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Stability of a Two-Dimensional Poiseuille-Type Flow for a Viscoelastic Fluid

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Abstract. A viscoelastic flow in a two-dimensional layer domain is considered. An L^2-stability of the Poiseuille-type flow is established provided that both Poiseuille flow and perturbation is sufficiently small. Our analysis is based on a stream function formulation introduced by Lin et al. (Commun Pure Appl Math 58(11):1437–1471, 2005).

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1. Introduction

This paper studies the stability of a Poiseuille-type flow for a viscoelastic fluid occupied in a two-dimensional layer domain $\Omega = \mathbb{R} \times (0, 1)$ with the adherence boundary condition. We describe the motion of viscoelastic fluids in Euler's coordinates as in [10]. We in particular consider the incompressible Hookean model introduced by Lin et al. [9], where they construct a local-in-time smooth solution in two or three dimensional bounded domains with smooth boundary as well as the whole space or periodic boxes. They moreover prove global-in-time existence of solutions with small initial data in a two-dimensional periodic box or the whole plane which also indicates some stability of the trivial steady motion (with zero velocity).

In this paper we consider a Poiseuille-type flow of the form $\overline{u}(t, x) = (\psi(t, x_2), 0)$, where x_2 is the vertical variable in $(0, 1)$. It turns out that the integral of ψ in time solves the viscous wave equation. We are interested in its stability as viscoelastic fluids. In fact, we prove that if both the Poiseuille-type flow and the initial perturbations are small, then it is exponentially stable as the time tends to infinity.

Our strategy to prove the stability is to use a stream function formulation due to [9] for a perturbed quantity from the Poiseuille flow, see (4.3). As in [9] the equation is parabolic for the velocity but not for the stream function. Moreover, since our basic flow is the Poiseuille flow, there is a new linear term of a perturbed stream function whose coefficient is not small in the momentum equation which is an extra difficulty compared with the situation in [9]. As in [9] we introduce a new velocity type variable generating dissipative effects and we fully take advantage of the structure of the system to obtain energy estimates. Since there are extra linear terms with non-small coefficients, we derive several energy estimates very carefully to cancel apparently uncontrollable terms. Except energy estimates, the way of construction is the Galerkin method which is the same as [9]. Thus we just concentrate on deriving energy estimates. We also established a non-trivial behavior of the Poiseuille

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flow, especially for higher spatial derivatives since spatial derivatives do not fulfill the boundary condition.

There is also a foregoing research by Giga et al. [3], in which the authors established L^p exponential stability for a small Poiseuille-type flow as well as local-in-time existence for non-small initial data if the layer is thin. Their method is completely different since they use L^p theory instead of L^2 theory developed in this paper. We do not assume that the thickness of the layer is small in this paper.

There is a global estimate result for incompressible viscoelastic flow subject to not necessarily Hookean elastic energy [8]. However, initial data is assumed to be close to a trivial solution. We wonder whether our stability results extends to such a situation but we do not pursue this problem in this paper.

The stability of the Poiseuille flow is an important topic in fluid mechanics. In fact, for the incompressible Navier–Stokes flow the stability of the Couette flow in a half space under small periodic perturbation is established even if the basic flow is large [4]; see also earlier work [11]. The compressible case is also discussed in [5], where stability of a small Couette-type flow is discussed. Moreover, the stability of small steady Poiseuille-type flows in a layer domain in $\mathbb{R}^2$ is discussed in [6] under low Mach numbers. It is actually unstable when the Mach number is not small as shown in a recent work by Kagei and Nishida [7].

For the future research, it is worth noting that this sort of stability problem for the special solutions of the viscoelastic model can be treated in the same way as in this paper. If an a priori estimate for the special solutions, that correspond to Proposition 3.1 in this paper, is proved, one can use the same estimates in this paper and obtain the stability of the perturbed flow.

In Sect. 2 we introduce the model of a viscoelastic fluid. In Sect. 3 we first introduce the Poiseuille-type flow in two dimensions. We then observe that the Poiseuille-type flow in two-dimension is reduced to the viscous wave equation in the $(0, 1)$ interval, and we investigate a priori estimates for the viscous wave system and state our main existence and stability result. Section 4 is devoted to the introduction of a system for the perturbed Poiseuille-type flow. In Sect. 5, we introduce our key notion of change of variables and discuss that the system has hidden dissipative structure. Finally in Sect. 6, we prove energy estimates and our main result. In Sect. 7, we state some basic properties of the Stokes operator that are used in this paper. Section 8 is dedicated to prove a priori estimates of viscous wave equation i.e. Proposition 3.1.

2. Deformation Tensor and Equations of Motion

Let Ω be a domain in $\mathbb{R}^2$ with smooth boundary and $T > 0$ be fixed time. We consider the viscoelastic fluid in Ω described by unknown variables:

- $F : (0, T) \times \Omega \to \mathbb{R}^2 \bigotimes \mathbb{R}^2$ is the deformation tensor,
- $\pi : (0, T) \times \Omega \to \mathbb{R}$ is the pressure,
- $u : (0, T) \times \Omega \to \mathbb{R}^2$ is the velocity of the fluid,
- $\mu > 0$ is the kinematic viscosity of the fluid,

in Eulerian description. The deformation tensor F in the Lagrangian coordinates is defined by $F_{ij} = \partial x_j / \partial X_i$ where X is the Lagrangian variables and $x = x(X, t)$ is the flow map. In the following we always describe F in the Eulerian coordinates. One should be careful to note our notation differs from that in [9], where the transpose of our F is used.

We consider the following two dimensional viscoelastic fluid system of the Oldroyd model with Dirichlet boundary condition of the form.

$$\begin{cases} \partial_t F + u \cdot \nabla F = F \nabla u & \text{in} \quad (0,T) \times \Omega, \\ \operatorname{div} u = 0 & \text{in} \quad (0,T) \times \Omega, \\ \partial_t u - \mu \Delta u + u \cdot \nabla u + \nabla \pi = \operatorname{div} F^T F & \text{in} \quad (0,T) \times \Omega, \\ F|_{t=0} = F_0 & \text{on} \quad \Omega, \\ u|_{t=0} = u_0 & \text{on} \quad \Omega, \\ u = 0 & \text{on} \quad (0,T) \times \partial\Omega \end{cases} \tag{2.1}$$

with the assumption $\det F|_{t=0} = 1$ and $\operatorname{div} F|_{t=0} = 0$. In this paper, we use the following notation.

- $\partial_t = (\partial/\partial t), \partial_i = (\partial/\partial x_i)$,
- $(\nabla u)_{ij} = \partial_j u_i$,
- $(\operatorname{div} G)_i = \sum_{j=1}^n \partial_j G_{ij}$.

Stream Function Formulation. One can show that $\operatorname{div} F$ is subject to advection with the flow, i.e.

$$\partial_t (\operatorname{div} F) + u \cdot \nabla (\operatorname{div} F) = 0.$$

Therefore $\operatorname{div} F_0 = 0$ implies $\operatorname{div} F = 0$ for all later times. Under this assumption, one can find an $\mathbb{R}^2$-valued stream function ζ_0 such that $F_0 = \nabla^\perp \zeta_0$ as in [9]. Moreover, if one lets ζ be the solution of the transport equation for a divergence free function u of the form

$$\begin{cases} \partial_t \zeta + u \cdot \nabla \zeta = 0, \\ \quad \zeta|_{t=0} = \zeta_0 \end{cases} \tag{2.2}$$

then one can find that for

$$F = \nabla^\perp \zeta = \begin{pmatrix} -\partial_2 \zeta^1 & \partial_1 \zeta^1 \\ -\partial_2 \zeta^2 & \partial_1 \zeta^2 \end{pmatrix},$$

the equation $F_t + u \cdot \nabla F = F \nabla u$ is fulfilled. It is much easier to consider the function ζ instead of F. In order to rewrite system (2.1) with respect to ζ, one can calculate

$$\operatorname{div} F^T F = \frac{1}{2} \nabla |\nabla \zeta|^2 - \Delta \zeta^1 \nabla \zeta^1 - \Delta \zeta^2 \nabla \zeta^2.$$

Note that the first term is a gradient that can be absorbed in pressure term. Thus we introduce a new variable $\tilde{\pi} = \pi - \frac{1}{2}|\nabla \zeta|^2$ and denote that by π again. We end up with the following new system for two dimensions with Dirichlet boundary condition.

$$\begin{cases} \partial_t \zeta + u \cdot \nabla \zeta = 0 & \text{in} \quad (0,T) \times \Omega, \\ \operatorname{div} u = 0 & \text{in} \quad (0,T) \times \Omega, \\ \partial_t u - \mu \Delta u + u \cdot \nabla u + \nabla \pi = -\sum_{k=1}^2 \Delta \zeta^k \nabla \zeta^k & \text{in} \quad (0,T) \times \Omega, \\ \zeta|_{t=0} = \zeta_0 & \text{on} \quad \Omega, \\ u|_{t=0} = u_0 & \text{on} \quad \Omega, \\ u = 0 & \text{on} \quad (0,T) \times \partial\Omega. \end{cases} \tag{2.3}$$

The corresponding assumption to the incompressibility condition $\det F|_{t=0} = 1$ is

$$\partial_1 \zeta_0^1 \partial_2 \zeta_0^2 - \partial_1 \zeta_0^2 \partial_2 \zeta_0^1 = 1. \tag{2.4}$$

Note that $\operatorname{div} F|_{t=0} = 0$ is satisfied by the construction of ζ.

Incompressibility. Considering fluids with a constant density, the incompressibility condition takes the form $\operatorname{div} u = 0$. It turns out that in terms of the deformation tensor, this means $\det F = 1$ if $\det F|_{t=0} = 1$ holds. Moreover, one can find that

$$\partial_t (\det F) + u \cdot \nabla (\det F) = 0.$$

Therefore if (2.4) holds, we have $\partial_1 \zeta^1 \partial_2 \zeta^2 - \partial_1 \zeta^2 \partial_2 \zeta^1 = 1$ for all time.

3. Poiseuille-Type Flow and Viscous Wave Equation

Let $\Omega = \mathbb{R} \times (0,1)$ i.e. a two-dimensional layer. This section aims to construct a suitable Poiseuille-type flow solution $\bar{u}$ to (2.1) or equivalently (2.3), i.e. a solution with horizontal flow-profile that is completely determined by the vertical component. Hence, we assume that $\bar{u}$ takes the form

$$\bar{u}(t,x) = \begin{pmatrix} \psi(t,x_2) \\ 0 \end{pmatrix}$$

with homogeneous Dirichlet boundary conditions. Then the divergence condition in (2.1) is trivially fulfilled.

In order to adequately determine the corresponding deformation tensor $\bar{F}$ or equivalently the corresponding stream function η, we introduce the flow map $x(t,X) = \big(x_1(t,X), x_2(t,X)\big)$, $0 \le t < T$ with $T > 0$, corresponding to Lagrangian coordinates X. These flow maps are given by the system of ordinary differential equations

$$\begin{cases} \dfrac{d}{dt} x_1(t,X) = \bar{u}^1(t, x_1(t,X), x_2(t,X)) = \psi(t, x_2(t,X)), & x_1(0) = X_1, \\ \dfrac{d}{dt} x_2(t,X) = \bar{u}^2(t, x_1(t,X), x_2(t,X)) = 0, & x_2(0) = X_2, \end{cases}$$

which can easily be solved by

$$\begin{cases} x_1(t,X) = X_1 + \displaystyle\int_0^t \psi(s, x_2(s,X))\,\mathrm{d}s = X_1 + \int_0^t \psi(s, X_2)\,\mathrm{d}s, \\ x_2(t,X) = X_2, \end{cases}$$

as long as ψ is sufficiently regular. Let us abbreviate

$$\phi(t,x_2) = \int_0^t \psi(s,x_2)\,\mathrm{d}s.$$

Then, we can calculate the deformation tensor and the resulting elastic force

$$\bar{F} = \begin{pmatrix} 1 & 0 \\ \partial_2\phi & 1 \end{pmatrix}, \quad \bar{F}^T\bar{F} = \begin{pmatrix} 1 + (\partial_2\phi)^2 & \partial_2\phi \\ \partial_2\phi & 1 \end{pmatrix} \quad \text{and} \quad \operatorname{div} \bar{F}^T\bar{F} = \begin{pmatrix} \partial_2^2\phi \\ 0 \end{pmatrix}.$$

Note here, that with $x_2(t,X) = X_2$ it is also $(\partial/\partial X_2) = (\partial/\partial x_2) = \partial_2$. Let us also remark at this point, that $\operatorname{div} \bar{F} = 0$.

The stream function η corresponding to $\bar{F}$ may be chosen as

$$\eta(t,x) = \begin{pmatrix} -x_2 \\ x_1 - \phi(t,x_2) \end{pmatrix} \tag{3.1}$$

solving the system

$$\begin{cases} \partial_t\eta + \bar{u}\cdot\nabla\eta = 0, & \text{in} \quad (0,T)\times\Omega, \\ \eta(0,x) = (-x_2, x_1)^T, & \text{for} \quad x \in \Omega. \end{cases}$$

Inserting the elastic force into the balance of momentum for $\bar{u}$, i.e.

$$\partial_t\bar{u} - \mu\Delta\bar{u} + \bar{u}\cdot\nabla\bar{u} + \nabla\bar{\pi} = \operatorname{div} \bar{F}^T\bar{F}, \quad \text{in} \quad (0,T)\times\Omega,$$

yields the equivalent formulation

$$\left.\begin{aligned} \partial_t\psi + \partial_1\bar{\pi} &= \mu\partial_2^2\psi + \partial_2^2\phi, \\ \partial_2\bar{\pi} &= 0. \end{aligned}\right\} \quad \text{in} \quad (0,T)\times\Omega.$$

We conclude from the second equation that the pressure is a function depending only on the horizontal variable $\bar{\pi} = \bar{\pi}(t, x_1)$. Since ψ and ϕ depend only on t and x_2, the first equation implies that $\partial_1 \bar{\pi}$ is a function of time only, i.e. $\partial_1 \bar{\pi}(t, x_1) = -h(t)$ for some function h. Inserting this into the system yields

$$\partial_t \psi - \partial_2^2 \phi = \mu \partial_2^2 \psi + h, \quad \text{in} \quad (0,T) \times (0,1).$$

Finally, by the definition of ϕ it is $\psi(t, x_2) = \partial_t \phi(t, x_2)$ and moreover, the homogeneous Dirichlet boundary conditions for $\bar{u}$ carry over to ϕ, i.e. $\phi(t, 0) = \phi(t, 1) = 0$. At initial time we have $\phi(0, x_2) = \int_0^0 \psi(s, x_2)\, \mathrm{d}s = 0$ and $\partial_t \phi(0, x_2) = \psi(0, x_2) = \psi_0(x_2)$ for some function ψ_0 that will be given satisfying homogeneous Dirichlet conditions.

