Landau Damping in Finite Regularity for Unconfined Systems with Screened Interactions

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Abstract

We prove Landau damping for the collisionless Vlasov equation with a class of L^1 interaction potentials (including the physical case of screened Coulomb interactions) on $\mathbb{R}^3_x \times \mathbb{R}^3_v$ for localized disturbances of an infinite, homogeneous background. Unlike the confined case $\mathbb{T}^3_x \times \mathbb{R}^3_v$, results are obtained for initial data in Sobolev spaces (as well as Gevrey and analytic classes). For spatial frequencies bounded away from 0, the Landau damping of the density is similar to the confined case. The finite regularity is possible due to an additional dispersive mechanism available on $\mathbb{R}^3_x$ that reduces the strength of the plasma echo resonance. © 2017 Wiley Periodicals, Inc.

1 Introduction

1.1 The Model

The collisionless Vlasov equation is a fundamental kinetic model for so-called hot plasmas and also arises elsewhere in physics, for example, in stellar dynamics [7, 27]. For single-species models, the unknown is the probability density, known as the distribution function $f(t,x,v)$ of particles in phase space. In this work, we consider the phase space $(x,v) \in \mathbb{R}^3_x \times \mathbb{R}^3_v$ and distribution functions of the form $f(t,x,v) = f^0(v) + h(t,x,v)$, where $f^0(v)$ is the infinitely extended, homogeneous equilibrium and $h(t,x,v)$ is the mean-zero fluctuation from equilibrium. Then, the Vlasov equations for the fluctuation are given by

$$\begin{cases} \partial_t h + v \cdot \nabla_x h + F(t,x) \cdot \nabla_v (f^0 + h) = 0, \\ F(t,x) := -\nabla_x W *_x \rho(t,x), \\ \rho(t,x) := \int_{\mathbb{R}^d} h(t,x,v) \mathrm{d}v, \\ h(t=0,x,v) = h_{\mathrm{in}}(x,v). \end{cases} \tag{1.1}$$

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The potential $W(x)$ describes the mean-field interaction between particles. In this paper we will be considering only $W \in L^1$ that satisfy (denoting $\langle x \rangle = (1 + |x|^2)^{1/2}$),

$$|\widehat{W}(k)| \lesssim \langle k \rangle^{-2}. \tag{1.2}$$

As we will see, one of the reasons for this assumption is that, together with a stability condition involving f^0 (see definition 2.6 below), (1.2) ensures that the linearized Vlasov equation behaves similarly to the free transport $\partial_t h + x \cdot \nabla_x h = 0$ (for long times) even at low spatial frequencies. Indeed, the results of [16, 17] show that this is not true in general if one allows Coulomb interactions $\widehat{W}(k) = |k|^{-2}$.

Screened Coulomb interactions provide a physically relevant setting that satisfies hypothesis (1.2) and the stability condition in definition (2.6) for a large class of f^0 (see Proposition 2.7 below). This model arises when considering the distribution function for ions in a plasma, after making the approximations of (1) that the electrons can be considered massless and reach thermal equilibrium on a much faster time scale than the ion evolution, (2) that the plasma is near equilibrium, (3) that an electrostatic approximation is suitable, and (4) that ion collisions can be neglected. In this case, the force field F satisfies (some physical parameters have been suppressed for notational convenience)

$$F = -\nabla \phi, \quad -\Delta \phi + \alpha \phi = \rho, \tag{1.3}$$

where the parameter $\alpha > 0$ accounts for the fact that the electrons equilibrate in a manner to shield the long-range effects of the electric field. The quantity $\alpha^{-1/2}$ has units of length and is proportional to the quantity known in plasma physics as the *Debye length*; it is the characteristic length scale of the mean-field interactions [7]. See [18–20] and the references therein for more details on the model (1.1) with (1.4) in the context of ion dynamics in quasi-neutral plasmas. In the case of (1.3), we have $F = -\nabla_x W *_x \rho$ with

$$\widehat{W}(k) = \frac{1}{\alpha + |k|^2}, \tag{1.4}$$

which satisfies (1.2).

1.2 Landau Damping and Existing Results

It was discovered by Landau [23] that the linearized Vlasov equations around homogeneous steady states satisfying certain stability conditions induce time decay on the nonzero modes of the spatial density. This decay, which is exponentially fast for analytic data, can be more easily deduced for the free transport evolution $\partial_t h + v \cdot \nabla_x h = 0$. For the free transport evolution, it becomes evident that the decay is due to to *mixing in phase space*; that is, spatial information is transferred to smaller scales in velocity, which are averaged away by the velocity integral for ρ (this appears to be first pointed out in [38]). The work of Landau can be summarized as asserting that the dynamics of the linearized Vlasov equations $\partial_t h + v \cdot \nabla_x h + F \cdot \nabla_v f^0 = 0$ are asymptotic to free transport in a suitably strong sense as $t \to \infty$. A number of other works regarding the linearized Vlasov equations followed, providing mathematically rigorous treatments, clarifications, and generalizations [1, 10, 16, 17, 29, 32, 38]. The phenomenon is now known as *Landau damping* and is a cornerstone of plasma physics in approximately collisionless regimes; see, e.g., [7, 34, 37].

The dynamics for each spatial mode decouples in the linearized Vlasov equations and the damping is derived in a relatively straightforward manner via the Laplace transform. In the nonlinear equations, there exist steady states and traveling waves with nontrivial densities [6, 25]; however, one can still hope for Landau damping in a perturbative nonlinear regime. In the perturbative nonlinear setting, the decoupling of Fourier modes of course ceases to hold, and it remained debated for decades whether or on which timescale the damping would hold (for example, the various discussions in [1, 32, 37]; see [30] for more information). The existence of analytic Landau damping solutions to the nonlinear Vlasov equations in $\mathbb{T}_x \times \mathbb{R}_v$ was first demonstrated in [8, 22], but only in [30] was there given a full proof of nonlinear stability with Landau damping in the nonlinear setting, and again in the confined case $\mathbb{T}_x^d \times \mathbb{R}_v^d$ and for smooth enough Gevrey [15] or analytic data. The proof was later simplified and the result improved to the "critical" Gevrey regularity in [4] by combining ideas of [30] and [3].

It is desirable for physical relevance to extend the theory to the *unconfined case*, i.e., when the phase space is $\mathbb{R}_x^d \times \mathbb{R}_v^d$. There are several issues with this even at the linear level. First, at low spatial frequencies, the decay due to mixing for free transport is very slow—there is an additional dispersive decay, but this is only t^{-d} in L^∞. Second, for Vlasov-Poisson, e.g., when the force field is given by $F = -\mu \nabla_x \Delta_x^{-1} \rho$ with $\mu \in \mathbb{R}$, it was shown in [16, 17] that the linearized Vlasov equations cannot be treated as a perturbation of free transport at low spatial frequencies. At the linear level, the modes decouple, so these issues only occur at low spatial modes; at higher spatial modes, the damping is the same as in the confined case. It is then natural to ask whether nonlinear stability in (1.1) still holds in a certain sense and that, at least, the decay of the spatial modes away from 0 (short waves) remains similar to the confined case. In this paper, we positively answer this question in the case that W satisfies (1.2) (and the linear stability condition in definition (2.6) below). These conditions precisely imply that the linearized Vlasov equation is close enough to free transport at low frequencies. Moreover, by taking advantage of a dispersive effect in *frequency* (see Section 1.4 and Section 3), we are able to get results in finite regularity.

Previous finite regularity Landau damping results have only been obtained for kinetic models in which $\widehat{W}$ has compact support, such as Vlasov-HMF [12] or the mean-field Kuramoto model [11, 13]. These results have been proved in the confined case; see Section 1.4 for more discussion on how finite regularity is obtained. A dispersive result in finite regularity for Vlasov-Poisson in the unconfined case $\mathbb{R}_x^3 \times \mathbb{R}_v^3$ without an infinite background density, that is $f^0(v) = 0$, was carried out in [2]. The lack of an infinite background greatly simplifies the setting:

the dynamics do not include the linearized Vlasov equations, and moreover, it is significantly easier to propagate moments in $x - tv$ on $f(t,x,v)$—an important aspect of [2] (propagating such moments seems very difficult even for the linearized Vlasov equations with $W \in L^1$ and f^0 very small). Moreover, the results of [2] do not directly extend to statements of the form (1.9b) or (1.10b), which quantify the fast decay of higher spatial modes (almost equivalently, the techniques seem ill-suited for deducing convergence in such strong norms as (1.9a) and (1.10a)).

1.3 Main Results

Our working norm in this paper is the weighted Sobolev norm:

$$\|h\|_{H^\sigma_M} = \sum_{|\alpha|\le M} \|\langle\nabla_{x,v}\rangle^\sigma (v^\alpha h)\|_{L^2},$$

where we define the Fourier multiplier

$$\widehat{\langle\nabla_{x,v}\rangle^\sigma f}(k,\eta) = \langle|k,\eta|\rangle^\sigma \hat{f}(k,\eta) := \langle k,\eta\rangle^\sigma \hat{f}(k,\eta). \tag{1.5}$$

Notice that

$$\|h\|_{H^\sigma_M} \approx_M \Big(\sum_{|\alpha|\le M} \|\langle\nabla_{x,v}\rangle^\sigma (v^\alpha h)\|^2_{L^2}\Big)^{1/2} \approx_{M,\sigma} \sum_{|\alpha|\le M} \|v^\alpha \langle\nabla_{x,v}\rangle^\sigma h\|_{L^2},$$

so that one may order the moments and derivatives in whichever order is most convenient.

The following linear stability condition is essentially an adaptation to finite regularity of the condition given in [30] (which is essentially the same as the Penrose condition [32]).

DEFINITION 1.1. Given a homogeneous distribution $f^0(v)$, we say that it satisfies the *stability condition* (L) if there exists constants $C_0, \kappa, \bar\sigma > 0$ with $\bar\sigma > \frac{3}{2}$ and an integer $M > \frac{3}{2}$ such that

$$\|f^0\|_{H^{\bar\sigma}_M} \le C_0 \tag{1.6}$$

and

$$\inf_{\xi\in\mathbb{C}:\mathrm{Re}\,\xi\le 0}\ \inf_{k\in\mathbb{R}^3} |\mathcal{L}(\xi,k) - 1| \ge \kappa, \tag{1.7}$$

where $\mathcal{L}$ is defined by the following ($\bar\xi$ denotes the complex conjugate of ξ):

$$\mathcal{L}(\xi,k) = -\int_0^\infty e^{\bar\xi t}\widehat{f^0}(kt)\widehat{W}(k)|k|^2 t\,\mathrm{d}t. \tag{1.8}$$

In Section 2.3 below, we discuss in detail how stringent the stability condition (L) is. We note here that if one takes power law interactions, $W(x) = \mu|x|^{-1}$ for any $\mu \in \mathbb{R}$, then (L) fails for *every* equilibrium $f^0 \in H^{3/2+}_2$; see [16, 17] (see Section 1.5 for the notation H^{p+}). A smallness condition on $\|W\|_{L^1}\|f^0\|_{H^{3/2+}_2}$

is sufficient to satisfy (1.7); however, it is not necessary. Indeed, we show in Proposition 2.7 below that (L) is satisfied for the screened Coulomb law (1.4), the fundamental solution to (1.3), for all $\alpha > 0$ and all rapidly decaying, radially symmetric equilibria f^0. The proof extends to any potential $\widehat{W}$ satisfying

$$0 \leq \widehat{W}(k) \lesssim \langle k \rangle^{-2},$$

and hence, a variety of large W and f^0 are permitted.

Our main result is the following:

THEOREM 1.2. *There exist universal constants $R_0 > 0$ and $c \in (0, R_0)$ such that if $\overline{\sigma} - 3 > \sigma > R_0$ and f^0 is given that satisfies stability condition* (L) *with constants M, C_0, κ, and $\overline{\sigma}$ and h_{in} is mean-zero and satisfies*

$$\sum_{|\alpha| \leq 2} \| z^\alpha h_{\mathrm{in}} \|_{H^\sigma_M} \leq \epsilon_0,$$

then there exists a mean-zero $h_\infty \in H^{\sigma - c}_M$ so that the solution $h(t, x, v)$ to (1.1) *satisfies the following for all $t \geq 0$:*

$$\| h(t, x + tv, v) - h_\infty(x, v) \|_{H^{\sigma-c}_M} \lesssim \frac{\epsilon}{\langle t \rangle^{3/2}}, \tag{1.9a}$$

$$|\widehat{\rho}(t, k)| \lesssim \epsilon \langle k, kt \rangle^{-(\sigma - c)}, \tag{1.9b}$$

$$\| \langle \nabla_x \rangle^{\sigma - c - 4} F(t) \|_{L^\infty} \lesssim \frac{\epsilon}{\langle t \rangle^4}. \tag{1.9c}$$

Remark 1.3. The proof shows that we may take $R_0 = 36$ and $c = 5$, although these are unlikely to be sharp.

Remark 1.4. Theorem 1.2 holds in all $d \geq 3$; in this case, R_0 depends in general on dimension.

Remark 1.5. An easier variant of our proof would yield a similar result in the case where $f^0 = 0$ (no homogeneous background). The linear stability condition is trivially satisfied then, and our nonlinear estimates adapt in a simpler way. However, as discussed briefly above, the results we obtain in Theorem 1.2 are significantly stronger, in certain ways, than the results of [2]. Specifically, (1.9a) gives "scattering" in much higher Sobolev norms, and (1.9b) gives fast decay of higher spatial modes of the density (as fast as if the problem were posed on $\mathbb{T}^3 \times \mathbb{R}^3$). It is not clear how the techniques employed in [2] can be adapted to deduce these higher regularity results.

Remark 1.6. That c and R_0 are taken independent of all parameters shows that regularity loss remains uniform even as $\sigma \to \infty$.

A natural question is whether one still observes exponential decay of $\widehat{\rho}(t, k)$ for k bounded away from 0 if the initial data is analytic. This is indeed the case, which is proved via an easy variation of the proof of Theorem 1.2 using some basic ideas from [4].

THEOREM 1.7. *Let f^0 be given which satisfies stability condition* (L) *with constants M, C_0, and κ, and is real analytic with*

$$\|e^{\overline{\lambda}\langle\nabla\rangle} f^0\|_{L^2_M} < \infty \quad \text{for some } \overline{\lambda} > 0.$$

Then there exists a $\lambda^\star \in (0, \overline{\lambda}]$ depending only on f^0 such that for all $0 < \lambda' < \lambda < \lambda^\star$, there exists an ϵ_0 such that if h_{in} is mean-zero and satisfies

$$\sum_{|\alpha|\leq 2} \|z^\alpha e^{\lambda\langle\nabla\rangle} h_{\mathrm{in}}\|_{L^2_M} \leq \epsilon_0,$$

then there exists a mean-zero, real analytic h_∞ satisfying for all $t \geq 0$,

$$\|e^{\lambda'\langle\nabla\rangle}(h(t, x + tv, v) - h_\infty(x, v))\|_{L^2_M} \lesssim \frac{\epsilon}{\langle t\rangle^{3/2}}, \tag{1.10a}$$

$$|\widehat{\rho}(t,k)| \lesssim \epsilon e^{-\lambda'\langle k,kt\rangle}, \tag{1.10b}$$

$$\|e^{\lambda'\langle\nabla_x, t\nabla_x\rangle} F(t)\|_{L^\infty} \lesssim \frac{\epsilon}{\langle t\rangle^4}. \tag{1.10c}$$

Remark 1.8. An analogue of Theorem 1.7 also holds for Gevrey initial data (see [4] for the Vlasov-Poisson systems with Coulomb-Newton potentials on $\mathbb{T}^d \times \mathbb{R}^d$ with Gevrey data).

1.4 Plasma Echoes and Dispersion in Frequency

As discussed in [4,30], the fundamental impediment to nonlinear Landau damping results in finite regularity are resonances known as plasma echoes, first discovered and isolated in the experiments [28]. During Landau damping, the force field is damped due to the transfer of $O(1)$ spatial information to small scales in the velocity distribution. However, mixing is time reversible, and hence unmixing creates (transient) growth in the force field. This effect is essentially the same as the analogous *Orr mechanism* in fluid mechanics, first identified in [31] (see [3] for more information). A plasma echo occurs when a nonlinear effect transfers information to modes that are *unmixing*, as this leads to a large force field in the future when that information reaches $O(1)$ spatial scales (hence "echo"). The plasma echo is a kind of nonlinear resonance, although associated with the transient unmixing in the linear problem rather than a true eigenvalue. These echoes can chain into a cascade, as demonstrated experimentally in the Vlasov equations [28] and two-dimensional Euler [40,41].

Mathematically, one must confront the echo resonance when attempting to close an estimate such as (1.9b). During the proof of (1.9b), one needs to get an $L^2_t L^2_k \to L^2_t L^2_k$ estimate on an integral operator that encodes the long-time interactions between the force field and the information that has already mixed (see Section 3). The primary new insight in our work is that, unlike in the confined case studied in [4,8,22,30], we can obtain these estimates in *finite regularity*. This is completely due to a dispersive mechanism that is present only in $\mathbb{R}^d_x$ for $d \geq 2$ (although it is too weak in $d = 2$ for our methods); it has little relevance to the periodically confined case $\mathbb{T}_x^d$ (although one could imagine attempting to recover it in a large box limit) and is quite distinct from the finite regularity results of [11–13].

We will capitalize on a dispersive effect in the free transport operator $\partial_t + v \cdot \nabla_x$, which on the *Fourier side* is of the form $\partial_t - k \cdot \nabla_\eta$. In order to lose a significant amount of regularity, one must chain a large number of echoes over a long period of time (see [4, 30]). Indeed, this is precisely why the finite results of [11–13] are possible: the models studied therein do not support infinite chains of echoes. For any spatial mode k, the set of possible "resonant" frequencies ℓ, the frequencies that can react strongly via a plasma echo with k, turns out to be those which are collinear with k. Indeed, if the two spatial modes are not collinear, then the velocity information in the two modes is moving in different directions in frequency (due to the dispersive effect of $\partial_t - k \cdot \nabla_\eta$) and is hence well-separated (in frequency) except for a limited amount of time. On the torus, the set of such resonant frequencies is of positive density in the lattice $\mathbb{Z}^d$ (for example, it suffices to consider modes that depend on only one coordinate), whereas in $\mathbb{R}_x^3$ the set of resonant frequencies is a one-dimensional line and is hence a very small set. Spatial localization implies that information in the Fourier transform cannot concentrate on small sets, which suggests that the resonance is weaker in $\mathbb{R}^3$ than in $\mathbb{T}^3$. This is indeed the case, as we show in Section 3. We remark that there may also be a link with the idea of space-time resonances in dispersive equations [14].

