epsilon) = g^2_\epsilon + R^2_\epsilon, \quad R^2_\epsilon := L^2_V(V_\epsilon) - J_\epsilon * L^2_V(V), \\ L^3_W(W_\epsilon) = g^3_\epsilon + R^3_\epsilon, \quad R^3_\epsilon := L^3_W(W_\epsilon) - J_\epsilon * L^3_W(W), \\ U_\epsilon \mid_{t=0} = U_{0,\epsilon}. \end{cases}$$

By Lemma 3.2 and Lemma 3.5, we can estimate the difference for any $\epsilon, \epsilon' > 0$,

$$\begin{aligned} \sup_{0\leqslant t\leqslant T} \|(U_\epsilon - U_{\epsilon'})(t)\|_{H^3} \leqslant e^{C(1+E)^2 T}\Bigg(\|(U_{0,\epsilon} - U_{0,\epsilon'})\|_{H^3} + \int_0^T \left\|(R^0_\epsilon, R^0_{\epsilon'})(\tau)\right\|_{H^3} d\tau \\ + \left(C\int_0^T \left\|(g^1_\epsilon - g^1_{\epsilon'}, g^2_\epsilon - g^2_{\epsilon'}, g^3_\epsilon - g^3_{\epsilon'})(\tau)\right\|^2_{H^2} + \sum_{i=1}^3 \left\|(R^i_\epsilon, R^i_{\epsilon'})(\tau)\right\|^2_{H^2} d\tau \right)^{1/2} \Bigg). \end{aligned}$$

By (3.6) of Lemma 3.2 and the property of mollifier, we have as $\epsilon, \epsilon' \to 0$,

$$\sup_{0\leqslant t\leqslant T} \|(U_\epsilon - U_{\epsilon'})(t)\|_{H^3} \to 0.$$

Hence, the solution sequence $\{U_\epsilon(t)\}$ has a limit $U(t) \in \mathcal{E}(0, T; H^3)$ as $\epsilon \to 0$.

Finally, we prove that the solution $U(t)$ is unique. We assume that there is another solution $\widetilde{U}(t) := (\tilde{\varrho}, \tilde{u}, \widetilde{V}, \widetilde{W})(t)$ satisfying (3.41)–(3.42). Recalling (3.47), we have

$$U_n(t) \to U(t) \text{ strongly in } \mathcal{C}^0(0, T; H^2),\ n \to \infty. \tag{3.48}$$

For such a sequence $\{U_n(t)\}$ and $\widetilde{U}(t)$, we consider

$$\begin{cases} L^0_{\varrho_n,u_n}(\varrho_n-\tilde{\varrho},u_n-\tilde{u})=L^0_{\tilde{\varrho},\tilde{u}}(\tilde{\varrho},\tilde{u})-L^0_{\varrho_n,u_n}(\tilde{\varrho},\tilde{u}),\\ L^1_{\varrho_n}(u_n-\tilde{u})=L^1_{\tilde{\varrho}}(\tilde{u})-L^1_{\varrho_n}(\tilde{u})+\left(g^1_n-\tilde{g}^1\right),\\ L^2_{V_n}(V_n-\widetilde{V})=L^2_{\widetilde{V}}(\widetilde{V})-L^2_{V_n}(\widetilde{V})+\left(g^2_n-\tilde{g}^2\right),\\ L^3_{W_n}(W_n-\widetilde{W})=L^3_{\widetilde{W}}(\widetilde{W})-L^3_{W_n}(\widetilde{W})+\left(g^3_n-\tilde{g}^3\right), \end{cases}$$

with the initial data

$$(U_n-\widetilde{U})\mid_{t=0}=0.$$

By Lemma 3.5, (3.7) of Lemma 3.2 and (3.46), we have,

$$\begin{aligned} \left\|(U_n-\widetilde{U})(t)\right\|^2_{H^2} &\leqslant e^{C(1+E_0)^2t}C(E_0)\int_0^t \left\|(U_n-\widetilde{U})(\tau)\right\|^2_{H^2}\,d\tau \\ &\leqslant e^{C(1+E_0)^2t}C(E_0)\int_0^t \left(\left\|(U_{n-1}-\widetilde{U})(\tau)\right\|^2_{H^2}+\|(U_n-U_{n-1})(\tau)\|^2_{H^2}\right)d\tau, \end{aligned}$$

which implies that

$$U_n(t)\to\widetilde{U}(t) \text{ strongly in } C^0(0,T;H^2),\ n\to\infty. \tag{3.49}$$

In light of (3.48) and (3.49), we have $U(t)\equiv\widetilde{U}(t)$ for any $t\in[0,T]$.

Hence, the proof of Proposition 3.11 is completed. □

4. A priori estimates

In this section, we will establish the a priori estimates. For this purpose, we need to derive the detailed lower- and higher-order dissipation estimates for ϱ, u, V, W, ϕ. Since the system (1.9) is a coupled complicated equations, there are some tricky nonlinear terms needed to be bounded carefully, for instance, see (4.21)–(4.22). In particular, some fine interpolation estimates are used. Unlike the pure Navier–Stokes–Poisson equations [55], we also need to deal carefully with the terms involved with the electrostatic potential ϕ. This is because that here the electrostatic potential ϕ is related to the microscopic charge density V, W by the Poisson equation

$$\Delta\phi=V-W.$$

However, for Navier–Stokes–Poisson equations, the relation is

$$\Delta\phi=\varrho. \tag{4.1}$$

In fact, the equation (4.1) is helpful to establish the lower-order dissipation estimates for ϱ, cf. [55]. But here it is impossible. This is also why we need to derive some extra estimates for the electric field $\nabla\phi$ in Lemma 5.1 so that we can prove the decay rates in Subsection 5.2.

After constructing the lower- and higher-order dissipation estimates for ϱ, u, V, W, ϕ as in Lemmas 4.1–4.3, it is easy to derive the a priori estimates given by Proposition 4.4.

To derive the a priori estimates effectively, we rewrite the system (1.9) as

$$\begin{cases} \varrho_t + \operatorname{div} u = -\operatorname{div}(\varrho u), \\ u_t + \nabla\varrho + \nabla V + \nabla W - \Delta u - \nabla \operatorname{div} u = F_1, \\ V_t - \Delta V + \operatorname{div} u + \Delta\phi = F_2, \\ W_t - \Delta W + \operatorname{div} u - \Delta\phi = F_3, \\ \Delta\phi = V - W, \\ (\varrho, u, V, W)\,|_{t=0} = (\varrho_0, u_0, V_0, W_0). \end{cases} \tag{4.2}$$

Here the above nonlinear terms are defined by

$$\begin{aligned} F_1 &:= -u\cdot\nabla u - f_1\,(\Delta u + \nabla \operatorname{div} u) - f_2\nabla\varrho - f_3\nabla V - f_4\nabla W + \frac{1}{\varrho+1}\Delta\phi\nabla\phi, \\ F_2 &:= -\operatorname{div}(Vu) - \operatorname{div}(V\nabla\phi) + \operatorname{div}\big((\varphi'(V+1)-1)\nabla V\big), \\ F_3 &:= -\operatorname{div}(Wu) + \operatorname{div}(W\nabla\phi) + \operatorname{div}\big((\psi'(W+1)-1)\nabla W\big), \end{aligned}$$

where

$$f_1 := \frac{\varrho}{\varrho+1}, \ \ f_2 := \frac{p'(\varrho+1)}{\varrho+1} - 1, \tag{4.3}$$

$$f_3 := \frac{\varphi'(V+1)}{\varrho+1} - 1, \ \ f_4 := \frac{\psi'(W+1)}{\varrho+1} - 1. \tag{4.4}$$

In the sequel, we will derive the a priori estimates for the solutions to the PNP-NS equations (4.2) by assuming that for some sufficiently small $\delta > 0$ and any $t \in [0, T]$ with $T > 0$,

$$\|U(t)\|_{H^3} = \|(\varrho, u, V, W)(t)\|_{H^3} < \delta, \tag{4.5}$$

which implies

$$1/2 \leqslant \varrho + 1, V + 1, W + 1 \leqslant 3/2 \tag{4.6}$$

and

$$\begin{cases} |f_1|, |f_2| \leqslant C|\varrho|, \\ |f_3|, |f_4| \leqslant C(|\varrho| + |V| + |W|), \end{cases} \tag{4.7}$$

by Sobolev's inequality and Taylor's expansion.

First, we derive the lower-order dissipation estimates for u, V, W and ϕ.