With this, we end up with a viscous wave equation in one dimension

$$\begin{cases} \partial_t^2 \phi - \partial_x^2 \phi = \mu \partial_t \partial_x^2 \phi + h, & \text{in} \quad (0,T) \times (0,1), \\ \phi(t,0) = \phi(t,1) = 0, & \text{for} \quad t \in (0,T), \\ \phi|_{t=0} = 0, \quad \partial_t \phi|_{t=0} = \psi_0, & \text{on} \quad (0,1). \end{cases} \tag{3.2}$$

for some $h = h(t)$ and initial data ψ_0. Note that we use ∂_x instead of ∂_2 since we consider ϕ is the function with two variables (t, x) here.

We state an a priori estimate for the Poiseuille-type flow in the following proposition. It enables us to control the norms of higher spatial derivatives of ϕ.

Proposition 3.1. *Let $T > 0$ and $\mu > 0$. For $\psi_0 \in H^3(0,1) \cap H_0^1(0,1)$ and $h \in H^1(0,T)$, there exists the unique solution $\phi \in C^3([0,T]; L^2(0,1)) \cap C^2([0,T), H^3(0,1)) \cap C([0,T]; H^4(0,1))$ of (3.2). Moreover, there is a constant C such that the solution satisfies*

$$\begin{aligned} &\|\partial_t \phi(t)\|_{H^3(0,1)} + \|\partial_x \phi(t)\|_{H^3(0,1)} + \|\partial_t^2 \phi(t)\|_{H^1(0,1)} \\ &\leq C \left(e^{-\min\left(2\mu\pi^2, \frac{1}{\mu}\right)t} \|\psi_0\|_{H^3(0,1)} + \sum_{k=1}^{4} \mu^{-k} \|h\|_{H^1(0,t)} \right) \end{aligned}$$

for $0 \leq t \leq T$. The constant C is independent of T and μ.

Proof. Combining Propositions 8.1 and 8.3 in Sect. 8, one can easily obtain the result. □

Notation of Spaces. In this paper, we write $\|f\|$ for $\|f\|_{L^2(U)}$ otherwise specified, and denote $H^k(U)$ by Sobolev space $W^{2,k}(U)$, equipped with the norm

$$\|f\|_{H^k(U)} = \sqrt{\sum_{|\alpha| \leq k} \|\partial^k f\|^2}$$

for some domain $U \subset \mathbb{R}^n$. We also define $H_0^k(U)$ by the closure of C_c^∞, the space of all smooth functions with compact support with respect to $\|\cdot\|_{H^k(U)}$; see [2, Section 5] for more detail.

Inserting the function $\psi = \partial_t \phi$ into the ansatz for $\bar{u}$, we receive a solution $(\bar{u}, \bar{\pi}, \eta)$ of the system

$$\begin{cases} \partial_t \eta + \bar{u} \cdot \nabla \eta = 0, & \text{in} \quad (0,T) \times \Omega, \\ \operatorname{div} \bar{u} = 0, & \text{in} \quad (0,T) \times \Omega, \\ \partial_t \bar{u} - \mu \Delta \bar{u} + \bar{u} \cdot \nabla \bar{u} + \nabla \bar{\pi} = -\Delta \eta^k \nabla \eta^k, & \text{in} \quad (0,T) \times \Omega, \\ \bar{u} = 0, & \text{on} \quad (0,T) \times \partial\Omega, \\ \eta(0,x) = (-x_2, x_1)^T, & \text{for} \quad x \in \Omega, \\ \bar{u}|_{t=0} = (\psi_0, 0)^T, \text{on} \quad \Omega, \end{cases}$$

where $-\Delta \eta^k \nabla \eta^k$ is a short notation for $\sum_{k=1}^2 -\Delta \eta^k \nabla \eta^k$. Note that we choose η by (3.1) and due to the homogeneous Dirichlet boundary conditions for ϕ, it is $\eta(t,x)|_{\partial\Omega} = (-x_2, x_1)^T$ for any $0 \leq t \leq T$.

We are now in a position to state our main result.

Theorem 3.2. *Let $\Omega = \mathbb{R} \times (0,1)$, $u_0 \in H^2(\Omega)$, $\zeta_0 \in H^3(\Omega)$, $\psi_0 \in H^3(0,1) \cap H_0^1(0,1)$, and $h \in H^1(0,\infty)$. There exist numbers $0 < \delta < 1$, $\kappa > 0$ such that if the following three conditions hold,*

1. *the smallness condition for the Poiseuille-type flow*

$$\|\psi_0\|_{H^3(0,1)} + \|h\|_{H^1(0,\infty)} \le \kappa,$$

2. *the smallness condition for the initial perturbation*

$$\|u_0 - \bar{u}_0\|_{H^3(\Omega)} + \|\zeta_0 - \eta_0\|_{H^3(\Omega)} \le \kappa,$$

3. *the compatibility conditions for the initial data*

$$\operatorname{div} u_0 = 0, \quad \zeta_0|_{\partial\Omega} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \quad \text{and} \quad \partial_1\zeta_0^1\partial_2\zeta_0^2 - \partial_1\zeta_0^2\partial_2\zeta_0^1 = 1$$

then there exists a smooth global solution (u, ζ, π) *of* (2.3) *with respect to initial data* (u_0, ζ_0) *satisfying*

$$\frac{d}{dt}\left(\|\partial_t(u-\bar{u})\|^2 + \delta\|\nabla(u-\bar{u})\|^2 + \delta\|\Delta(\zeta-\eta)\|^2 + \delta\|\nabla\Delta(\zeta-\eta)\|^2 + \|\partial_t(\zeta-\eta)\|^2\right)$$
$$+ \mu\delta\left(\|\partial_t\nabla(u-\bar{u})\|^2 + \|A(u-\bar{u})\|^2 + \frac{1}{\mu^2}\|\Delta(\zeta-\eta)\|^2 + \frac{1}{\mu^2}\|\nabla\Delta(\zeta-\eta)\|^2\right) \le 0$$

for all times $t \ge 0$. *Here,* $A = -P\Delta$ *is the Stokes operator in* Ω*; see Sect. 7.*

Integrating the last differential inequality over $(0, t)$ implies

$$\delta\|\nabla(u-\bar{u})\|^2(t) + \mu\delta\int_0^t \|A(u-\bar{u})\|^2(s)\,\mathrm{d}s \le C_\kappa$$

with C_κ which tends to zero as $\kappa \to 0$. By (7.1) we in particular obtain

$$\|\nabla(u-\bar{u})\|^2(t) + \int_0^t C\mu\|\nabla(u-\bar{u})\|^2(s)\,\mathrm{d}s \le C_\kappa$$

which implies

$$\|\nabla(u-\bar{u})\|^2(t) \le C_\kappa e^{-C\mu t}$$

by the Gronwall inequality. By the Poincaré inequality this implies that $\bar{u}$ is exponentially stable in H^1 sense. Similar stability holds for η.

4. Perturbation of the Flow Through the Layer

We are interested in the solution (u, π, ζ) of the system (2.1) and its stability around the Poiseuille-type flow $(\bar{u}, \bar{\pi}, \eta)$. Let (u_0, ζ_0) satisfies the compatibility conditions,

$$\operatorname{div} u_0 = 0, \quad \zeta_0|_{\partial\Omega} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \quad \text{and} \quad \partial_1\zeta_0^1\partial_2\zeta_0^2 - \partial_1\zeta_0^2\partial_2\zeta_0^1 = 1. \tag{4.1}$$

The second condition together with the homogeneous Dirichlet boundary condition of u guarantees $\zeta|_{\partial\Omega} = (-x_2, x_1)^T$ for all times. The third condition is a reformulation of the incompressibility condition as we discussed in Sect. 2 and that holds for any $t \ge 0$.

Now let us introduce the perturbation

$$(v, p, \alpha) = (u, \pi, \zeta) - (\bar{u}, \bar{\pi}, \eta).$$

By a simple calculation one can show that the decomposition $\zeta = \alpha + \eta$ implies

$$\partial_2\alpha^1 - \partial_1\alpha^2 = \partial_1\alpha^1\partial_2\alpha^2 - \partial_2\alpha^1\partial_1\alpha^2 - \partial_2\phi\partial_1\alpha^1. \tag{4.2}$$

This will be a crucial identity later on.

Then for given $(\bar{u}, \bar{\pi}, \eta)$, (v, p, α) solves

$$\begin{cases} \partial_t \alpha + v \cdot \nabla \alpha + \bar{u} \cdot \nabla \alpha = -v \cdot \nabla \eta & \text{in} \quad (0,T) \times \Omega, \\ \operatorname{div} v = 0 & \text{in} \quad (0,T) \times \Omega, \\ \partial_t v - \mu \Delta v + v \cdot \nabla v + v \cdot \nabla \bar{u} + \bar{u} \cdot \nabla v + \nabla p & \\ \quad = -\Delta \alpha^k \nabla \alpha^k - \Delta \eta^k \nabla \alpha^k - \Delta \alpha^k \nabla \eta^k & \text{in} \quad (0,T) \times \Omega, \\ v = 0, & \text{on} \quad (0,T) \times \partial\Omega, \\ \alpha(0,x) = \zeta_0(x) - (-x_2, x_1)^T & \text{for} \quad x \in \Omega, \\ v|_{t=0} = u_0 - (\psi_0, 0)^T & \text{in} \quad \Omega \end{cases} \tag{4.3}$$

Note that it is $\alpha|_{\partial\Omega} = 0$ for all times since $\zeta|_{\partial\Omega} = \eta|_{\partial\Omega} = (-x_2, x_1)$.

The stream function of the Poiseuille-type flow is given by $\eta(t,x) = (-x_2, x_1 - \phi(t,x_2))$. We note that derivatives of η contain constant parts which may not be small even if ϕ is small. Let us rewrite the right-hand side of the momentum equation as

$$\begin{aligned} -\Delta \alpha^k \nabla \eta^k - \Delta \eta^k \nabla \alpha^k &= \begin{pmatrix} -\Delta \alpha^2 + \partial_1(\partial_2^2 \phi \alpha^2) \\ \Delta \alpha^1 + \Delta(\partial_2 \phi \alpha^2) - \partial_2^3 \phi \alpha^2 - \partial_2^2 \phi \partial_2 \alpha^2 \end{pmatrix} \\ &= \Delta \begin{pmatrix} -\alpha^2 \\ \alpha^1 \end{pmatrix} + \partial_2^2 \phi \nabla \alpha^2 + \nabla \phi \Delta \alpha^2. \end{aligned} \tag{4.4}$$

Therefore the momentum equation is rewritten as follows.

$$\begin{aligned} &\partial_t v - \mu \Delta v + v \cdot \nabla v + v \cdot \nabla \bar{u} + \bar{u} \cdot \nabla v + \nabla p \\ &\quad = -\Delta \alpha^k \nabla \alpha^k + \Delta \begin{pmatrix} -\alpha^2 \\ \alpha^1 \end{pmatrix} + \nabla \phi \Delta \alpha^2 + \partial_2^2 \phi \nabla \alpha^2. \end{aligned} \tag{4.5}$$

5. Change of Variables and Dissipation

Observing the momentum equation in (4.5), one may notice that terms like $v \cdot \nabla \bar{u}$ or $\bar{u} \cdot \nabla v$ can be handled through Proposition 3.1 if the Poiseuille-type flow is sufficiently small. On the other hand, $\Delta(-\alpha_2, \alpha_1)$ causes a problem. Although α seems to have no dissipative structure so far, this term produces linear terms. That calls a particular method.

Taking a closer look at right-hand side in (4.5), one can find another dissipative structure. Let us focus on the term $\Delta(-\alpha^2, \alpha^1)^T$ in (4.4), and rewrite whole equation as follows

$$\begin{aligned} &\partial_t v - \mu \Delta \left(v + \frac{1}{\mu} \begin{pmatrix} -\alpha^2 \\ \alpha^1 \end{pmatrix} \right) + v \cdot \nabla v + v \cdot \nabla \bar{u} + \bar{u} \cdot \nabla v + \nabla p \\ &\quad = -\Delta \alpha^k \nabla \alpha^k + \nabla \phi \Delta \alpha^2 + \partial_2^2 \phi \nabla \alpha^2. \end{aligned} \tag{5.1}$$

Now we will introduce a new dependent variable as in [9]:

$$w = v + \frac{1}{\mu} \begin{pmatrix} -\alpha^2 \\ \alpha^1 \end{pmatrix} \quad \text{or equivalently,} \quad \alpha = \mu \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (w - v).$$

Let us rewrite the transport equation of α in (4.3), i.e.

$$\partial_t \alpha + v \cdot \nabla \alpha + \bar{u} \cdot \nabla \alpha = -v \cdot \nabla \eta. \tag{5.2}$$

In the right-hand side, one can find

$$\begin{aligned} -v \cdot \nabla \eta &= \begin{pmatrix} v^2 \\ -v^1 \end{pmatrix} + v^2 \nabla \phi \\ &= \begin{pmatrix} w^2 \\ -w^1 \end{pmatrix} - \frac{1}{\mu} \alpha + v^2 \nabla \phi. \end{aligned}$$

Therefore the transport equation can be rewritten as follows

$$\partial_t \alpha + \frac{1}{\mu}\alpha + v \cdot \nabla\alpha + \bar{u} \cdot \nabla\alpha = \begin{pmatrix} w_2 \\ -w_1 \end{pmatrix} + v_2 \nabla\phi. \tag{5.3}$$

Hence we can see that α has dissipative structure. However, we must control the w term in the right-hand side. The question is how to introduce estimates for w or v? The idea is that we regard (4.5) as a perturbed Stokes system of w and p, i.e.

$$-\mu\Delta w + \nabla p = -\partial_t v - v \cdot \nabla v - v \cdot \nabla u - \Delta\alpha^k \nabla\alpha^k + \partial_2^2\phi\nabla\alpha^2 + \nabla\phi\Delta\alpha^2 \tag{5.4}$$

and invoke a higher regularity estimates of the Stokes system (Lemma 7.2). For this purpose, we need to calculate $\operatorname{div} w$ first.

Divergence of w and Higher Order Estimate. Let us note that w is not divergence free in general. However, its divergence is quadratic in α and ϕ as the following calculation shows:

$$\begin{aligned} \operatorname{div} w &= \operatorname{div} v + \frac{1}{\mu}\operatorname{div}\begin{pmatrix} -\alpha^2 \\ \alpha^1 \end{pmatrix} \\ &= \frac{1}{\mu}(\partial_2\alpha^1 - \partial_1\alpha^2 + \partial_2\phi\partial_1\alpha^1 - \partial_2\phi\partial_1\alpha^1) \\ &= \frac{1}{\mu}(\det G - \partial_2\phi\partial_1\alpha^1) \\ &= \frac{1}{\mu}(\partial_1\alpha^1\partial_2\alpha^2 - \partial_2\alpha^1\partial_1\alpha^2 - \partial_2\phi\partial_1\alpha^1). \end{aligned} \tag{5.5}$$

Note that we used the incompressibility property (4.2).