1.5 Notation and Conventions

We denote $\mathbb{N} = \{0, 1, 2, \dots\}$ (including 0) and $\mathbb{Z}_* = \mathbb{Z} \setminus \{0\}$. For $\xi \in \mathbb{C}$ we use $\overline{\xi}$ to denote the complex conjugate. We denote

$$\langle v \rangle = (1 + |v|^2)^{1/2}.$$

We use the multi-index notation: given $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ and $v = (v_1, \dots, v_d) \in \mathbb{R}^d$, then

$$v^\alpha = v_1^{\alpha_1} \cdots v_d^{\alpha_d}, \quad D_\eta^\alpha = (i\partial_{\eta_1})^{\alpha_1} \cdots (i\partial_{\eta_d})^{\alpha_d}.$$

We denote Lebesgue norms for $p, q \in [1, \infty]$ and $a, b \in \mathbb{R}^3$ as

$$\begin{aligned} \|f\|_{L_a^p L_b^q} &= \left(\int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} |f(a,b)|^q \, \mathrm{d}b \right)^{p/q} \mathrm{d}a \right)^{1/p} \\ &= \left(\int \left(\int |f(a,b)|^q \, \mathrm{d}b \right)^{p/q} \mathrm{d}a \right)^{1/p} \end{aligned}$$

and Sobolev norms (usually applied to Fourier transforms) as

$$\|\widehat{f}\|_{H_\eta^M}^2 = \sum_{\alpha \in \mathbb{N}^d : |\alpha| \le M} \|D_\eta^\alpha \widehat{f}\|_{L_\eta^2}^2.$$

We will often use the short-hand $\|\cdot\|_2$ for $\|\cdot\|_{L_{z,v}^2}$ or $\|\cdot\|_{L_v^2}$ depending on the context. Finally, we use the notation $f \in H^{s+}$ as shorthand to denote that $f \in H^{s+\delta}$

for all $\delta > 0$. Similarly, the quantity $\|f\|_{H^{s+}}$ is meant to satisfy $\|f\|_{H^{s+}} \lesssim_\delta \|f\|_{H^{s+\delta}}$ for all $\delta > 0$ (where the constant in general blows up as $\delta \to 0$).

For a function $g = g(z, v)$ we write its Fourier transform $\widehat{g}_k(\eta)$ where $(k, \eta) \in \mathbb{R}^3 \times \mathbb{R}^3$ with

$$\widehat{g}_k(\eta) := \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} e^{-izk - iv\eta} g(z, v) \mathrm{d}z \, \mathrm{d}v,$$
$$g(z, v) := \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} e^{izk + iv\eta} \widehat{g}_k(\eta) \mathrm{d}k \, \mathrm{d}\eta.$$

We use an analogous convention for Fourier transforms to functions of x or v alone. With these conventions we have the following relations:

$$\begin{cases} \int_{\mathbb{R}^3 \times \mathbb{R}^3} g(z, v) \overline{g}(z, v) \mathrm{d}z \, \mathrm{d}v = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \widehat{g}(k, \eta) \overline{\widehat{g}}(k, \eta) \mathrm{d}k \, \mathrm{d}\eta, \\ \widehat{g^1 g^2} = \frac{1}{(2\pi)^3} \widehat{g^1} * \widehat{g^2}, \\ (\widehat{\nabla g})(k, \eta) = (ik, i\eta) \widehat{g}(k, \eta), \\ (\widehat{v^\alpha g})(k, \eta) = D_\eta^\alpha \widehat{g}(k, \eta). \end{cases}$$

By convention, we use Greek letters such as η and ξ to denote velocity frequencies, and lowercase Latin characters such as k and ℓ to denote spatial frequencies.

We use the notation $f \lesssim g$ when there exists a constant $C > 0$ independent of the parameters of interest such that $f \le Cg$ (we analogously define $f \gtrsim g$). Similarly, we use the notation $f \approx g$ when there exists $C > 0$ such that $C^{-1} g \le f \le Cg$. We sometimes use the notation $f \lesssim_\alpha g$ if we want to emphasize that the implicit constant depends on some parameter α.

2 Outline of the Proof

2.1 Local-in-Time Well-Posedness

The following standard lemma provides local existence of a classical solution that remains classical as long as a suitable Sobolev norm remains finite. The propagation of regularity can be proved by a variant of the arguments in, e.g., [24], along with the inequality

$$\|B(t, \nabla_x, \nabla_x t)\rho(t)\|_2 \lesssim \sum_{\alpha \le M} \|v^\alpha B(t, \nabla_x, \nabla_v) h(t)\|_2 \tag{2.1}$$

for all Fourier multipliers B and all integers $M > \frac{3}{2}$.

LEMMA 2.1 (Local existence and propagation of regularity). *Let $M > \frac{3}{2}$ be an integer and $h_{\mathrm{in}} \in H^\alpha_M$ for $\alpha > 4$. Then there exists some $T_0 > 0$ such that for all $T < T_0$, there exists a unique solution $g(t) \in C([0, T]; H^\alpha_M)$ to (2.3) on $[0, T]$. Moreover, if for some $T \le T_0$ and σ' with $\sigma \ge \sigma' > 4$, there holds $\limsup_{t \nearrow T} \|g(t)\|_{H^{\sigma'}_M} < \infty$, then $T < T_0$.*

Remark 2.2. Finite energy, strong solutions are well-known to be global in time in $\mathbb{T}^3_x \times \mathbb{R}^3_v$ or on $\mathbb{R}^3_x \times \mathbb{R}^3_v$ if there is no homogeneous background [5, 21, 26, 33, 35]; however, to the authors' knowledge, there is no global existence theory that covers the entire range of Theorem 1.2. However, Theorem 1.2 shows that in the perturbative regime, solutions are global.

2.2 Coordinate Shift

As the solution in Theorem 1.2 is asymptotic to free transport, it makes sense to begin (as in [4, 8, 22]) by modding out by this evolution:

$$z := x - tv, \tag{2.2a}$$

$$g(t,z,v) := h(t, z + tv, v). \tag{2.2b}$$

From (2.2) and (1.1) we derive the system

$$\begin{cases} \partial_t g + F(t, z + vt) \cdot (\nabla_v - t\nabla_z) g + F(t, z + vt) \cdot \nabla_v f^0 = 0, \\ g(t = 0, z, v) = h_{\mathrm{in}}(z, v), \\ \widehat{\rho}(t,k) = \widehat{g}(t,k,kt). \end{cases} \tag{2.3}$$

As in [4], we derive from (2.3) the following system on the Fourier side:

$$\begin{aligned} \partial_t \widehat{g}(t,k,\eta) = {} & -\widehat{\rho}(t,k)\widehat{W}(k)k \cdot (\eta - tk)\widehat{f}^0(\eta - kt) \\ & - \int_{\mathbb{R}^3} \widehat{\rho}(t,\ell)\widehat{W}(\ell)\ell \cdot [\eta - tk]\widehat{g}(t, k-\ell, \eta - t\ell)\mathrm{d}\ell \end{aligned} \tag{2.4a}$$

$$\begin{aligned} \widehat{\rho}(t,k) = {} & \widehat{f_{\mathrm{in}}}(k,kt) - \int_0^t \widehat{\rho}(\tau,k)\widehat{W}(k)k \cdot k(t-\tau) f^0(k(t-\tau))\mathrm{d}\tau \\ & - \int_0^t \int_{\mathbb{R}^3} \widehat{\rho}(\tau,\ell)\widehat{W}(\ell)\ell \cdot k(t-\tau)\widehat{g}(\tau, k-\ell, kt - \tau\ell)\mathrm{d}\ell\, \mathrm{d}\tau. \end{aligned} \tag{2.4b}$$

2.3 Linear Landau Damping in $\mathbb{R}^3_x \times \mathbb{R}^3_v$

The first step in proving Theorem 1.2 is understanding the linear term in (2.4b). In particular, we need estimates on the linear Volterra equation

$$\phi(t,k) = H(t,k) + \int_0^t K^0(t-\tau,k)\phi(\tau,k)\mathrm{d}\tau, \tag{2.5}$$

where $K^0(t,k) := -\widehat{f}^0(kt)\widehat{W}(k)|k|^2 t$ and $H(t,k)$ has sufficiently rapid decay. Recall that by definition, $\mathcal{L}$ is the Fourier-Laplace transform of the kernel K^0:

$$\mathcal{L}(\xi,k) = \int_0^\infty e^{\bar{\xi}t} K^0(t,k)\mathrm{d}t = -\int_0^\infty e^{\bar{\xi}t}|k|^2 t\widehat{W}(k)\widehat{f^0}(kt)\mathrm{d}t. \tag{2.6}$$

We begin by proving that (L) implies Landau damping for (2.5). See Appendix 5.3 for the proof, which is a variation of the arguments in [4, 30].

PROPOSITION 2.3 (Linear L^2_t control). *Let f^0 satisfy the condition* (L) *with constants $C_0, \kappa > 0$. Let α be arbitrary and $s \geq 0$ be an arbitrary integer. Let $H(t,k)$*

and $T^\star > 0$ be given such that, if we denote $I = [0, T^\star)$, then we take $H(t,k) = 0$ for $t > T_\star$ and

$$\| |k|^\alpha \langle k, kt\rangle^s H(t,k)\|_{L^2_t(I)}^2 < \infty.$$

Then there exists a constant $C_{LD} = C_{LD}(C_0, s, \overline{\sigma}, \kappa)$ such that the solution $\phi(t,k)$ to the system (2.5) *satisfies the pointwise-in-k estimate,*

$$\| |k|^\alpha \langle k, kt\rangle^s \phi(t,k)\|_{L^2_t(I)} \leq C_{LD} \| |k|^\alpha \langle k, kt\rangle^s H(t,k)\|_{L^2_t(I)}^2. \tag{2.7}$$

Remark 2.4. As long as condition (L) is satisfied, there is no difference between $x \in \mathbb{T}^d$ and $x \in \mathbb{R}^d$ for the purposes of Proposition 2.3. In [16, 17], the convergence rates are degraded due to the lack of (L).

Remark 2.5. In fact, Proposition 2.3 holds for any $s \in \mathbb{R}_+$; however, the integer case is simpler. Once the integer case is solved, a decomposition argument based on almost-orthogonality is applied to reach fractional s; see, e.g., [4] for an analogous argument (although the finite regularity setting is easier).

It is important to discuss how restrictive the linear stability condition (L) is. The proofs can be found in Appendix 5.3. The first observation is that a smallness condition on the interaction is sufficient to imply stability (this follows immediately from Lemma A.1).

PROPOSITION 2.6. *There exists a universal $c > 0$ such that if $\|W\|_{L^1} \|f^0\|_{H^{3/2+}_2} < c$, then f^0,W satisfy the linear stability condition* (L) *for some C_0, κ, and $\overline{\sigma}$.*

As discussed above, if one takes the interaction potential $W(x) = \mu |x|^{-1}$ for any $\mu \in \mathbb{R}$, then (L) fails for every equilibrium considered here [16,17]. However, the screened Coulomb law (1.4) does not have this problem. Indeed, we have linear stability for all $\alpha > 0$.

PROPOSITION 2.7. *Let W be* (1.4)*, the fundamental solution to* (1.3)*. Then for any strictly positive, radially symmetric equilibrium $f^0 \in H^{3/2+}_2$ with $|f^0(v)| + |\nabla f^0(v)| \lesssim \langle v\rangle^{-4}$ for $|v|$ large and all $\alpha > 0$, W and f^0 satisfy* (L) *for some constants C_0, κ, and $\overline{\sigma}$. In fact, the same applies to any potential W satisfying*

$$0 \leq \widehat{W}(k) \lesssim \langle k\rangle^{-2}.$$

Remark 2.8. Of course, the constant κ in (L) blows up as $\alpha \to 0$.

2.4 Nonlinear Energy Estimates

Next we set up the continuity argument we use to derive a uniform bound on g via the system (2.4). Define the following, which is convenient when considering the density: for any $s > 0$,

$$A_s(t,k) = |k|^{1/2} \langle k, tk\rangle^s.$$

We will employ the same notation for the corresponding Fourier multiplier:

$$A_s(t,\nabla_z) = |\nabla_z|^{1/2}\langle \nabla_z, t\nabla_z\rangle^s;$$

we hope there will be no confusion.

Fix regularity levels $\overline{\sigma} > \sigma_4 > \sigma_3 > \sigma_2 > \sigma_1 \geq 11$ and constants $K_i \geq 1$ determined by the proof. Let $I = [0, T^\star]$ be the largest connected interval containing 0 such that the following bootstrap controls hold:

$$\|\langle t\nabla_z, \nabla_v\rangle g(t)\|^2_{H^{\sigma_4}_M} \leq 4K_1\langle t\rangle^5\epsilon^2, \tag{2.8a}$$

$$\|A_{\sigma_4}\widehat{\rho}\|^2_{L^2_t L^2_k} \leq 4K_2\epsilon^2, \tag{2.8b}$$

$$\||\nabla_z|^\delta g(t)\|^2_{H^{\sigma_3}_M} \leq 4K_3\epsilon^2, \tag{2.8c}$$

$$\|A_{\sigma_2}\widehat{\rho}\|_{L^\infty_k L^2_t} \leq 4K_4\epsilon^2 \tag{2.8d}$$

$$\|\widehat{\langle\nabla\rangle^{\sigma_1} g}\|_{L^\infty_{k,\eta}} \leq 4K_5\epsilon^2. \tag{2.8e}$$

Remark 2.9. A close reading of the proof suggests that one can take $\sigma_i - \sigma_{i-1} = 6$ and $\overline{\sigma} - \sigma_4 = 6$, although this seems far from optimal. This technically brings the regularity requirement given by the proof to 35; however, we did not attempt to optimize this number.

Remark 2.10. The constants K_2 and K_4 are determined only by the properties of the linearized Vlasov equations (hence they depend only on f^0 and W), and the constants K_1, K_3, K_5 are fixed independently, depending only on K_2, K_4, and universal constants.

Remark 2.11. Notice the order $L^\infty_k L^2_t$ in the estimate (2.8d). This norm is reminiscent of the norms used by Chemin and Lerner in [9].

PROPOSITION 2.12 (Bootstrap). *Let* (2.8) *be satisfied for all* $t \in [0, T^\star]$ *with* $T^\star < T^0$ *(*T^0 *defined in* (2.1)*). Then for* ϵ *chosen sufficiently small, the estimates* (2.8) *all hold with* 4 *replaced with* 2*.*

Proposition 2.12 comprises the main step of the proof of Theorem 1.2 (see Proposition 2.17 below).

2.5 Useful Toolbox

First, we observe the following, which at least shows that the norms employed to measure ρ in (2.8) are natural.

LEMMA 2.13. *Define*

$$\rho_0(t,k) = \widehat{h_{\mathrm{in}}}(k, kt).$$

For all $s > 4$*, there holds (recall the notation* H^{s+} *from Section* 1.5*),*

$$\|A_s\rho_0\|_{L^2_t L^2_k} \lesssim \sum_{|\alpha|\leq 2}\|z^\alpha h_{\mathrm{in}}\|_{H^{s+2+}_M}, \quad \|A_s\rho_0\|_{L^\infty_k L^2_t} \lesssim \sum_{|\alpha|\leq 2}\|z^\alpha h_{\mathrm{in}}\|_{H^{s+1+}_M}.$$

PROOF. Proving the first estimate is straightforward by the $H^{3/2+}(\mathbb{R}^3) \hookrightarrow C^0$ Sobolev embedding applied on the Fourier side and $M \geq 2$,

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^3} |k| \langle k, kt \rangle^{2s} |\widehat{h_{\rm in}}(k, kt)|^2 \, {\rm d}k \, {\rm d}t \\
&\quad \lesssim \left(\int_0^\infty \int_{\mathbb{R}^3} \frac{|k|}{\langle k, kt \rangle^{4+}} \, {\rm d}k \, {\rm d}t \right) \Big(\sum_{|\alpha| \leq 2} \| z^\alpha h_{\rm in} \|^2_{H^{s+2+}_M} \Big) \\
&\quad \lesssim \left(1 + \int_1^\infty \frac{1}{t^4} \int_\zeta \frac{|\zeta|}{\langle \zeta \rangle^{4+}} \, {\rm d}\zeta \, {\rm d}t \right) \Big(\sum_{|\alpha| \leq 2} \| z^\alpha h_{\rm in} \|^2_{H^{s+2+}_M} \Big) \\
&\quad \lesssim \sum_{|\alpha| \leq 2} \| z^\alpha h_{\rm in} \|^2_{H^{\sigma+2+}_M}.
\end{aligned}$$

The second estimate follows slightly differently. For all $k \in \mathbb{R}^3$ we have

$$\begin{aligned}
\int_0^\infty |k| \langle k, kt \rangle^{s} |\widehat{h_{\rm in}}(k, kt)|^2 \, {\rm d}t &\lesssim \left(\int_0^\infty \frac{|k|}{\langle k, kt \rangle^{1+}} \, {\rm d}t \right) \Big(\sum_{|\alpha| \leq 2} \| z^\alpha h_{\rm in} \|^2_{H^{s+1+}_M} \Big) \\
&\lesssim \Big(\sum_{|\alpha| \leq 2} \| z^\alpha h_{\rm in} \|^2_{H^{s+1+}_M} \Big). \qquad \square
\end{aligned}$$

Next, let us point out a consequence of estimate (2.8e), which provides the dispersive decay of the density and force field.

LEMMA 2.14. *Under the bootstrap hypotheses, for all* $0 \leq \alpha < \sigma_1 - \gamma - 3$*, there holds*

$$\| |\partial_z|^\alpha \langle \partial_z, t\partial_z \rangle^\gamma \rho \|_{L^\infty} \lesssim \int_{\mathbb{R}^3} |k|^\alpha \langle k, kt \rangle^\gamma |\widehat{\rho}(t, k)| {\rm d}k \lesssim K_5 \epsilon \langle t \rangle^{-3-\alpha}. \tag{2.9}$$

PROOF. This follows immediately from (2.8e) (and recalling that $\widehat{g}(t, k, kt) = \widehat{\rho}(t, k)$ from (2.3)),

$$\begin{aligned}
\int_{\mathbb{R}^3} |k|^\alpha \langle k, kt \rangle^\gamma |\widehat{\rho}(t, k)| {\rm d}k &\lesssim \epsilon \int_{\mathbb{R}^3} |k|^\alpha \langle k, kt \rangle^{\gamma - \sigma_1} \, {\rm d}k \\
&\lesssim K_5 \langle t \rangle^{-\alpha} \epsilon \int_{\mathbb{R}^3} \langle k, kt \rangle^{\gamma - \sigma_1 + \alpha} \, {\rm d}k \lesssim K_5 \epsilon \langle t \rangle^{-3-\alpha}. \square
\end{aligned}$$

Let us also record a few simple inequalities that will be used a few times in what follows (on the Fourier side).