Lemma 4.1. *It holds that*

$$\frac{d}{dt}\|U\|_{L^2}^2 + C\,\|\nabla(u, V, W)\|_{L^2}^2 + C\,\|\Delta\phi\|_{L^2}^2 \lesssim \delta\,\|\nabla\varrho\|_{L^2}^2. \tag{4.8}$$

Proof. Multiplying the first four equations of (4.2) by ϱ, u, V and W, respectively, summing them up and then integrating over $\mathbb{R}^3$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|U\|_{L^2}^2+\|\nabla(u,V,W)\|_{L^2}^2+\|\operatorname{div}u\|_{L^2}^2+\|\Delta\phi\|_{L^2}^2\\&\quad=-\langle\operatorname{div}(\varrho u),\varrho\rangle+\langle F_1,u\rangle+\langle F_2,V\rangle+\langle F_3,W\rangle.\end{aligned}\tag{4.9}$$

We now estimate the right-hand side of (4.9). By Hölder's, Sobolev's and Cauchy's inequalities, the integration by parts, (4.5)–(4.7) and Lemma A.2, we can estimate some typical terms as

$$\begin{aligned}&-\langle f_1u,\Delta u\rangle=\langle\nabla(f_1u),\nabla u\rangle\\&\quad\lesssim\left(\|\nabla f_1\|_{L^2}\|u\|_{L^\infty}+\|f_1\|_{L^\infty}\|\nabla u\|_{L^2}\right)\|\nabla u\|_{L^2}\lesssim\delta\|\nabla(\varrho,u)\|_{L^2}^2;\end{aligned}\tag{4.10}$$

$$\left\langle\frac{1}{\varrho+1},\Delta\phi\nabla\phi\cdot u\right\rangle\lesssim\|\Delta\phi\|_{L^2}\|\nabla\phi\|_{L^6}\|u\|_{L^3}\lesssim\delta\|\Delta\phi\|_{L^2}^2;\tag{4.11}$$

$$-\langle f_3\nabla V,u\rangle\lesssim\||\varrho|+|V|+|W|\|_{L^3}\|\nabla V\|_{L^2}\|u\|_{L^6}\lesssim\delta\|\nabla(u,V,W)\|_{L^2}^2;\tag{4.12}$$

$$\begin{aligned}&-\langle\operatorname{div}(V\nabla\phi),V\rangle=\langle V\nabla\phi,\nabla V\rangle\\&\quad\lesssim\|V\|_{L^3}\|\nabla\phi\|_{L^6}\|\nabla V\|_{L^2}\lesssim\delta\left(\|\nabla V\|_{L^2}^2+\|\Delta\phi\|_{L^2}^2\right);\end{aligned}\tag{4.13}$$

$$\begin{aligned}&\left\langle\operatorname{div}\left((\varphi'(V+1)-1)\nabla V\right),V\right\rangle=-\left\langle\varphi'(V+1)-1,|\nabla V|^2\right\rangle\\&\quad\lesssim\left\|\varphi'(V+1)-1\right\|_{L^\infty}\|\nabla V\|_{L^2}^2\lesssim\delta\|\nabla V\|_{L^2}^2.\end{aligned}\tag{4.14}$$

Then, the other terms could be estimated as (4.10)–(4.14). Plugging all the estimates for the right-hand side into (4.9), since δ is small, we deduce (4.8). □

Next, we derive the higher-order dissipation estimates for u, V, W and ϕ.

Lemma 4.2. *Let $k\geqslant 3$. Then we have for $3\leqslant\ell\leqslant k$,*

$$\begin{aligned}&\frac{d}{dt}\left\|\nabla^\ell U\right\|_{L^2}^2+C\left\|\nabla^{\ell+1}(u,V,W)\right\|_{L^2}^2+C\left\|\nabla^\ell\Delta\phi\right\|_{L^2}^2\\&\quad\lesssim\delta\left(\left\|\nabla^\ell(\varrho,V,W)\right\|_{L^2}^2+\left\|\nabla^{\ell-1}\Delta\phi\right\|_{L^2}^2\right).\end{aligned}\tag{4.15}$$

Proof. Let $k\geqslant 3$. For $3\leqslant\ell\leqslant k$, applying ∇^ℓ to the first four equations of (4.2) and then multiplying the resulting identities by $\nabla^\ell\varrho$, $\nabla^\ell u$, $\nabla^\ell V$ and $\nabla^\ell W$, respectively, summing them up and then integrating over $\mathbb{R}^3$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\nabla^\ell U\right\|_{L^2}^2+\left\|\nabla^{\ell+1}(u,V,W)\right\|_{L^2}^2+\left\|\nabla^\ell\operatorname{div}u\right\|_{L^2}^2+\left\|\nabla^\ell\Delta\phi\right\|_{L^2}^2\\&\quad=-\left\langle\nabla^\ell\operatorname{div}(\varrho u),\nabla^\ell\varrho\right\rangle+\left\langle\nabla^\ell F_1,\nabla^\ell u\right\rangle+\left\langle\nabla^\ell F_2,\nabla^\ell V\right\rangle+\left\langle\nabla^\ell F_3,\nabla^\ell W\right\rangle.\end{aligned}\tag{4.16}$$

Next, we only estimate some typical terms on the right-hand side of (4.16). Then, the remaining terms could be estimated in the same way. First, we split the following term as

$$-\left\langle \nabla^\ell \operatorname{div}(\varrho u), \nabla^\ell \varrho \right\rangle = -\left\langle \nabla^\ell \left(\varrho \operatorname{div} u\right), \nabla^\ell \varrho \right\rangle - \left\langle \nabla^\ell \left(u \cdot \nabla \varrho\right), \nabla^\ell \varrho \right\rangle := J_1 + J_2. \quad (4.17)$$

By the product estimates (A.11), we obtain

$$\begin{aligned} J_1 &\lesssim \left\| \nabla^\ell \left(\varrho \operatorname{div} u\right) \right\|_{L^2} \left\| \nabla^\ell \varrho \right\|_{L^2} \\ &\lesssim \left(\left\| \nabla^\ell \varrho \right\|_{L^2} \|\operatorname{div} u\|_{L^\infty} + \|\varrho\|_{L^\infty} \left\| \nabla^\ell \operatorname{div} u \right\|_{L^2} \right) \left\| \nabla^\ell \varrho \right\|_{L^2} \\ &\lesssim \delta \left(\left\| \nabla^\ell \varrho \right\|_{L^2}^2 + \left\| \nabla^{\ell+1} u \right\|_{L^2}^2 \right). \end{aligned} \quad (4.18)$$

By the commutator estimates (A.10) and the integration by parts, we obtain

$$\begin{aligned} J_2 &= -\left\langle \left[\nabla^\ell, u\right] \cdot \nabla \varrho, \nabla^\ell \varrho \right\rangle - \left\langle u \cdot \nabla \nabla^\ell \varrho, \nabla^\ell \varrho \right\rangle \\ &\leqslant C \left(\|\nabla u\|_{L^\infty} \left\| \nabla^\ell \varrho \right\|_{L^2} + \left\| \nabla^\ell u \right\|_{L^6} \|\nabla \varrho\|_{L^3} \right) \left\| \nabla^\ell \varrho \right\|_{L^2} + \frac{1}{2} \left\langle \operatorname{div} u, \left| \nabla^\ell \varrho \right|^2 \right\rangle \\ &\lesssim \left(\|\nabla u\|_{L^\infty} \left\| \nabla^\ell \varrho \right\|_{L^2} + \left\| \nabla^{\ell+1} u \right\|_{L^2} \|\nabla \varrho\|_{L^3} \right) \left\| \nabla^\ell \varrho \right\|_{L^2} + \|\operatorname{div} u\|_{L^\infty} \left\| \nabla^\ell \varrho \right\|_{L^2}^2 \\ &\lesssim \delta \left(\left\| \nabla^\ell \varrho \right\|_{L^2}^2 + \left\| \nabla^{\ell+1} u \right\|_{L^2}^2 \right). \end{aligned} \quad (4.19)$$

In light of (4.17)–(4.19), we obtain

$$-\left\langle \nabla^\ell \operatorname{div}(\varrho u), \nabla^\ell \varrho \right\rangle \lesssim \delta \left(\left\| \nabla^\ell \varrho \right\|_{L^2}^2 + \left\| \nabla^{\ell+1} u \right\|_{L^2}^2 \right). \quad (4.20)$$