Now let f and g be the right-hand sides of (5.4), (5.5) respectively. If w satisfies appropriate conditions, we can invoke Lemma 7.2 and obtain

$$\mu\|w\|_{H^3(\Omega)} + \|\nabla p\|_{H^1(\Omega)} \le C(\mu\|g\|_{H^2(\Omega)} + \|f\|_{H^1(\Omega)}). \tag{5.6}$$

We can easily obtain the estimate for v by the definition of w. We will state the result of these estimates in the next section as a proposition.

6. A Priori Estimate with Energy Method

The existence of approximate solutions to (4.3) may be proved using a Galerkin approximation scheme similarly to [9]. Since compact embeddings are required for this approach in order to pass to the limit, the problem then needs to be considered on a sequence of domains $\Omega^M = (-M, M) \times (0, 1)$.

We impose $v = 0$ on the artificial left and right boundaries. For the stream function α no boundary conditions may be imposed and it will in general not vanish on the artificial boundaries. It vanishes on the lower and upper boundary however, since by definition $\alpha = \zeta - \eta$ and the stream functions η and ζ are transported by $\bar{u}$ and u which vanish on the upper and lower boundary (but not on the artificial boundaries). Hence, for v as well as α the Poincaré inequality is still applicable. Since all the estimates do not depend on the horizontal size of the domain, one can let M tend to infinity to receive a solution of (4.3).

The a priori estimates for the approximate solutions of the Galerkin-scheme are of the same structure as for the original equations. Let us therefore concentrate on the formal a priori estimates for system (4.3).

Let Ω be a layer domain $\mathbb{R} \times (0, 1)$ in this section.

Notation. In order to simplify the notation, we now introduce variables corresponding to the data, the time-derivative part and the dissipative part of the estimates respectively. We write

$$\begin{aligned} X(t) &= \|\partial_t \phi\|_{H^3(0,1)} + \|\partial_2 \phi\|_{H^3(0,1)} + \|\partial_t^2 \phi\|_{H^1(0,1)}, \\ Y(t) &= \|\partial_t v\| + \|\nabla v\| + \|\Delta\alpha\| + \|\nabla\Delta\alpha\|, \\ Z(t) &= \|\partial_t \nabla v\|^2 + \|Av\|^2 + \frac{1}{\mu^2}\|\Delta\alpha\|^2 + \frac{1}{\mu^2}\|\nabla\Delta\alpha\|^2. \end{aligned}$$

With the definition of $Y(t)$ and $Z(t)$ as it is, by the Poincaré inequality we find

$$Y^2(t) \le C(1+\mu^2)Z(t).$$

Here, A in $Z(t)$ is the Stokes operator. We use Av instead of Δv to annihilate the pressure term in some energy estimates with aid of the regularity of the Stokes operator $\|v\|_{H^2_2(\Omega)} \le C\|Av\|$.

In this section, we derive five energy estimates. Then combining these results, we obtain the strong stability inequality stated in Theorem 3.2. To begin with, we need to calculate (5.6) for higher order estimates.

6.1. Spatial Estimate of the Artificial Variable w and v

It is difficult to estimate higher spatial derivatives directly. We cannot use integration by parts since higher spatial derivatives would not vanish on the boundary in general. Therefore we consider a priori estimates for higher order terms in time, and then transfer them into spatial estimates using the regularity of the Stokes system Lemma 7.2.

Proposition 6.1. *Let $\Omega = \mathbb{R} \times (0,1)$, $u_0 \in H^2(\Omega)$, $\zeta_0 \in H^3(\Omega)$, $\psi_0 \in H^3(0,1) \cap H^1_0(0,1)$, $h \in H^1(0,\infty)$, and (u,π,ζ) be a solution of (2.3). If (u_0,ζ_0) satisfies the compatibility conditions*

$$\operatorname{div} u_0 = 0, \quad \zeta_0|_{\partial\Omega} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \quad \text{and} \quad \partial_1\zeta_0^1\partial_2\zeta_0^2 - \partial_1\zeta_0^2\partial_2\zeta_0^1 = 1,$$

then there exists a numerical constant $C > 0$ such that for $(v,p,\alpha) = (u,\pi,\zeta) - (\bar{u},\bar{\pi},\eta)$ and $w = v - \frac{1}{\mu}(-\alpha^2,\alpha^1)^T$, the estimates

$$\begin{aligned} \|v\|_{H^3(\Omega)} &\le C\big(\|\partial_t \nabla v\| + (1+\mu^2)Z + X(\|Av\| + \|\Delta\alpha\| + \|\nabla\Delta\alpha\|)\big) \\ &\quad + \frac{C}{\mu}(\|\Delta\alpha\| + \|\nabla\Delta\alpha\|). \\ \|w\|_{H^3(\Omega)} &\le C\big(\|\partial_t \nabla v\| + (1+\mu^2)Z + X(\|Av\| + \|\Delta\alpha\| + \|\nabla\Delta\alpha\|)\big) \end{aligned}$$

hold for all $t \ge 0$.

Proof. Inserting (5.4) and (5.5) to (5.6), we have

$$\begin{aligned} &\mu\|w\|_{H^3(\Omega)} + \|\nabla p\|_{H^1(\Omega)} \\ &\le C\Big(\big\|\partial_t v - v\cdot\nabla v - v\cdot\nabla\bar{u} - \bar{u}\cdot\nabla v \\ &\quad - \Delta\alpha^k\nabla\alpha^k + \nabla\phi\Delta\alpha^2 + \partial_2^2\phi\nabla\alpha^2\big\|_{H^1(\Omega)} \\ &\quad + \big\|\partial_1\alpha^1\partial_2\alpha^2 - \partial_2\alpha^1\partial_1\alpha^2 - \partial_2\phi\partial_1\alpha^1\big\|_{H^2(\Omega)}\Big) \\ &\le C\Big(\|\partial_t v\|_{H^1(\Omega)} + \|v\cdot\nabla v\|_{H^1(\Omega)} + \|v\cdot\nabla\bar{u}\|_{H^1(\Omega)} + \|\bar{u}\cdot\nabla v\|_{H^1(\Omega)} \\ &\quad + \|\Delta\alpha^k\nabla\alpha^k\|_{H^1(\Omega)} + \|\nabla\phi\Delta\alpha^2\|_{H^1(\Omega)} + \|\partial_2^2\phi\nabla\alpha^2\|_{H^1(\Omega)} \\ &\quad + \|\partial_1\alpha^1\partial_2\alpha^2\|_{H^2(\Omega)} + \|\partial_2\alpha^1\partial_1\alpha^2\|_{H^2(\Omega)} + \|\partial_2\phi\partial_1\alpha^1\|_{H^2(\Omega)}\Big). \end{aligned}$$

We investigate these terms one by one, beginning with

$$\|\partial_t v\|_{H^1(\Omega)} \leq \|\partial_t \nabla v\| \tag{6.1}$$

by the Poincaré inequality. We next observe that

$$\begin{aligned}
\|v \cdot \nabla v\|_{H^1(\Omega)} &\leq \|v \cdot \nabla v\| + \|\nabla(v \cdot \nabla v)\| \\
&\leq C\big(\|v\|_{L^\infty(\Omega)}\|\nabla v\| + \|v\|_{L^\infty(\Omega)}\|\nabla^2 v\| + \|\nabla v\|^2_{L^4(\Omega)}\big) \\
&\leq C\|v\|^2_{H^2(\Omega)} \\
&\leq C\|Av\|^2.
\end{aligned}$$

We have invoked embeddings in Lemma 7.5 and the regularity of the Stokes operator (7.1).

Similarly,

$$\begin{aligned}
\|v \cdot \nabla \bar{u}\|_{H^1(\Omega)} &\leq C\big(\|v\|_{L^\infty(\Omega)}\|\nabla \bar{u}\| + \|\partial_2 \bar{u}\|\|\nabla^2 v\| + \|\bar{u}\|_{L^\infty(\Omega)}\|\nabla^2 v\|\big) \\
&\leq C\|Av\|\|\partial_t \phi\|_{H^2(0,1)} \\
\|\bar{u} \cdot \nabla v\|_{H^1(\Omega)} &\leq C\big(\|\bar{u}\|_{L^\infty(\Omega)}\|\nabla v\| + \|\partial_2 \bar{u}\|\|\nabla^2 v\| + \|\bar{u}\|_{L^\infty(\Omega)}\|\nabla^2 v\|\big) \\
&\leq C\|Av\|\|\partial_t \phi\|_{H^2(0,1)}.
\end{aligned}$$

Here, we have invoked the 1D-2D product estimate Lemma 7.6. Note that $\bar{u}$ is a function of one space variable.

Finally, we have

$$\begin{aligned}
&\|\Delta\alpha^k \nabla\alpha^k\|_{H^1(\Omega)} + \|\nabla\phi\Delta\alpha^2\|_{H^1(\Omega)} + \|\partial_2^2\phi\nabla\alpha^2\|_{H^1(\Omega)} \\
&\quad\leq C\big(\|\Delta\alpha\|\|\nabla\alpha\|_{L^\infty(\Omega)} + \|\nabla\Delta\alpha\|\|\nabla\alpha\|_{L^\infty(\Omega)} \\
&\qquad + \|\Delta\alpha\|_{L^4(\Omega)}\|\nabla^2\alpha\|_{L^4(\Omega)} + \|\partial_2\phi\|_{H^3(0,1)}\|\nabla\alpha\|_{H^3(\Omega)}\big) \\
&\quad\leq C\big(\|\nabla\Delta\alpha\|^2 + \|\Delta\alpha\|^2 + \|\partial_2\phi\|_{H^3(0,1)}(\|\nabla\Delta\alpha\| + \|\Delta\alpha\|)\big). \\
&\qquad \times \|\partial_1\alpha^1\partial_2\alpha^2\|_{H^2(\Omega)} + \|\partial_2\alpha^1\partial_1\alpha^2\|_{H^2(\Omega)} + \|\partial_2\phi\partial_1\alpha^1\|_{H^2(\Omega)} \\
&\quad\leq C\big(\|\nabla\alpha\|_{H^2(\Omega)}\|\nabla\alpha\|_{L^\infty(\Omega)} + \|\Delta\alpha\|^2_{L^4(\Omega)} \\
&\qquad + \|\partial_2\phi\|_{L^\infty(\Omega)}\|\nabla\alpha\|_{H^2(\Omega)} + \|\partial_2^2\phi\|_{L^\infty(\Omega)}\|\nabla\alpha\|_{H^1(\Omega)} + \|\partial_2^3\phi\|\|\nabla\alpha\|_{L^\infty(\Omega)}\big) \\
&\quad\leq C\big(\|\nabla\Delta\alpha\|^2 + \|\Delta\alpha\|^2 + \|\partial_2\phi\|_{H^3(0,1)}(\|\nabla\Delta\alpha\| + \|\Delta\alpha\|)\big).
\end{aligned}$$

Combining these results, we obtain,

$$\begin{aligned}
&\mu\|w\|_{H^3(\Omega)} + \|\nabla p\|_{H^1(\Omega)} \\
&\quad\leq C\big(\|\partial_t\nabla v\|^2 + \|Av\|^2 + \|\Delta\alpha\|^2 + \|\nabla\Delta\alpha\|^2 \\
&\qquad + (\|\partial_t\phi\|_{H^2(0,1)} + \|\partial_2\phi\|_{H^3(0,1)})(\|Av\| + \|\Delta\alpha\| + \|\nabla\Delta\alpha\|)\big) \\
&\quad\leq C(\|\partial_t\nabla v\| + (1+\mu^2)Z + X(\|Av\| + \|\Delta\alpha\| + \|\nabla\Delta\alpha\|)\big)
\end{aligned} \tag{6.2}$$

We immediately find a similar estimate for higher regularity of v with $v = w - \frac{1}{\mu}(-\alpha^2, \alpha^1)^T$ i.e.

$$\begin{aligned}
\|v\|_{H^3(\Omega)} &\leq C\big(\|\partial_t\nabla v\| + (1+\mu^2)Z + X(\|Av\| + \|\Delta\alpha\| + \|\nabla\Delta\alpha\|)\big) \\
&\quad + \frac{C}{\mu}(\|\Delta\alpha\| + \|\nabla\Delta\alpha\|).
\end{aligned} \tag{6.3}$$

□

Later on, we will use these results to estimate higher derivatives in time as mentioned above.

6.2. Summary of the Energy Estimates

Let us state the result of the energy estimates first:

Proposition 6.2. *Under the same assumption as in Proposition 6.1, we have the following estimates:*

$$\frac{1}{2}\frac{d}{dt}\|\nabla v\|^2 + \mu\|Av\|^2 \leq \|\Delta\alpha\|\|Av\| + C(1+\mu)(X+Y)Z, \tag{6.4}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial_t v\|^2 + \mu\|\partial_t\nabla v\|^2 \leq & -\left(\partial_t\nabla\begin{pmatrix}-\alpha^2\\ \alpha^1\end{pmatrix}, \partial_t\nabla v\right) \\ & + C(1+\mu)(1+X+Y)(X+Y)Z,\end{aligned} \tag{6.5}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta\alpha\|^2 + \frac{1}{\mu}\|\Delta\alpha\|^2 \leq & C\frac{1}{\mu}\|\partial_t\nabla v\|\|\Delta\alpha\| \\ & + C\left(\frac{1}{\mu}+1+\mu+\mu^2\right)(X+Y)Z,\end{aligned} \tag{6.6}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\Delta\alpha\|^2 + \frac{1}{\mu}\|\nabla\Delta\alpha\|^2 \leq & C\frac{1}{\mu}\|\partial_t\nabla v\|\|\nabla\Delta\alpha\| \\ & + C\left(\frac{1}{\mu}+1+\mu+\mu^2\right)(X+Y)Z,\end{aligned} \tag{6.7}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial_t\nabla\alpha\|^2 \leq & \left(\partial_t\nabla\begin{pmatrix}-\alpha^2\\ \alpha^1\end{pmatrix}, \partial_t\nabla v\right) \\ & + C\left(\frac{1}{\mu}+1+\mu+\mu^2\right)(1+X+Y)^3(1+Y)Z.\end{aligned} \tag{6.8}$$

Here, $C>0$ is a numerical constant (independent of μ).

The estimate (6.4) is obtained by taking the inner product of momentum equation for v with Av, for short we write (v-momentum, Av). Let us summarize the estimates and corresponding inner products in the table below.

Estimate	Inner product	Corresponding subsection
(6.4)	(v-momentum, Av)	6.3
(6.5)	(∂_t(v-momentum), $\partial_t v$)	6.4
(6.6)	(Δ(α-transport), $\Delta\alpha$)	6.5
(6.7)	($\nabla\Delta$(α-transport), $\nabla\Delta\alpha$)	6.6
(6.8)	($\partial_t\nabla$(α-transport), $\partial_t\alpha$)	6.7

In the following subsections, we shall show these estimates one by one. Before going into the detail, let us note what are the aims of each estimate. The first estimate (6.4) is the core estimate, although the estimate produces a linear term $\|\Delta\alpha\|\|\Delta Av\|$. This problem will lead us to the energy estimates of α i.e. (6.6) and (6.7). One can immediately notice that these estimates produce the term $\|\partial_t\nabla v\|$ in the right-hand side. In order to manage these terms, we derive the estimate (6.5) to absorb $\|\partial_t\nabla v\|$ in the right-hand side of (6.6) and (6.7). However, we receive another linear term again as one can see in (6.5). Therefore we derive another estimate (6.8) to cancel out this linear term.