LEMMA 2.15 (L^2 trace). *Let* $g \in H^s(\mathbb{R}^d)$ *with* $s > (d-1)/2$ *and* $C \subset \mathbb{R}^d$ *be an arbitrary straight line. Then there holds*

$$\| g \|_{L^2(C)} \lesssim_s \| g \|_{H^s}.$$

LEMMA 2.16.

(a) *Let* $g^1, g^2 \in L^2(\mathbb{R}^d_k \times \mathbb{R}^d_\eta)$ *and* $r \in L^1(\mathbb{R}^d_\eta)$. *Then*

$$\left|\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d} g^1(k,\eta)r(\ell)g^2(k-\ell,\eta-t\ell)\mathrm{d}\ell\,\mathrm{d}k\,\mathrm{d}\eta\right| \lesssim \|g^1\|_{L^2_{k,\eta}}\|g^2\|_{L^2_{k,\eta}}\|r\|_{L^1_\eta}. \tag{2.10}$$

(b) *Let* $g^1 \in L^2(\mathbb{R}^d_k \times \mathbb{R}^d_\eta)$, $g^2 \in L^1(\mathbb{R}^d_k; L^2(\mathbb{R}^d_\eta))$, *and* $r \in L^2(\mathbb{R}^d)$. *Then*

$$\left|\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d} g^1(k,\eta)r(\ell)g^2(k-\ell,\eta-t\ell)\mathrm{d}\ell\,\mathrm{d}k\,\mathrm{d}\eta\right| \lesssim \left\|g^1\right\|_{L^2_{k,\eta}}\left\|g^2\right\|_{L^1(\mathbb{R}^d_k;L^2(\mathbb{R}^d_\eta))}\|r\|_{L^2_\eta}. \tag{2.11}$$

As a result, if $s > d/2$, *there also holds*

$$\left|\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d} g^1(k,\eta)r(\ell)g^2(k-\ell,\eta-t\ell)\mathrm{d}\ell\,\mathrm{d}k\,\mathrm{d}\eta\right| \lesssim_{d,s} \|g^2\|_{L^2_{k,\eta}}\|g^2\|_{L^2_{k,\eta}}\|\langle\cdot\rangle^s r(\cdot)\|_{L^2_\eta} \tag{2.12}$$

$$\left|\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d} g^1(k,\eta)r(\ell)g^2(k-\ell,\eta-t\ell)\mathrm{d}\ell\,\mathrm{d}k\,\mathrm{d}\eta\right| \lesssim_{d,s} \left\|g^1\right\|_{L^2_{k,\eta}}\left\|\langle k\rangle^s g^2\right\|_{L^2_{k,\eta}}\|r\|_{L^2_\eta}. \tag{2.13}$$

As a straightforward application of the above lemmas, we show that Proposition 2.12 implies Theorem 1.2.

PROPOSITION 2.17. *Proposition* 2.12 *implies Theorem* 1.2.

PROOF. Estimate (2.8e) directly implies (1.9b) by $\widehat{\rho}(t,k) = \widehat{g}(t,k,kt)$, while (1.9c) follows by (2.9) above (also a direct consequence of (2.8e)).

To deduce (1.9a), begin by applying $\langle k,\eta\rangle^{\sigma_0} D^\alpha_\eta$ for a multi-index $|\alpha| \le M$ and $\sigma_0 < \sigma_1 - \frac{5}{2}$ and integrating (2.4a):

$$\begin{aligned}
&\langle k,\eta\rangle^{\sigma_0} D^\alpha_\eta \widehat{g}(t,k,\eta)\\
&\quad= \langle k,\eta\rangle^{\sigma_0} D^\alpha_\eta \widehat{g}_{\mathrm{in}}(k,\eta)\\
&\qquad- \int_0^t \langle k,\eta\rangle^{\sigma_0} D^\alpha_\eta\big(\widehat{\rho}(\tau,k)\widehat{W}(k)k\cdot(\eta-\tau k)\widehat{f}^0(\eta-k\tau)\big)\mathrm{d}\tau\\
&\qquad- \int_0^t\int_{\mathbb{R}^3} \langle k,\eta\rangle^{\sigma_0} D^\alpha_\eta\big(\widehat{\rho}(\tau,\ell)\widehat{W}(\ell)\ell\cdot[\eta-\tau k]\widehat{g}(\tau,k-\ell,\eta-t\ell)\big)\mathrm{d}\ell\,\mathrm{d}\tau\\
&\quad= \langle k,\eta\rangle^{\sigma_0} D^\alpha_\eta \widehat{g}_{\mathrm{in}} - \int_0^t L\,\mathrm{d}\tau - \int_0^t NL\,\mathrm{d}\tau.
\end{aligned}$$

By Proposition 2.12 there holds (using that $\overline{\sigma}$ is sufficiently large),

$$\begin{aligned}\|L\|_{L^2_{k,\eta}}^2 &\lesssim \int_{\mathbb{R}^3\times\mathbb{R}^3} |k|^2\langle k,kt\rangle^{2\sigma_0}|\rho(t,k)|^2\langle \eta-kt\rangle^{2\sigma_0}\left|D_\eta^\alpha\big((\eta-kt)\widehat{f^0}(\eta-kt)\big)\right|^2 \mathrm{d}k\,\mathrm{d}\eta\\ &\lesssim \int_{\mathbb{R}^3}|k|^2\langle k,kt\rangle^{2\sigma_0}|\rho(t,k)|^2\,\mathrm{d}k\\ &\lesssim \epsilon^2\int_{\mathbb{R}^3}|k|^2\langle k,kt\rangle^{2\sigma_0-2\sigma_1}\,\mathrm{d}k\\ &\lesssim \epsilon^2\langle t\rangle^{-5}.\end{aligned}$$

Similarly,

$$\begin{aligned}\|NL\|_{L^2_{k,\eta}}^2 &\lesssim \langle t\rangle^2\||\partial_z|^\delta f\|_{L^{\sigma_0}_M}^2\left(\int_{\mathbb{R}^3}\frac{|\ell|}{|k-\ell|^\delta}\langle \ell,\ell t\rangle^{\sigma_0}|\widehat{\rho}(t,\ell)|\mathrm{d}\ell\right)^2\\ &\lesssim \epsilon^4\langle t\rangle^{2\delta-6}.\end{aligned}$$

Therefore, both of the time integrals in (2.5) are absolutely convergent in $L^2_{k,\eta}$ (recall $0<\delta\ll\frac{1}{2}$). Hence, define

$$\begin{aligned}\widehat{h_\infty}(k,\eta) := \widehat{g}_{\mathrm{in}}(k,\eta) &- \int_0^\infty \widehat{\rho}(\tau,k)\widehat{W}(k)k\cdot(\eta-\tau k)\widehat{f}^0(\eta-k\tau)\mathrm{d}\tau\\ &- \int_0^\infty\int_{\mathbb{R}^3}\widehat{\rho}(\tau,\ell)\widehat{W}(\ell)\ell\cdot[\eta-\tau k]\widehat{g}(\tau,k-\ell,\eta-t\ell)\mathrm{d}\ell\,\mathrm{d}\tau.\end{aligned}$$

Inequality (1.9a) then follows from the decay estimates on the integrands and the definition of g. □

3 Plasma Echoes in Finite Regularity

As discussed in Section 1.4, the plasma echo effect is the main difficulty in deducing Landau damping. When attempting the estimate (2.8b), one must get an $L^2_tL^2_k \to L^2_tL^2_k$ estimate on the integral operator:

$$\phi(t,k)\mapsto \int_0^t\int_{\mathbb{R}^3}\phi(\tau,\ell)\bar{K}(t,\tau,k,\ell)\mathrm{d}\ell\,\mathrm{d}\tau, \tag{3.1}$$

where the so-called *time-response kernel* is given by

$$\bar{K}(t,\tau,k,\ell) = \frac{|k|^{1/2}|\ell|^{1/2}|k(t-\tau)|}{\langle \ell\rangle^2}|\widehat{g}(\tau,k-\ell,kt-\ell\tau)|, \tag{3.2}$$

as will be derived in Section 4.1 below. This kernel measures the maximal strength at which the ℓ^{th} mode of the density at time τ can force the k^{th} mode of the density at time t through the nonlinear interaction with g at mode $k-\ell, kt-\ell\tau$ at time τ. By (2.8e), we estimate

$$\bar{K}(t,\tau,k,\ell)\lesssim \sqrt{K_2}\epsilon\frac{|k|^{1/2}|\ell|^{1/2}\langle\tau\rangle}{\langle\ell\rangle^2\langle k-\ell,kt-\ell\tau\rangle^{\sigma_1-1}}; \tag{3.3}$$

for notational convenience we define $\beta := \sigma_1 - 1$. By Schur's test, it suffices to bound the supremum of the row sums and the supremum of the column sums of (3.3) in order to show that the integral operator (3.1) is bounded. This is the content of this section, proved below in Lemmas 3.1 and 3.2. Similar time-response kernels arose in [4] and [30]—the primary new insight here is the fact that we can prove Lemmas 3.1 and 3.2 in finite regularity.

It is clear that the row and column sums of (3.3) are dominated by contributions from large τ and where $kt - \ell\tau$ is small, which is only possible when k and ℓ are nearly collinear. On $\mathbb{T}_x^d$, the one-dimensional reductions used in the proofs of the analogous lemmas in [4, 30] are essentially reductions to collinear resonant frequencies. In the proofs of Lemmas 3.1 and 3.2, we will separate the approximately collinear "resonant" frequencies from the "nonresonant" frequencies with a time-varying cutoff. The fact that we can take the cutoff shrinking in time is due to the dispersion encoded in the free transport on the frequency side, $\partial_t + k \cdot \nabla_\eta$. We will then use that the resonant frequencies comprise a small set that shrinks in time, whereas on the nonresonant frequencies, $\bar{K}$ has much better estimates. The cutoff is then chosen to balance both requirements; it is in this balance where $d \geq 3$ is used.

LEMMA 3.1 (Time response estimate I). *Under the bootstrap hypotheses* (2.8)*, there holds*

$$\sup_{t\in[0,T^\star]} \sup_{k\in\mathbb{R}^3} \int_0^t \int_{\mathbb{R}^3} \bar{K}(t,\tau,k,\ell) \mathrm{d}\tau \, \mathrm{d}\ell \lesssim \sqrt{K_2}\epsilon.$$

PROOF. First, we eliminate irrelevant early times: for $\beta > 4$ we have

$$\int_0^{\min(1,t)} \int_{\mathbb{R}^3} \frac{\langle\tau\rangle |\ell|^{1/2} |k|^{1/2}}{\langle\ell\rangle^2 \langle k-\ell, kt-\ell\tau\rangle^\beta} \mathrm{d}\ell \, \mathrm{d}\tau \lesssim 1.$$

For a fixed $k \in \mathbb{R}^3$ and all $\ell \in \mathbb{R}^3$, define

$$\ell_{||} = \frac{k\cdot\ell}{|k|^2} k \quad \text{and} \quad \ell_\perp = \ell - \ell_{||},$$

the collinear and perpendicular components. Define the following parameters:

$$\zeta \in \left(\tfrac{4}{5}, 1\right), \tag{3.4a}$$

$$b = \beta^{-1}, \tag{3.4b}$$

where we will choose β such that at least $b < \frac{1}{6}$. Define the two subregions of resonant ℓ and nonresonant ℓ:

$$I_R = I_R(\tau,k) = \left\{\ell : |\ell_\perp| < (1+\tau)^{-\zeta} |k|^b\right\},$$
$$I_{NR} = I_{NR}(\tau,k) = \left\{\ell : |\ell_\perp| \geq (1+\tau)^{-\zeta} |k|^b\right\}.$$

The set I_R denotes the frequencies that can resonate strongly with frequency k and is a cylinder around the line containing k, which is shrinking in time. Physically,

I_R is restricted to frequencies that spend a long time sufficiently aligned with k. The dispersive effect is highlighted due to the fact that we can shrink the cross-sectional area of the cylinder in time. In I_R, for each ℓ with $\ell = \ell_{||}$ we can associate a disk of radius $(1+\tau)^{-\zeta}|k|^b$ that lies in the resonant region. We first integrate over this two-dimentional disk; this is where we are going to exploit that $\ell \in \mathbb{R}^d$ with $d = 3$ (note also that we have used the inequality $|x+y|^{1/2} \leq |x|^{1/2} + |y|^{1/2}$ for $x, y > 0$):

$$
\begin{aligned}
&\int_{\min(1,t)}^{t} \int_{I_R} \frac{\langle \tau \rangle |\ell|^{1/2} |k|^{1/2}}{\langle \ell \rangle^2 \langle k-\ell, kt-\ell\tau\rangle^{\beta}} \, \mathrm{d}\ell \, \mathrm{d}\tau \\
&\quad \lesssim \int_{\min(1,t)}^{t} \int_{I_R} \frac{\langle \tau \rangle |k|^{1/2} (|\ell_{||}|^{1/2} + |\ell_{\perp}|^{1/2})}{\langle \ell_{||} \rangle^2 \langle k-\ell, kt-\ell\tau\rangle^{\beta}} \, \mathrm{d}\ell \, \mathrm{d}\tau \\
&\quad \lesssim \int_{\min(1,t)}^{t} \int_{\mathbb{R}} \frac{\langle \tau \rangle |k|^{1/2}}{\langle \ell_{||} \rangle^2 \langle k-\ell_{||}, kt-\ell_{||}\tau\rangle^{\beta}} \frac{|k|^{2b}}{(1+\tau)^{2\zeta}} \left(|\ell_{||}|^{1/2} + \frac{|k|^{b/2}}{(1+\tau)^{\zeta/2}} \right) \mathrm{d}\ell_{||} \, \mathrm{d}\tau \\
&\quad = \mathcal{I}_1 + \mathcal{I}_2.
\end{aligned} \tag{3.5}
$$

In particular, $|k|^{2b}(1+\tau)^{-2\zeta} = |k|^{(d-1)b}(1+\tau)^{-(d-1)\zeta}$, and hence the argument extends to all $d \geq 3$. Then, requiring that $\zeta > \frac{1}{2}$, $2b + \frac{1}{2} < 2$, and $\beta > 4$ we get

$$
\begin{aligned}
\mathcal{I}_1 &\lesssim \int_{\min(1,t)}^{t} \int_{\mathbb{R}} \frac{|\ell_{||}|^{1/2}}{\langle k-\ell_{||}, kt-\ell_{||}\tau\rangle^{\beta-1/2-2b}} \, \mathrm{d}\ell_{||} \, \mathrm{d}\tau \\
&\lesssim_{\beta} \int_{\mathbb{R}} \frac{1}{|\ell_{||}|^{1/2} \langle k-\ell_{||}\rangle^{\beta-3}} \, \mathrm{d}\ell_{||} \\
&\lesssim_{\beta} 1,
\end{aligned}
$$

which completes the first term in (3.5). For the second term in (3.5) we require $\zeta > \frac{4}{5}$ and $1 + 5b < 4$:

$$
\begin{aligned}
\mathcal{I}_2 &\lesssim \int_{\min(1,t)}^{t} \int_{\mathbb{R}} \frac{|k|^{1/2+5b/2}}{\langle \ell_{||} \rangle^2 \langle k-\ell_{||}, kt-\ell_{||}\tau\rangle^{\beta} \langle \tau \rangle^{5\zeta/2-1}} \, \mathrm{d}\ell_{||} \, \mathrm{d}\tau \\
&\lesssim_{\beta} \int_{\mathbb{R}} \frac{1}{\langle k-\ell_{||}\rangle^{\beta-1/2-3b}} \, \mathrm{d}\ell_{||} \\
&\lesssim_{\beta} 1.
\end{aligned}
$$

This completes the treatment of the resonant region in (3.5).

Turn next to the I_{NR} region. In this region,

$$
|kt - \ell\tau| \gtrsim \tau |\ell_{\perp}| \geq \frac{\tau}{(1+\tau)^{\zeta}} |k|^b.
$$

Therefore, using that $b = \beta^{-1}$

$$\begin{aligned}
&\int_{\min(1,t)}^{t} \int_{I_{NR}} \frac{|k|^{1/2}|\ell|^{1/2}\langle\tau\rangle}{\langle\ell\rangle^2\langle k-\ell, kt-\ell\tau\rangle^{\beta}} \,\mathrm{d}\ell\,\mathrm{d}\tau \\
&\quad\lesssim \int_{\min(1,t)}^{t} \int_{I_{NR}} \frac{\langle\tau\rangle|k|^{1/2}|\ell|^{1/2}}{\langle\ell\rangle^2\langle k-\ell, kt-\ell\tau\rangle^{\beta/2}|k|^{\beta b/2}\langle\tau\rangle^{\beta/2(1-\zeta)}} \,\mathrm{d}\ell\,\mathrm{d}\tau \\
&\quad\lesssim \int_{\min(1,t)}^{t} \int_{I_{NR}} \frac{|\ell|^{1/2}}{\langle\ell\rangle^2\langle k-\ell, kt-\ell\tau\rangle^{\frac{\beta}{2}}\langle\tau\rangle^{\frac{\beta}{2}(1-\zeta)-1}} \,\mathrm{d}\ell\,\mathrm{d}\tau \\
&\quad\lesssim \int_{\min(1,t)}^{t} \int_{\mathbb{R}^3} \frac{|\ell|^{1/2}}{\langle k-\ell, kt-\ell\tau\rangle^{\frac{\beta}{2}-1}\langle\tau\rangle^{\frac{\beta}{2}(1-\zeta)-1}} \,\mathrm{d}\ell\,\mathrm{d}\tau.
\end{aligned}$$

This integral is uniformly bounded provided that (using that the dimension is 3),

$$\frac{\beta}{2} - 1 > 4 \quad \text{and} \quad \frac{\beta}{2}(1-\zeta) > 1,$$

which, using that $1 - \zeta < \frac{1}{5}$ (and otherwise ζ is arbitrary), gives the regularity requirement $\beta > 10$, which is also sufficiently large to satisfy all of the other conditions above as well. □

The next estimate is in some sense the dual of Lemma 3.1, and the proof is analogous.