By the integration by parts, the product estimates (A.11), Lemma A.2 and Young's inequality, we obtain

$$\begin{aligned} -\left\langle \nabla^\ell \left(u \cdot \nabla u\right), \nabla^\ell u \right\rangle &= \left\langle \nabla^{\ell-1} \left(u \cdot \nabla u\right), \nabla^{\ell+1} u \right\rangle \lesssim \left\| \nabla^{\ell-1} \left(u \cdot \nabla u\right) \right\|_{L^2} \left\| \nabla^{\ell+1} u \right\|_{L^2} \\ &\lesssim \left(\|u\|_{L^3} \left\| \nabla^\ell u \right\|_{L^6} + \left\| \nabla^{\ell-1} u \right\|_{L^6} \|\nabla u\|_{L^3} \right) \left\| \nabla^{\ell+1} u \right\|_{L^2} \\ &\lesssim \left(\|u\|_{L^3} \left\| \nabla^{\ell+1} u \right\|_{L^2} + \|u\|_{L^2}^{\frac{1}{\ell+1}} \left\| \nabla^{\ell+1} u \right\|_{L^2}^{\frac{\ell}{\ell+1}} \left\| \nabla^{\frac{\ell+1}{2\ell}} u \right\|_{L^2} \left\| \nabla^{\ell+1} u \right\|_{L^2}^{\frac{1}{\ell+1}} \right) \left\| \nabla^{\ell+1} u \right\|_{L^2} \\ &\lesssim \delta \left\| \nabla^{\ell+1} u \right\|_{L^2}^2; \end{aligned} \quad (4.21)$$

$$\begin{aligned} -\left\langle \nabla^\ell \left(f_3 \nabla V\right), \nabla^\ell u \right\rangle &\lesssim \left\| \nabla^\ell \left(f_3 \nabla V\right) \right\|_{L^{6/5}} \left\| \nabla^\ell u \right\|_{L^6} \\ &\lesssim \left(\|f_3\|_{L^3} \left\| \nabla^{\ell+1} V \right\|_{L^2} + \left\| \nabla^\ell f_3 \right\|_{L^2} \|\nabla V\|_{L^3} \right) \left\| \nabla^{\ell+1} u \right\|_{L^2} \\ &\lesssim \|(\varrho, V)\|_{L^3} \left\| \nabla^{\ell+1} V \right\|_{L^2} \left\| \nabla^{\ell+1} u \right\|_{L^2} \end{aligned}$$

$$
\begin{aligned}
&+\left(\left\|\nabla^{\ell}\left(\frac{\varphi'(V+1)-1}{\varrho+1}\right)\right\|_{L^2}+\left\|\nabla^{\ell}\left(\frac{\varrho}{\varrho+1}\right)\right\|_{L^2}\right)\|\nabla V\|_{L^3}\left\|\nabla^{\ell+1}u\right\|_{L^2}\\
&\lesssim \|(\varrho,V)\|_{L^3}\left\|\nabla^{\ell+1}V\right\|_{L^2}\left\|\nabla^{\ell+1}u\right\|_{L^2}+\left\|\nabla^{\ell}\varrho\right\|_{L^2}\|\nabla V\|_{L^3}\left\|\nabla^{\ell+1}u\right\|_{L^2}\\
&\quad+\left\|\nabla^{\ell}\varrho\right\|_{L^2}\|V\|_{L^\infty}\|\nabla V\|_{L^3}\left\|\nabla^{\ell+1}u\right\|_{L^2}+\left\|\nabla^{\ell}V\right\|_{L^2}\|\nabla V\|_{L^3}\left\|\nabla^{\ell+1}u\right\|_{L^2}\\
&\lesssim\left(\|(\varrho,V)\|_{L^3}\left\|\nabla^{\ell+1}V\right\|_{L^2}+\left\|\nabla^{\ell}\varrho\right\|_{L^2}\|\nabla V\|_{L^3}+\left\|\nabla^{\ell}\varrho\right\|_{L^2}\|V\|_{L^\infty}\|\nabla V\|_{L^3}\right)\left\|\nabla^{\ell+1}u\right\|_{L^2}\\
&\quad+\|V\|_{L^2}^{\frac{1}{\ell+1}}\left\|\nabla^{\ell+1}V\right\|_{L^2}^{\frac{\ell}{\ell+1}}\left\|\nabla^{\frac{\ell+1}{2\ell}}V\right\|_{L^2}^{\frac{\ell}{\ell+1}}\left\|\nabla^{\ell+1}V\right\|_{L^2}^{\frac{1}{\ell+1}}\left\|\nabla^{\ell+1}u\right\|_{L^2}\\
&\lesssim\delta\left(\left\|\nabla^{\ell}\varrho\right\|_{L^2}^2+\left\|\nabla^{\ell+1}(u,V)\right\|_{L^2}^2\right);
\end{aligned}
\tag{4.22}
$$

$$
\begin{aligned}
&\left\langle\nabla^{\ell}\left(\frac{1}{\varrho+1}\Delta\phi\nabla\phi\right),\nabla^{\ell}u\right\rangle=-\left\langle\nabla^{\ell-1}\left(\frac{1}{\varrho+1}\Delta\phi\nabla\phi\right),\nabla^{\ell+1}u\right\rangle\\
&\quad\lesssim\left\|\nabla^{\ell-1}\left(\frac{1}{\varrho+1}\Delta\phi\nabla\phi\right)\right\|_{L^2}\left\|\nabla^{\ell+1}u\right\|_{L^2}\\
&\quad\lesssim\left(\left\|\nabla^{\ell-1}\left(\frac{1}{\varrho+1}\right)\right\|_{L^6}\|\Delta\phi\nabla\phi\|_{L^3}+\left\|\frac{1}{\varrho+1}\right\|_{L^\infty}\left\|\nabla^{\ell-1}(\Delta\phi\nabla\phi)\right\|_{L^2}\right)\left\|\nabla^{\ell+1}u\right\|_{L^2}\\
&\quad\lesssim\left\|\nabla^{\ell}\varrho\right\|_{L^2}\|\Delta\phi\|_{L^3}\|\nabla\phi\|_{L^\infty}\left\|\nabla^{\ell+1}u\right\|_{L^2}+\left\|\nabla^{\ell-1}(\Delta\phi\nabla\phi)\right\|_{L^2}\left\|\nabla^{\ell+1}u\right\|_{L^2}\\
&\quad\lesssim\delta\left\|\nabla^{\ell}\varrho\right\|_{L^2}\left\|\nabla^{\ell+1}u\right\|_{L^2}+\left(\left\|\nabla^{\ell-1}\Delta\phi\right\|_{L^2}\|\nabla\phi\|_{L^\infty}+\|\Delta\phi\|_{L^3}\left\|\nabla^{\ell-1}\nabla\phi\right\|_{L^6}\right)\left\|\nabla^{\ell+1}u\right\|_{L^2}\\
&\quad\lesssim\delta\left(\left\|\nabla^{\ell}\varrho\right\|_{L^2}^2+\left\|\nabla^{\ell-1}\Delta\phi\right\|_{L^2}^2+\left\|\nabla^{\ell+1}u\right\|_{L^2}^2\right);
\end{aligned}
\tag{4.23}
$$

$$
\begin{aligned}
&\left\langle\nabla^{\ell}\operatorname{div}\left((\varphi'(V+1)-1)\nabla V\right),\nabla^{\ell}V\right\rangle=-\left\langle\nabla^{\ell}\left((\varphi'(V+1)-1)\nabla V\right),\nabla^{\ell+1}V\right\rangle\\
&\quad\lesssim\left\|\nabla^{\ell}\left((\varphi'(V+1)-1)\nabla V\right)\right\|_{L^2}\left\|\nabla^{\ell+1}V\right\|_{L^2}\\
&\quad\lesssim\left(\left\|\nabla^{\ell}\left(\varphi'(V+1)-1\right)\right\|_{L^6}\|\nabla V\|_{L^3}+\|\varphi'(V+1)-1\|_{L^\infty}\left\|\nabla^{\ell+1}V\right\|_{L^2}\right)\left\|\nabla^{\ell+1}V\right\|_{L^2}\\
&\quad\lesssim\delta\left\|\nabla^{\ell+1}V\right\|_{L^2}^2.
\end{aligned}
\tag{4.24}
$$

Here we shall carefully analyze the following two terms

$$
-\left\langle\nabla^{\ell}\operatorname{div}(V\nabla\phi),\nabla^{\ell}V\right\rangle+\left\langle\nabla^{\ell}\operatorname{div}(W\nabla\phi),\nabla^{\ell}W\right\rangle:=\tilde{S}.
$$

We want to control $\tilde{S}$ in terms of $(\ell+1)$-order of V and W, that is

$$\tilde{S} \lesssim \delta\left(\left\|\nabla^{\ell}\Delta\phi\right\|_{L^2}^2 + \left\|\nabla^{\ell+1}(V,W)\right\|_{L^2}^2\right). \tag{4.25}$$

However, it cannot be done. In fact, we only could estimate

$$\tilde{S} \lesssim \delta\left(\left\|\nabla^{\ell}(V,W,\Delta\phi)\right\|_{L^2}^2 + \left\|\nabla^{\ell+1}(V,W)\right\|_{L^2}^2\right). \tag{4.26}$$

If we additionally assume that the initial electric filed $\|\nabla\phi_0\|_{L^2}$ is sufficiently small, then we could control $\tilde{S}$ like (4.25) (see Lemma 5.2).