6.3. A Priori Estimate for the Velocity Gradient

For receiving an estimate for spatial derivatives, we want to test the equation with second derivatives of v. A simple way would be using $-\Delta v$ which, unfortunately, does not vanish on the boundary. Hence, the pressure term would not vanish in the estimate and must be estimated explicitly. We will therefore employ Av instead of $-\Delta v$ in the estimate.

Taking the inner product of the momentum equation in (4.5), i.e.

$$\begin{aligned}&\partial_t v - \mu\Delta v + v\cdot\nabla v + v\cdot\nabla\bar{u} + \bar{u}\cdot\nabla v + \nabla p\\ &\quad = -\Delta\alpha^k\nabla\alpha^k + \Delta\begin{pmatrix}-\alpha^2\\ \alpha^1\end{pmatrix} + \partial_2^2\phi\nabla\alpha^2 + \nabla\phi\Delta\alpha^2,\end{aligned}$$

with Av. We can use the boundary condition for $\partial_t v$ to integrate by parts. Since the Helmholtz-projection is self-adjoint, we have

$$\begin{aligned}(\partial_t v, Av) &= -(\partial_t v, P\Delta v) = -(P\partial_t v, \Delta v)\\ &= -(\partial_t v, \Delta v) = (\partial_t \nabla v, \nabla v) = \frac{1}{2}\frac{d}{dt}\|\nabla v\|^2,\end{aligned}$$

and

$$(-\mu\Delta v, Av) = \mu\|Av\|^2.$$

For the convection terms we use the embedding $H^2(\Omega)\hookrightarrow L^\infty(\Omega)$, the 1D-2D product estimates and the usual Stokes regularity to estimate

$$\begin{aligned}|(v\cdot\nabla v, Av)| &\le \|v\|_{L^\infty(\Omega)}\|\nabla v\|\|Av\| \le C\|v\|_{H^2(\Omega)}\|\nabla v\|\|Av\|\\ &\le C\|\nabla v\|\|Av\|^2 \le CYZ,\\ |(v\cdot\nabla\bar{u}, Av)| &\le C\|\nabla v\|\|\nabla\bar{u}\|\|Av\| \le C\|\partial_t\phi\|_{H^1(0,1)}\|Av\|^2 \le CXZ,\end{aligned}$$

and

$$|(\bar{u}\cdot\nabla v, Av)| \le \|\bar{u}\|_{L^\infty(\Omega)}\|\nabla v\|\|Av\| \le C\|\partial_t\phi\|_{H^1(0,1)}\|Av\|^2 \le CXZ.$$

Since the Stokes operator maps into $L^2_\sigma(\Omega)$, the L^2 closure of all smooth solenoidal vector fields with compact support in Ω, the pressure term ∇p vanishes in the a priori estimate.

The quadratic form in α gives

$$\begin{aligned}|(-\Delta\alpha^k\nabla\alpha^k, Av)| &\le \|\Delta\alpha\|\|\nabla\alpha\|_{L^\infty(\Omega)}\|Av\|\\ &\le C\|\Delta\alpha\|(\|\Delta\alpha\| + \|\nabla\Delta\alpha\|)\|Av\|\\ &\le C\mu YZ,\end{aligned}$$

and for the last terms it is

$$\begin{aligned}\left|\left(\partial_2^2\phi\nabla\alpha^2 + \nabla\phi\Delta\alpha^2, -Av\right)\right| &\le C(\|\partial_2^2\phi\|\|\nabla^2\alpha\| + \|\partial_2\phi\|_{L^\infty(\Omega)}\|\Delta\alpha\|)\|Av\|\\ &\le C\|\partial_2\phi\|_{H^1(0,1)}\|\Delta\alpha\|\|Av\|\\ &\le C\mu XZ.\end{aligned}$$

Linear Term. The term $\Delta\begin{pmatrix}-\alpha^2\\ \alpha^1\end{pmatrix}$ contains linear parts with non-small coefficients. Due to the presence of the Helmholtz-projection, it is not possible to cancel this term with a corresponding term $(v^2, -v^1)^T$ in the α-estimate. We have

$$\left|\left(\Delta\begin{pmatrix}-\alpha^2\\ \alpha^1\end{pmatrix}, Av\right)\right| \le \|\Delta\alpha\|\|Av\|.$$

Altogether we have for the estimate of ∇v

$$\frac{1}{2}\frac{d}{dt}\|\nabla v\|^2 + \mu\|Av\|^2 \le \|\Delta\alpha\|\|Av\| + C(1+\mu)(X+Y)Z.$$

6.4. A Priori Estimate for the Time Derivative of the Velocity

Here, we investigate higher derivatives in time, since we are able to transfer higher time regularity to higher space regularity by using the Stokes regularity.

We apply ∂_t to equation (4.5) and take the inner product with $\partial_t v$. Since v vanishes on the boundary, so does $\partial_t v$ and integration by parts gives the terms

$$(\partial_t^2 v, \partial_t v) = \frac{1}{2}\frac{d}{dt}\|\partial_t v\|^2 \quad \text{and} \quad (-\mu\partial_t\Delta v, \partial_t v) = \mu\|\partial_t\nabla v\|^2.$$

With $\operatorname{div}\partial_t v = 0$ and $\partial_t v = 0$ on the boundary, we find $(v\cdot\nabla\partial_t v, \partial_t v) = 0$ and therefore the Poincaré inequality as well as the usual Stokes regularity give

$$\begin{aligned}(\partial_t(v\cdot\nabla v), \partial_t v) &= (\partial_t v\cdot\nabla v, \partial_t v)\\ &\le \|\partial_t v\|_{L^4(\Omega)}\|\nabla v\|_{L^4(\Omega)}\|\partial_t v\|\\ &\le C\|\partial_t v\|_{H^1(\Omega)}\|\nabla v\|_{H^1(\Omega)}\|\partial_t v\|\\ &\le C\|\partial_t\nabla v\|\|Av\|\|\partial_t v\|\\ &\le CYZ.\end{aligned}$$

Similarly, we have after employing the product estimates for one- and two-dimensional functions

$$\begin{aligned}(\partial_t(\bar{u}\cdot\nabla v), \partial_t v) &= (\partial_t\bar{u}\cdot\nabla v, \partial_t v)\\ &\le \|\partial_t\bar{u}\|_{L^\infty(\Omega)}\|\nabla v\|\|\partial_t v\|\\ &\le C\|\partial_t^2\phi\|_{H^1(0,1)}\|Av\|\|\partial_t\nabla v\|\\ &\le CXZ\end{aligned}$$

and

$$\begin{aligned}(\partial_t(v\cdot\nabla\bar{u}), \partial_t v) &\le (\|\partial_t v\cdot\nabla\bar{u}\| + \|v\cdot\nabla\partial_t\bar{u}\|)\|\partial_t v\|\\ &\le C(\|\partial_t\nabla v\|\|\nabla\bar{u}\| + \|\nabla v\|\|\partial_t\nabla\bar{u}\|)\|\partial_t\nabla v\|\\ &\le C(\|\partial_t\phi\|_{H^1} + \|\partial_t^2\phi\|_{H^1})(\|\partial_t\nabla v\| + \|Av\|)\|\partial_t\nabla v\|\\ &\le CXZ.\end{aligned}$$

The pressure term vanishes due to $\operatorname{div}\partial_t v = 0$ and $\partial_t v = 0$ on the boundary. In the linear term $\left(\partial_t\Delta\begin{pmatrix}-\alpha^2\\ \alpha^1\end{pmatrix}, \partial_t v\right)$ we integrate by parts once, using $\partial_t v = 0$ on the boundary:

$$\left(\partial_t\Delta\begin{pmatrix}-\alpha^2\\ \alpha^1\end{pmatrix}, \partial_t v\right) = -\left(\partial_t\nabla\begin{pmatrix}-\alpha^2\\ \alpha^1\end{pmatrix}, \partial_t\nabla v\right).$$

This term is going to be absorbed in the estimate for the time derivative of the gradient of the stream function α. Considering the quadratic α-term, we note with Einstein's sum convention

$$[\Delta\alpha^k\nabla\alpha^k]_i = \partial_l(\partial_l\alpha^k\partial_i\alpha^k) - \partial_l\alpha^k\partial_i\partial_l\alpha^k = [\operatorname{div}(\nabla\alpha^k\otimes\nabla\alpha^k)]_i - [\nabla(|\nabla\alpha^k|^2)]_i$$

and therefore in the a priori estimate, the second part vanishes being a gradient and we have after integrating by parts

$$\begin{aligned}(-\partial_t(\Delta\alpha^k\nabla\alpha^k), \partial_t v) &= -(\partial_t\operatorname{div}(\nabla\alpha^k\otimes\nabla\alpha^k), \partial_t v) + (\partial_t\nabla(|\nabla\alpha^k|^2), \partial_t v)\\ &= (\partial_t(\nabla\alpha^k\otimes\nabla\alpha^k), \partial_t\nabla v)\\ &\le C\|\partial_t\nabla\alpha\|\|\nabla\alpha\|_{L^\infty(\Omega)}\|\partial_t\nabla v\|\\ &\le C\|\partial_t\nabla\alpha\|(\|\Delta\alpha\| + \|\nabla\Delta\alpha\|)\|\partial_t\nabla v\|\\ &\le C\mu\|\partial_t\nabla\alpha\|Z.\end{aligned}$$

In the remaining term, we estimate after integrating by parts

$$
\begin{aligned}
&\big(\partial_t(\partial_2^2\phi\nabla\alpha^2+\nabla\phi\Delta\alpha^2),\partial_t v\big)\\
&\quad=\left(\partial_t\begin{pmatrix}\partial_1(\partial_2^2\phi\alpha^2)\\ \operatorname{div}(\partial_2\phi\nabla\alpha^2)\end{pmatrix},\partial_t v\right)\\
&\quad\le C(\|\partial_t(\partial_2^2\phi\alpha^2)\|+\|\partial_t(\partial_2\phi\nabla\alpha^2)\|)\|\partial_t\nabla v\|\\
&\quad\le C\big(\|\partial_2^2\partial_t\phi\|\|\nabla\alpha\|+\|\partial_2^2\phi\|\|\partial_t\nabla\alpha\|\\
&\qquad+\|\partial_2\partial_t\phi\|\|\nabla^2\alpha\|+\|\partial_2\phi\|_{L^\infty(\Omega)}\|\partial_t\nabla\alpha\|\big)\|\partial_t\nabla v\|\\
&\quad\le C(\|\partial_t\phi\|_{H^2(0,1)}\|\Delta\alpha\|+\|\partial_2\phi\|_{H^1(0,1)}\|\partial_t\nabla\alpha\|)\|\partial_t\nabla v\|\\
&\quad\le C\|\partial_t\phi\|_{H^2(0,1)}\left(\frac{1}{2\mu}\|\Delta\alpha\|^2+\frac{\mu}{2}\|\partial_t\nabla v\|^2\right)\\
&\qquad+C\|\partial_2\phi\|_{H^1(0,1)}\|\partial_t\nabla\alpha\|\|\partial_t\nabla v\|\\
&\quad\le C\mu XZ+C\|\partial_2\phi\|_{H^1(0,1)}\|\partial_t\nabla\alpha\|\|\partial_t\nabla v\|.
\end{aligned}
$$

For the estimation of the remaining terms including $\|\partial_t\nabla\alpha\|$ in the two foregoing estimates, we employ the transport equation for the stream function α in (4.3). Note, that one can write $-v\cdot\nabla\eta=(v^2,-v^1)^T+(0,\partial_2\phi v^2)^T$ and hence

$$
\begin{aligned}
\|\partial_t\nabla\alpha\|&\le\|\nabla(v\cdot\nabla\alpha)\|+\|\nabla(\bar u\cdot\nabla\alpha)\|+\|\nabla(v\cdot\nabla\eta)\|\\
&\le\|\nabla v\|\|\nabla\alpha\|_{L^\infty(\Omega)}+\|v\|_{L^4(\Omega)}\|\nabla^2\alpha\|_{L^4(\Omega)}+\|\nabla\bar u\|\|\nabla^2\alpha\|+\|\bar u\|_{L^\infty(\Omega)}\|\nabla^2\alpha\|\\
&\quad+\|\nabla v\|+\|\partial_2^2\phi\|\|\nabla v\|+\|\partial_2\phi\|_{L^\infty(\Omega)}\|\nabla v\|\\
&\le\|\nabla v\|+C(\|\Delta\alpha\|+\|\nabla\Delta\alpha\|+\|\partial_2\phi\|_{H^1}+\|\partial_t\phi\|_{H^1})(\|\nabla v\|+\|\Delta\alpha\|)\\
&\le C(1+X+Y)Y.
\end{aligned}
\tag{6.9}
$$

Applying this inequality yields

$$
\begin{aligned}
(-\partial_t(\Delta\alpha^k\nabla\alpha^k),\partial_t v)&\le C\mu\|\partial_t\nabla\alpha\|Z\\
&\le C\mu(1+X+Y)YZ
\end{aligned}
$$

and with $\|\partial_t\nabla v\|Y\le C(1+\mu)Z$, we have

$$
\begin{aligned}
&\big(\partial_t(\partial_2^2\phi\nabla\alpha^2+\nabla\phi\Delta\alpha^2),\partial_t v\big)\\
&\quad\le C\mu XZ+C\|\partial_2\phi\|_{H^1(0,1)}\|\partial_t\nabla v\|(1+X+Y)Y\\
&\quad\le C(1+\mu)(1+X+Y)XZ.
\end{aligned}
$$

Summarizing the foregoing estimates, we receive

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_t v\|^2+\mu\|\partial_t\nabla v\|^2+\left(\partial_t\nabla\begin{pmatrix}-\alpha^2\\ \alpha^1\end{pmatrix},\partial_t\nabla v\right)\\
&\quad\le C(1+\mu)(1+X+Y)(X+Y)Z.
\end{aligned}
$$

6.5. A Priori Estimate for the Laplacian of the Stream Function

We aim to control the $H^3(\Omega)$-norm of α with an a priori estimate of the Laplacian (Lemma 7.4) and it is therefore necessary to estimate $\Delta\alpha$ as well as $\nabla\Delta\alpha$. We will be able to produce a regularizing term $\|\Delta\alpha\|^2$ on the left-hand side. This, however, comes at the cost of linear error terms involving the artificial variable $w=v+\frac{1}{\mu}(-\alpha^2,\alpha^1)^T$. These error terms will later on be handled with higher estimates of w (6.2).