LEMMA 3.2 (Time response estimate II). *Under the bootstrap hypotheses* (2.8) *there holds*

$$\sup_{\tau\in[0,T^\star]} \sup_{\ell\in\mathbb{R}^d} \int_{\tau}^{T^\star} \int_{\mathbb{R}^d} \bar{K}_{k,\ell}(t,\tau)\mathrm{d}t\,\mathrm{d}k \lesssim \sqrt{K_2}\,\epsilon.$$

PROOF. As above, we eliminate irrelevant early times: for $\beta > 4$ we have

$$\int_{\tau}^{\min(1,T^\star)} \int_{\mathbb{R}^d} \frac{\langle\tau\rangle|k|^{1/2}|\ell|^{1/2}}{\langle\ell\rangle^2\langle k-\ell, kt-\ell\tau\rangle^{\beta}} \,\mathrm{d}t\,\mathrm{d}k \lesssim 1.$$

For a fixed $\ell \in \mathbb{R}^3$ and all $k \in \mathbb{R}^3$ define

$$k_{||} = \frac{k\cdot\ell}{|\ell|^2}\ell \quad \text{and} \quad k_\perp = k - k_{||},$$

the collinear and perpendicular components. Fix as in the proof of Lemma 3.1.

$$\zeta \in (\tfrac{4}{5}, 1), \tag{3.6a}$$

$$b = \beta^{-1}. \tag{3.6b}$$

Define the two subregions:

$$\begin{aligned}
I_R &= I_R(t,\ell) = \{k : |k_\perp| < (1+t)^{-\zeta}|\ell|^b\}, \\
I_{NR} &= I_{NR}(t,\ell) = \{k : |k_\perp| \geq (1+t)^{-\zeta}|\ell|^b\}.
\end{aligned}$$

As above, I_R is cutting out a shrinking cylinder around the line containing ℓ and is restricted to the set of frequencies that can create strong echo cascades over the time interval of interest. Integrating over the two-dimensional disk as above,

$$\begin{aligned}
&\int_{\min(1,T^\star)}^{T^\star}\int_{I_R} \frac{\langle t\rangle |\ell|^{1/2}|k|^{1/2}}{\langle \ell\rangle^2 \langle k-l, kt-l\tau\rangle^\beta}\,\mathrm{d}t\,\mathrm{d}k \\
&\quad\lesssim \int_{\min(1,T^\star)}^{T^\star}\int_{I_R} \frac{\langle t\rangle |\ell|^{1/2}|k|^{1/2}}{\langle \ell\rangle^2 \langle k_{||}-\ell, k_{||}t-\ell\tau\rangle^\beta}\,\mathrm{d}t\,\mathrm{d}k \\
&\quad\lesssim \int_{\min(1,T^\star)}^{T^\star}\int_{\mathbb{R}} \frac{\langle t\rangle |\ell|^{1/2}}{\langle \ell\rangle^2 \langle k_{||}-\ell, k_{||}t-\ell\tau\rangle^\beta}\frac{|\ell|^{2b}}{(1+t)^{2\zeta}}\left(\left|k_{||}\right|^{1/2} + \frac{|\ell|^{b/2}}{(1+t)^{\zeta/2}}\right)\mathrm{d}t\,\mathrm{d}k_{||} \\
&\quad=: \mathcal{I}_1 + \mathcal{I}_2.
\end{aligned} \tag{3.7}$$

Then using that $\zeta > \frac{1}{2}$ and β is sufficiently large (equivalently, b is sufficiently small),

$$\begin{aligned}
\mathcal{I}_1 &\lesssim \int_{\min(1,T^\star)}^{T^\star}\int_{\mathbb{R}} \frac{|k_{||}|^{1/2}}{\langle k_{||}-\ell, k_{||}t-\ell\tau\rangle^{\beta-1}}\,\mathrm{d}t\,\mathrm{d}k_{||} \\
&\lesssim_b \int_{\mathbb{R}} \frac{1}{|k_{||}|^{1/2}\langle k_{||}-\ell\rangle^{\beta-3}}\,\mathrm{d}k_{||} \lesssim_\beta 1,
\end{aligned}$$

For the other contribution in (3.7) we use $\zeta > \frac{4}{5}$ and β is sufficiently large (equivalently, b is sufficiently small),

$$\begin{aligned}
\mathcal{I}_2 &\lesssim \int_{\min(1,T^\star)}^{T^\star}\int_{\mathbb{R}} \frac{1}{\langle k_{||}-\ell, k_{||}t-\ell\tau\rangle^{\beta-1-3b}(1+t)^{5\zeta/2-1}}\,\mathrm{d}t\,\mathrm{d}k_{||} \\
&\lesssim \int_{\mathbb{R}} \frac{1}{\langle k_{||}-\ell\rangle^{\beta-1-3b}}\,\mathrm{d}k_{||} \lesssim 1.
\end{aligned}$$

This completes the treatment of the resonant region in (3.7).

Turn now to the nonresonant I_{NR} region. On the support of the integrand, notice that

$$|kt-\ell\tau| \gtrsim t|k_\perp| \geq \frac{t}{(1+t)^\zeta}|\ell|^b.$$

Recalling the choice $b = \beta^{-1}$ we get

$$\begin{aligned}
&\int_{\min(1,T^\star)}^{T^\star}\int_{I_{NR}} \frac{|k|^{1/2}|\ell|^{1/2}\langle t\rangle}{\langle \ell\rangle^2 \langle k-\ell, kt-\ell\tau\rangle^\beta}\,\mathrm{d}t\,\mathrm{d}k \\
&\quad\lesssim \int_{\min(1,T^\star)}^{T^\star}\int_{I_{NR}} \frac{\langle t\rangle |k|^{1/2}|\ell|^{1/2}}{\langle \ell\rangle^2 \langle k-\ell, kt-\ell\tau\rangle^{\beta/2}|\ell|^{1/2}\langle t\rangle^{\beta/2(1-\zeta)}}\,\mathrm{d}t\,\mathrm{d}k \\
&\quad\lesssim \int_{\min(1,T^\star)}^{T^\star}\int_{I_{NR}} \frac{|k|^{1/2}}{\langle \ell\rangle^{3/2} \langle k-\ell, kt-\ell\tau\rangle^{(\beta-1)/2}\langle t\rangle^{(\beta/2)(1-\zeta)-1}}\,\mathrm{d}t\,\mathrm{d}k.
\end{aligned}$$

This integral is uniformly bounded in ℓ and τ if (using that the dimension is 3),

$$\frac{\beta}{2} - 1 > 4 \quad \text{and} \quad \frac{\beta}{2}(1-\zeta) > 1,$$

as in Lemma 3.1 above. □

4 Nonlinear Energy Estimates on $\rho(t,k)$

4.1 L^2_k Estimates on $\widehat{\rho}$

From (2.4b) and the linearized damping inequality, Proposition 2.3, we have (recall $A_\sigma = |k|^{1/2}\langle k, kt\rangle^\sigma$),

$$\begin{aligned}
&\|A_{\sigma_4}\widehat{\rho}\|^2_{L^2_t(I)} \\
&\lesssim \|A_{\sigma_4}\widehat{h_{\mathrm{in}}}(k,k\cdot)\|^2_{L^2_t(I)} \\
&\quad + \int_0^{T_\star}\left[A_{\sigma_4}(t,k)\int_0^t\int_{\mathbb{R}^3}\widehat{\rho}(\tau,\ell)\widehat{W}(\ell)\ell\cdot k(t-\tau)\widehat{g}(\tau,k-\ell,kt-\ell\tau)\mathrm{d}\tau\,\mathrm{d}\ell\right]^2\mathrm{d}t.
\end{aligned} \tag{4.1}$$

To improve the L^2_k estimate (2.8b), we integrate in k to yield

$$\begin{aligned}
&\|A_{\sigma_4}\widehat{\rho}\|^2_{L^2_tL^2_k(I\times\mathbb{R}^3)} \\
&\lesssim \int_0^{T_\star}\int_{\mathbb{R}^3}|A_{\sigma_4}(k,kt)\widehat{h_{\mathrm{in}}}(k,kt)|^2\,\mathrm{d}t\,\mathrm{d}k \\
&\quad + \int_0^{T_\star}\int_{\mathbb{R}^3}\left[A_{\sigma_4}(t,k)\int_0^t\int_{\mathbb{R}^3}\widehat{\rho}(\tau,\ell)\widehat{W}(\ell)\ell\cdot k(t-\tau)\widehat{g}(\tau,k-\ell,kt-\ell\tau)\mathrm{d}\tau\,\mathrm{d}\ell\right]^2\mathrm{d}t\,\mathrm{d}k.
\end{aligned} \tag{4.2}$$

As in Lemma 2.13, we have

$$\int_0^{T_\star}\int_{\mathbb{R}^3}||k|\langle k,kt\rangle^{2\sigma_4}\widehat{h_{\mathrm{in}}}(k,kt)|^2\,\mathrm{d}t\,\mathrm{d}k \lesssim \epsilon^2. \tag{4.3}$$

It remains to see how to deal with the nonlinear contributions in (4.2). By the triangle inequality and (1.2):

$$\begin{aligned}
&\|A_{\sigma_4}\widehat{\rho}\|^2_{L^2_tL^2_k(I\times\mathbb{R}^3)} \\
&\lesssim \epsilon^2 + \int_0^{T_\star}\int_{\mathbb{R}^3}\left[\int_0^t\int_{\mathbb{R}^3}\langle k-\ell,kt-\ell\tau\rangle^{\sigma_4}|k|\right. \\
&\qquad\qquad \left.\left|\widehat{\rho}(\tau,\ell)\frac{\ell}{\langle\ell\rangle^2}\cdot k(t-\tau)\widehat{g}(\tau,k-\ell,kt-\ell\tau)\right|\mathrm{d}\tau\,\mathrm{d}\ell\right]^2\mathrm{d}t\,\mathrm{d}k \\
&\quad + \int_0^{T_\star}\int_{\mathbb{R}^3}\left[\int_0^t\int_{\mathbb{R}^3}\langle\ell,\ell\tau\rangle^{\sigma_4}|k|\left|\widehat{\rho}(\tau,\ell)\frac{\ell}{\langle\ell\rangle^2}\cdot k(t-\tau)\widehat{g}(\tau,k-\ell,kt-\ell\tau)\right|\mathrm{d}\tau\,\mathrm{d}\ell\right]^2\mathrm{d}t\,\mathrm{d}k \\
&= \epsilon^2 + T + R,
\end{aligned} \tag{4.4}$$

where we refer to T and R as *transport* and *reaction* as they are analogous to the corresponding terms named similarly in [4] (the terminology *reaction* goes back to [30] and *transport* goes back to [3]).

Transport

The purpose of this section is to prove the following:

$$T \lesssim K_2 K_3 \epsilon^4, \tag{4.5}$$

which is consistent with Proposition 2.12 provided ϵ is chosen sufficiently small. By Cauchy-Schwarz,

$$\begin{aligned}
T &\lesssim \int_0^{T^\star}\int_{\mathbb{R}^3}\Big[\int_0^t\int_{\mathbb{R}^3}|k(t-\tau)\langle k-\ell, kt-\ell\tau\rangle^{\sigma_4}\widehat{g}(\tau,k-\ell,kt-\ell\tau)||k|^{1/2} \\
&\qquad\qquad \left|\frac{\ell}{\langle\ell\rangle^2}\widehat{\rho}(\tau,\ell)\right|\mathrm{d}\tau\,\mathrm{d}\ell\Big]^2\mathrm{d}t\,\mathrm{d}k \\
&\lesssim \int_0^{T^\star}\int_{\mathbb{R}^3}\Big[\int_0^t\int_{\mathbb{R}^3}\langle\tau\rangle^{5/2}\frac{|\ell|}{\langle\ell\rangle^2}|\widehat{\rho}(\tau,\ell)|\mathrm{d}\tau\,\mathrm{d}\ell\Big] \\
&\qquad \Big[\int_0^t\int_{\mathbb{R}^3}|k(t-\tau)\langle k-\ell, kt-\ell\tau\rangle^{\sigma_4}\widehat{g}(\tau,k-\ell,kt-\ell\tau)|^2 \\
&\qquad |k|\langle\tau\rangle^{-5/2}\frac{|\ell|}{\langle\ell\rangle^2}|\widehat{\rho}(\tau,\ell)|\mathrm{d}\tau\,\mathrm{d}\ell\Big]\mathrm{d}t.
\end{aligned}$$

Using (2.9),

$$\begin{aligned}
T &\lesssim \sqrt{K_5}\epsilon\int_0^{T^\star}\int_{\mathbb{R}^3}\left(\int_0^t\int_{\mathbb{R}^3}\left|(|k(t-\tau)|\langle k-\ell,kt-\ell\tau\rangle^{\sigma_4}\widehat{g}(\tau,k-\ell,kt-\ell\tau)\right|^2\right. \\
&\qquad \left.|k|\langle\tau\rangle^{-5/2}\frac{|\ell|}{\langle\ell\rangle^2}|\widehat{\rho}(\tau,\ell)|\mathrm{d}\tau\,\mathrm{d}\ell\right)\mathrm{d}t\,\mathrm{d}k \\
&\le \sqrt{K_5}\epsilon\int_0^{T^\star}\int_{\mathbb{R}^3}\left(\iint_{\mathbb{R}\times\mathbb{R}^3}|k|\left|k(t-\tau)\langle k-\ell,kt-\ell\tau\rangle^{\sigma_4}\widehat{g}(\tau,k-\ell,kt-\ell\tau)\right|^2\mathrm{d}t\,\mathrm{d}k\right) \\
&\qquad \langle\tau\rangle^{-5/2}\frac{|\ell|}{\langle\ell\rangle^2}|\widehat{\rho}(\tau,\ell)|\mathrm{d}\tau\,\mathrm{d}\ell \\
&\lesssim K_5\epsilon^2\sup_{\tau\ge 0}\sup_{\ell\in\mathbb{R}^3}\langle\tau\rangle^{-5}\left(\iint_{\mathbb{R}\times\mathbb{R}^3}|k|\big(|(kt-\ell\tau)-\tau(k-\ell)|\langle k-\ell,kt-\ell\tau\rangle^{\sigma_4}\right. \\
&\qquad \left.|\widehat{g}(\tau,k-\ell,kt-\ell\tau)|\big)^2\mathrm{d}t\,\mathrm{d}k\right) \\
&= K_5\epsilon^2\sup_{\tau\ge 0}\sup_{\ell\in\mathbb{R}^3}\langle\tau\rangle^{-5}\left(\iint_{\mathbb{R}\times\mathbb{R}^3}\left|\widehat{(\langle t\nabla_z,\nabla_v\rangle\langle\nabla_{z,v}\rangle^{\sigma_4}g)}\left(\tau,k-\ell,\frac{k}{|k|}\zeta-\ell\tau\right)\right|^2\mathrm{d}\zeta\,\mathrm{d}k\right) \\
&\le K_5\epsilon^2\sup_{\tau\ge 0}\sup_{\ell\in\mathbb{R}^3}\langle\tau\rangle^{-5}\left(\int_{\mathbb{R}^3}\left(\sup_{\omega\in\mathbb{S}^2}\int_{\mathbb{R}}\left|\widehat{(\langle t\nabla_z,\nabla_v\rangle\langle\nabla_{z,v}\rangle^{\sigma_4}g)}(\tau,k-\ell,\omega\zeta-\ell\tau)\right|^2\mathrm{d}\zeta\right)\mathrm{d}k\right) \\
&\le K_5\epsilon^2\sup_{\tau\ge 0}\sup_{\ell\in\mathbb{R}^3}\langle\tau\rangle^{-5}\left(\int_{\mathbb{R}^3}\left(\sup_{y\in\mathbb{R}^3}\sup_{\omega\in\mathbb{S}^2}\int_{\mathbb{R}}\left|\widehat{(\langle t\nabla_z,\nabla_v\rangle\langle\nabla_{x,v}\rangle^{\sigma_4}g)}(\tau,k,\omega\zeta-y)\right|^2\mathrm{d}\zeta\right)\mathrm{d}k\right).
\end{aligned}$$

The inside factor is an L^2 norm along a line in $\mathbb{R}^3$, maximized over all possible lines; therefore, by the Sobolev trace lemma, Lemma 2.15, we have from (2.8a)

$$T \lesssim K_5\epsilon^2\sup_{\tau\ge 0}\langle\tau\rangle^{-5}\sum_{|\alpha|\le M}\|v^\alpha\langle t\nabla_z,\nabla_v\rangle\langle\nabla\rangle^{\sigma_4}g\|_2^2 \lesssim K_5K_1\epsilon^4,$$

as stated in (4.5). By choosing $\epsilon^2 \ll K_1 K_5$, this is consistent with Proposition 2.12.

Reaction

For the reaction term we will prove

$$R \lesssim K_3 \epsilon^2 \|A_{\sigma_4} \widehat{\rho}\|^2_{L^2_t L^2_k}, \tag{4.6}$$

since for ϵ chosen sufficiently small, this contribution can then be absorbed on the left-hand side of (4.4). By (1.2),

$$R \lesssim \int_0^{T^\star} \int_{\mathbb{R}^3} \left[\int_0^t \int_{\mathbb{R}^3} |k|^{1/2} \left| \widehat{g}(\tau, k-\ell, kt - \ell\tau) \frac{|\ell \cdot k(t-\tau)|}{\langle \ell \rangle^2} \langle \ell, \ell\tau \rangle^{\sigma_4} \widehat{\rho}(\tau, \ell) \right| \mathrm{d}\tau\, \mathrm{d}\ell \right]^2 \mathrm{d}t\, \mathrm{d}k.$$

By Schur's test, it follows that

$$\begin{aligned} R \lesssim & \left(\sup_{t \geq 0} \sup_{k \in \mathbb{R}^3} \int_0^t \int_{\mathbb{R}^3} \bar{K}(t, \tau, k, \ell) \mathrm{d}\tau\, \mathrm{d}\ell \right) \\ & \left(\sup_{\tau \geq 0} \sup_{\ell \in \mathbb{R}^3} \int_\tau^{T^\star} \int_{\mathbb{R}^3} \bar{K}(t, \tau, k, \ell) \mathrm{d}t\, \mathrm{d}k \right) \|A_{\sigma_4} \widehat{\rho}\|^2_{L^2_t L^2_k}, \end{aligned} \tag{4.7}$$

where the time-response kernel $\bar{K}(t, \tau, k, \ell)$ is given in (3.2). Therefore, Lemmas 3.1 and 3.2 imply (4.6). Putting the previous estimates together and choosing ϵ sufficiently small implies (2.8b).