Given the estimates (4.20)–(4.26), the right-hand side of (4.16) could be bounded by

$$\delta\left(\left\|\nabla^{\ell}(\varrho,V,W,\Delta\phi)\right\|_{L^2}^2 + \left\|\nabla^{\ell+1}(u,V,W)\right\|_{L^2}^2 + \left\|\nabla^{\ell-1}\Delta\phi\right\|_{L^2}^2\right). \tag{4.27}$$

Plugging (4.27) into (4.16), since δ is small, we obtain (4.15). □

Next, we will derive the dissipation estimates for ϱ.

Lemma 4.3. *Let $k \geqslant 3$ and $3 \leqslant \ell \leqslant k$. Then we have for $0 \leqslant l \leqslant \ell-1$,*

$$\frac{d}{dt}\left\langle\nabla^l u, \nabla\nabla^l\varrho\right\rangle + C\left\|\nabla^{l+1}\varrho\right\|_{L^2}^2 \lesssim \left\|\nabla^{l+1}(u,V,W)\right\|_{L^2}^2 + \left\|\nabla^l\Delta\phi\right\|_{L^2}^2 + \left\|\nabla^{l+2}u\right\|_{L^2}^2. \tag{4.28}$$

Proof. For $0 \leqslant l \leqslant \ell-1$, applying ∇^l to Eq. $(4.2)_2$ and then multiplying the resulting identity by $\nabla\nabla^l\varrho$, and integrating over $\mathbb{R}^3$, by the integration by parts, the Poisson equation $(4.2)_5$ and the product estimates (A.11), we obtain

$$\begin{aligned}\left\|\nabla^{l+1}\varrho\right\|_{L^2}^2 \leqslant& -\left\langle\nabla^l u_t, \nabla\nabla^l\varrho\right\rangle + C\left\|\nabla^{l+1}(V,W)\right\|_{L^2}\left\|\nabla^{l+1}\varrho\right\|_{L^2}\\ &+ C\left\|\nabla^{l+2}u\right\|_{L^2}\left\|\nabla^{l+1}\varrho\right\|_{L^2} + C\left\|\nabla^l F_1\right\|_{L^2}\left\|\nabla^{l+1}\varrho\right\|_{L^2}.\end{aligned} \tag{4.29}$$

By the integration by parts, Eq. $(4.2)_1$ and the product estimates (A.11), we obtain

$$\begin{aligned}-\left\langle\nabla^l u_t, \nabla\nabla^l\varrho\right\rangle &= -\frac{d}{dt}\left\langle\nabla^l u, \nabla\nabla^l\varrho\right\rangle - \left\langle\nabla^l \operatorname{div} u, \nabla^l\varrho_t\right\rangle\\ &\leqslant -\frac{d}{dt}\left\langle\nabla^l u, \nabla\nabla^l\varrho\right\rangle + C\left\|\nabla^{l+1}u\right\|_{L^2}^2 + \delta\left\|\nabla^{l+1}\varrho\right\|_{L^2}^2\end{aligned} \tag{4.30}$$

and

$$\left\|\nabla^l F_1\right\|_{L^2} \lesssim \delta\left(\left\|\nabla^{l+1}(\varrho,u,V,W)\right\|_{L^2} + \left\|\nabla^l\Delta\phi\right\|_{L^2} + \left\|\nabla^{l+2}u\right\|_{L^2}\right). \tag{4.31}$$

Plugging (4.30)–(4.31) into (4.29), by Cauchy's inequality, since δ is small, we obtain (4.28). □

Now, we could use Lemmas 4.1–4.3 to show the a priori estimates.

Proposition 4.4 *(A priori estimates). Let $T > 0$ and $k \geqslant 3$. Assume that for some sufficiently small $\delta > 0$,*

$$\sup_{t\in[0,T]} \|U(t)\|_{H^3} < \delta.$$

Then, we have for any $t \in [0, T]$ and $3 \leqslant \ell \leqslant k$,

$$\|U(t)\|_{H^\ell} + \left(\int_0^t \|\nabla\varrho(\tau)\|_{H^{\ell-1}}^2 + \|\nabla(u, V, W)(\tau)\|_{H^\ell}^2 + \|\Delta\phi(\tau)\|_{H^\ell}^2 \, d\tau \right)^{1/2} \leqslant \tilde{c}\, \|U_0\|_{H^\ell}, \tag{4.32}$$

where $\tilde{c}$ is some fixed positive constant.

Proof. Let $k \geqslant 3$ and $3 \leqslant \ell \leqslant k$. Adding the estimates (4.8) of Lemma 4.1 to the estimates (4.15) of Lemma 4.2 and using the interpolation estimates and Young's inequality, since δ is small, we obtain

$$\frac{d}{dt} \|U\|_{H^\ell}^2 + C_1 \left(\|\nabla(u, V, W)\|_{H^\ell}^2 + \|\Delta\phi\|_{H^\ell}^2 \right) \leqslant C_2\delta \|\nabla\varrho\|_{H^{\ell-1}}^2, \tag{4.33}$$

where we have used the equivalent relation

$$\|h\|_{H^\ell}^2 \sim \|h\|_{L^2}^2 + \left\|\nabla^\ell h\right\|_{L^2}^2.$$

Summing up the estimates (4.28) of Lemma 4.3 from $l = 0$ to $\ell - 1$, we obtain

$$\begin{aligned} &\frac{d}{dt} \sum_{0\leqslant l\leqslant \ell-1} \left\langle \nabla^l u, \nabla\nabla^l \varrho \right\rangle + C_3 \|\nabla\varrho\|_{H^{\ell-1}}^2 \\ &\quad \leqslant C_4 \left(\|\nabla u\|_{H^\ell}^2 + \|\nabla(V, W)\|_{H^{\ell-1}}^2 + \|\Delta\phi\|_{H^{\ell-1}}^2 \right). \end{aligned} \tag{4.34}$$

Multiplying (4.34) by $2C_2\delta/C_3$, and then adding it to (4.33), since $\delta > 0$ is small, we deduce that there exists a constant $c > 0$ such that

$$\begin{aligned} &\frac{d}{dt} \left(\|U\|_{H^\ell}^2 + \frac{2C_2\delta}{C_3} \sum_{0\leqslant l\leqslant \ell-1} \left\langle \nabla^l u, \nabla\nabla^l \varrho \right\rangle \right) \\ &\quad + c \left(\|\nabla\varrho\|_{H^{\ell-1}}^2 + \|\nabla(u, V, W)\|_{H^\ell}^2 + \|\Delta\phi\|_{H^\ell}^2 \right) \leqslant 0. \end{aligned} \tag{4.35}$$

Next, we define $\mathcal{E}_0^\ell(t)$ to be c^{-1} times the expression under the time derivative in (4.35). Observe that since δ is small $\mathcal{E}_0^\ell(t)$ is equivalent to $\|U\|_{H^\ell}^2$, that is, there exists a constant $\tilde{c} > 0$ such that

$$\frac{1}{\tilde{c}} \|U(t)\|_{H^\ell}^2 \leqslant \mathcal{E}_0^\ell(t) \leqslant \tilde{c}\, \|U(t)\|_{H^\ell}^2. \tag{4.36}$$

Then we could rewrite (4.35) as that

$$\frac{d}{dt}\mathcal{E}_0^\ell(t)+\|\nabla\varrho\|_{H^{\ell-1}}^2+\|\nabla(u,V,W)\|_{H^\ell}^2+\|\Delta\phi\|_{H^\ell}^2\leqslant 0. \tag{4.37}$$