We apply Δ to the transport equation of the stream function (5.3), i.e.

$$\partial_t \alpha + \frac{1}{\mu}\alpha + v \cdot \nabla\alpha + \bar{u} \cdot \nabla\alpha = \begin{pmatrix} w^2 \\ -w^1 \end{pmatrix} + v^2 \nabla\phi.$$

and take the inner product with $\Delta\alpha$. Then the time derivative gives $(\partial_t \Delta\alpha, \Delta\alpha) = \frac{1}{2}\frac{d}{dt}\|\Delta\alpha\|^2$. The second term gives $(\Delta\alpha, \Delta\alpha) = \|\Delta\alpha\|^2$. For the third term we note $(v \cdot \nabla\Delta\alpha, \Delta\alpha) = 0$ and therefore with Einstein's sum convention

$$\begin{aligned}(\Delta(v \cdot \nabla\alpha), \Delta\alpha) &= (\Delta v \cdot \nabla\alpha + 2\partial_i v \cdot \nabla\partial_i\alpha, \Delta\alpha) \\ &\le C(\|\Delta\alpha\|\|\nabla\alpha\|_{L^\infty(\Omega)} + \|\nabla v\|_{L^4(\Omega)}\|\nabla^2\alpha\|_{L^4(\Omega)})\|\Delta\alpha\| \\ &\le C\|Av\|(\|\Delta\alpha\| + \|\nabla\Delta\alpha\|)\|\Delta\alpha\| \\ &\le C(\|\Delta\alpha\| + \|\nabla\Delta\alpha\|)\left(\frac{\mu}{2}\|Av\|^2 + \frac{1}{2\mu}\|\Delta\alpha\|^2\right) \\ &\le C\mu YZ.\end{aligned}$$

Similarly the second advection term yields

$$\begin{aligned}(\Delta(\bar{u} \cdot \nabla\alpha), \Delta\alpha) &= (\Delta\bar{u} \cdot \nabla\alpha + 2\partial_i\bar{u} \cdot \nabla\partial_i\alpha, \Delta\alpha) \\ &\le C(\|\partial_2^2\partial_t\phi\|\|\nabla^2\alpha\| + \|\partial_2\partial_t\phi\|_{L^\infty(\Omega)}\|\nabla^2\alpha\|)\|\Delta\alpha\| \\ &\le C\|\partial_t\phi\|_{H^2(0,1)}\|\Delta\alpha\|^2 \\ &\le C\mu^2 XZ.\end{aligned}$$

Let us take care of the right-hand side.

$$\begin{aligned}\Big(-\Delta\Big(\begin{pmatrix} w^2 \\ -w^1 \end{pmatrix} + v^2\nabla\phi\Big), \Delta\alpha\Big) &= -\Big(\Delta\begin{pmatrix} w^2 \\ -w^1 \end{pmatrix}, \Delta\alpha\Big) + (\Delta(\partial_2\phi v^2), \Delta\alpha^2) \\ &\le \|\Delta w\|\|\Delta\alpha\| + C\|\partial_2\phi\|_{H^2(0,1)}\|Av\|\|\Delta\alpha\| \\ &\le \|\Delta w\|\|\Delta\alpha\| + C\mu XZ.\end{aligned}$$

Now invoking the estimate in Proposition 6.1, we have

$$\begin{aligned}\|\Delta w\|\|\Delta\alpha\| &\le C\|\Delta\alpha\|\frac{1}{\mu}\big(\|\partial_t\nabla v\| + (1+\mu^2)Z + X(\|Av\| + \|\Delta\alpha\| + \|\nabla\Delta\alpha\|)\big) \\ &\le C\frac{1}{\mu}\|\Delta\alpha\|\|\partial_t\nabla v\| + C\frac{1}{\mu}(1+\mu^2)YZ + C\frac{1}{\mu}(\mu+\mu^2)XZ \\ &\le C\frac{1}{\mu}\|\Delta\alpha\|\|\partial_t\nabla v\| + C\left(\frac{1}{\mu} + 1 + \mu\right)(X+Y)Z.\end{aligned}$$

The complete estimate is of the form

$$\frac{1}{2}\frac{d}{dt}\|\Delta\alpha\|^2 + \frac{1}{\mu}\|\Delta\alpha\|^2 \le C\frac{1}{\mu}\|\partial_t\nabla v\|\|\Delta\alpha\| + C\left(\frac{1}{\mu} + 1 + \mu + \mu^2\right)(X+Y)Z.$$

6.6. A Priori Estimate for Gradient of the Laplacian of the Stream Function

We now want to estimate terms $\|\nabla\Delta\alpha\|$ such that together with the foregoing estimate, we can control the $H^3(\Omega)$-norm of α. Once again, we receive error terms including higher space derivatives of the artificial variable w.

For the corresponding estimate we apply $\nabla\Delta$ to (5.3) and then take the L^2-inner product with $\nabla\Delta\alpha$. Then the first two terms give us

$$(\partial_t\nabla\Delta\alpha + \nabla\Delta\alpha, \nabla\Delta\alpha) = \frac{1}{2}\frac{d}{dt}\|\nabla\Delta\alpha\|^2 + \|\nabla\Delta\alpha\|^2.$$

We employ the identity $(v\cdot\nabla\nabla\Delta\alpha,\nabla\Delta\alpha)=0$ and estimate (using Einstein's sum convention)

$$\begin{aligned}
&(\nabla\Delta(v\cdot\nabla\alpha),\nabla\Delta\alpha)\\
&\quad=(\partial_l(\Delta v\cdot\nabla\alpha+2\partial_i v\cdot\nabla\partial_i\alpha+v\cdot\nabla\Delta\alpha),\partial_l\Delta\alpha)\\
&\quad=(\partial_l\Delta v\cdot\nabla\alpha+\Delta v\cdot\nabla\partial_l\alpha+2\partial_l\partial_i v\cdot\nabla\partial_i\alpha+2\partial_i v\cdot\nabla\partial_l\partial_i\alpha+\partial_l v\cdot\nabla\Delta\alpha,\partial_l\Delta\alpha)\\
&\quad\le(\|\nabla\Delta v\|\|\nabla\alpha\|_{L^\infty(\Omega)}+3\|\nabla^2 v\|_{L^4(\Omega)}\|\nabla^2\alpha\|_{L^4(\Omega)}+2\|\nabla v\|_{L^\infty(\Omega)}\|\nabla^3\alpha\|)\|\nabla\Delta\alpha\|\\
&\quad\le C\|\nabla v\|_{H^2(\Omega)}\|\nabla\alpha\|_{H^2(\Omega)}\|\nabla\Delta\alpha\|\\
&\quad\le C\|\nabla v\|_{H^2(\Omega)}(\|\Delta\alpha\|^2+\|\nabla\Delta\alpha\|^2)\\
&\quad\le C\|\nabla v\|_{H^2(\Omega)}Y^2.
\end{aligned}$$

This is the first place, where higher spacial derivatives of v are appearing as error terms. Once again using $(\bar u\cdot\nabla\nabla\Delta\alpha,\nabla\Delta\alpha)=0$ we obtain more easily

$$\begin{aligned}
&(\nabla\Delta(\bar u\cdot\nabla\alpha),\nabla\Delta\alpha)\\
&\quad=(\partial_l\Delta\bar u\cdot\nabla\alpha+\Delta\bar u\cdot\nabla\partial_l\alpha+2\partial_l\partial_i\bar u\cdot\nabla\partial_i\alpha+2\partial_i\bar u\cdot\nabla\partial_l\partial_i\alpha+\partial_l\bar u\cdot\nabla\Delta\alpha,\partial_l\Delta\alpha)\\
&\quad\le C(\|\partial_2^3\partial_t\phi\|\|\nabla^2\alpha\|+\|\partial_2^2\partial_t\phi\|_{L^\infty(\Omega)}\|\nabla^2\alpha\|+\|\partial_2\partial_t\phi\|_{L^\infty(\Omega)}\|\nabla^3\alpha\|)\|\nabla\Delta\alpha\|\\
&\quad\le C\|\partial_t\phi\|_{H^3(0,1)}\|\nabla\Delta\alpha\|(\|\Delta\alpha\|+\|\nabla\Delta\alpha\|)\\
&\quad\le C\mu^2XZ.
\end{aligned}$$

Linear Term. Now let us focus on the right-hand side. Once again, we receive higher spatial derivatives of w and v.

$$\begin{aligned}
&\Big(-\nabla\Delta\Big(\begin{pmatrix}w^2\\-w^1\end{pmatrix}+v^2\nabla\phi\Big),\nabla\Delta\alpha\Big)\\
&\quad=\Big(\nabla\Delta\begin{pmatrix}w^2\\-w^1\end{pmatrix},\nabla\Delta\alpha\Big)+(\nabla\Delta(\partial_2\phi v^2),\nabla\Delta\alpha)\\
&\quad\le\|\nabla\Delta w\|\|\nabla\Delta\alpha\|+C\|\partial_2\phi\|_{H^3(0,1)}\|\nabla v\|_{H^2(\Omega)}\|\nabla\Delta\alpha\|\\
&\quad\le\|\nabla\Delta w\|\|\nabla\Delta\alpha\|+C\|\nabla v\|_{H^2(\Omega)}XY.
\end{aligned}$$

Here again, we have invoked Proposition 6.1 for the first term

$$\begin{aligned}
&\|\nabla\Delta w\|\|\nabla\Delta\alpha\|\\
&\quad\le C\|\nabla\Delta\alpha\|\frac{1}{\mu}\left(\|\partial_t\nabla v\|+(1+\mu^2)Z+X(\|Av\|+\|\Delta\alpha\|+\|\nabla\Delta\alpha\|)\right)\\
&\quad\le C\frac{1}{\mu}\|\nabla\Delta\alpha\|\|\partial_t\nabla v\|+C\left(\frac{1}{\mu}+1+\mu\right)(X+Y)Z.
\end{aligned}$$

We deal with the second term in the same way.

$$\begin{aligned}
&\|\nabla v\|_{H^2(\Omega)}Y\\
&\quad\le C\big(\|\partial_t\nabla v\|Y+(1+\mu^2)YZ+XY(\|Av\|+\|\Delta\alpha\|+\|\nabla\Delta\alpha\|)\big)\\
&\qquad+\frac{C}{\mu}(\|\Delta\alpha\|+\|\nabla\Delta\alpha\|)Y\\
&\quad\le C\Big((1+\mu)Z+(1+\mu^2)YZ+(1+\mu)XZ\Big)\\
&\quad\le C(1+\mu+\mu^2)(1+X+Y)Z. \qquad (6.10)
\end{aligned}$$

We summarize the above estimates to obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla\Delta\alpha\|^2 + \frac{1}{\mu}\|\nabla\Delta\alpha\|^2 \leq C\frac{1}{\mu}\|\partial_t\nabla v\|\|\nabla\Delta\alpha\| + C\left(\frac{1}{\mu} + 1 + \mu + \mu^2\right)(1 + X + Y)(X + Y)Z.$$

6.7. A Priori Estimate for the Time Derivative of the Gradient of the Stream Function

The following estimate has the role of absorbing the linear error term that appeared when estimating the time-derivative of v. In contrast to the two foregoing estimates on $\Delta\alpha$ and $\nabla\Delta\alpha$ we will not produce a stabilizing term on the left-hand side of the estimate since this would come at the cost of a linear error term $\partial_t\nabla w$. For that term, the higher Stokes-regularity result in Proposition 6.1 is not applicable. The result would not be easier to estimate than $\partial_t\nabla\alpha$ itself. Therefore, the main goal of the following estimate is simply to absorb the linear remainder from the $\partial_t v$-estimate.

Application of $\partial_t\nabla$ to (5.2) and taking the inner product with $\partial_t\nabla\alpha$ yields for the first term

$$(\partial_t^2\nabla\alpha, \partial_t\alpha) = \frac{1}{2}\frac{d}{dt}\|\partial_t\nabla\alpha\|^2.$$

In the following, we need to estimate the term $\|\partial_t\nabla\alpha\|$ several times. We remind, that as calculated in (6.9), we have

$$\|\partial_t\nabla\alpha\| \leq C(1 + X + Y)Y.$$

Similarly to the foregoing a priori estimates for α, the term $(v\cdot\nabla\partial_t\nabla\alpha, \partial_t\nabla\alpha)$ vanishes. This way, it is

$$\begin{aligned}
&(\partial_t\nabla(v\cdot\nabla\alpha), \partial_t\nabla\alpha)\\
&\quad= (\partial_t\partial_i v\cdot\nabla\alpha + \partial_i v\cdot\nabla\partial_t\alpha + \partial_t v\cdot\nabla\partial_i\alpha, \partial_t\partial_i\alpha)\\
&\quad\leq C(\|\partial_t\nabla v\|\|\nabla\alpha\|_{L^\infty(\Omega)} + \|\nabla v\|_{L^\infty(\Omega)}\|\partial_t\nabla\alpha\| + \|\partial_t v\|_{L^4(\Omega)}\|\nabla^2\alpha\|_{L^4(\Omega)})\|\partial_t\nabla\alpha\|\\
&\quad\leq C(\|\partial_t\nabla v\|\|\nabla\alpha\|_{H^2(\Omega)}\|\partial_t\nabla\alpha\| + \|\nabla v\|_{H^2(\Omega)}\|\partial_t\nabla\alpha\|^2)\\
&\quad\leq C\|\partial_t\nabla v\|(\|\Delta\alpha\| + \|\nabla\Delta\alpha\|)(1 + X + Y)Y\\
&\qquad+ C\|\nabla v\|_{H^2(\Omega)}(1 + X + Y)^2Y^2\\
&\quad\leq C\mu(1 + X + Y)YZ + C(1 + \mu + \mu^2)(1 + X + Y)^3YZ.
\end{aligned}$$

Note that in the last line, the estimate of ∇v (6.10) was invoked.