4.2 $L^\infty_k L^2_t$ Estimate on $\widehat{\rho}$

Next, it remains to see how we can get the requisite $L^\infty_k L^2_t$ estimate on $\widehat{\rho}$. For this, we will rely on the higher regularity controls (2.8b) and (2.8c). Fix k arbitrary. As in (4.1) above, applying Lemma 2.3 to (2.4b) implies

$$\begin{aligned} & \|A(\cdot, k)_{\sigma_2} \widehat{\rho}(\cdot, k)\|^2_{L^2_t(I)} \\ & \quad \lesssim \left\| |k|^{1/2} \langle k, k\cdot \rangle^{\sigma_2} \widehat{h_{\text{in}}}(k, k\cdot) \right\|^2_{L^2_t(I)} \\ & \qquad + \int_0^{T_\star} \left[|k|^{1/2} \langle k, kt \rangle^{\sigma_2} \int_0^t \int_{\mathbb{R}^3} \widehat{\rho}(\tau, \ell) \widehat{W}(\ell) \ell \cdot k(t-\tau) \widehat{g}(\tau, k-\ell, kt - \ell\tau) \mathrm{d}\tau\, \mathrm{d}\ell \right]^2 \mathrm{d}t \\ & \quad = L(k) + NL(k). \end{aligned}$$

From (2.13) it follows that $L(k) \lesssim \epsilon^2$, so it suffices to consider the nonlinear term. We begin by dividing into two contributions via the triangle inequality:

$$\begin{aligned} NL \lesssim & \int_0^\infty \Bigg(|k|^{1/2} \int_0^t \int_{\mathbb{R}^3} [\langle \ell, \ell\tau \rangle^{\sigma_2} + \langle k-\ell, kt - \ell\tau \rangle^{\sigma_2}] \\ & \qquad \left| \widehat{\rho}(\tau, \ell) \ell \widehat{W}(\ell) \cdot k(t-\tau) \widehat{g}(k-\ell, kt - \ell\tau) \right| \mathrm{d}\tau\, \mathrm{d}\ell \Bigg)^2 \mathrm{d}t \\ \lesssim & \; R + T. \end{aligned}$$

For the R term we start with Cauchy-Schwarz followed by (2.8b) and (2.8e) for some $\alpha > 6$ depending on σ_i:

$$
\begin{aligned}
R &\lesssim \int_0^{+\infty} \left(\int_0^t\!\!\int_{\mathbb{R}^3} \langle \ell, \ell\tau\rangle^{2\sigma_4} |\ell| |\widehat{\rho}(\tau,\ell)|^2 \, \mathrm{d}\tau \, \mathrm{d}\ell \right) \\
&\qquad \left(\int_0^t\!\!\int_{\mathbb{R}^3} \frac{|\ell|}{\langle \ell\rangle^4} \frac{|k|^3 |t-\tau|^2}{\langle \ell, \ell\tau\rangle^{2\sigma_4 - 2\sigma_2}} |\widehat{g}(k-\ell, kt-\ell\tau)|^2 \, \mathrm{d}\tau \, \mathrm{d}\ell \right) \mathrm{d}t \\
&\lesssim K_2 K_5 \epsilon^4 \int_0^{+\infty}\!\!\int_0^t\!\!\int_{\mathbb{R}^3} \frac{|\ell|}{\langle \ell\rangle^4} \frac{|k|^3|t-\tau|^2}{\langle \ell,\ell\tau\rangle^{2\sigma_4-2\sigma_2} \langle k-\ell, kt-\ell\tau\rangle^{2\sigma_1}} \, \mathrm{d}t \, \mathrm{d}\ell \, \mathrm{d}\tau \\
&\lesssim K_2 K_5 \epsilon^4 \int_0^{+\infty}\!\!\int_0^t \frac{|k|^3 |t-\tau|^2}{\langle\tau\rangle \langle k\rangle^4 \langle kt\rangle^{\alpha/2}} \left(\int_{\mathbb{R}^3} \frac{1}{\langle \ell\rangle^3 \langle \ell,\ell\tau\rangle^{\alpha/2}} \, \mathrm{d}\ell \right) \mathrm{d}t \, \mathrm{d}\tau \\
&\lesssim K_2 K_5 \epsilon^4 \int_0^{+\infty}\!\!\int_0^t \frac{|k|^3 |t-\tau|^2}{\langle\tau\rangle \langle k\rangle^4 \langle kt\rangle^{\alpha/2}} \frac{1}{\langle\tau\rangle^{3-2\delta}} \, \mathrm{d}t \, \mathrm{d}\tau \\
&\lesssim K_2 K_5 \epsilon^4.
\end{aligned}
$$

Consider next T, using $\widehat{\rho}(\tau,\ell) = \widehat{g}(\tau,\ell,\ell\tau)$ and (2.8e), followed by Cauchy-Schwarz in ℓ and (2.8c), again for some $\alpha > 3$ depending on the gaps between σ_i:

$$
\begin{aligned}
T &\lesssim K_5 \epsilon^2 \int_0^{+\infty} \Bigg(\int_0^t\!\!\int_{\mathbb{R}^3} \frac{|k|^{1/2}|\ell|}{\langle \ell\rangle^2 \langle \ell,\ell\tau\rangle^{\sigma_1}} |k(t-\tau)| \langle k-\ell, kt-\ell\tau\rangle^{\sigma_2} \\
&\qquad\qquad |\widehat{g}(\tau, k-\ell, kt-\ell\tau)| \mathrm{d}\tau \, \mathrm{d}\ell \Bigg)^2 \mathrm{d}t \\
&\lesssim K_5 K_3 \epsilon^4 \int_0^{+\infty} \frac{|k|}{\langle kt\rangle^\alpha \langle k\rangle} \left(\int_0^t \langle\tau\rangle \left(\int_{\mathbb{R}^3} \frac{|\ell|^2}{\langle \ell\rangle^3 \langle \ell,\ell\tau\rangle^\alpha} \frac{1}{|k-\ell|^{2\delta}} \, \mathrm{d}\ell \right)^{1/2} \mathrm{d}\tau \right)^2 \mathrm{d}t \\
&\lesssim K_5 K_3 \epsilon^4 \int_0^{+\infty} \frac{|k|}{\langle kt\rangle^\alpha \langle k\rangle} \left(\int_0^t \langle\tau\rangle \left(\frac{1}{\langle\tau\rangle^{5-4\delta}} \right)^{1/2} \mathrm{d}\tau \right)^2 \mathrm{d}t \\
&\lesssim K_5 K_3 \epsilon^4 \int_0^{+\infty} \frac{|k|}{\langle kt\rangle^\alpha \langle k\rangle} \, \mathrm{d}t \\
&\lesssim K_5 K_3 \epsilon^4.
\end{aligned}
$$

This completes the estimate $L^\infty_k L^2_t$ estimate on $\widehat{\rho}$.

5 Nonlinear Energy Estimates on g

5.1 High Norm Estimates

Estimate on $\|\langle\nabla_v\rangle g\|_{H^{\sigma_4}_M}$

First consider the velocity high norm estimate on g stated in Proposition 2.12. Let $\alpha \in \mathbb{N}^3$ be a multi-index. An energy estimate yields

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\langle\nabla_v\rangle v^\alpha g\|^2_{H^{\sigma_4}_M}\\
&\quad = -\iint_{\mathbb{R}^3\times\mathbb{R}^3} \langle\eta\rangle\langle k,\eta\rangle^{\sigma_4} D^\alpha_\eta \widehat{g}(k,\eta)\langle\eta\rangle\langle k,\eta\rangle^{\sigma_4}\\
&\qquad\qquad \big(\widehat{\rho}(t,k)\widehat{W}(k)k\cdot D^\alpha_\eta\big((\eta-kt)\widehat{f}^0(\eta-kt)\big)\big)\mathrm{d}k\,\mathrm{d}\eta\\
&\qquad - \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} \langle\eta\rangle\langle k,\eta\rangle^{\sigma_4} D^\alpha_\eta \widehat{g}(k,\eta)\langle\eta\rangle\langle k,\eta\rangle^{\sigma_4}\\
&\qquad\qquad \big(\widehat{\rho}(t,\ell)\widehat{W}(\ell)\ell\cdot D^\alpha_\eta((\eta-kt)\widehat{g}(k-\ell,\eta-\ell t))\big)\mathrm{d}\ell\,\mathrm{d}k\,\mathrm{d}\eta\\
&\quad = L + NL.
\end{aligned}$$

Consider first the linear term L. We have

$$\begin{aligned}
L &\lesssim \int_{\mathbb{R}^3\times\mathbb{R}^3} \langle\eta\rangle\langle k,\eta\rangle^{\sigma_4}\big|D^\alpha_\eta\widehat{g}(k,\eta)\big|\langle kt\rangle\langle k,kt\rangle^{\sigma_4}\frac{|k|}{\langle k\rangle^2}\\
&\qquad |\widehat{\rho}(t,k)|\langle\eta-kt\rangle^{\sigma_4}\big|D^\alpha_\eta((\eta-kt)\widehat{f}^0(\eta-kt))\big|\mathrm{d}k\,\mathrm{d}\eta\\
&\lesssim \int_{\mathbb{R}^3\times\mathbb{R}^3} \langle\eta\rangle\langle k,\eta\rangle^{\sigma_4}\big|D^\alpha_\eta\widehat{g}(k,\eta)\big|\langle k,kt\rangle^{\sigma_4} t|k|^{1/2}\\
&\qquad |\widehat{\rho}(t,k)|\langle\eta-kt\rangle^{\sigma_4}\big|D^\alpha_\eta((\eta-kt)\widehat{f}^0(\eta-kt))\big|\mathrm{d}k\,\mathrm{d}\eta\\
&\lesssim \langle t\rangle\|\langle\nabla_v\rangle v^\alpha g\|_{H^{\sigma_4}_0}\|A_{\sigma_4}\widehat{\rho}\|_{L^2_k}\\
&\lesssim \frac{\delta'}{\langle t\rangle}\|\langle\nabla_v\rangle v^\alpha g\|^2_{H^{\sigma_4}_0} + \frac{\langle t\rangle^3}{\delta'}\|A_{\sigma_4}\widehat{\rho}\|^2_{L^2_k},
\end{aligned}$$

which for δ' sufficiently small depending only on universal constants and f^0 and K_1 sufficiently large depending only on δ' and K_2 is consistent with Proposition 2.12.

Turn next to the nonlinear term NL. It is here where the full $\langle t\rangle^{5/2}$ growth is observed. First, we commute the moments and the differentiation in the transport operator:

$$
\begin{aligned}
NL &= -\int_{\mathbb{R}^3\times\mathbb{R}^3} \langle\nabla_v\rangle\langle\nabla\rangle^{\sigma_4}(v^\alpha g)\langle\nabla_v\rangle\langle\nabla\rangle^{\sigma_4} \\
&\qquad\qquad \big(E(t,z+tv,v)\cdot(\nabla_v - t\nabla_z)(v^\alpha g)\big)\mathrm{d}v\,\mathrm{d}z \\
&\quad - \int_{\mathbb{R}^3\times\mathbb{R}^3} \langle\nabla_v\rangle\langle\nabla\rangle^{\sigma_4}(v^\alpha g)\langle\nabla_v\rangle\langle\nabla\rangle^{\sigma_4} \\
&\qquad\qquad \big(E(t,z+tv,v)\cdot\nabla_v(v^\alpha)g\big)\mathrm{d}v\,\mathrm{d}z \\
&= NL_0 + NL_M.
\end{aligned}
$$

First consider the leading-order NL_0 term. As is standard when treating transport equations, we use the following integration-by-parts trick:

$$
\begin{aligned}
NL_0 = -\int_{\mathbb{R}^3\times\mathbb{R}^3} \langle\nabla_v\rangle\langle\nabla\rangle^{\sigma_4}(v^\alpha g)\big[&\langle\nabla_v\rangle\langle\nabla\rangle^{\sigma_4}\big(E(t,z+tv,v)\cdot(\nabla_v - t\nabla_z)(v^\alpha g)\big) \\
&- E(t,z+tv,v)\cdot(\nabla_v - t\nabla_z)\langle\nabla_v\rangle\langle\nabla\rangle^{\sigma_4}(v^\alpha g)\big]\mathrm{d}v\,\mathrm{d}z.
\end{aligned}
$$

On the frequency side this becomes the following, which we decompose based on which frequencies are dominant:

$$
\begin{aligned}
NL_0 &= -\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} \langle\eta\rangle\langle k,\eta\rangle^{\sigma_4} D_\eta^\alpha \overline{\widehat{g}}(k,\eta)\big(\langle\eta\rangle\langle k,\eta\rangle^{\sigma_4} - \langle\eta - t\ell\rangle\langle k-\ell,\eta-t\ell\rangle^{\sigma_4}\big) \\
&\qquad \big(\widehat{\rho}(t,\ell)\widehat{W}(\ell)\ell\cdot(\eta - kt)D_\eta^\alpha \widehat{g}(k-\ell,\eta-\ell t)\big)\mathrm{d}\ell\,\mathrm{d}k\,\mathrm{d}\eta \\
&= -\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} \big(\mathbf{1}_{|\ell,\ell t|\geq|k-\ell,\eta-t\ell|} + \mathbf{1}_{|\ell,\ell t|\leq|k-\ell,\eta-t\ell|}\big)\langle\eta\rangle\langle k,\eta\rangle^{\sigma_4} D_\eta^\alpha\overline{\widehat{g}}(k,\eta) \\
&\qquad \big(\langle\eta\rangle\langle k,\eta\rangle^{\sigma_4} - \langle\eta-t\ell\rangle\langle k-\ell,\eta-t\ell\rangle^{\sigma_4}\big) \\
&\qquad \big(\widehat{\rho}(t,\ell)\widehat{W}(\ell)\ell\cdot(\eta-kt)D_\eta^\alpha\widehat{g}(k-\ell,\eta-\ell t)\big)\mathrm{d}\ell\,\mathrm{d}k\,\mathrm{d}\eta \\
&= R + T,
\end{aligned}
$$

where the terminology is again *reaction* and *transport* in analogy with (4.4) (and [3]). Consider the leading-order R:

$$
\begin{aligned}
R &\lesssim \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} \mathbf{1}_{|\ell,\ell t|\geq|k-\ell,\eta-t\ell|}\langle\eta\rangle\langle k,\eta\rangle^{\sigma_4}\left|D_\eta^\alpha\widehat{g}(k,\eta)\right|\langle\ell t\rangle\langle\ell,\ell t\rangle^{\sigma_4} \\
&\qquad |\widehat{\rho}(t,\ell)|\frac{|\ell|}{\langle\ell\rangle^2}|\eta - kt||\widehat{g}(k-\ell,\eta-\ell t)|\mathrm{d}\ell\,\mathrm{d}k\,\mathrm{d}\eta \\
&\lesssim \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} \mathbf{1}_{|\ell,\ell t|\geq|k-\ell,\eta-t\ell|}\langle\eta\rangle\langle k,\eta\rangle^{\sigma_4}\left|D_\eta^\alpha\widehat{g}(k,\eta)\right|\frac{\langle\ell t\rangle|\ell t|}{\langle\ell\rangle^2}\langle\ell,\ell t\rangle^{\sigma_4} \\
&\qquad |\widehat{\rho}(t,\ell)|\left|\widehat{(\langle\nabla\rangle g)}(k-\ell,\eta-\ell t)\right|\mathrm{d}\ell\,\mathrm{d}k\,\mathrm{d}\eta \\
&\lesssim \epsilon\langle t\rangle^2\|\langle\nabla_v\rangle v^\alpha g\|_{H_0^{\sigma_4}}\|A_{\sigma_4}\widehat{\rho}\|_{L_k^2} \\
&\lesssim \frac{\delta'}{\langle t\rangle}\|\langle\nabla_v\rangle v^\alpha g\|_{H_0^\sigma}^2 + \epsilon^2\frac{\langle t\rangle^5}{\delta'}\|A_{\sigma_4}\widehat{\rho}\|_{L_k^2}^2,
\end{aligned}
$$

which, for δ' sufficiently small and depending only on universal constants and ϵ sufficiently small, is consistent with Proposition 2.12. This completes the treatment of the reaction term R.

Turn next to the transport term. We use the following inequality, which follows from the mean value theorem and holds on the support of the integrand:

$$
\begin{aligned}
&\langle\eta\rangle\langle k,\eta\rangle^{\sigma} - \langle\eta - t\ell\rangle\langle k-\ell,\eta-t\ell\rangle^{\sigma} \\
&\quad = \langle\eta\rangle(\langle k,\eta\rangle^{\sigma} - \langle k-\ell,\eta-t\ell\rangle^{\sigma}) + (\langle\eta\rangle - \langle\eta-t\ell\rangle)\langle k-\ell,\eta-t\ell\rangle^{\sigma} \\
&\quad \lesssim \langle\eta\rangle\langle k-\ell,\eta-t\ell\rangle^{\sigma-1}|\ell,\ell t| + \langle t\ell\rangle\langle k-\ell,\eta-t\ell\rangle^{\sigma} \\
&\quad \lesssim \langle\ell,\ell t\rangle^{2}\big(\langle\eta-t\ell\rangle\langle k-\ell,\eta-t\ell\rangle^{\sigma-1} + \langle k-\ell,\eta-t\ell\rangle^{\sigma}\big).
\end{aligned}
$$

Applying this to T implies

$$
\begin{aligned}
T \lesssim \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} & \mathbf{1}_{|\ell,\ell t|\le|k-\ell,\eta-t\ell|}\langle\eta\rangle\langle k,\eta\rangle^{\sigma_4}\big|D^{\alpha}_{\eta}\widehat{g}(k,\eta)\big| \\
& \big(\langle\eta-t\ell\rangle\langle k-\ell,\eta-t\ell\rangle^{\sigma_4-1} + \langle k-\ell,\eta-t\ell\rangle^{\sigma_4}\big) \\
& |\eta-kt|\big|D^{\alpha}_{\eta}\widehat{g}(k-\ell,\eta-\ell t)\big||\ell|\langle\ell,\ell t\rangle^{2}|\widehat{\rho}(t,\ell)|\mathrm{d}\ell\,\mathrm{d}k\,\mathrm{d}\eta \\
= T_1 + T_2. &
\end{aligned}
$$

For T_1 we use (2.9):

$$
\begin{aligned}
T_1 &\lesssim \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} \langle\eta\rangle\langle k,\eta\rangle^{\sigma_4}\big|D^{\alpha}_{\eta}\widehat{g}(k,\eta)\big|\langle\eta-t\ell\rangle\langle k-\ell,\eta-t\ell\rangle^{\sigma_4} \\
&\qquad \big|D^{\alpha}_{\eta}\widehat{g}(k-\ell,\eta-\ell t)\big||\ell|\langle t\rangle\langle\ell,\ell t\rangle^{2}|\widehat{\rho}(t,\ell)|\mathrm{d}\ell\,\mathrm{d}k\,\mathrm{d}\eta \\
&\lesssim \|\langle\nabla_v\rangle v^{\alpha} g\|^2_{H^{\sigma}_{M}} \int_{\mathbb{R}^3}|\ell|\langle t\rangle\langle\ell,\ell t\rangle^{2}|\widehat{\rho}(t,\ell)|\mathrm{d}\ell \\
&\lesssim \frac{\epsilon}{\langle t\rangle^{3}}\|\langle\nabla_v\rangle g\|^2_{H^{\sigma}_{M}}.
\end{aligned}
$$

For T_2 we similarly use

$$
\begin{aligned}
T_2 &\lesssim \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} \langle\eta\rangle\langle k,\eta\rangle^{\sigma}\big|D^{\alpha}_{\eta}\widehat{g}(k,\eta)\big|\langle k-\ell,\eta-t\ell\rangle^{\sigma}\langle t(k-\ell),\eta-\ell t\rangle \\
&\qquad \big|D^{\alpha}_{\eta}\widehat{g}(k-\ell,\eta-\ell t)\big||\ell|\langle\ell,\ell t\rangle^{2}|\widehat{\rho}(t,\ell)|\mathrm{d}\ell\,\mathrm{d}k\,\mathrm{d}\eta \\
&\lesssim \frac{\epsilon}{\langle t\rangle^{4}}\|\langle\nabla_v\rangle g\|_{H^{\sigma}_{M}}\|\langle t\nabla_z,\nabla_v\rangle g\|_{H^{\sigma}_{M}},
\end{aligned}
$$

which for ϵ sufficiently small is consistent with Proposition 2.12. This completes the leading-order T term.