Integrating (4.37) in time, by (4.36), we obtain (4.32). □

5. Proof of Theorems 1.1 and 1.2

In this section, we will prove the main results Theorems 1.1 and 1.2. In fact, we have proved Theorem 1.1 about the local solution in Section 3. Then, we can make use of the local solution and the a priori estimates to prove Theorem 1.2 about the global small solution only if we do a continuous argument as in Subsection 5.1. To prove the decay rates in Theorem 1.2, we need to derive more detailed energy estimates given in Lemmas 5.1 and 5.2 under the assumption that the initial $\left\|\nabla^{-1}(V_0-W_0)\right\|_{L^2}$ is sufficiently small. Here, we find that the electrostatic potential ϕ satisfies the special equation (5.12), which allows us to derive some effective estimates for the electric field $\nabla\phi$. On this premise, we can obtain the ℓ-order differential inequality (5.19) for $0\leqslant\ell\leqslant k-1$ in Lemma 5.2. Roughly speaking, the inequality (5.19) looks like

$$\frac{d}{dt}\mathcal{E}_\ell^k(t)+\mathcal{D}_{\ell+1}^k(t)\leqslant 0,$$

where

$$\mathcal{E}_\ell^k(t)\sim\left\|\nabla^\ell(\varrho,u,V,W,\nabla\phi)\right\|_{H^{k-\ell}}^2.$$

Noting that the dissipation $\mathcal{D}_{\ell+1}^k(t)$ lacks the ℓ-order of (ϱ,u,V,W). This is why we cannot obtain the exponential decay rates. Fortunately, we can remedy the dissipation $\mathcal{D}_{\ell+1}^k(t)$ in terms of the energy $\mathcal{E}_\ell^k(t)$ by using the negative Besov estimate (5.29) in Lemma 5.3 and the interpolation estimate (5.30), that is, for $0\leqslant\ell\leqslant k-1$ and $0<s\leqslant 3/2$,

$$\mathcal{D}_{\ell+1}^k(t)\geqslant C_0\left(\mathcal{E}_\ell^k(t)\right)^{1+\frac{1}{\ell+s}}.$$

Thus, we deduce

$$\frac{d}{dt}\mathcal{E}_\ell^k(t)+C_0\left(\mathcal{E}_\ell^k(t)\right)^{1+\frac{1}{\ell+s}}\leqslant 0.$$

By solving the above differential inequality, we can obtain the algebraic decay rates of the solution.

Next, we first prove the local solution and global solution.

5.1. Local solution and global solution

First, Theorem 1.1 can be directly obtained by Proposition 3.11. Next, we combine Proposition 3.11 (local solution) and Proposition 4.4 (a priori estimates) to prove the global solution in Theorem 1.2.

Assume $U_0 \in H^k$ with an integer $k \geqslant 3$. And we set

$$\|U_0\|_{H^3} < \min\left\{\frac{\delta}{C_1}, \frac{\delta}{C_1\tilde{c}}\right\}, \tag{5.1}$$

where $C_1 > 1, \tilde{c} > 0, \delta > 0$ are given by Proposition 3.11 and Proposition 4.4. If

$$\|U_0\|_{H^3} < \frac{\delta}{C_1},$$

by Proposition 3.11, the Cauchy problem (1.9) has a unique local solution

$$U(t) \in \mathcal{E}(0, T_1; H^3)$$

with $T_1 > 0$ such that for any $t \in [0, T_1]$

$$\|U(t)\|_{H^3} \leqslant C_1 \|U_0\|_{H^3} < \delta. \tag{5.2}$$

By (5.2) and Proposition 4.4, we obtain for any $t \in [0, T_1]$ and $3 \leqslant \ell \leqslant k$,

$$\|U(t)\|_{H^\ell} \leqslant \tilde{c}\, \|U_0\|_{H^\ell}\,. \tag{5.3}$$

In particular, by (5.1), we have for any $t \in [0, T_1]$,

$$\|U(t)\|_{H^3} \leqslant \tilde{c}\, \|U_0\|_{H^3} < \frac{\delta}{C_1}. \tag{5.4}$$

By (5.3) and (5.4), we have

$$U(T_1) \in H^3 \text{ and } \|U(T_1)\|_{H^3} < \frac{\delta}{C_1}. \tag{5.5}$$

Then, by Proposition 3.11 again, the Cauchy problem (1.9) has a unique local solution

$$U(t) \in \mathcal{E}(T_1, 2T_1; H^3)$$

such that for any $t \in [T_1, 2T_1]$

$$\|U(t)\|_{H^3} \leqslant C_1 \|U(T_1)\|_{H^3} < \delta. \tag{5.6}$$

By (5.2), (5.6) and Proposition 4.4, we obtain for any $t \in [0, 2T_1]$ and $3 \leqslant \ell \leqslant k$,

$$\|U(t)\|_{H^\ell} \leqslant \tilde{c}\, \|U_0\|_{H^\ell}\,. \tag{5.7}$$

By repeating the procedures (5.2)–(5.7), we could extend the local solution to the global one only if we assume that the initial data belonging to $H^k (k \geqslant 3)$ satisfy $\|U_0\|_{H^3}$ is sufficiently small, as (5.1). The energy estimates (1.10) could be obtained by Proposition 4.4. Hence, the proof of the global solution in Theorem 1.2 is completed.

5.2. Decay rates

To obtain the decay rates of the global solution in Theorem 1.2, we need to prove some extra energy estimates showed in the following.

5.2.1. Energy estimates

Assume that $(U, \nabla\phi)(t)$ is the solution to the PNP-NS equations (4.2) proved in the above. By (1.10) with $\ell = 3$, we know that for any $t \geqslant 0$ and some sufficiently small $\delta_0 > 0$,

$$\|U(t)\|_{H^3}, \left(\int_0^t \|\nabla\varrho(\tau)\|_{H^2}^2 + \|\nabla(u, V, W)(\tau)\|_{H^3}^2 + \|\Delta\phi(\tau)\|_{H^3}^2 \, d\tau\right)^{1/2} < \delta_0 \quad (5.8)$$

if the initial H^3 norm is sufficiently small. And then the estimates (4.6)–(4.7) also follow.

Lemma 5.1. *Let $0 \leqslant l \leqslant k$. If $\nabla^{-1}(V_0 - W_0) \in L^2$, then it holds that for any $t \geqslant 0$ and some constant $\alpha > 0$,*

$$\left\|\nabla^l \nabla\phi(t)\right\|_{L^2}^2 + \alpha \int_0^t \left\|\nabla^l \nabla\phi(\tau)\right\|_{L^2}^2 d\tau \leqslant C_0 \quad (5.9)$$

and

$$\left\|\nabla^l \nabla\phi(t)\right\|_{L^2}^2 \leqslant C_0 e^{-\alpha t} + C \int_0^t e^{-\alpha(t-\tau)} \left\|\nabla^{l+1}(u, V, W)(\tau)\right\|_{L^2}^2 d\tau. \quad (5.10)$$

In particular, if $\left\|\nabla^{-1}(V_0 - W_0)\right\|_{L^2}$ is sufficiently small, then for any $t \geqslant 0$ and some sufficiently small $\delta_0' > 0$,

$$\|\nabla\phi(t)\|_{L^2} < \delta_0'. \quad (5.11)$$

Proof. Subtracting Eq. $(4.2)_4$ from Eq. $(4.2)_3$ and using the Poisson equation $(4.2)_5$, we obtain

$$\Delta\phi_t + 2\Delta\phi - \Delta\Delta\phi = F_2 - F_3. \quad (5.12)$$

For $0 \leqslant l \leqslant k$, applying ∇^l to Eq. (5.12) and then multiplying the resulting identity by $-\nabla^l\phi$, and integrating over $\mathbb{R}^3$, by the integration by parts and the Poisson equation $(4.2)_5$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\nabla^l\nabla\phi\right\|_{L^2}^2 + 2\left\|\nabla^l\nabla\phi\right\|_{L^2}^2 + \left\|\nabla^l\Delta\phi\right\|_{L^2}^2 \\
&\quad = -\left\langle \nabla^l(\Delta\phi u) + \nabla^l(V\nabla\phi) + \nabla^l(W\nabla\phi), \nabla^l\nabla\phi \right\rangle \\
&\qquad + \left\langle \nabla^l\left((\varphi'(V+1) - 1)\nabla V\right), \nabla^l\nabla\phi \right\rangle - \left\langle \nabla^l\left((\psi'(W+1) - 1)\nabla W\right), \nabla^l\nabla\phi \right\rangle. \quad (5.13)
\end{aligned}$$