The second advection term can be estimated similarly with $Y^2 \leq C(1 + \mu^2)Z$

$$\begin{aligned}
&(\partial_t\nabla(\bar{u}\cdot\nabla\alpha), \partial_t\nabla\alpha)\\
&\quad= (\partial_t\partial_i\bar{u}\cdot\nabla\alpha + \partial_i\bar{u}\cdot\nabla\partial_t\alpha + \partial_t\bar{u}\cdot\nabla\partial_i\alpha, \partial_t\partial_i\alpha)\\
&\quad\leq C(\|\partial_2\partial_t^2\phi\|\|\nabla^2\alpha\| + \|\partial_2\partial_t\phi\|_{L^\infty(\Omega)}\|\partial_t\nabla\alpha\| + \|\partial_t^2\phi\|_{L^\infty(\Omega)}\|\nabla^2\alpha\|)\|\partial_t\nabla\alpha\|\\
&\quad\leq C(\|\partial_t^2\phi\|_{H^1(0,1)}\|\Delta\alpha\|\|\partial_t\nabla\alpha\| + \|\partial_t\phi\|_{H^2(0,1)}\|\partial_t\nabla\alpha\|^2)\\
&\quad\leq C\big(\|\partial_t^2\phi\|_{H^1(0,1)}\|\Delta\alpha\|(1 + X + Y)Y\\
&\qquad+ \|\partial_t\phi\|_{H^2(0,1)}(1 + X + Y)^2Y^2\big)\\
&\quad\leq C(1 + \mu^2)(1 + X + Y)^2XZ.
\end{aligned}$$

Linear Term. We split the linear term into

$$-v \cdot \nabla \eta = \begin{pmatrix} v^2 \\ -v^1 \end{pmatrix} + \begin{pmatrix} 0 \\ \partial_2 \phi v^2 \end{pmatrix}$$

and estimate the second part by

$$\begin{aligned}
&(\partial_t \nabla(\partial_2 \phi v^2), \partial_t \nabla \alpha^2) \\
&\quad = (\partial_t \partial_2^2 \phi v^2 + \partial_2^2 \phi \partial_t v^2 + \partial_t \partial_2 \phi \nabla v^2 + \partial_2 \phi \partial_t \nabla v^2, \partial_t \nabla \alpha^2) \\
&\quad \leq C(\|\partial_t \phi\|_{H^2(0,1)} \|\nabla v\| + \|\partial_2 \phi\|_{H^1(0,1)} \|\partial_t \nabla v\|)(1 + X + Y)Y \\
&\quad \leq C(1 + X + Y)XY^2 + C(1 + \mu)(1 + X + Y)XZ \\
&\quad \leq C(1 + \mu + \mu^2)(1 + X + Y)XZ.
\end{aligned}$$

The remaining linear term is used to cancel a corresponding term appearing in the estimate of the time-derivative of v. It is

$$\begin{aligned}
\left(\partial_t \nabla \begin{pmatrix} v^2 \\ -v^1 \end{pmatrix}, \partial_t \nabla \alpha\right) &= (\partial_t \nabla v^2, \partial_t \nabla \alpha^1) - (\partial_t \nabla v^1, \partial_t \nabla \alpha^2) \\
&= \left(\partial_t \nabla \begin{pmatrix} -\alpha^2 \\ \alpha^1 \end{pmatrix}, \partial_t \nabla v\right).
\end{aligned}$$

Hence, the combined estimate is

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_t \nabla \alpha\|^2 - \left(\partial_t \nabla \begin{pmatrix} -\alpha^2 \\ \alpha^1 \end{pmatrix}, \partial_t \nabla v\right) \\
&\quad \leq C(1 + \mu + \mu^2)(1 + X + Y)^3(1 + Y)Z.
\end{aligned}$$

6.8. Combining the Estimates

In this subsection, we are going to combine all the estimates in Proposition 6.2 and derive the key estimate for the stability argument.

With Eq. (6.4) we proceed using Young's inequality to absorb the term $\|Av\|$ in the right-hand side. This yields

$$\frac{1}{2}\frac{d}{dt}\|\nabla v\|^2 + \frac{\mu}{2}\|Av\|^2 \leq \frac{1}{2\mu}\|\Delta \alpha\|^2 + C(1 + \mu)(X + Y)Z.$$

Applying Young's inequality for (6.6) and (6.7) in a similar way, we receive

$$\frac{1}{2}\frac{d}{dt}\|\Delta \alpha\|^2 + \frac{3}{4\mu}\|\Delta \alpha\|^2 \leq \frac{C}{\mu}\|\partial_t \nabla v\|^2 + C\left(\frac{1}{\mu} + 1 + \mu + \mu^2\right)(X + Y)Z, \tag{6.11}$$

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla \Delta \alpha\|^2 + \frac{1}{2\mu}\|\nabla \Delta \alpha\|^2 &\leq \frac{C}{\mu}\|\partial_t \nabla v\|^2 \\
&\quad + C\left(\frac{1}{\mu} + 1 + \mu + \mu^2\right)(1 + X + Y)(X + Y)Z.
\end{aligned} \tag{6.12}$$

Now we can add these three inequalities above to find a combined estimate

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla v\|^2 + \|\Delta \alpha\|^2 + \|\nabla \Delta \alpha\|^2\right) + \frac{\mu}{2}\|Av\|^2 + \frac{1}{4\mu}\|\Delta \alpha\|^2 + \frac{1}{2\mu}\|\nabla \Delta \alpha\|^2 \\
&\quad \leq \frac{C}{\mu}\|\partial_t \nabla v\|^2 + C\left(\frac{1}{\mu} + 1 + \mu + \mu^2\right)(1 + X + Y)(X + Y)Z.
\end{aligned} \tag{6.13}$$

We now turn to (6.5) and (6.8). Adding these inequalities leads to a cancellation of the remainders of linear terms of the time-derivative estimates for v and α. The resulting estimate is

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\partial_t v\|^2+\|\partial_t\nabla\alpha\|^2)+\mu\|\partial_t\nabla v\|^2\\&\quad\le C\left(\frac{1}{\mu}+1+\mu+\mu^2\right)(1+X+Y)^3(X+Y)Z.\end{aligned}\tag{6.14}$$

We aim to absorb the remaining quadratic term $C\|\partial_t\nabla v\|^2/\mu$ in the right-hand side of (6.13) with the one on the left-hand side of (6.14). For that it is necessary to multiply (6.13) by a sufficiently small constant $\delta>0$.

The sum of (6.14) with $\delta\times$(6.13) gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\partial_t v\|^2+\delta\|\nabla v\|^2+\delta\|\Delta\alpha\|^2+\delta\|\nabla\Delta\alpha\|^2+\|\partial_t\nabla\alpha\|^2\right)\\&\quad+\left(\mu-\delta\frac{C}{\mu}\right)\|\partial_t\nabla v\|^2+\delta\frac{\mu}{2}\|Av\|^2+\delta\frac{1}{4\mu}\|\Delta\alpha\|^2+\delta\frac{1}{2\mu}\|\nabla\Delta\alpha\|^2\\&\quad\le C(1+\delta)\left(\frac{1}{\mu}+1+\mu+\mu^2\right)(1+X+Y)^3(X+Y)Z.\end{aligned}$$

We choose $0<\delta<1$ such that $\mu-\delta\frac{C}{\mu}\ge\frac{\mu}{2}$ and hence receive

$$\begin{aligned}&\frac{d}{dt}\left(\|\partial_t v\|^2+\delta\|\nabla v\|^2+\delta\|\Delta\alpha\|^2+\delta\|\nabla\Delta\alpha\|^2+\|\partial_t\nabla\alpha\|^2\right)\\&\quad+\mu\|\partial_t\nabla v\|^2+\delta\mu\|Av\|^2+\frac{\delta}{2\mu}\|\Delta\alpha\|^2+\frac{\delta}{\mu}\|\nabla\Delta\alpha\|^2\\&\quad\le C(1+\delta)\left(\frac{1}{\mu}+1+\mu+\mu^2\right)(1+X+Y)^3(X+Y)Z.\end{aligned}$$

Finally we modify the left-hand side so that Z appears, and put $C_{\mu,\delta}=C(1+\delta)(1/\mu+1+\mu+\mu^2)$ to simplify the estimate. We obtain

$$\begin{aligned}&\frac{d}{dt}\left(\|\partial_t v\|^2+\delta\|\nabla v\|^2+\delta\|\Delta\alpha\|^2+\delta\|\nabla\Delta\alpha\|^2+\|\partial_t\nabla\alpha\|^2\right)+2\mu\delta Z\\&\quad\le C_{\mu,\delta}(1+X+Y)^3(X+Y)Z.\end{aligned}\tag{6.15}$$

6.9. Stability Argument

In this subsection, we give a proof of the estimate in our main result Theorem 3.2 with the aid of estimate (6.15). As we mentioned at the beginning of Sect. 6, the actual existence proof is by the Galerkin method which we skipped in this paper. We also note that the solution is unique, which is similarly proved by the estimate in this paper.

Proof. It is obvious that Proposition 6.2 applies under the assumption of Theorem 3.2. Thus we can employ (6.15). Fix some $a>0$ such that $C_{\mu,\delta}(1+a)^3a<\mu\delta$. Note that as long as $Y(t),X(t)\le a/2$ hold for such a, we have

$$\frac{d}{dt}\left(\|\partial_t v\|^2+\delta\|\nabla v\|^2+\delta\|\Delta\alpha\|^2+\delta\|\nabla\Delta\alpha\|^2+\|\partial_t\nabla\alpha\|^2\right)(t)+\mu\delta Z(t)\le 0.$$

Thus we can conclude corresponding norm of the solution is decreasing except the case where all X,Y,Z equal zero which is a trivial case.

Therefore, the proof is reduced to show that $Y(t),X(t)\le a/2$ hold for all $t\ge 0$ if we choose initial data and flow data sufficiently small.

We now invoke the a priori estimate for the Poiseuille flow Proposition 3.1 and obtain

$$X(t) \leq C\big(\|\psi_0\|_{H^3(0,1)} + \|h\|_{H^1(0,\infty)}\big)$$

for some $C = C_\mu$. Therefore if we choose initial data ψ_0 and pressure data h sufficiently small, we have $X(t) < a/2$ for all $t \geq 0$. In what follows, we may assume $X(t) < a/2$ for all $t \geq 0$.

Now let us focus on $Y(t)$. For this, we investigate the initial data $Y(0)$ first.

$$Y(0) = \|\partial_t v(0)\| + \|\nabla v_0\| + \|\Delta\alpha_0\| + \|\nabla\Delta\alpha_0\|.$$

Let us recall the momentum equation (4.5) and find

$$\begin{aligned}\partial_t v(0) = \mu\Delta v_0 - v_0\cdot\nabla v_0 - v_0\cdot\nabla\bar{u}(0) - \bar{u}(0)\cdot\nabla v_0 - \nabla p\\ + \Delta\begin{pmatrix}-\alpha_0^2\\ \alpha_0^1\end{pmatrix} - \Delta\alpha_0^k\nabla\alpha_0^k + \partial_2^2\phi(0)\nabla\alpha_0^2 + \nabla\phi(0)\Delta\alpha_0^2.\end{aligned}$$

Applying the Helmholtz projection to remove the pressure term, we obtain

$$\begin{aligned}\|\partial_t v(0)\| &\leq C(\mu\|\Delta v_0\| + \|v_0\cdot\nabla v_0\| + \|v_0\cdot\nabla\bar{u}\|\\ &\quad + \|\bar{u}_0\cdot\nabla v_0\| + \|\Delta\alpha_0^k\nabla\alpha_0^k\| + \|\partial_2^2\phi(0)\nabla\alpha_0^2\|)\\ &\leq C(\mu + \|\nabla v_0\| + \|\psi_0\|_{H^1(0,1)})\|Av_0\|\\ &\quad + C(1 + \|\Delta\alpha_0\| + \|\nabla\Delta\alpha_0\|)\|\Delta\alpha_0\|.\end{aligned}$$

Similarly, we recall the transport equation of α (5.2) for $\partial_t\nabla\alpha(0)$.

$$\begin{aligned}\|\partial_t\nabla\alpha(0)\| &\leq \|\nabla(v_0\cdot\nabla\alpha_0)\| + \|\nabla(\bar{u}(0)\cdot\nabla\alpha_0)\| + \|\nabla(v_0\cdot\nabla\eta_0)\|\\ &\leq \|\nabla^2\alpha_0\|\|v_0\|_{L^\infty(\Omega)} + \|\nabla v_0\|_{L^4(\Omega)}\|\nabla\alpha_0\|_{L^4(\Omega)} + \|\nabla^2\alpha_0\|\|\bar{u}(0)\|_{L^\infty(\Omega)}\\ &\quad + \|\nabla\bar{u}(0)\|_{L^4(\Omega)}\|\nabla\alpha_0\|_{L^4(\Omega)} + \|\nabla v_0\|\|\nabla\eta_0\|_{L^\infty(\Omega)}\\ &\leq C(\|\nabla v_0\| + (\|Av_0\| + \|\psi_0\|_{H^1(0,1)})\|\Delta\alpha_0\|).\end{aligned}$$

Therefore we choose v_0, α_0 and retake ψ_0 if necessary so small that

$$Y(0),\ \|\partial_t\nabla\alpha(0)\| < \frac{\sqrt{\delta}a}{8}$$

holds. Then we define a time $T_* = \inf\{t > 0; Y(t) > a/2\}$. To prove $T_* = \infty$, suppose $T_* < \infty$ and seek a contradiction. $Y(t) \leq a/2$ for all $t \in [0, T_*]$ holds, since Y is continuous. Thus,

$$\frac{d}{dt}\left(\|\partial_t v\|^2 + \delta\|\nabla v\|^2 + \delta\|\Delta\alpha\|^2 + \delta\|\nabla\Delta\alpha\|^2 + \|\partial_t\nabla\alpha\|^2\right)(t) + \mu\delta Z(t) \leq 0$$

holds in the interval $[0, T_*]$. Integrating over $[0, T_*]$, we receive

$$\begin{aligned}\|\partial_t v(T_*)\|^2 + \delta\|\nabla v(T_*)\|^2 + \delta\|\Delta\alpha(T_*)\|^2 + \delta\|\nabla\Delta\alpha(T_*)\|^2 + \|\partial_t\nabla\alpha(T_*)\|^2\\ \leq \|\partial_t v(0)\|^2 + \delta\|\nabla v_0\|^2 + \delta\|\Delta\alpha_0\|^2 + \delta\|\nabla\Delta\alpha_0\|^2 + \|\partial_t\nabla\alpha_0\|^2.\end{aligned}$$

Now evaluating $Y(T_*)$ with the above estimate yields,

$$\begin{aligned}(Y(T_*))^2 &= \left(\|\partial_t v(T_*)\| + \|\nabla v(T_*)\| + \|\Delta\alpha(T_*)\| + \|\nabla\Delta\alpha(T_*)\|\right)^2 \\ &\le \frac{4}{\delta}\left(\|\partial_t v(T_*)\|^2 + \delta\|\nabla v(T_*)\|^2 + \delta\|\Delta\alpha(T_*)\|^2 + \delta\|\nabla\Delta\alpha(T_*)\|^2\right) \\ &\le \frac{4}{\delta}\left(\|\partial_t v(0)\|^2 + \delta\|\nabla v_0\|^2 + \delta\|\Delta\alpha_0\|^2 + \delta\|\nabla\Delta\alpha_0\|^2 + \|\partial_t\nabla\alpha_0\|^2\right) \\ &< \frac{4}{\delta}\left((Y(0))^2 + \|\partial_t\nabla\alpha(0)\|^2\right) \\ &\le \frac{4}{\delta}\left(\frac{\sqrt{\delta}Y_0}{8}\right)^2 \times 2 \\ &\le \frac{Y_0^2}{8}.\end{aligned}$$

This leads to a contradiction to the definition of T_* and therefore $T_* = \infty$. □

7. Basic Properties of the Stokes Operator

In this section, we recall some basic estimates for the Stokes and the Laplacian operator, which are frequently used in this paper for the reader's convenience.

Assume that Ω is either the layer $\mathbb{R} \times (0,1)$ or one of the approximations $(-M, M) \times (0,1)$ in this section.

7.1. Stokes Operator

Definition 7.1. Let P be the Helmholtz projection on Ω. The Stokes operator A is defined as a closed linear operator in $L^2_\sigma(\Omega)$ with domain $D(A) = H^2(\Omega) \cap H^1_0(\Omega) \cap L^2_\sigma(\Omega)$ such that $Au = -P\Delta$.