Turn next to the moment term NL_M (recall (5.1)), which we divide into high- and low-frequency contributions similar to NL_0:

$$NL_M \lesssim \sum_{|\beta|=|\alpha|-1} \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} \left(\mathbf{1}_{|\ell,\ell t|\geq|k-\ell,\eta-t\ell|} + \mathbf{1}_{|\ell,\ell t|\leq|k-\ell,\eta-t\ell|}\right)\langle\eta\rangle\langle k,\eta\rangle^{\sigma_4}\left|D_\eta^\alpha \widehat{g}(k,\eta)\right| \langle\eta\rangle\langle k,\eta\rangle^{\sigma_4}\left|\widehat{\rho}(t,\ell)\widehat{W}(\ell)\ell D_\eta^\beta \widehat{g}(k-\ell,\eta-\ell t)\right| \mathrm{d}\ell\,\mathrm{d}k\,\mathrm{d}\eta$$

$$= R_M + T_M.$$

The R_M term is treated in essentially the same way as R above:

$$\begin{aligned} R_M &\lesssim \sum_{|\beta|=|\alpha|-1} \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} \mathbf{1}_{|\ell,\ell t|\geq|k-\ell,\eta-t\ell|}\langle\eta\rangle\langle k,\eta\rangle^{\sigma_4}\left|D_\eta^\alpha \widehat{g}(k,\eta)\right| \langle\ell t\rangle\langle\ell,\ell t\rangle^{\sigma_4}|\widehat{\rho}(t,\ell)|\frac{|\ell|}{\langle\ell\rangle^2}\left|D_\eta^\beta \widehat{g}(k-\ell,\eta-\ell t)\right| \mathrm{d}\ell\,\mathrm{d}k\,\mathrm{d}\eta \\ &\lesssim \epsilon\langle t\rangle \|\langle\nabla_v\rangle v^\alpha g\|_{H_0^{\sigma_4}} \|A_{\sigma_4}\widehat{\rho}\|_{L_k^2} \\ &\lesssim \frac{\delta'}{\langle t\rangle}\|\langle\nabla_v\rangle v^\alpha g\|_{H_0^{\sigma_4}}^2 + \frac{\epsilon^2\langle t\rangle^3}{\delta'}\|A_{\sigma_4}\widehat{\rho}\|_{L_k^2}^2, \end{aligned}$$

which for δ' sufficiently small depending only on universal constants and ϵ sufficiently small, is consistent with an improvement to (2.8a). For the T_M treatment, we use the following, applying (2.9):

$$\begin{aligned} T_M &\lesssim \sum_{|\beta|\leq|\alpha|-1} \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} \langle\eta\rangle\langle k,\eta\rangle^{\sigma_4}\left|D_\eta^\alpha \widehat{g}(k,\eta)\right|\frac{|\ell|}{\langle\ell\rangle^2} |\widehat{\rho}(t,\ell)|\langle\eta-t\ell\rangle\langle k-\ell,\eta-t\ell\rangle^{\sigma_4}\left|D_\eta^\beta \widehat{g}(k-\ell,\eta-\ell t)\right| \mathrm{d}\ell\,\mathrm{d}k\,\mathrm{d}\eta \\ &\lesssim \frac{\epsilon}{\langle t\rangle^4}\|\langle\nabla_v\rangle g\|_{H_M^{\sigma_4}} \|\langle t\nabla_z,\nabla_v\rangle g\|_{H_M^{\sigma_4}}, \end{aligned}$$

which is sufficient as in T_2 above. This completes the estimate on $\|\langle\nabla_v\rangle g\|_{H_M^\sigma}$.

Estimate on $\|\langle\nabla_z\rangle g\|_{H_M^{\sigma_4}}$

Next turn to the estimate on $\|\langle\nabla_z\rangle g\|_{H_M^{\sigma_4}}$, which proceeds similarly to that on $\|\langle\nabla_v\rangle g\|_{H_M^{\sigma_4}}$. Let $\alpha\in\mathbb{N}^3$ be a multi-index. An energy estimate yields

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\langle\nabla_z\rangle v^\alpha g\|_{H_M^{\sigma_4}}^2 &= -\int_{\mathbb{R}^3\times\mathbb{R}^3} \langle k\rangle\langle k,\eta\rangle^{\sigma_4} D_\eta^\alpha \widehat{g}(k,\eta)\langle k\rangle\langle k,\eta\rangle^{\sigma_4} \left(\widehat{\rho}(t,k)\widehat{W}(k)k\cdot D_\eta^\alpha((\eta-kt)\widehat{f}^0(\eta-kt))\right)\mathrm{d}k\,\mathrm{d}\eta \\ &\quad - \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} \langle k\rangle\langle k,\eta\rangle^{\sigma_4} D_\eta^\alpha \widehat{g}(k,\eta)\langle k\rangle\langle k,\eta\rangle^{\sigma_4} \left(\widehat{\rho}(t,\ell)\widehat{W}(\ell)\ell\cdot D_\eta^\alpha((\eta-kt)\widehat{g}(k-\ell,\eta-\ell t))\right)\mathrm{d}\ell\,\mathrm{d}k\,\mathrm{d}\eta \\ &= L + NL. \end{aligned}$$

The linear term is treated as in Section 5.1:

$$\begin{aligned}L &\lesssim \int_{\mathbb{R}^3\times\mathbb{R}^3} \langle k\rangle\langle k,\eta\rangle^{\sigma_4}\left|D_\eta^\alpha \widehat{g}(k,\eta)\right|\langle k\rangle\langle k,kt\rangle^{\sigma_4}\frac{|k|}{\langle k\rangle^2} \\ &\qquad |\widehat{\rho}(t,k)|\langle \eta-kt\rangle^{\sigma_4}\left|D_\eta^\alpha((\eta-kt)\widehat{f}^0(\eta-kt))\right|\mathrm{d}k\,\mathrm{d}\eta \\ &\lesssim \int_{\mathbb{R}^3\times\mathbb{R}^3} \langle k\rangle\langle k,\eta\rangle^{\sigma_4}\left|D_\eta^\alpha \widehat{g}(k,\eta)\right|\langle k,kt\rangle^{\sigma_4}|k|^{1/2} \\ &\qquad |\widehat{\rho}(t,k)|\langle \eta-kt\rangle^{\sigma_4}\left|D_\eta^\alpha((\eta-kt)\widehat{f}^0(\eta-kt))\right|\mathrm{d}k\,\mathrm{d}\eta \\ &\lesssim \frac{\delta'}{\langle t\rangle}\|\langle\nabla_v\rangle v^\alpha g\|^2_{H_0^{\sigma_4}} + \frac{\langle t\rangle}{\delta'}\|A_{\sigma_4}\widehat{\rho}\|^2_{L^2_k},\end{aligned}$$

which, for δ' sufficiently small and depending only on universal constants and f^0, is consistent with an improvement on (2.8b) provided K_1 is chosen sufficiently large (depending on δ' and K_2) and K_2 is consistent with an improvement on (2.8c).

As above in Section 5.1, we commute the moments and the differentiation in the transport operator:

$$\begin{aligned}NL &= -\int_{\mathbb{R}^3\times\mathbb{R}^3}\langle\nabla_z\rangle\langle\nabla\rangle^{\sigma_4}(v^\alpha g)\langle\nabla_z\rangle\langle\nabla\rangle^{\sigma_4}(E(t,z+tv,v)\cdot(\nabla_v - t\nabla_z)(v^\alpha g))\mathrm{d}v\,\mathrm{d}z \\ &\quad -\int_{\mathbb{R}^3\times\mathbb{R}^3}\langle\nabla_z\rangle\langle\nabla\rangle^{\sigma_4}(v^\alpha g)\langle\nabla_z\rangle\langle\nabla\rangle^{\sigma_4}(E(t,z+tv,v)\cdot\nabla_v(v^\alpha)g)\mathrm{d}v\,\mathrm{d}z \\ &= NL_0 + NL_M.\end{aligned} \tag{5.1}$$

First consider the leading-order NL_0 term, which we begin as above with an integration by parts and subdivide based on which frequencies are dominant:

$$\begin{aligned}NL_0 &= -\int_{\mathbb{R}^3\times\mathbb{R}^3}\langle\nabla_z\rangle\langle\nabla\rangle^{\sigma_4}(v^\alpha g)\big[\langle\nabla_z\rangle\langle\nabla\rangle^{\sigma_4}(E(t,z+tv,v)\cdot(\nabla_v-t\nabla_z)(v^\alpha g)) \\ &\qquad\qquad - E(t,z+tv,v)\cdot(\nabla_v-t\nabla_z)\langle\nabla_z\rangle\langle\nabla\rangle^{\sigma_4}(v^\alpha g)\big]\mathrm{d}v\,\mathrm{d}z \\ &= -\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3}\big(\mathbf{1}_{|\ell,\ell t|\geq|k-\ell,\eta-t\ell|} + \mathbf{1}_{|\ell,\ell t|\leq|k-\ell,\eta-t\ell|}\big)\langle k\rangle\langle k,\eta\rangle^{\sigma_4}D_\eta^\alpha\overline{\widehat{g}}(k,\eta) \\ &\qquad (\langle k\rangle\langle k,\eta\rangle^{\sigma_4} - \langle k-\ell\rangle\langle k-\ell,\eta-t\ell\rangle^{\sigma_4}) \\ &\qquad \big(\widehat{\rho}(t,\ell)\widehat{W}(\ell)\ell\cdot(\eta-kt)D_\eta^\alpha\widehat{g}(k-\ell,\eta-\ell t)\big)\mathrm{d}\ell\,\mathrm{d}k\,\mathrm{d}\eta \\ &= R + T.\end{aligned}$$

The reaction R is treated similarly to the treatment in Section 5.1 (note that there is one less power of t lost),

$$R \lesssim \epsilon\langle t\rangle\|\langle\nabla_v\rangle v^\alpha g\|_{H_0^{\sigma_4}}\|A_{\sigma_4}\widehat{\rho}\|_{L^2_k} \lesssim \frac{\delta'}{\langle t\rangle}\|\langle\nabla_v\rangle v^\alpha g\|^2_{H_0^{\sigma_4}} + \epsilon^2\frac{\langle t\rangle^3}{\delta'}\|A_{\sigma_4}\widehat{\rho}\|^2_{L^2_k},$$

which for δ' sufficiently small depending only on universal constants is consistent with an improvement on (2.8a).

Similar to the analogous estimate in Section 5.1, we have

$$T \lesssim \int \mathbf{1}_{|\ell,\ell t| \le |k-\ell,\eta - t\ell|} \langle k \rangle \langle k,\eta\rangle^{\sigma_4} \left| D^\alpha_\eta \widehat{g}(k,\eta)\right| \\ \left(\langle k-\ell\rangle\langle k-\ell,\eta - t\ell\rangle^{\sigma_4 - 1} + \langle k-\ell,\eta-t\ell\rangle^{\sigma_4}\right) \\ |\eta - kt| \left| D^\alpha_\eta \widehat{g}(k-\ell,\eta-\ell t)\right| |\ell| \langle \ell,\ell t\rangle^2 |\widehat{\rho}(t,\ell)| \mathrm{d}\ell \, \mathrm{d}k \, \mathrm{d}\eta \\ = T_1 + T_2.$$

The first term, T_1, is treated as above.

$$T_1 \lesssim \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} \langle k \rangle \langle k,\eta\rangle^{\sigma_4} \left| D^\alpha_\eta \widehat{g}(k,\eta)\right| \langle k-\ell\rangle \langle k-\ell,\eta - t\ell\rangle^{\sigma_4} \\ \left| D^\alpha_\eta \widehat{g}(k-\ell,\eta-\ell t)\right| |\ell| \langle t \rangle \langle \ell,\ell t\rangle^2 |\widehat{\rho}(t,\ell)| \mathrm{d}\ell \, \mathrm{d}k \, \mathrm{d}\eta \\ \lesssim \| \langle \nabla_z \rangle v^\alpha g\|^2_{H^\sigma_0} \int_{\mathbb{R}^3} |\ell| \langle t \rangle \langle \ell,\ell t\rangle^2 |\widehat{\rho}(t,\ell)| \mathrm{d}\ell \\ \lesssim \frac{\epsilon}{\langle t \rangle^3} \| \langle \nabla_z \rangle f\|^2_{H^\sigma_M}.$$

The second is treated with a slight variation:

$$T_2 \lesssim \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} \langle k \rangle \langle k,\eta\rangle^{\sigma} \left| D^\alpha_\eta \widehat{g}(k,\eta)\right| \langle k-\ell,\eta - t\ell\rangle^{\sigma} \langle t(k-\ell), \eta - \ell t\rangle \\ \left| D^\alpha_\eta \widehat{g}(k-\ell,\eta-\ell t)\right| |\ell| \langle \ell,\ell t\rangle^2 |\widehat{\rho}(t,\ell)| \mathrm{d}\ell \, \mathrm{d}k \, \mathrm{d}\eta \\ \lesssim \frac{\epsilon}{\langle t \rangle^4} \| \langle \nabla_z \rangle f\|_{H^\sigma_M} \| \langle t \nabla_z, \nabla_v \rangle f\|_{H^\sigma_M},$$

which is still consistent with the final estimate provided $\delta < \frac{1}{2}$.

The lower-order moment term NL_M is treated as in Section 5.1 and is omitted here for brevity. After collecting all the above estimates and choosing ϵ small, this completes the improvement of (2.8a).

5.2 The $L^\infty_t H^{\sigma_3}_M$ Estimate

In this section we improve the estimate (2.8c) as stated in Proposition 2.12. For $\alpha \in \mathbb{N}^d$, an energy estimate yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \left\| |\partial_z|^\delta \langle \nabla \rangle^{\sigma_3} v^\alpha g \right\|^2_{L^2} \\ = - \int_{\mathbb{R}^3\times\mathbb{R}^3} |k|^\delta \langle k,\eta\rangle^{\sigma_3} D^\alpha_\eta \overline{\widehat{g}}(k,\eta) |k|^\delta \langle k,\eta\rangle^{\sigma_3} \\ \widehat{\rho}(t,k)\widehat{W}(k) k \cdot D^\alpha_\eta ((\eta - kt)\widehat{f}^0(\eta - kt)) \mathrm{d}k \, \mathrm{d}\eta \\ - \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} |k|^\delta \langle k,\eta\rangle^{\sigma_3} D^\alpha_\eta \overline{\widehat{g}}(k,\eta) |k|^\delta \langle k,\eta\rangle^{\sigma_3} \\ \left(\widehat{\rho}(t,\ell)\widehat{W}(\ell)\ell \cdot D^\alpha_\eta ((\eta - kt)\widehat{g}(k-\ell,\eta-\ell t))\right) \mathrm{d}\ell \, \mathrm{d}k \, \mathrm{d}\eta \\ = L + NL.$$

The linear term L is treated as follows:

$$\begin{aligned} L &\lesssim \int_{\mathbb{R}^3\times\mathbb{R}^3} |k|^\delta \langle k,\eta\rangle^{\sigma_3} \left|D_\eta^\alpha \widehat{g}(k,\eta)\right| \langle k,kt\rangle^{\sigma_3} \frac{|k|^{1+\delta}}{\langle k\rangle^2} \\ &\qquad |\widehat{\rho}(t,k)| \langle \eta-kt\rangle^{\sigma_3} \left|D_\eta^\alpha((\eta-kt)\widehat{f}^0(\eta-kt))\right| \mathrm{d}k\,\mathrm{d}\eta \\ &\lesssim \int_{\mathbb{R}^3\times\mathbb{R}^3} |k|^\delta \langle k,\eta\rangle^{\sigma_3} \left|D_\eta^\alpha \widehat{g}(k,\eta)\right| \frac{|k|^{1+\delta}}{\langle k\rangle^2 \langle k,kt\rangle^{\sigma_4-\sigma_3}} \langle k,kt\rangle^{\sigma_4} |\widehat{\rho}(t,k)| \\ &\qquad \langle \eta-kt\rangle^{\sigma_3} \left|D_\eta^\alpha((\eta-kt)\widehat{f}^0(\eta-kt))\right| \mathrm{d}k\,\mathrm{d}\eta \\ &\lesssim \frac{1}{\langle t\rangle^{1/2+\delta}} \int_{\mathbb{R}^3\times\mathbb{R}^3} |k|^\delta \langle k,\eta\rangle^{\sigma_3} \left|D_\eta^\alpha \widehat{g}(k,\eta)\right| \langle k,kt\rangle^{\sigma_4} |k|^{1/2} \\ &\qquad |\widehat{\rho}(t,k)| \langle \eta-kt\rangle^{\sigma_3} \left|D_\eta^\alpha((\eta-kt)\widehat{f}^0(\eta-kt))\right| \mathrm{d}k\,\mathrm{d}\eta \\ &\lesssim \frac{1}{\langle t\rangle^{1/2+\delta}} \left\| |\partial_z|^\delta g \right\|_{H_M^{\sigma_3}} \|A_{\sigma_4}\widehat{\rho}\|_{L_k^2}, \end{aligned}$$

which is sufficient to deduce $\||\partial_z|^\delta g\|_{H_M^{\sigma_3}} \lesssim_\delta K_2\epsilon^2$ via (2.8b). This is consistent with an improvement of (2.8c) by choosing K_3 sufficiently large relative to K_2 (see Remark 2.10).