If $l = 0$, by (5.8), (5.13), Hölder's, Sobolev's and Cauchy's inequalities, we obtain for some constant $\alpha > 0$,

$$\frac{d}{dt}\|\nabla\phi\|_{L^2}^2 + \alpha\|\nabla\phi\|_{L^2}^2 \leqslant C\|\nabla(u, V, W)\|_{L^2}^2. \tag{5.14}$$

Integrating (5.14) in time and using (5.8) and Eq. $(4.2)_5$, we obtain

$$\|\nabla\phi(t)\|_{L^2}^2 + \alpha\int_0^t \|\nabla\phi(\tau)\|_{L^2}^2\, d\tau \leqslant \|\nabla\phi_0\|_{L^2}^2 + C\delta_0^2, \tag{5.15}$$

which implies (5.9) with $l = 0$ and (5.11) if $\left\|\nabla^{-1}(V_0 - W_0)\right\|_{L^2}$ is sufficiently small. For $1 \leqslant l \leqslant k$, by (5.13), (5.15), Hölder's, Sobolev's and Cauchy's inequalities and the product estimates (A.11), we obtain for any $\varepsilon > 0$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\nabla^l\nabla\phi\right\|_{L^2}^2 + 2\left\|\nabla^l\nabla\phi\right\|_{L^2}^2 + \left\|\nabla^l\Delta\phi\right\|_{L^2}^2 \\
&\quad\leqslant (\delta_0 + \varepsilon)\left(\left\|\nabla^l\Delta\phi\right\|_{L^2}^2 + \left\|\nabla^l\nabla\phi\right\|_{L^2}^2\right) + C_\varepsilon\left\|\nabla^{l+1}(u, V, W)\right\|_{L^2}^2.
\end{aligned}$$

Taking the above $\varepsilon > 0$ properly small, since δ_0 is small, we deduce that there exists some constant $\alpha > 0$ such that for $1 \leqslant l \leqslant k$,

$$\frac{d}{dt}\left\|\nabla^l\nabla\phi\right\|_{L^2}^2 + \alpha\left\|\nabla^l\nabla\phi\right\|_{L^2}^2 \leqslant C\left\|\nabla^{l+1}(u, V, W)\right\|_{L^2}^2. \tag{5.16}$$

By (1.10) with $\ell = k$, we have

$$\int_0^t \|\nabla(u, V, W)(\tau)\|_{H^k}^2\, d\tau \leqslant C_0. \tag{5.17}$$

Integrating (5.16) in time and using (5.17) and Eq. $(4.2)_5$, we obtain for $1 \leqslant l \leqslant k$,

$$\left\|\nabla^l\nabla\phi(t)\right\|_{L^2}^2 + \alpha\int_0^t \left\|\nabla^l\nabla\phi(\tau)\right\|_{L^2}^2\, d\tau \lesssim \left\|\nabla^l\nabla\phi_0\right\|_{L^2}^2 + C_0, \tag{5.18}$$

So, the estimates (5.9) with $1 \leqslant l \leqslant k$ follow from (5.18). And the estimates (5.10) follow from (5.14) and (5.16) together with Gronwall's inequality. □

Lemma 5.2. *Let $k \geqslant 3$ and $0 \leqslant \ell \leqslant k - 1$. If $\left\|\nabla^{-1}(V_0 - W_0)\right\|_{L^2} \leqslant \delta_0$, then there exists a functional $\mathcal{E}_\ell^k(t)$ which is equivalent to $\left\|\nabla^\ell(U, \nabla\phi)\right\|_{H^{k-\ell}}^2$ such that*

$$\frac{d}{dt}\mathcal{E}_\ell^k(t) + \left\|\nabla^{\ell+1}\varrho\right\|_{H^{k-\ell-1}}^2 + \left\|\nabla^{\ell+1}(u, V, W)\right\|_{H^{k-\ell}}^2 + \left\|\nabla^\ell\nabla\phi\right\|_{H^{k-\ell+1}}^2 \leqslant 0. \tag{5.19}$$

Proof. Let $k \geqslant 3$. For the case $\ell = 0$, the inequality (5.19) is trivial by (4.37), (5.14) and (5.16). We will prove (5.19) for $1 \leqslant \ell \leqslant k-1$. We notice

$$\begin{aligned} -\left\langle \nabla^l \operatorname{div}(V\nabla\phi), \nabla^l V\right\rangle &= \left\langle \nabla^l (V\nabla\phi), \nabla^{l+1} V\right\rangle \\ &\lesssim \left(\|V\|_{L^3}\left\|\nabla^l\nabla\phi\right\|_{L^6} + \left\|\nabla^l V\right\|_{L^6}\|\nabla\phi\|_{L^3}\right)\left\|\nabla^{l+1}V\right\|_{L^2} \\ &\lesssim (\delta_0+\delta_0')\left(\left\|\nabla^l\Delta\phi\right\|_{L^2}^2 + \left\|\nabla^{l+1}V\right\|_{L^2}^2\right), \end{aligned} \tag{5.20}$$

where we have used $\|\nabla\phi(t)\|_{L^2} < \delta_0'$. Then, the estimate of the form as (4.25) follows. Thus, the terms $\|\nabla^k(V,W)\|_{L^2}^2$ do not appear on the right-hand side of the estimates (4.15) in Lemma 4.2 with $l = k$, i.e.,

$$\begin{aligned} &\frac{d}{dt}\left\|\nabla^k U\right\|_{L^2}^2 + C\left\|\nabla^{k+1}(u,V,W)\right\|_{L^2}^2 + C\left\|\nabla^k\Delta\phi\right\|_{L^2}^2 \\ &\quad\lesssim (\delta_0+\delta_0')\left(\left\|\nabla^k\varrho\right\|_{L^2}^2 + \left\|\nabla^k\nabla\phi\right\|_{L^2}^2\right). \end{aligned} \tag{5.21}$$

And in the proof of Lemma 4.2, using some different interpolation estimates, we easily obtain for $1 \leqslant l \leqslant k-1$,

$$\begin{aligned} &\frac{d}{dt}\left\|\nabla^l U\right\|_{L^2}^2 + C\left\|\nabla^{l+1}(u,V,W)\right\|_{L^2}^2 + C\left\|\nabla^l\Delta\phi\right\|_{L^2}^2 \\ &\quad\lesssim (\delta_0+\delta_0')\left(\left\|\nabla^{l+1}\varrho\right\|_{L^2}^2 + \left\|\nabla^l\nabla\phi\right\|_{L^2}^2\right). \end{aligned} \tag{5.22}$$

By (5.14), (5.16) and (5.21)–(5.22), since δ_0, δ_0' are small, we easily obtain for $1 \leqslant l \leqslant k-1$,

$$\frac{d}{dt}\left\|\nabla^l(U,\nabla\phi)\right\|_{L^2}^2 + C\left\|\nabla^{l+1}(u,V,W)\right\|_{L^2}^2 + C\left\|\nabla^l\nabla\phi\right\|_{H^1}^2 \lesssim (\delta_0+\delta_0')\left\|\nabla^{l+1}\varrho\right\|_{L^2}^2 \tag{5.23}$$

and

$$\frac{d}{dt}\left\|\nabla^k(U,\nabla\phi)\right\|_{L^2}^2 + C\left\|\nabla^{k+1}(u,V,W)\right\|_{L^2}^2 + C\left\|\nabla^k\nabla\phi\right\|_{H^1}^2 \lesssim (\delta_0+\delta_0')\left\|\nabla^k\varrho\right\|_{L^2}^2. \tag{5.24}$$

Summing up the estimates (5.23) from $l = \ell$ to $k-1$ and then adding it to (5.24), we obtain

$$\begin{aligned} &\frac{d}{dt}\left\|\nabla^\ell(U,\nabla\phi)\right\|_{H^{k-\ell}}^2 + C_1\left(\left\|\nabla^{\ell+1}(u,V,W)\right\|_{H^{k-\ell}}^2 + \left\|\nabla^\ell\nabla\phi\right\|_{H^{k-\ell+1}}^2\right) \\ &\quad\leqslant C_2(\delta_0+\delta_0')\left\|\nabla^{\ell+1}\varrho\right\|_{H^{k-\ell-1}}^2. \end{aligned} \tag{5.25}$$

