Hybrid Robust Minimum-time Control for a Class of Non-Exponentially Unstable Planar Systems*

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Abstract— **This paper deals with robust minimum-time control of a class of asymptotically null-controllable with bounded input planar systems. A hybrid controller is proposed to robustly achieve global finite time stability of a set of points wherein the plant state is zero. The resulting controller provides time optimal response from initial conditions in a certain subset of the state space, and finite time convergence elsewhere. Finally, the effectiveness of the proposed methods is demonstrated in two numerical examples.**

I. Introduction

The problem of minimum-time control consists of transferring the state of a dynamical system from one point to another in the shortest amount of time, while possibly ensuring the satisfaction of certain constraints. Such a problem, due to its relevance in numerous applications, has attracted the attention of researchers since the 17th century. The first minimum-time control problem can be traced back to 1697 when Johann Bernoulli formulated in the *Acta Eruditorum* the well-known *brachistochrone* problem. Since then, minimum-time control has received much attention and different scenarios have been considered; see [6], [7], [1], [9]. A key result in this context is *Pontryagin's Maximum Principle* [9], which provides necessary conditions for a constrained control to be an open-loop optimal control.

Due to their importance in engineering applications, particular attention has been devoted to finding solutions to minimum-time control problems characterized by single input second-order linear time-invariant plants (LTI) with an input constraint, *i.e.*,

$$\begin{cases} \dot{x}_{p1} = a_{11}x_{p1} + a_{12}x_{p2} + b_1 u \\ \dot{x}_{p2} = a_{21}x_{p1} + a_{22}x_{p2} + b_2 u \\ u \in [-M, M] \end{cases}$$

where $M > 0$. In this setting, minimum-time transferring from any initial condition to a given point, without loss of generality, the origin, can be accomplished by a control input taking values in $\{-M, M\}$ if and only if the plant is asymptotically null-controllable with bounded input, *i.e.*, if its eigenvalues are contained in the closed left-half plane; [11], [9]. Furthermore, if one further restricts the attention to the case of plants with either real or purely imaginary eigenvalues, then, a (discontinuous) state-feedback $\kappa\colon \mathbb{R}^2 \to \{-M, M\}$ such that solutions to the resulting closed-loop system converge, from any initial condition, and in minimum-time, to the origin can be explicitly obtained; see [3], [1]. Although following this approach provides a viable solution to the optimal control problem of the considered class of plants, the adoption of a discontinuous law may induce a lack of robustness for the resulting closed-loop system. Indeed, it is well-known that discontinuous controllers are very sensitive to (small) measurement noise which renders their implementation in practice somewhat delicate; see [2], [4], [5] just to cite a few. This drawback is well known by the community and for this reason researchers have provided different approaches to avoid the use of discontinuous laws in an attempt to achieve a trade-off between robustness and optimality; [10], [8], [3].

In this paper we pursue a different approach. By relying on the framework for hybrid systems in [4], we design a hybrid feedback controller ensuring robust minimum-time convergence from certain points of the state space. In particular, by restricting the attention to a class of planar systems for which a closed-form expression of a static time-optimal feedback controller is available, we propose a hybrid controller ensuring time-optimal convergence to a set given by the origin of the plant (when projected to the plant state space) for a set of initial conditions for the closed-loop system and finite time convergence elsewhere. The applicability of the proposed construction is shown in two examples of practical interests: the double integrator and the harmonic oscillator.

The remainder of the paper is organized as follows. Section I-A presents some preliminaries on hybrid systems. Section II-A presents some background on time-optimal control. Section II.B is dedicated to the problem statement. Section III is devoted to the main results of our paper. Finally, Section IV shows the effectiveness of the results presented in two case studies.