As in Section 5.1, we begin the nonlinear estimate by commuting the moments and the differentiation:

$$\begin{aligned} NL &= -\int_{\mathbb{R}^3\times\mathbb{R}^3} |\partial_z|^\delta \langle\nabla\rangle^{\sigma_3}\big(v^\alpha g\big) |\partial_z|^\delta \langle\nabla\rangle^{\sigma_3} \\ &\qquad [E(t,z+tv,v)\cdot(\nabla_v - t\nabla_z)(v^\alpha g)]\mathrm{d}v\,\mathrm{d}z \\ &\quad -\int_{\mathbb{R}^3\times\mathbb{R}^3} |\partial_z|^\delta \langle\nabla\rangle^{\sigma_3}\big(v^\alpha g\big) |\partial_z|^\delta \langle\nabla\rangle^{\sigma_3} \\ &\qquad (E(t,z+tv,v)\cdot\nabla_v(v^\alpha)g)\mathrm{d}v\,\mathrm{d}z \\ &= NL_0 + NL_M. \end{aligned} \tag{5.2}$$

For the leading-order term, as above in Section 5.1, we use the following via integration by parts and subdividing based on frequency:

$$\begin{aligned} NL_0 &= -\int_{\mathbb{R}^3\times\mathbb{R}^3} |\partial_z|^\delta \langle\nabla\rangle^{\sigma_3}(v^\alpha g)\big[|\partial_z|^\delta \langle\nabla\rangle^{\sigma_3}(E(t,z+tv,v)\cdot(\nabla_v - t\nabla_z)(v^\alpha g)) \\ &\qquad\qquad \big[-E(t,z+tv,v)\cdot(\nabla_v - t\nabla_z)|\partial_z|^\delta\langle\nabla\rangle^{\sigma_3}(v^\alpha g)\big]\mathrm{d}v\,\mathrm{d}z \\ &= -\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} \big(\mathbf{1}_{|\ell,\ell t|\geq|k-\ell,\eta-t\ell|} + \mathbf{1}_{|\ell,\ell t|\leq|k-\ell,\eta-t\ell|}\big)|k|^\delta\langle k,\eta\rangle^{\sigma_3} D_\eta^\alpha \overline{\widehat{g}}(k,\eta) \\ &\qquad \big(|k|^\delta\langle k,\eta\rangle^{\sigma_3} - |k-\ell|^\delta\langle k-\ell,\eta-t\ell\rangle^{\sigma_3}\big) \\ &\qquad \big(\widehat{\rho}(t,\ell)\widehat{W}(\ell)\ell\cdot(\eta-kt)D_\eta^\alpha\widehat{g}(k-\ell,\eta-\ell t)\big)\mathrm{d}\ell\,\mathrm{d}k\,\mathrm{d}\eta \\ &= R + T. \end{aligned}$$

In the reaction term R, we use that on the support of the integrand there holds (using $\delta < 1$),

$$\left| |k|^{\delta} \langle k, \eta \rangle^{\sigma_3} - |k-\ell|^{\delta} \langle k-\ell, \eta - t\ell \rangle^{\sigma_3} \right| \lesssim (|\ell|^{\delta} + |k-\ell|^{\delta}) \langle \ell, \ell t \rangle^{\sigma_3}. \tag{5.3}$$

Hence

$$\begin{aligned} R \lesssim & \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} |k|^{\delta} \langle k, \eta \rangle^{\sigma_3} \left| D_\eta^\alpha \widehat{g}(k,\eta) \right| \frac{|\ell|^{1+\delta}}{\langle \ell \rangle^2} \langle \ell, \ell t \rangle^{\sigma_3} |\widehat{\rho}(t,\ell)| \\ & \qquad (|\eta - \ell t| + t|k-\ell|) \left| D_\eta^\alpha \widehat{g}(k-\ell, \eta - \ell t) \right| \mathrm{d}\ell \, \mathrm{d}k \, \mathrm{d}\eta \\ & + \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} |k|^{\delta} \langle k, \eta \rangle^{\sigma_3} \left| D_\eta^\alpha \widehat{g}(k,\eta) \right| \frac{|\ell|}{\langle \ell \rangle^2} \langle \ell, \ell t \rangle^{\sigma_3} |\widehat{\rho}(t,\ell)| \\ & \qquad |k-\ell|^{\delta} (|\eta - \ell t| + t|k-\ell|) \left| D_\eta^\alpha \widehat{g}(k-\ell, \eta - \ell t) \right| \mathrm{d}\ell \, \mathrm{d}k \, \mathrm{d}\eta \\ = & \; R_{1;V} + R_{1;Z} + R_{2;V} + R_{2;Z}, \end{aligned}$$

where the subdivisions $R_{i;V}$ versus $R_{i;Z}$ denote the terms involving $|\eta - \ell t|$ and $t|k-\ell|$, respectively. The first contribution we treat in a manner analogous to the treatment of the L-term above (using $|\ell|^{1/2+\delta} \langle \ell, \ell t \rangle^{-1/2-\delta} \lesssim \langle t \rangle^{-1/2-\delta}$, Lemma 2.16, and Cauchy-Schwarz):

$$\begin{aligned} R_{1;V} \lesssim & \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} |k|^{\delta} \langle k, \eta \rangle^{\sigma_3} \left| D_\eta^\alpha \widehat{g}(k,\eta) \right| \frac{|\ell|^{1/2}}{\langle \ell \rangle^2 \langle t \rangle^{1/2+\delta}} \langle \ell, \ell t \rangle^{\sigma_3} \\ & \qquad \left| \widehat{\rho}(t,\ell) \right| \eta - \ell t \left| D_\eta^\alpha \widehat{g}(k-\ell, \eta - \ell t) \right| \mathrm{d}\ell \, \mathrm{d}k \, \mathrm{d}\eta \\ \lesssim & \frac{1}{\langle t \rangle^{1/2+\delta}} \left\| |\partial_z|^{\delta} g \right\|_{H^{\sigma_3}_M} \left\| |\partial_z|^{1/2} \langle \partial_z, \partial_z t \rangle^{\sigma_3} \rho \right\|_{L^2} \int_{\mathbb{R}^3} \left\| \langle \eta \rangle D_\eta^\alpha \widehat{g}(t,\ell,\cdot) \right\|_{L^2_\eta} \mathrm{d}\ell \\ \lesssim & \frac{1}{\langle t \rangle^{1/2+\delta}} \left\| |\partial_z|^{\delta} g \right\|_{H^{\sigma_3}_M}^2 \left\| |\partial_z|^{1/2} \langle \partial_z, \partial_z t \rangle^{\sigma_3} \rho \right\|_{L^2}. \end{aligned}$$

This estimate is sufficient to improve (2.8c) for $\delta > 0$ and ϵ sufficiently small by (2.8b).

Turn next to R_Z, which is treated with a slight variation (using (2.8c)):

$$\begin{aligned} R_{1;Z} \lesssim & \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} |k|^{\delta} \langle k, \eta \rangle^{\sigma_3} \left| D_\eta^\alpha \widehat{g}(k,\eta) \right| \frac{|\ell|^{1+\delta} t}{\langle \ell \rangle^2} \langle \ell, \ell t \rangle^{\sigma_3} \\ & \qquad |\widehat{\rho}(t,\ell)| |\eta - \ell t| |k-\ell| D_\eta^\alpha \widehat{g}(k-\ell, \eta - \ell t) | \mathrm{d}\ell \, \mathrm{d}k \, \mathrm{d}\eta \\ \lesssim & \; t \left\| |\partial_z|^{\delta} g \right\|_{H^{\sigma_3}_M}^2 \int_{\mathbb{R}^3} \frac{|\ell|^{1+\delta}}{\langle \ell \rangle^2} \langle \ell, \ell t \rangle^{\sigma_3} |\widehat{\rho}(t,\ell)| \mathrm{d}\ell \\ \lesssim & \; t \left\| |\partial_z|^{\delta} g \right\|_{H^{\sigma_3}_M}^2 \left(\int_{\mathbb{R}^3} \frac{|\ell|^{1+2\delta}}{\langle \ell \rangle^4} \langle \ell, \ell t \rangle^{-2\sigma_4 + 2\sigma_3} \mathrm{d}\ell \right)^{1/2} \left\| |\partial_z|^{1/2} \langle \partial_z, \partial_z t \rangle^{\sigma_4} \rho \right\|_{L^2} \\ \lesssim & \frac{1}{\langle t \rangle^{1+\delta}} \left\| |\partial_z|^{\delta} g \right\|_{H^{\sigma_3}_M}^2 \left\| |\partial_z|^{1/2} \langle \partial_z, \partial_z t \rangle^{\sigma_4} \rho \right\|_{L^2}, \end{aligned} \tag{5.4}$$

which is sufficient to improve (2.8c) for ϵ sufficiently small. The terms $R_{2;Z} + R_{2;V}$ are treated in the same manner as $R_{1;Z}$; the details are omitted for brevity:

$$R_{1;V} + R_{1;Z} \lesssim \frac{1}{\langle t \rangle} \left\| |\partial_z|^{\delta} g \right\|_{H^{\sigma_3}_M}^2 \left\| |\partial_z|^{1/2} \langle \partial_z, \partial_z t \rangle^{\sigma_4} \rho \right\|_{L^2},$$

which is sufficient to improve (2.8c) for ϵ sufficiently small.

Turn next to the transport term T. On the support of the integrand there holds (from the mean value theorem)

$$\big||k|^{\delta}\langle k,\eta\rangle^{\sigma_3} - |k-\ell|^{\delta}\langle k-\ell,\eta-t\ell\rangle^{\sigma_3}\big| \lesssim$$
$$|k-\ell|^{\delta}|\ell,\ell t|\langle k-\ell,\eta-t\ell\rangle^{\sigma_3-1} + \langle k-\ell,\eta-t\ell\rangle^{\sigma_3}\big||k|^{\delta}-|k-\ell|^{\delta}\big|.$$

Therefore,

$$\begin{aligned} T &\lesssim \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} |k|^{\delta}\langle k,\eta\rangle^{\sigma_3}\big|D_\eta^\alpha \widehat{g}(k,\eta)\big|\frac{|\ell||\ell,\ell t|}{\langle\ell\rangle^2}|\widehat{\rho}(t,\ell)||\eta-kt| \\ &\qquad |k-\ell|^{\delta}\langle k-\ell,\eta-\ell t\rangle^{\sigma_3-1}\big|D_\eta^\alpha\widehat{g}(k-\ell,\eta-\ell t)\big|\mathrm{d}\ell\,\mathrm{d}k\,\mathrm{d}\eta \\ &+ \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} |k|^{\delta}\langle k,\eta\rangle^{\sigma_3}\big|D_\eta^\alpha \widehat{g}(k,\eta)\big|\frac{|\ell|}{\langle\ell\rangle^2}|\widehat{\rho}(t,\ell)| \\ &\qquad |\eta-kt|\big||k|^{\delta}-|k-\ell|^{\delta}\big|\langle k-\ell,\eta-\ell t\rangle^{\sigma_3}\big|D_\eta^\alpha\widehat{g}(k-\ell,\eta-\ell t)\big|\mathrm{d}\ell\,\mathrm{d}k\,\mathrm{d}\eta \\ &= T_1 + T_2. \end{aligned}$$

First consider T_1. Using $|\eta-kt| \lesssim \langle t\rangle\langle k-\ell,\eta-t\ell\rangle$ and (2.9), there holds

$$\begin{aligned} T_1 &\lesssim \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} |k|^{\delta}\langle k,\eta\rangle^{\sigma_3}\big|D_\eta^\alpha \widehat{g}(k,\eta)\big|\frac{|\ell|\langle t\rangle|\ell,\ell t|}{\langle\ell\rangle^2} \\ &\qquad |\widehat{\rho}(t,\ell)||k-\ell|^{\delta}\langle k-\ell,\eta-\ell t\rangle^{\sigma_3}\big|D_\eta^\alpha\widehat{g}(k-\ell,\eta-\ell t)\big|\mathrm{d}\ell\,\mathrm{d}k\,\mathrm{d}\eta \\ &\lesssim \big\||\partial_z|^{\delta}g\big\|_{H_M^{\sigma_3}}^2 \int_{\mathbb{R}^3}\frac{|\ell|\langle t\rangle|\ell,\ell t|}{\langle\ell\rangle^2}|\rho(t,\ell)|\mathrm{d}\ell \\ &\lesssim \frac{\epsilon}{\langle t\rangle^3}\Big\||\partial_z|^{\delta}g\Big\|_{H_M^{\sigma_3}}^2. \end{aligned}$$

For T_2 we instead have the following, using $||k|^{\delta}-|k-\ell|^{\delta}| \leq |\ell|^{\delta}$ and (2.9):

$$\begin{aligned} T_2 &\lesssim \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} |k|^{\delta}\langle k,\eta\rangle^{\sigma_3}\big|D_\eta^\alpha \widehat{g}(k,\eta)\big|\frac{|\ell|^{1+\delta}}{\langle\ell\rangle^2}|\widehat{\rho}(t,\ell)| \\ &\qquad |\eta-kt|\langle k-\ell,\eta-\ell t\rangle^{\sigma_3}\big|D_\eta^\alpha\widehat{g}(k-\ell,\eta-\ell t)\big|\mathrm{d}\ell\,\mathrm{d}k\,\mathrm{d}\eta \\ &\lesssim \big\||\partial_z|^{\delta}f\big\|_{H_M^{\sigma_3}}\big(\langle t\rangle^{-5/2}\|\langle\nabla_z t,\nabla_v\rangle g\|_{H_M^{\sigma_4}}\big)\langle t\rangle^{5/2}\int_{\mathbb{R}^3}\frac{|\ell|^{1+\delta}}{\langle\ell\rangle^2}|\rho(t,\ell)|\mathrm{d}\ell \\ &\lesssim \frac{\epsilon}{\langle t\rangle^{3/2+\delta}}\big\||\partial_z|^{\delta}g\big\|_{H_M^{\sigma_3}}\big(\langle t\rangle^{-5/2}\|\langle\nabla_z t,\nabla_v\rangle g\|_{H_M^{\sigma_4}}\big), \end{aligned}$$

which suffices to improve (2.8c) for ϵ sufficiently small by (2.8a).

Turn to the lower-order moments. We divide the treatment into reaction and transport as above:

$$\begin{aligned} NL_M \leq \sum_{|\beta|=|\alpha|-1} \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} & \left(\mathbf{1}_{|\ell,\ell t|\geq|k-\ell,\eta-t\ell|} + \mathbf{1}_{|\ell,\ell t|\leq|k-\ell,\eta-t\ell|}\right) \\ & |k|^\delta \langle k,\eta\rangle^{\sigma_3} \left|D_\eta^\alpha \widehat{g}(k,\eta)\right| \\ & |k|^\delta \langle k,\eta\rangle^{\sigma_3} \left|\widehat{\rho}(t,\ell)\widehat{W}(\ell)\ell D_\eta^\beta \widehat{g}(k-\ell,\eta-\ell t)\right| \mathrm{d}\ell\, \mathrm{d}k\, \mathrm{d}\eta \\ = R_M + T_M. & \end{aligned}$$

For the R_M term, we may treat as $R_{1;V}$ and $R_{1;Z}$ above. Indeed, using (5.3) and treating the resulting two terms as in (5.4) and (5.4), respectively,

$$\begin{aligned} R_M &\lesssim \sum_{|\beta|=|\alpha|-1} \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} |k|^\delta \langle k,\eta\rangle^{\sigma_3} \left|D_\eta^\alpha \widehat{g}(k,\eta)\right| \frac{|\ell|^{1+\delta} + |\ell||k-\ell|^\delta}{\langle \ell\rangle^2} \langle \ell,\ell t\rangle^{\sigma_4} \\ &\qquad |\widehat{\rho}(t,\ell)| \left|D_\eta^\beta \widehat{g}(k-\ell,\eta-\ell t)\right| \mathrm{d}\ell\, \mathrm{d}k\, \mathrm{d}\eta \\ &\lesssim \frac{1}{\langle t\rangle^{1/2+\delta}} \left\| |\partial_z|^\delta g \right\|_{H_M^{\sigma_3}}^2 \left\| |\partial_z|^{1/2} \langle \partial_z, \partial_z t\rangle^{\sigma_4} \rho \right\|_{L^2}, \end{aligned}$$

which suffices to improve (2.8c) for ϵ sufficiently small by (2.8b). For T_M we can use a treatment as in T_2:

$$\begin{aligned} T_M &\lesssim \sum_{|\beta|=|\alpha|-1} \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} |k|^\delta \langle k,\eta\rangle^{\sigma_3} \left|D_\eta^\alpha \widehat{g}(k,\eta)\right| \frac{|\ell|^{1+\delta}}{\langle \ell\rangle^2} \\ &\qquad |\widehat{\rho}(t,\ell)| \langle k-\ell,\eta-\ell t\rangle^{\sigma_3} \left|D_\eta^\beta \widehat{g}(k-\ell,\eta-\ell t)\right| \mathrm{d}\ell\, \mathrm{d}k\, \mathrm{d}\eta \\ &\lesssim \left\| |\partial_z|^\delta g \right\|_{H_M^{\sigma_3}} \left(\langle t\rangle^{-5/2} \| \langle \nabla_z t, \nabla_v\rangle g\|_{H_M^{\sigma_3}}\right) \langle t\rangle^{5/2} \int_{\mathbb{R}^3} \frac{|\ell|^{1+\delta}}{\langle \ell\rangle^2} |\rho(t,\ell)| \mathrm{d}\ell \\ &\lesssim \frac{\epsilon}{\langle t\rangle^{3/2+\delta}} \left\| |\partial_z|^\delta g \right\|_{H_M^{\sigma_3}} \left(\langle t\rangle^{-5/2} \| \langle \nabla_z t, \nabla_v\rangle g\|_{H_M^{\sigma_4}}\right), \end{aligned}$$

which, as above, is sufficient to improve (2.8c) for ϵ sufficiently small.