In addition, Lemma 4.3 gives for $1 \leqslant l \leqslant k-1$,

$$\frac{d}{dt}\left\langle\nabla^l u,\nabla\nabla^l\varrho\right\rangle+C\left\|\nabla^{l+1}\varrho\right\|_{L^2}^2\lesssim\left\|\nabla^{l+1}(u,V,W)\right\|_{L^2}^2+\left\|\nabla^l\Delta\phi\right\|_{L^2}^2+\left\|\nabla^{l+2}u\right\|_{L^2}^2. \tag{5.26}$$

Summing up the estimates (5.26) from $l=\ell$ to $k-1$, we obtain

$$\begin{aligned}&\frac{d}{dt}\sum_{\ell\leqslant l\leqslant k-1}\left\langle\nabla^l u,\nabla\nabla^l\varrho\right\rangle+C_3\left\|\nabla^{\ell+1}\varrho\right\|_{H^{k-\ell-1}}^2\\&\leqslant C_4\left(\left\|\nabla^{\ell+1}u\right\|_{H^{k-\ell}}^2+\left\|\nabla^{\ell+1}(V,W)\right\|_{H^{k-\ell-1}}^2+\left\|\nabla^{\ell}\Delta\phi\right\|_{H^{k-\ell-1}}^2\right).\end{aligned} \tag{5.27}$$

Multiplying (5.27) by $2C_2(\delta_0+\delta_0')/C_3$, and then adding it to (5.25), since δ_0,δ_0' are small, we deduce that there exists a constant $c>0$ such that for $1\leqslant\ell\leqslant k-1$,

$$\begin{aligned}&\frac{d}{dt}\left(\left\|\nabla^\ell(U,\nabla\phi)\right\|_{H^{k-\ell}}^2+\frac{2C_2(\delta_0+\delta_0')}{C_3}\sum_{\ell\leqslant l\leqslant k-1}\left\langle\nabla^l u,\nabla\nabla^l\varrho\right\rangle\right)\\&+c\left(\left\|\nabla^{\ell+1}\varrho\right\|_{H^{k-\ell-1}}^2+\left\|\nabla^{\ell+1}(u,V,W)\right\|_{H^{k-\ell}}^2+\left\|\nabla^\ell\nabla\phi\right\|_{H^{k-\ell+1}}^2\right)\leqslant 0.\end{aligned} \tag{5.28}$$

By defining $\mathcal{E}_\ell^k(t)$ to be c^{-1} times the expression under the time derivative in (5.28), we obtain (5.19). □

Next, we give the evolution of the negative Besov norms of $U(t)$.

Lemma 5.3. *For $s\in(0,3/2]$, we have*

$$\|U(t)\|_{\dot B^{-s}_{2,\infty}}\leqslant C_0. \tag{5.29}$$

Proof. Similar to (1.8) of Theorem 1.2 in [54], we omit the details. □

5.2.2. Proof of decay rates

In this subsection, we will prove the decay rates by using the previous estimates.

By Lemma A.6 and (5.29) of Lemma 5.3, we have for $0\leqslant\ell\leqslant k-1$,

$$\left\|\nabla^\ell U\right\|_{L^2}\leqslant\|U\|_{\dot B^{-s}_{2,\infty}}^{\frac{1}{\ell+1+s}}\left\|\nabla^{\ell+1}U\right\|_{L^2}^{\frac{\ell+s}{\ell+1+s}}\leqslant C_0\left\|\nabla^{\ell+1}U\right\|_{L^2}^{\frac{\ell+s}{\ell+1+s}}. \tag{5.30}$$

Using (1.10) and (5.30), we obtain that for $0\leqslant\ell\leqslant k-1$,

$$\begin{aligned}&\left\|\nabla^{\ell+1}\varrho\right\|_{H^{k-\ell-1}}^2+\left\|\nabla^{\ell+1}(u,V,W)\right\|_{H^{k-\ell}}^2+\left\|\nabla^\ell\nabla\phi\right\|_{H^{k-\ell+1}}^2\\&\geqslant C_0\left(\left\|\nabla^\ell(U,\nabla\phi)\right\|_{H^{k-\ell}}^2\right)^{1+\frac{1}{\ell+s}}.\end{aligned} \tag{5.31}$$

In view of (5.19) of Lemma 5.2 and (5.31), we obtain for $0\leqslant\ell\leqslant k-1$,

$$\frac{d}{dt}\mathcal{E}_\ell^k(t)+C_0\left(\mathcal{E}_\ell^k(t)\right)^{1+\frac{1}{\ell+s}}\leqslant 0. \tag{5.32}$$

We solve this inequality directly to obtain for $0\leqslant\ell\leqslant k-1$,

$$\mathcal{E}_\ell^k(t)\leqslant C_0(1+t)^{-\ell-s}, \tag{5.33}$$

which implies (1.11). Then, (5.10) of Lemma 5.1 and (1.11) give the higher decay (1.12) about the electric field $\nabla\phi$. So, the proof of decay rates is completed.

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Appendix A. Some analytic tools

In this appendix, we give some useful lemmas which are often used in the previous sections. First, we recall the Gagliardo–Nirenberg inequality of Sobolev type.

Lemma A.1. *Let $2\leqslant p\leqslant\infty$ and $\alpha,\beta,\gamma\geqslant 0$. Then we have*

$$\left\|\nabla^\alpha f\right\|_{L^p}\lesssim\left\|\nabla^\beta f\right\|_{L^2}^{1-\theta}\left\|\nabla^\gamma f\right\|_{L^2}^{\theta}. \tag{A.1}$$

Here $0\leqslant\theta\leqslant 1$ (if $p=\infty$, then we require that $0<\theta<1$) and α satisfy

$$\alpha+3\left(\frac{1}{2}-\frac{1}{p}\right)=\beta(1-\theta)+\gamma\theta. \tag{A.2}$$

Proof. We can refer to Theorem (p.125) in [42] or Lemma A.1 in [26]. □

Lemma A.2. *Assume that $\|\varrho\|_{L^\infty}\leqslant 1$ and $\|V\|_{L^\infty}\leqslant 1$. Let $g(\varrho,V)$ be a smooth function of ϱ, V with bounded derivatives of any order, then for any integer $k\geqslant 1$ and $2\leqslant p\leqslant\infty$, we have*

$$\left\|\nabla^k(g(\varrho,V))\right\|_{L^p}\lesssim\left\|\nabla^k\varrho\right\|_{L^p}+\left\|\nabla^k V\right\|_{L^p}. \tag{A.3}$$

Proof. For $k\geqslant 1$, we have

$$\begin{aligned}\nabla^k(g(\varrho,V))=&\sum_{\alpha_1+\cdots+\alpha_j=k}\partial^{\alpha_1,\dots,\alpha_j}g(\varrho,V)\nabla^{\alpha_1}\varrho\nabla^{\alpha_2}\varrho\cdots\nabla^{\alpha_j}\varrho\\&+\sum_{\beta_1+\cdots+\beta_m=k}\partial^{\beta_1,\dots,\beta_m}g(\varrho,V)\nabla^{\beta_1}V\nabla^{\beta_2}V\cdots\nabla^{\beta_m}V\\&+\sum_{\gamma_1+\cdots+\gamma_l=k}\partial^{\gamma_1,\dots,\gamma_l}g(\varrho,V)\nabla^{\gamma_1}\varrho\cdots\nabla^{\gamma_s}\varrho\nabla^{\gamma_{s+1}}V\cdots\nabla^{\gamma_l}V\\&:=Q_1+Q_2+Q_3.\end{aligned} \tag{A.4}$$

By Hölder's and Gagliardo–Nirenberg's inequalities, we obtain

$$
\begin{aligned}
\|Q_1\|_{L^p} &\lesssim \left\|\nabla^{\alpha_1}\varrho\nabla^{\alpha_2}\varrho\cdots\nabla^{\alpha_j}\varrho\right\|_{L^p} \\
&\lesssim \left\|\nabla^{\alpha_1}\varrho\right\|_{L^{\frac{kp}{\alpha_1}}}\left\|\nabla^{\alpha_2}\varrho\right\|_{L^{\frac{kp}{\alpha_2}}}\cdots\left\|\nabla^{\alpha_j}\varrho\right\|_{L^{\frac{kp}{\alpha_j}}} \\
&\lesssim \|\varrho\|_{L^\infty}^{1-\frac{\alpha_1}{k}}\left\|\nabla^k\varrho\right\|_{L^p}^{\frac{\alpha_1}{k}}\|\varrho\|_{L^\infty}^{1-\frac{\alpha_2}{k}}\left\|\nabla^k\varrho\right\|_{L^p}^{\frac{\alpha_2}{k}}\cdots\|\varrho\|_{L^\infty}^{1-\frac{\alpha_j}{k}}\left\|\nabla^k\varrho\right\|_{L^p}^{\frac{\alpha_j}{k}} \\
&\lesssim \|\varrho\|_{L^\infty}^{j-1}\left\|\nabla^k\varrho\right\|_{L^p}. \qquad \text{(A.5)}
\end{aligned}
$$