Note that $0 \in \rho(A)$ and $D(A) = H^2(\Omega) \cap H^1_0(\Omega) \cap L^2_\sigma(\Omega)$ in either case that Ω is layer domain or bounded domain [12, III.2]. Therefore we have the estimate

$$\|v\|_{H^2(\Omega)} \le C\|Av\| \tag{7.1}$$

for $v \in H^2(\Omega)$ with $v = 0$ on the boundary and $\operatorname{div} v = 0$.

7.2. Stokes System

We use the regularity of Stokes system to obtain higher spatial estimates.

Lemma 7.2. *Let $\Omega = \mathbb{R} \times (0,1)$ or $\Omega = (-M, M) \times (0,1)$ for $M > 1$. Let $f \in H^1(\Omega)$, $g \in H^2(\Omega)$ and (u,p) solve*

$$\begin{cases} -\mu\Delta u + \nabla p = f, & \text{in } \Omega, \\ \operatorname{div} u = g, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{7.2}$$

Then there holds

$$\mu\|u\|_{H^3(\Omega)} + \|\nabla p\|_{H^1(\Omega)} \le C(\mu\|g\|_{H^2(\Omega)} + \|f\|_{H^1(\Omega)}).$$

The constant C is independent of the assumption to Ω.

Proof. See [12, Theorem 1.5.3]. □

7.3. Elliptic Regularity and Implications

Lemma 7.3. *Under the same assumptions for Ω of Lemma 7.2, there exists a constant C such that*

$$\|f\|_{H^2(\Omega)} \leq C\|\Delta f\| \tag{7.3}$$

holds for all $f \in H^2(\Omega)$ which vanish on both upper and lower boundaries.

Proof. It follows by the elliptic regularity. For more details, see [1, Cor 6.31]. □

Moreover, by standard elliptic regularity arguments, we can estimate the $H^3(\Omega)$-norm by only terms of the Laplacian as stated below.

Lemma 7.4. *Assume the same hypotheses of Lemma 7.2. Then there exists constant C such that*

$$C^{-1}\|f\|_{H^3(\Omega)} \leq \|\Delta f\| + \|\nabla\Delta f\| \leq C\|f\|_{H^3(\Omega)} \tag{7.4}$$

holds for all $f \in H^3(\Omega)$ which vanishes on the upper and lower boundary.

Proof. See [2, Section 6.3]. □

We will use some embeddings to deal with products of functions.

Lemma 7.5. *Assume the same hypotheses of Lemma 7.2. The following embeddings hold:*

$$H^2(\Omega) \hookrightarrow L^\infty(\Omega),\ H^1(\Omega) \hookrightarrow L^4(\Omega),\ \text{and} \quad H^1(0,1) \hookrightarrow L^\infty(0,1). \tag{7.5}$$

Proof. See [2, Section 5.6] □

The following lemma are used to control the order of estimates.

Lemma 7.6. *Assume the same hypotheses of Lemma 7.2. Let $g \in H^1(\Omega)$ vanish on the upper and lower boundaries and $f \in L^2(0,1)$, one can estimate using the embedding above,*

$$\|fg\|_{L^2(\Omega)} \leq C\|f\|_{L^2(0,1)}\|\nabla g\|_{L^2(\Omega)}. \tag{7.6}$$

Proof. We invoke the embeddings Lemma 7.5 to conclude

$$\begin{aligned}
\|fg\|_{L^2(\Omega)} &= \left(\int_{\mathbb{R}}\int_{(0,1)} |f(y)|^2|g(x,y)|^2\,\mathrm{d}y\,\mathrm{d}x\right)^{\frac{1}{2}} \\
&\leq \left(\int_{\mathbb{R}} \|g(x,\cdot)\|^2_{L^\infty(0,1)}\int_{(0,1)}|f(y)|^2\,\mathrm{d}y\,\mathrm{d}x\right)^{\frac{1}{2}} \\
&\leq \|f\|_{L^2(0,1)}\left(\int_{\mathbb{R}} C^2\|g(x,\cdot)\|^2_{H^1(0,1)}\,\mathrm{d}x\right)^{\frac{1}{2}} \\
&\leq C\|f\|_{L^2(0,1)}\|\nabla g\|_{L^2(\Omega)}.
\end{aligned} \tag{7.7}$$

□

8. Viscous Wave Equation

This section is dedicated to a proof of Proposition 3.1. We split the initial-boundary value problem (3.2) into two parts for sharper estimation. The first part is the homogeneous case, i.e.,

$$\begin{cases}
\partial_t^2\phi_1 - \partial_x^2\phi_1 = \mu\partial_t\partial_x^2\phi_1, & \text{in} \quad (0,1), \\
\phi_1(t,0) = \phi_1(t,1) = 0, & \text{for} \quad t \in (0,T), \\
\phi_1(0) = 0, \quad \partial_t\phi_1(0) = \psi_0, & \text{in} \quad (0,1).
\end{cases} \tag{8.1}$$

The second one is the inhomogeneous case, i.e.,

$$\begin{cases} \partial_t^2\phi_2 - \partial_x^2\phi_2 = \mu\partial_t\partial_x^2\phi_2 + h, & \text{in} \quad (0,1), \\ \phi_2(t,0) = \phi_2(t,1) = 0, & \text{for} \quad t \in (0,T), \\ \phi_2(0) = 0, \quad \partial_t\phi_2(0) = 0, & \text{in} \quad (0,1). \end{cases} \tag{8.2}$$

Again, here h is some given function which depends only on t. We shall show a priori estimates for each of them. Note that we only need estimates for

$$\partial_x^4\phi_i,\ \partial_t\partial_x^3\phi_i,\ \partial_t^2\partial_x\phi_i \qquad i = 1, 2$$

in both cases by the Poincaré inequality.

8.1. Homogeneous Case

In this case, we can use the separation of variables method and derive the solution explicitly. For the readability, let us denote the solution ϕ_1 of (8.1) by ϕ in this subsection.

Separation of Variables. To begin with, we consider the simple ansatz with the form $\phi(t,x) = T(t)X(x)$ with the boundary condition $X(0) = X(1) = 0$. Then inserting this ansatz in the system (8.1) yields,

$$X(x)T''(t) - X''(x)T(t) = \mu X''(x)T'(t), \quad x \in (0,1), \quad t > 0.$$

This leads to the following equation

$$\frac{T''(t)}{\mu T'(t) + T(t)} = \frac{X''(x)}{X(x)} = \lambda$$

for some $\lambda \in \mathbb{R}$ unless $X(x)$ and $\mu T'(t) + T(t)$ vanish.

Let us focus on $X(x)$ first. The equation $X''(x) = \lambda X(t)$ with initial condition $X(0) = X(1) = 0$ gives us a solution $X_n(x) = a_n \sin(n\pi x)$ and here $\lambda = -(n\pi)^2$. We simply regard $a_n = 1$ and take care of those coefficients in $T(t)$ side.

Now let us turn to $T_n''(t) + \mu(n\pi)^2 T_n'(t) + (n\pi)^2 T_n(t) = 0$. Solving the characteristic equation $y^2 + \mu(n\pi)^2 y + (n\pi)^2 = 0$ yields

$$y_n^{\pm} = -\frac{\mu}{2}(n\pi)^2 \pm n\pi\sqrt{\frac{\mu^2}{4}(n\pi)^2 - 1}.$$

We set $B_n = n\pi\sqrt{\left|\frac{\mu^2}{4}(n\pi)^2 - 1\right|}$ for simplicity. Then $y_n^{\pm}$ is written as

$$y_n^{\pm} = \begin{cases} -\frac{\mu}{2}(n\pi)^2 \pm iB_n, & n < \frac{2}{\mu\pi}, \\ -\frac{\mu}{2}(n\pi)^2, & n = \frac{2}{\mu\pi}, \\ -\frac{\mu}{2}(n\pi)^2 \pm B_n, & n > \frac{2}{\mu\pi}. \end{cases}$$

Now we assume $N := \frac{2}{\mu\pi} \in \mathbb{N}$ in the following. Otherwise, we just ignore the terms with respect to $y_N^{\pm} = -\mu(N\pi)^2/2$. Hence the solution $T_n(t)$ is written as

$$T_n(t) = \begin{cases} e^{-\frac{\mu}{2}(n\pi)^2 t}(a_n \sin(B_n t) + b_n \cos(B_n t)), & n < \frac{2}{\mu\pi}, \\ e^{-\frac{\mu}{2}(n\pi)^2 t}(ta_n + b_n), & n = \frac{2}{\mu\pi}, \\ e^{-\frac{\mu}{2}(n\pi)^2 t}(\frac{a_n}{2}e^{B_n t} + \frac{b_n}{2}e^{-B_n t}), & n > \frac{2}{\mu\pi} \end{cases}$$

for some coefficients $a_n, b_n \in \mathbb{R}$.

Determining the Coefficients. We now consider the solution ansatz for (8.1) of the form

$$\begin{aligned}\phi(t,x) &= \sum_{n=1}^{N-1} \sin(n\pi x) e^{-\frac{\mu}{2}(n\pi)^2 t}(a_n \sin(B_n t) + b_n \cos(B_n t)) \\ &\quad + \sin(N\pi x) e^{-\frac{\mu}{2}(N\pi)^2 t}(t a_N + b_N) \\ &\quad + \sum_{n=N+1}^{\infty} \sin(n\pi x) e^{-\frac{\mu}{2}(n\pi)^2 t} \left(\frac{a_n}{2} e^{B_n t} + \frac{b_n}{2} e^{-B_n t} \right).\end{aligned}$$

We determine a_n, b_n so that $\phi(0,x) = 0$ and $\phi_t(0,x) = \psi_0$ are satisfied.

Decay in Time. At this point, we would like to remark, that each of the functions T_n decay exponentially in time. This observation is directly clear in the cases $n \leq \frac{2}{\mu\pi}$ and in the latter case, we note $B_n < \frac{\mu}{2}(n\pi)^2$ for $n > \frac{2}{\mu\pi}$. In fact, it is

$$\begin{aligned}-\frac{\mu}{2}(n\pi)^2 + B_n &= -\frac{\mu}{2}(n\pi)^2 + n\pi \sqrt{\frac{\mu^2}{4}(n\pi)^2 - 1} \\ &= -n\pi \left(\sqrt{\frac{\mu^2}{4}(n\pi)^2} - \sqrt{\frac{\mu^2}{4}(n\pi)^2 - 1} \right) \\ &= -\frac{\pi}{\sqrt{\frac{\mu^2}{4}\pi^2} + \sqrt{\frac{\mu^2}{4}\pi^2 - \frac{1}{n^2}}} \\ &\leq -\frac{\pi}{2\sqrt{\frac{\mu^2}{4}\pi^2}} \\ &= -\frac{1}{\mu}. \end{aligned} \tag{8.3}$$

Hence, the solution decays exponentially to zero at infinity at least as fast as $e^{-\frac{1}{\mu}t}$ for $n > \frac{2}{\mu\pi}$.

We superpose these solutions for $n \in \mathbb{N}$ receiving the solution ansatz for ϕ of the form

$$\begin{aligned}\phi(t,x) &= \sum_{n=1}^{N-1} \sin(n\pi x) e^{-\frac{\mu}{2}(n\pi)^2 t}(a_n \sin(B_n t) + b_n \cos(B_n t)) \\ &\quad + \sin(N\pi x) e^{-\frac{\mu}{2}(N\pi)^2 t}(t a_N + b_N) \\ &\quad + \sum_{n=N+1}^{\infty} \sin(n\pi x) e^{-\frac{\mu}{2}(n\pi)^2 t} \left(\frac{a_n}{2} e^{B_n t} + \frac{b_n}{2} e^{-B_n t} \right).\end{aligned}$$

Determining the Coefficients. Our next step is to exploit the initial conditions on ϕ to determine the proper values for the coefficients a_n and b_n. It is

$$\begin{aligned}0 &= \phi(0,x) \\ &= \sum_{n=1}^{N-1} \sin(n\pi x) b_n + \sin(N\pi x) b_N + \sum_{n=N+1}^{\infty} \sin(n\pi x) \left(\frac{a_n + b_n}{2} \right) \\ &= \sum_{n=1}^{\infty} \sin(n\pi x) \left(b_n + \chi_{\{N+1,\dots\}}(n) \left(\frac{a_n - b_n}{2} \right) \right).\end{aligned}$$

It is then easy to conclude that for $n = 1, \ldots, N$ we have $b_n = 0$ and for $n = N+1, \ldots$ it is $b_n = -a_n$ and therefore

$$\begin{aligned}\phi(t,x) = &\sum_{n=1}^{N-1} \sin(n\pi x) e^{-\frac{\mu}{2}(n\pi)^2 t}(a_n \sin(B_n t)) \\ &+ \sin(N\pi x) e^{-\frac{\mu}{2}(N\pi)^2 t}(t a_N) \\ &+ \sum_{n=N+1}^{\infty} \sin(n\pi x) e^{-\frac{\mu}{2}(n\pi)^2 t} \frac{a_n}{2}\left(e^{B_n t} - e^{-B_n t}\right).\end{aligned}$$

The other initial condition $\phi_t(0,x) = \psi_0(x)$ enables us to uniquely determine the remaining coefficients via Fourier-series. We calculate the derivative in time

$$\begin{aligned}\phi_t(t,x) = &\sum_{n=1}^{N-1} \sin(n\pi x)\Big(\left(-\frac{\mu}{2}(n\pi)^2\right) e^{-\frac{\mu}{2}(n\pi)^2 t}(a_n \sin(B_n t)) \\ &+ e^{-\frac{\mu}{2}(n\pi)^2 t}(a_n B_n \cos(B_n t))\Big) \\ &+ \sin(N\pi x)\left(\left(-\frac{\mu}{2}(N\pi)^2\right) e^{-\frac{\mu}{2}(N\pi)^2 t}(t a_N) + e^{-\frac{\mu}{2}(N\pi)^2 t} a_N\right) \\ &+ \sum_{n=N+1}^{\infty} \sin(n\pi x)\Big(\left(-\frac{\mu}{2}(n\pi)^2\right) e^{-\frac{\mu}{2}(n\pi)^2 t} \frac{a_n}{2}\left(e^{B_n t} - e^{-B_n t}\right) \\ &+ e^{-\frac{\mu}{2}(n\pi)^2 t} \frac{a_n B_n}{2}\left(e^{B_n t} + e^{-B_n t}\right)\Big).\end{aligned} \tag{8.4}$$

At $t = 0$ then holds with $B_N = 0$

$$\begin{aligned}\psi_0(x) = \phi_t(0,x) &= \sum_{n=1}^{N-1} \sin(n\pi x) a_n B_n + \sin(N\pi x) a_N + \sum_{n=N+1}^{\infty} \sin(n\pi x) a_n B_n \\ &= \sum_{n=1}^{\infty} \sin(n\pi x)(a_n B_n + \chi_{\{N\}}(n) a_N).\end{aligned}$$