5.3 The $L_t^\infty L_{k,\eta}^\infty$ Estimate

In this section we improve (2.8e). Integrating (2.4a) gives

$$\begin{aligned} &\langle k,\eta\rangle^{\sigma_1} |\widehat{g}(T,k,\eta)| \\ &\quad \leq \langle k,\eta\rangle^{\sigma_1} \left|\widehat{h_{\mathrm{in}}(k,\eta)}\right| + \langle k,\eta\rangle^{\sigma_1} \int_0^T \left|\widehat{\rho}(t,k) k W(k)\cdot(\eta-kt)\widehat{f}^0(\eta-kt)\right| \mathrm{d}t \\ &\qquad + \langle k,\eta\rangle^{\sigma_1} \int_0^T \int_{\mathbb{R}^3} |\widehat{\rho}(t,\ell)\ell W(\ell)\cdot(\eta-kt)\widehat{g}(t,k-\ell,\eta-\ell t)| \mathrm{d}\ell\, \mathrm{d}t \\ &\quad = I + L + NL. \end{aligned}$$

For the linear term we have (using $|\widehat{f}^0(\eta - kt)| \lesssim \langle \eta - kt\rangle^{-\sigma_1-3/2}$)

$$L \lesssim \left(\int_0^T \langle k, kt\rangle^{2\sigma_2} |\widehat{\rho}(t,k)|^2 |k| \mathrm{d}t\right)^{1/2}$$
$$\left(\int_0^T |k| \langle \eta - kt\rangle^{2\sigma_1} |W(k)(\eta - kt)|^2 \left|\widehat{f}^0(\eta - kt)\right|^2 \mathrm{d}t\right)^{1/2}$$
$$\lesssim \epsilon.$$

For the nonlinear term, we use a more sophisticated estimate. Write

$$NL \lesssim \int_0^T\int_{\mathbb{R}^3} |k|^\delta \left(\langle \ell, \ell t\rangle^{\sigma_1} + \langle k-\ell, \eta - \ell t\rangle^{\sigma_1}\right)$$
$$|\widehat{\rho}(t,\ell)| \frac{|\ell|}{\langle \ell\rangle^2} |\eta - kt| |\widehat{g}(t, k-\ell, \eta - \ell t)| \mathrm{d}t\, \mathrm{d}\ell$$
$$= NL_{HL} + NL_{LH}.$$

The easier is NL_{HL}, which is handled via the following by (2.8d) and (2.8e) (also using that $\sigma_2 - \sigma_1$ and σ_1 are sufficiently large),

$$NL_{HL} \lesssim \int_{\mathbb{R}^3} \left(\int_0^T |\rho(t,\ell)|^2 |\ell| \langle \ell, \ell t\rangle^{2\sigma_2} \mathrm{d}t\right)^{1/2}$$
$$\left(\int_0^T \frac{|\ell| |\eta - kt|^2}{\langle \ell\rangle^4 \langle \ell, \ell t\rangle^{2(\sigma_2-\sigma_1)} \langle k-\ell, \eta - t\ell\rangle^{2\sigma_1}}\right.$$
$$\left.\langle k-\ell, \eta - t\ell\rangle^{2\sigma_1} |\widehat{g}(t, k-\ell, \eta - t\ell)|^2 \mathrm{d}t\right)^{1/2} \mathrm{d}\ell$$
$$\lesssim \epsilon^2 \int_{\mathbb{R}^3} \left(\int_0^T \frac{|\ell| |\eta - kt|^2}{\langle \ell\rangle^4 \langle \ell, \ell t\rangle^{2(\sigma_2-\sigma_1)} \langle k-\ell, \eta - t\ell\rangle^{2\sigma_1}} \mathrm{d}t\right)^{1/2} \mathrm{d}\ell$$
$$\lesssim \epsilon^2 \int_{\mathbb{R}^3} \left(\frac{|\ell|}{|\ell|^3 \langle \ell\rangle^6}\right)^{1/2} \mathrm{d}\ell$$
$$\lesssim \epsilon^2.$$

Now turn to the NL_{LH} term. First, we use that $\rho(t,k) = \widehat{g}(t,k,kt)$, (2.8e), and (2.8c); second (using that the dimension is $d = 3$)

$$NL_{LH} \lesssim \int_0^T\int_{\mathbb{R}^3} |\widehat{g}(\ell, \ell t)| \frac{|\ell|}{\langle \ell\rangle^2} |\eta - kt| \langle k-\ell, \eta - \ell t\rangle^{\sigma_1} |\widehat{g}(k-\ell, \eta - \ell t)| \mathrm{d}t\, \mathrm{d}\ell$$
$$\lesssim \epsilon \int_0^T\int_{\mathbb{R}^3} \frac{|\ell|}{\langle \ell, \ell t\rangle^{\sigma_1} \langle \ell\rangle^2} |\eta - kt| \langle k-\ell, \eta - \ell t\rangle^{\sigma_1} |\widehat{g}(k-\ell, \eta - \ell t)| \mathrm{d}t\, \mathrm{d}\ell$$
$$\lesssim \epsilon \int_0^T \langle t\rangle \left(\int_{\mathbb{R}^3} \frac{|\ell|^2}{\langle \ell, \ell t\rangle^{2\sigma_1} \langle \ell\rangle^4 |k-\ell|^{2\delta}} \mathrm{d}\ell\right)^{1/2}$$
$$\left(\int_{\mathbb{R}^3} |k-\ell|^{2\delta} \langle k-\ell, \eta - \ell t\rangle^{2\sigma_3} |\widehat{g}(k-\ell, \eta - \ell t)|^2 \mathrm{d}\ell\right)^{1/2} \mathrm{d}t$$

$$
\begin{aligned}
&\lesssim \epsilon^2 \int_0^T \langle t\rangle \left(\int_{\mathbb{R}^3} \frac{|\ell|^2}{\langle \ell, \ell t\rangle^{2\sigma_1}\langle \ell\rangle^4 |k-\ell|^{2\delta}} \mathrm{d}\ell\right)^{1/2} \mathrm{d}t \\
&\lesssim \epsilon^2 \int_0^T \langle t\rangle \left(\frac{1}{\langle t\rangle^{2-2\delta+3}}\right)^{1/2} \mathrm{d}t \\
&\lesssim \epsilon^2.
\end{aligned}
$$

which, by choosing ϵ small enough, completes the improvement of the $L^\infty_t L^\infty_{\mathbb{R}^3\times\mathbb{R}^3}$ estimate (2.8e). As this is the last estimate, this also completes the proof of Proposition 2.12.

Appendix: Details Regarding the Linear Problem

First, we state an important lemma regarding $\mathcal{L}$.

LEMMA A.1. *Recall the definition of $\mathcal{L}$ in* (2.6). *For $0 \le j \le \sigma$ and any $\zeta > 0$*

$$
|k|^j \left|\partial_\omega^j \mathcal{L}(i\omega, k)\right| \lesssim_\zeta \|W\|_{L^1} \|f^0\|_{H_2^{j+3/2+\zeta}}. \tag{A.1}
$$

PROOF. By the regularity requirement $f^0 \in H_M^{\sigma+3/2+0}$ and the Sobolev trace Lemma 2.15, we have

$$
\begin{aligned}
\left|\partial_\omega^j \mathcal{L}(i\omega, k)\right| &\le \int_0^{+\infty} |kt|^j |\widehat{W}(k)||k|^2 t \left|\widehat{f^0}(kt)\right| \mathrm{d}t \\
&= \int_0^{+\infty} |\widehat{W}(k)| s^{j+1} \left|\widehat{f^0}\left(\frac{k}{|k|}s\right)\right| \mathrm{d}s \\
&\lesssim_\zeta \|W\|_{L^1} \left(\int_0^{+\infty} s^{2j+2}\langle s\rangle^{1+2\zeta} \left|\widehat{f^0}\left(\frac{k}{|k|}s\right)\right|^2 \mathrm{d}s\right)^{1/2} \\
&\lesssim \|W\|_{L^1} \|f^0\|_{H_2^{j+3/2+\zeta}}.
\end{aligned}
$$

□

Next, we prove Proposition 2.7.

PROOF OF PROPOSITION 2.7. Let us recall the following formula from [30] (essentially from [32]; see also [16]), adapted here to our slightly different definition of $\mathcal{L}$, which neatly divides $\mathcal{L}$ into real and imaginary parts:

$$
\mathcal{L}(i\omega, k) = \widehat{W}(k)\left(\text{p.v.} \int_{\mathbb{R}} \frac{(f_k^0)'(r)}{r - \omega|k|^{-1}} \mathrm{d}r - i\pi \left(f_k^0\right)'\left(\frac{\omega}{|k|}\right)\right),
$$

where

$$
f_k^0(r) := \int_{\frac{k}{|k|}r + k_\perp} f^0(x)\mathrm{d}x.
$$

Next note that when f^0 is radially symmetric, f_k^0 does not depend on k. Further, recall that if f^0 is radially symmetric and f^0 is strictly positive, then $(f_k^0)' < 0$ by

$v \in \mathbb{R}^3$ (see, e.g., [30]). Further, observe that, by Sobolev embedding, $f^0 \in C^{0,\gamma}$ for some $\gamma \in (0,1)$, and hence the real part of $\mathcal{L}$ is also a $C^{0,\gamma}$ function of $\omega|k|^{-1}$ (since the Hilbert transform maps $C^{0,\gamma} \mapsto C^{0,\gamma}$ for $\gamma \in (0,1)$ [36]). Next note that

$$\text{p.v.} \int_{\mathbb{R}} \frac{\left(f_k^0\right)'(r)}{r}\, \mathrm{d}r \leq 0,$$

and hence by the Hölder continuity, there is an m depending only on f^0 and α such that, for all $\omega|k|^{-1} < m$, there holds

$$\text{p.v.} \int_{\mathbb{R}} \frac{\left(f_k^0\right)'(r)}{r - \omega|k|^{-1}}\, \mathrm{d}r < \frac{1}{2\alpha}.$$

As $0 \leq \widehat{W}(k) \leq \frac{1}{\alpha}$ (recall (1.4)), it follows that for $\omega|k|^{-1} < m$, $|\mathcal{L} - 1| \geq \frac{1}{2}$.

For $\omega|k|^{-1} > M$, for M to be chosen below sufficiently large, write

$$\begin{aligned}
&\text{p.v.} \int_{\mathbb{R}} \frac{\left(f_k^0\right)'(r)}{r - \omega|k|^{-1}}\, \mathrm{d}r \\
&\quad = \int_{|r| \leq \frac{1}{2}\omega|k|^{-1}} \frac{\left(f_k^0\right)'(r)}{r - \omega|k|^{-1}}\, \mathrm{d}r + \text{p.v.} \int_{|r| > \frac{1}{2}\omega|k|^{-1}} \frac{\left(f_k^0\right)'(r)}{r - \omega|k|^{-1}}\, \mathrm{d}r \\
&\quad = -\frac{|k|}{\omega} \int_{|r| \leq \frac{1}{2}\omega|k|^{-1}} \left(f_k^0\right)'(r) \sum_{n=0}^{\infty} \left(\frac{r|k|}{\omega}\right)^n \mathrm{d}r \\
&\qquad + \text{p.v.} \int_{|r| > \frac{1}{2}\omega|k|^{-1}} \frac{\left(f_k^0\right)'(r)}{r - \omega|k|^{-1}}\, \mathrm{d}r \\
&\quad = L_I + L_O.
\end{aligned}$$

For f^0 and ∇f^0 rapidly decaying, the outer integral satisfies

$$L_O \lesssim \frac{|k|^3}{\omega^3}.$$

Since $(f_k^0)'$ has zero average and is rapidly decaying, the leading-order contribution to the inner integral is also decaying rapidly:

$$-\frac{|k|}{\omega} \int_{|r| \leq \frac{1}{2}\omega|k|^{-1}} \left(f_k^0\right)'(r) \mathrm{d}r \lesssim \frac{|k|^3}{\omega^3}.$$

Therefore, the next-order contribution is

$$L_I = -\frac{|k|^2}{\omega^2} \int_{\mathbb{R}} \left(f_k^0\right)'(r) r\, \mathrm{d}r + O\left(\frac{|k|^3}{\omega^3}\right).$$

It follows that for $\omega|k|^{-1} > M$,

$$\operatorname{Re}\mathcal{L}(i\omega,k) = -\frac{|k|^2}{(\alpha+|k|^2)\omega^2}\int_{\mathbb{R}}\big(f_k^0\big)'(r)r\,\mathrm{d}r + O\bigg(\frac{|k|^3}{\omega^3}\bigg).$$

Therefore, for M chosen sufficiently large (depending on α and f^0), there holds

$$|\operatorname{Re}\mathcal{L}(i\omega,k) - 1| \geq \frac{1}{2}.$$

On the other hand, for all $0 < m < M < \infty$, due to the assumptions on f_k^0, the imaginary part of $\mathcal{L}$ is bounded uniformly away from 0 over $m < \omega|k|^{-1} < M$; that is, there exists a $\kappa = \kappa(m,M) > 0$ such that

$$\inf_{m\leq\omega|k|^{-1}\leq M}|\operatorname{Im}\mathcal{L}(i\omega,k)| \geq \kappa.$$

The result then follows. □

Next we prove Proposition 2.3.

PROOF OF PROPOSITION 2.3. Note that ϕ will not generally be compactly supported in time but obviously

$$\big\||k|^\alpha\langle k,kt\rangle^\sigma\phi(t,k)\big\|_{L_t^2(I)} \leq \big\||k|^\alpha\langle k,kt\rangle^\sigma\phi(t,k)\big\|_{L_t^2(\mathbb{R}_+)}.$$

Step 1. *A priori estimate for integer σ.* Define

$$\Phi(t,k) = |k|^\alpha\phi(t,k), \quad H'(t,k) = |k|^\alpha H(t,k),$$

and multiply both sides of equation (2.5) by $|k|^\alpha$ to derive

$$\Phi(t,k) = H'(t,k) + \int_0^t K^0(t-\tau,k)\Phi(\tau,k)\mathrm{d}\tau. \tag{A.2}$$

If we assume a priori that all the quantities involved are L^2 integrable in time, then we can take the Fourier transform in time (extending as 0 for $t<0$ and extending H by 0 for $t > T_\star$), and we have for $\omega\in\mathbb{R}$,

$$\widetilde{\Phi}(\omega,k) = \widetilde{H'}(\omega,k) + \widetilde{K^0}(\omega,k)\widetilde{\Phi}(\omega,k),$$

where $\widetilde{\Phi}(\omega,k)$, $\widetilde{H'}(\omega,k)$, and $\widetilde{K^0}(\omega,k)$ is the Fourier transform in time of $\Phi(t,k)$, $H'(t,k)$, and $K^0(t,k)$, respectively, after extending by 0 for negative times. Now we note that

$$\widetilde{K^0}(\omega,k) = \mathcal{L}(i\omega,k). \tag{A.3}$$

Regularity estimates in ω imply decay in t, so let us prove H^σ estimates in ω. Taking β derivatives, where $0\leq\beta\leq\sigma$, and multiplying by $|k|^\beta\langle k\rangle^\gamma$ for $0\leq\gamma\leq$

σ gives

$$|k|^{\beta}\partial_{\omega}^{\beta}\langle k\rangle^{\gamma}\widetilde{\Phi}(\omega,k) = |k|^{\beta}\langle k\rangle^{\gamma}\partial_{\omega}^{\beta}\widetilde{H'}(\omega,k) + |k|^{\beta}\langle k\rangle^{\gamma}\sum_{j=0}^{\beta}\binom{\beta}{j}\partial_{\omega}^{\beta-j}\mathcal{L}(i\omega,k)\partial_{\omega}^{j}\widetilde{\Phi}(\omega,k).$$

By taking L^2_ω norms and using the stability condition we then have

$$\begin{aligned}&|k|^{\beta}\langle k\rangle^{\gamma}\left\|\partial_{\omega}^{\beta}\widetilde{\Phi}(\cdot,k)\right\|_{L^2_\omega}\\ &\quad\lesssim_{\kappa,\beta} |k|^{\beta}\langle k\rangle^{\gamma}\left\|\partial_{\omega}^{\beta}\widetilde{H'}(\cdot,k)\right\|_{L^2_\omega}\\ &\qquad + \langle k\rangle^{\gamma}\sum_{j=0}^{\beta-1}\left\||k|^{\beta-j}\partial_{\omega}^{\beta-j}\mathcal{L}(i\cdot,k)\right\|_{L^\infty_\omega}|k|^{j}\left\|\partial_{\omega}^{j}\widetilde{\Phi}(\cdot,k)\right\|_{L^2_\omega}.\end{aligned}$$

Then using (A.1) and induction on β, we get for all β, $0\le\beta\le\sigma$, and all s for $0\le s\le\sigma$,

$$\||kt|^{\beta}\langle k\rangle^{s}\Phi(t,k)\|_{L^2_t(I)} \lesssim_{s,\beta} \kappa^{-\beta}\langle k\rangle^{s}\|\langle kt\rangle^{\beta}H'(t,k)\|_{L^2_t(I)}. \tag{A.4}$$

Now we apply $\langle k,kt\rangle^{\sigma}\approx\langle k\rangle^{\sigma}+|kt|^{\sigma}$ and use (A.4) with $\beta=0$, $s=\sigma$, and $\beta=\sigma$, $s=0$, to conclude the a priori estimate (2.7).

Step 2. *Justifying a priori estimate for integer σ.* Recall that this argument assumes a priori that we already have sufficiently rapid decay on ϕ. In order to make this argument rigorous, one may use the technique described in [4,39] which is for all $\delta>0$, define $\eta_\delta(t)=e^{-\delta t^2/2}$, and choose $\mu\le 0$ to be a real number; then study

$$\Phi^{\delta}(t,k)=e^{\mu t}\eta_\delta(t)\Phi(t,k).$$

It is straightforward to show that for C sufficiently large $|\Phi(t,k)|\lesssim e^{Ct}$ and hence for $\mu<-C$, one goes through the derivations above and derives:

$$\widetilde{\Phi^{\delta}}(\omega,k)=\widetilde{\eta_\delta}*\left(\frac{\widetilde{H}(\cdot,k)}{1-\mathcal{L}(\mu+i\cdot,k)}\right)(\omega).$$

Moreover, this function depends analytically on μ as long as we stay away from a singularity where $\mathcal{L}=1$. By analytic continuation, we may hence deduce that this formula holds all the way for all $\mu\le 0$. From there, one may proceed by taking derivatives in ω on $\widetilde{\Phi}^{\delta}(\omega,k)$ and then passing to the limit $\delta\to 0$ to deduce the desired estimate (2.7). □

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