Similarly, we have

$$
\|Q_2\|_{L^p} \lesssim \|V\|_{L^\infty}^{m-1}\left\|\nabla^k V\right\|_{L^p}. \tag{A.6}
$$

For the third term Q_3, by Hölder's, Gagliardo–Nirenberg's and Young's inequalities, we obtain

$$
\begin{aligned}
\|Q_3\|_{L^p} &\lesssim \left\|\nabla^{\gamma_1}\varrho\cdots\nabla^{\gamma_s}\varrho\nabla^{\gamma_{s+1}}V\cdots\nabla^{\gamma_l}V\right\|_{L^p} \\
&\lesssim \left\|\nabla^{\gamma_1}\varrho\right\|_{L^{\frac{kp}{\gamma_1}}}\cdots\left\|\nabla^{\gamma_s}\varrho\right\|_{L^{\frac{kp}{\gamma_s}}}\left\|\nabla^{\gamma_{s+1}}V\right\|_{L^{\frac{kp}{\gamma_{s+1}}}}\cdots\left\|\nabla^{\gamma_l}V\right\|_{L^{\frac{kp}{\gamma_l}}} \\
&\lesssim \|\varrho\|_{L^\infty}^{1-\frac{\gamma_1}{k}}\left\|\nabla^k\varrho\right\|_{L^p}^{\frac{\gamma_1}{k}}\cdots\|\varrho\|_{L^\infty}^{1-\frac{\gamma_s}{k}}\left\|\nabla^k\varrho\right\|_{L^p}^{\frac{\gamma_s}{k}}\|V\|_{L^\infty}^{1-\frac{\gamma_{s+1}}{k}}\left\|\nabla^k V\right\|_{L^p}^{\frac{\gamma_{s+1}}{k}}\cdots\|V\|_{L^\infty}^{1-\frac{\gamma_l}{k}} \\
&\quad\times\left\|\nabla^k V\right\|_{L^p}^{\frac{\gamma_l}{k}} \\
&\lesssim \|n\|_{L^\infty}^{s-\frac{\gamma_1+\cdots+\gamma_s}{k}}\left\|\nabla^k\varrho\right\|_{L^p}^{\frac{\gamma_1+\cdots+\gamma_s}{k}}\|V\|_{L^\infty}^{l-s-\frac{\gamma_{s+1}+\cdots+\gamma_l}{k}}\left\|\nabla^k V\right\|_{L^p}^{\frac{\gamma_{s+1}+\cdots+\gamma_l}{k}} \\
&\lesssim \|(\varrho,V)\|_{L^\infty}^{l-1}\left(\left\|\nabla^k\varrho\right\|_{L^p}+\left\|\nabla^k V\right\|_{L^p}\right). \qquad \text{(A.7)}
\end{aligned}
$$

Substituting (A.5)–(A.7) into (A.4), we deduce (A.3) since $\|\varrho\|_{L^\infty}\leqslant 1$ and $\|V\|_{L^\infty}\leqslant 1$. □

As a byproduct of Lemma A.2, we immediately have

Corollary A.3. *Assume that $\|\varrho\|_{L^\infty}\leqslant 1$. Let $g(\varrho)$ be a smooth function of ϱ with bounded derivatives of any order, then for any integer $k\geqslant 1$ and $2\leqslant p\leqslant\infty$, we have*

$$
\left\|\nabla^k(g(\varrho))\right\|_{L^p}\lesssim\left\|\nabla^k\varrho\right\|_{L^p}. \tag{A.8}
$$

We then recall the following commutator and product estimates:

Lemma A.4. *Let $l\geqslant 1$ be an integer and define the commutator*

$$
\left[\nabla^l, g\right]h=\nabla^l(gh)-g\nabla^l h. \tag{A.9}
$$

Then we have

$$\left\| \left[\nabla^l, g \right] h \right\|_{L^{p_0}} \lesssim \|\nabla g\|_{L^{p_1}} \left\| \nabla^{l-1} h \right\|_{L^{p_2}} + \left\| \nabla^l g \right\|_{L^{p_3}} \|h\|_{L^{p_4}} . \tag{A.10}$$

In addition, we have that for $l \geqslant 0$,

$$\left\| \nabla^l (gh) \right\|_{L^{p_0}} \lesssim \|g\|_{L^{p_1}} \left\| \nabla^l h \right\|_{L^{p_2}} + \left\| \nabla^l g \right\|_{L^{p_3}} \|h\|_{L^{p_4}} . \tag{A.11}$$

In the above, $p_0, p_1, p_2, p_3, p_4 \in [1, +\infty]$ *such that*

$$\frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Proof. Referring to Lemma 3.4 (p. 129) in [38], we give a complete and simple proof in the following. We first prove (A.11). Let $p_0, p_1, p_2, p_3, p_4 \in [1, +\infty]$ such that

$$\frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Assume $\ell = 0, 1, \dots, l$. We choose q_1, q_2 by

$$\frac{1}{q_1} = \frac{1}{p_1}\left(1 - \frac{\ell}{l}\right) + \frac{1}{p_3}\frac{\ell}{l}, \quad \frac{1}{q_2} = \frac{1}{p_2}\left(1 - \frac{\ell}{l}\right) + \frac{1}{p_4}\frac{\ell}{l}.$$

Thus, we have

$$\frac{1}{p_0} = \frac{1}{q_1} + \frac{1}{q_2}.$$

By Hölder's, Gagliardo–Nirenberg's and Young's inequalities, we have for $l \geqslant 0$,

$$\begin{aligned}
\left\| \nabla^l (gh) \right\|_{L^{p_0}} &= \left\| \sum_{\ell=0}^{l} \nabla^\ell g \nabla^{l-\ell} h \right\|_{L^{p_0}} \\
&\lesssim \sum_{\ell=0}^{l} \left\| \nabla^\ell g \right\|_{L^{q_1}} \left\| \nabla^{l-\ell} h \right\|_{L^{q_2}} \\
&\lesssim \|g\|_{L^{p_1}}^{1-\frac{\ell}{l}} \left\| \nabla^l g \right\|_{L^{p_3}}^{\frac{\ell}{l}} \|h\|_{L^{p_4}}^{\frac{\ell}{l}} \left\| \nabla^l h \right\|_{L^{p_2}}^{1-\frac{\ell}{l}} \\
&= \left(\|g\|_{L^{p_1}} \left\| \nabla^l h \right\|_{L^{p_2}} \right)^{1-\frac{\ell}{l}} \left(\left\| \nabla^l g \right\|_{L^{p_3}} \|h\|_{L^{p_4}} \right)^{\frac{\ell}{l}} \\
&\lesssim \|g\|_{L^{p_1}} \left\| \nabla^l h \right\|_{L^{p_2}} + \left\| \nabla^l g \right\|_{L^{p_3}} \|h\|_{L^{p_4}} .
\end{aligned} \tag{A.12}$$

Note that for $l \geqslant 1$,

$$\left[\nabla^l, g \right] h = \sum_{\ell=1}^{l} \nabla^\ell g \nabla^{l-\ell} h.$$

We can prove (A.10) in the same way as (A.12). □

We have the following L^p embedding lemmas:

Lemma A.5. *Let* $1 \leqslant p < 2$, *and* $1/2 + s/3 = 1/p$, *then* $0 < s \leqslant 3/2$ *and*

$$\|f\|_{\dot{B}^{-s}_{2,\infty}} \lesssim \|f\|_{L^p}. \tag{A.13}$$

Proof. See Lemma 4.1 in [52]. □

We will give the special interpolation estimate:

Lemma A.6. *Let* $k \geqslant 0$ *and* $s > 0$, *then we have*

$$\left\|\nabla^k f\right\|_{L^2} \lesssim \left\|\nabla^{k+1} f\right\|_{L^2}^{1-\theta} \|f\|_{\dot{B}^{-s}_{2,\infty}}^{\theta}, \text{ where } \theta = \frac{1}{\ell + 1 + s}. \tag{A.14}$$

Proof. We refer to Lemma 4.2 in [52] by noting that $\dot{B}^{-s}_{2,p} \subset \dot{B}^{-s}_{2,q}$ for $p \leqslant q$. □

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