Now we will assume that $\psi_0 \in H^3(0,1)$ with $\psi_0(0) = \psi_0(1) = 0$ and therefore we can write it as a Fourier-series

$$\begin{aligned}\psi_0(x) &= \frac{\beta_0}{2} + \sum_{k=1}^{\infty} \alpha_k \sin(2\pi k x) + \beta_k \cos(2\pi k x) \\ &= \sum_{k=1}^{\infty} \alpha_k \sin(2\pi k x),\end{aligned}$$

where the coefficients β_k vanish due to the boundary conditions. Moreover, $\psi \in H^3(0,1)$ implies with Plancherel's identity, that the series $((2k\pi)^3 \alpha_k)_{k \in \mathbb{N}}$ is in l^2. Comparing this representation with condition (8.4), the uniqueness of the Fourier-series implies that $a_{2k+1} = 0$ and

$$a_{2k} = \begin{cases} B_{2k}^{-1} \alpha_k = B_{2k}^{-1} \int_0^1 \psi_0(x) \sin(2\pi k x)\, \mathrm{d}x, & 2k \neq N, \\ \alpha_k = \int_0^1 \psi_0(x) \sin(2\pi k x)\, \mathrm{d}x, & 2k = N. \end{cases}$$

Altogether, for $N = \frac{2}{\mu\pi} \in \mathbb{N}$ even, the solution ϕ of system (8.1) is given by

$$\begin{aligned}\phi(t,x) = &\sum_{k=1}^{\frac{N}{2}-1} \alpha_k B_{2k}^{-1} \sin(2k\pi x) e^{-\frac{\mu}{2}(2k\pi)^2 t} \sin(B_{2k} t)\\ &+ \alpha_{\frac{N}{2}} \sin(N\pi x) e^{-\frac{\mu}{2}(N\pi)^2 t}(t a_N)\\ &+ \sum_{k=\frac{N}{2}+1}^{\infty} \frac{\alpha_k}{2} B_{2k}^{-1} \sin(2k\pi x) e^{-\frac{\mu}{2}(2k\pi)^2 t}\Big(e^{B_{2k}t} - e^{-B_{2k}t}\Big),\end{aligned}$$

where $\alpha_k = \int_0^1 \psi_0(x)\sin(2\pi k x)\,\mathrm{d}x$. In the case, where N is odd, the second term vanishes, more precisely

$$\begin{aligned}\phi(t,x) = &\sum_{k=1}^{\frac{N-1}{2}} \alpha_k B_{2k}^{-1} \sin(2k\pi x) e^{-\frac{\mu}{2}(2k\pi)^2 t} \sin(B_{2k} t)\\ &+ \sum_{k=\frac{N+1}{2}}^{\infty} \frac{\alpha_k}{2} B_{2k}^{-1} \sin(2k\pi x) e^{-\frac{\mu}{2}(2k\pi)^2 t}\Big(e^{B_{2k}t} - e^{-B_{2k}t}\Big).\end{aligned}$$

Continuity of the Solution. Looking at the coefficients, we see that $\frac{\mu}{\sqrt{8}}(k\pi)^2 \le B_{2k} \le \frac{\mu}{2}(k\pi)^2$ for n such that $1 \le \frac{\mu^2}{8}(\pi n)^2$. Therefore, the term B_{2k} may absorb $(2\pi k)^2$ coming from a time-derivative or two derivatives in space. With $\psi_0 \in H^3(0,1)$ we see that $\phi_{txxx}(t)$ is continuous in time (up to zero) as a function taking values in $L^2(0,1)$ and we can estimate with Plancherel's identity

$$\begin{aligned}&\|\phi_{txxx}(t)\|^2\\ &\le \sum_{k=1}^{\frac{N-1}{2}} \Big|\alpha_k B_{2k}^{-1}(2k\pi)^3 e^{-\frac{\mu}{2}(2k\pi)^2 t}\big(B_{2k}\cos(B_{2k}t) - \frac{\mu}{2}(2k\pi)^2 \sin(B_{2k}t)\big)\Big|^2\\ &\quad + \sum_{k=\frac{N+1}{2}}^{\infty} \Big|\frac{\alpha_k}{2} B_{2k}^{-1}(2k\pi)^3 e^{-\frac{\mu}{2}(2k\pi)^2 t}\\ &\quad \times \Big(B_{2k}(e^{B_{2k}t} + e^{-B_{2k}t}) - \frac{\mu}{2}(2k\pi)^2(e^{B_{2k}t} - e^{-B_{2k}t})\Big)\Big|^2\\ &\le \sum_{k=1}^{\frac{N-1}{2}} \Big|\alpha_k (2k\pi)^3 e^{-\frac{\mu}{2}(2k\pi)^2 t}\big(\cos(B_{2k}t) - \frac{\mu}{2}\frac{(2k\pi)^2}{B_{2k}} \sin(B_{2k}t)\big)\Big|^2\\ &\quad + \sum_{k=\frac{N+1}{2}}^{\infty} \Big|\frac{\alpha_k}{2}(2k\pi)^3 e^{-\frac{\mu}{2}(2k\pi)^2 t}\\ &\quad \times \Big((e^{B_{2k}t} + e^{-B_{2k}t}) - \frac{\mu}{2}\frac{(2k\pi)^2}{B_{2k}}(e^{B_{2k}t} - e^{-B_{2k}t})\Big)\Big|^2\\ &\le C e^{-\frac{\mu}{2}(2\pi)^2 t}\sum_{k=1}^{\infty}\Big|\alpha_k(2k\pi)^3\Big|^2 + C e^{-\frac{1}{\mu}t}\sum_{k=1}^{\infty}\Big|\alpha_k(2k\pi)^3\Big|^2\\ &\le C e^{-\min(2\mu\pi^2, \frac{1}{\mu})t}\|\psi_0\|^2_{H^3(0,1)}.\end{aligned}$$

Here we used the exponential decay in time for high frequency part $n > \frac{2}{\mu\pi}$ calculated in (8.3).

With the same arguments we find similar estimates for the case where N is even as well as ϕ_{xxxx} and ϕ_{ttx}. We have now proved the following Proposition:

Proposition 8.1. *For $\psi_0 \in H^3(0,1) \cap H_0^1(0,1)$ there exists a unique solution $\phi \in C^2([0,\infty); H^1(0,1)) \cap C^1([0,\infty), H^3(0,1)) \cap C([0,\infty); H^4(0,1))$ of (8.1). The solution satisfies*

$$\|\phi_t(t)\|_{H^3(0,1)} + \|\phi_x(t)\|_{H^3(0,1)} + \|\phi_{tt}(t)\|_{H^1(0,1)} \leq Ce^{-\min(2\mu\pi^2, \frac{1}{\mu})t}\|\psi_0\|_{H^3(0,1)}.$$

for $t \geq 0$. The constant $C > 0$ is independent of t and μ.

8.2. Inhomogeneous Case

Now let us consider inhomogeneous case (8.2) and its solution ϕ_2. Note that for the readability, we denote ϕ_2 by ϕ again.

Instead of solving the equation explicitly, we take advantage of the structure of the equations. First, we shall show the following point wise estimates in time.

Proposition 8.2 (A priori estimate in time). *Let $k \in \mathbb{N}$, $T > 0$ and ϕ be the solution in $C^{k+1}([0,T); H^3(0,1))$of system (8.2) with some given $h \in H_2^k(0,T)$. Then for each $t \in [0,T]$ the following estimates holds.*

$$\|\partial_t^k \phi_t(t)\|^2_{L^2(0,1)} + \|\partial_t^k \phi_x(t)\|^2_{L^2(0,1)} + \mu \int_0^t \|\partial_t^k \phi_{tx}(s)\|^2_{L^2(0,1)}\,\mathrm{d}s$$
$$\leq \|\partial_t^k \phi_t(0)\|_{L^2(0,1)} + \|\partial_t^k \phi_x(0)\|_{L^2(0,1)} + \frac{1}{3\mu}\|\partial_t^k h\|_{L^2(0,T)}$$

and

$$\|\partial_t^k \phi_{tx}(t)\|^2_{L^2(0,1)} + \|\partial_t^k \phi_{xx}(t)\|^2_{L^2(0,1)} + \mu \int_0^t \|\partial_t^k \phi_{txx}(s)\|^2_{L^2(0,1)}\,\mathrm{d}s$$
$$\leq \|\partial_t^k \phi_{tx}(0)\|_{L^2(0,1)} + \|\partial_t^k \phi_{xx}(0)\|_{L^2(0,1)} + \frac{1}{\mu}\|\partial_t^k h\|_{L^2(0,T)}.$$

Proof. Let us focus on the first estimate. We take derivatives ∂_t^k of (8.2) and take inner products with $\partial_t^{k+1}\phi(s)$ in $L^2(0,1)$. Then we have,

$$\int_0^1 \partial_t^{k+1}\phi_t(s,x)\partial_t^k\phi_t(s,x)\,\mathrm{d}x - \int_0^1 \partial_t^k\phi_{xx}(s,x)\partial_t^{k+1}\phi(s,x)\,\mathrm{d}x$$
$$= \mu\int_0^1 \partial_t^{k+1}\phi_{xx}(s,x)\partial_t^{k+1}\phi(s,x)\,\mathrm{d}x + \partial_t^k h(s)\int_0^1 \partial_t^{k+1}\phi(s,x)\,\mathrm{d}x.$$

We can employ integration by parts in the second, third and fourth term since $\partial_t^k\phi(s,0) = \partial_t^k\phi(s,1) = 0$, and that gives us

$$\frac{1}{2}\frac{d}{dt}\left(\|\partial_t^k\phi_t(s)\|^2_{L^2(0,1)} + \|\partial_t^k\phi_x(s)\|^2_{L^2(0,1)}\right)$$
$$= -\mu\|\partial_t^k\phi_{tx}(s)\|^2_{L^2(0,1)} - \partial_t^k h(s)\int_0^1 x\ \partial_t^{k+1}\phi_x(s,x)\,\mathrm{d}x.$$

We estimate the fourth term by means of the Cauchy-Schwarz inequality and Young's inequality:

$$\partial_t^k h(s)\int_0^1 x\ \partial_t^{k+1}\phi_x(s,x)\,\mathrm{d}x \leq |\partial_t^k h(s)|\ \left(\int_0^1 x^2\,\mathrm{d}x\right)^{\frac{1}{2}}\ \|\partial_t^k\phi_{tx}(s)\|_{L^2(0,1)}$$
$$\leq \frac{1}{6\mu}|\partial_t^k h(s)|^2 + \frac{\mu}{2}\|\partial_t^k\phi_{tx}(s)\|^2_{L^2(0,1)}.$$

Absorbing the last term and integrating from 0 to some $t > 0$ leads to the intended result:

$$\|\partial_t^k \phi_t(t)\|_{L^2(0,1)}^2 + \|\partial_t^k \phi_x(t)\|_{L^2(0,1)}^2 + \mu \int_0^t \|\partial_t^k \phi_{tx}(s)\|_{L^2(0,1)}^2 \, \mathrm{d}s$$
$$\leq \|\partial_t^k \phi_t(0)\|_{L^2(0,1)}^2 + \|\partial_t^k \phi_x(0)\|_{L^2(0,t)}^2 + \frac{1}{3\mu}\|\partial_t^k h\|_{L^2(0,t)}^2.$$

One can obtain the second estimate by taking inner products with $\partial_t^{k+1}\partial_x^2\phi$ instead of $\partial_t^{k+1}\phi$ in the above calculation. □

We somehow need to introduce a spatial a priori estimate. However, the same trick we used in the proposition above doesn't work since there is no boundary condition given for higher space-derivatives of ϕ. Therefore we take advantage of the structure of the equation and eliminate temporal derivatives. Let us introduce new variable $z = \phi + \mu\phi_t$ for that purpose. Then z satisfies the following.

$$\begin{cases} \partial_t z = \mu \partial_x^2 z + \dfrac{z}{\mu} - \dfrac{\phi}{\mu} + \mu h, & \text{in} \quad (0,1), \\ z(t,0) = z(t,1) = 0, & \text{for} \quad t \in (0,T), \\ z(0) = 0, & \text{in} \quad (0,1). \end{cases} \tag{8.5}$$

Additionally, by solving $z = \phi + \mu\phi_t$ for ϕ, we have

$$\phi(t) = \frac{1}{\mu}\int_0^t e^{-\frac{t-s}{\mu}} z(s)\, \mathrm{d}s. \tag{8.6}$$

Estimate of $\partial_x^4\phi$. Taking derivatives ∂_x^2 in (8.5), we have

$$\partial_x^4 z = \frac{1}{\mu}\left(\partial_t \partial_x^2 z - \frac{\partial_x^2 z}{\mu} - \frac{\partial_x^2 \phi}{\mu}\right).$$

Inserting this into (8.6) will produce

$$\partial_x^4\phi(t) = \frac{1}{\mu}\int_0^t e^{-\frac{t-s}{\mu}}\left(\partial_t\partial_x^2 z(s) - \frac{\partial_x^2 z(s)}{\mu} - \frac{\partial_x^2\phi(s)}{\mu}\right)\mathrm{d}s$$
$$= \frac{1}{\mu^2}\int_0^t e^{-\frac{t-s}{\mu}}\left(-\frac{2}{\mu}\partial_x^2 z(s) - \frac{\partial_x^2\phi(s)}{\mu}\right)\mathrm{d}s + \partial_x^2 z(t) - e^{-\frac{t}{\mu}}\partial_x^2 z(0).$$

Note that we used integration by parts for $\partial_t\partial_x^2 z$ to eliminate the derivative in time. Finally we take norm in $L^2(0,1)$ in both sides.

$$\|\partial_x^4\phi(t)\|_{L^2(0,1)} \leq C\left(1 + \frac{1}{\mu^2}\right)\sup_{0\leq s\leq t}\|\partial_x^2 z(s)\|_{L^2(0,1)} + \frac{C}{\mu^2}\sup_{0\leq s\leq t}\|\partial_x^2\phi(s)\|_{L^2(0,1)}$$
$$\leq C\left(1 + \frac{1}{\mu^2}\right)\left(\sup_{0\leq s\leq t}\|\partial_t\partial_x^2\phi(s)\|_{L^2(0,1)} + \sup_{0\leq s\leq t}\|\partial_x^2\phi(s)\|_{L^2(0,1)}\right).$$

Finally, we invoke the a priori estimate in time above and obtain the intended result.

With the same arguments we find similar estimates for $\partial_t\partial_x^3\phi,\ \partial_t^2\partial_x\phi$.

Proposition 8.3. *Let $T > 0$. For $h \in H^1(0,T)$, there exists the unique solution $\phi \in C^2([0,\infty); H^1(0,1)) \cap C^1([0,\infty), H^3(0,1)) \cap C([0,\infty); H^4(0,1))$ of the system (8.2). The solution satisfies*

$$\|\partial_t\phi(t)\|_{H^3(0,1)} + \|\partial_x\phi(t)\|_{H^3(0,1)} + \|\partial_t^2\phi(t)\|_{H^1(0,1)}$$
$$\leq C\sum_{k=1}^{4}\mu^{-k}\|h\|_{H^1(0,t)}$$

for $0 \leq t \leq T$. The constant C is independent of μ and t.

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