Notation: The set $\mathbb{N}$ is the set of strictly positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}_{\geq 0}$ represents the set of non-negative real scalars, and $\mathbb{C}_{-}$ is the set of complex numbers with negative real part. For a vector $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm, while x_i denotes its i-th entry, and $\mathbf{1}_n$ denotes the vector in $\mathbb{R}^n$ whose entries are equal to one. Given two vectors x, y, we denote $(x, y) = [x'\ y']'$. Given a vector $x \in \mathbb{R}^n$ and a closed set $\mathcal{A}$, the distance of x to $\mathcal{A}$ is defined as $|x|_{\mathcal{A}} = \inf_{y \in \mathcal{A}} |x - y|$. Given a set S, we denote $\overline{S}$ the closure of S. Given $I \subset \mathbb{R}$, the set $\mathcal{L}^{\infty}_{loc}(I) \subset \mathbb{R}$ is the set of Lebesgue-measurable and locally essentially bounded functions from I to $\mathbb{R}$. Given a matrix $A \in \mathbb{R}^{n \times n}$, $\sigma(A)$ denotes the spectrum of A. Given a function $f\colon X \to Y$, $\operatorname{rge} f$ denotes the range of f.

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A. Preliminaries on Hybrid Systems

In this paper, we adopt the framework for hybrid systems in [4]. Next, we give some basic notions on hybrid systems and we refer the reader to [4] for more details on hybrid systems.

A hybrid dynamical system $\mathcal{H}$ with state $x \in \mathbb{R}^n$ is a tuple (C, f, D, g), where $C, D \subset \mathbb{R}^n$ are, respectively, the *flow set* and the *jump set*, while $f\colon \mathbb{R}^n \to \mathbb{R}^n$ and $g\colon \mathbb{R}^n \to \mathbb{R}^n$ are, respectively, the *flow map* and the *jump map*. The flow map f describes the continuous evolution (flow) of $\mathcal{H}$, while the jump map g describes how instantaneous changes (jumps) occur. The flow set C indicates the set wherein continuous evolution is allowed, while the jump set D indicates the set wherein instantaneous changes may take place. A hybrid time domain is a subset of $\mathbb{R}_{\geq 0} \times \mathbb{N}_0$. Given a hybrid time domain E, we denote $\sup E = (\sup_t E, \sup_j E)$, where $\sup_t E$ and $\sup_j E$ are, respectively, the supremum of the projection of E onto $\mathbb{R}_{\geq 0}$ and the supremum of the projection of E onto $\mathbb{N}_0$. A solution $\mathcal{H}$ is any hybrid arc defined over a hybrid time domain that satisfies the dynamics of $\mathcal{H}$. A solution is said to be *complete* if its domain is unbounded. A solution is *maximal* if it is not the truncation of another solution. Given a set S, we denote $\mathcal{S}_{\mathcal{H}}(S)$ the set of all maximal solutions ϕ to $\mathcal{H}$ with $\phi(0,0) \in S$. Given a set $S \subset \mathbb{R}^n$, we say that S is strongly forward invariant for $\mathcal{H}$, if each $\phi \in \mathcal{S}_{\mathcal{H}}(S)$ is complete and one has $\operatorname{rge} \phi \subset S$. Given $\mathcal{H} = (C, f, D, g)$, we say that $\mathcal{H}$ satisfies the *hybrid basic conditions* ([4]) if: C and D are closed, and $f\colon C \to \mathbb{R}^n$ and $g\colon D \to \mathbb{R}^n$ are continuous.

Definition 1: Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$, a compact set $\mathcal{A} \subset \mathbb{R}^n$, and an open neighborhood $\mathcal{S}$ of $\mathcal{A}$. The set $\mathcal{A}$ is said to be

- *stable* for $\mathcal{H}$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{A} + \delta\mathbb{B})$, one has that $|\phi(t,j)|_{\mathcal{A}} \leq \epsilon$ for every $(t,j) \in \operatorname{dom} \phi$.

Definition 2: Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$, a compact set $\mathcal{A} \subset \mathbb{R}^n$, an open neighborhood $\mathcal{N}$ of $\mathcal{A}$, and a function $\mathcal{T}\colon \mathcal{N} \to \mathbb{R}_{\geq 0}$, called the settling-time function. The set $\mathcal{A}$ is said to be

- *finite time attractive* for $\mathcal{H}$ if for each $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{N})$, $\sup\{t + j\colon (t,j) \in \operatorname{dom}\phi\} \geq \mathcal{T}(\phi(0,0))$ and
$$\lim_{(t,j)\in \operatorname{dom}\phi\colon\, t+j \uparrow \mathcal{T}(\phi(0,0))} |\phi(t,j)|_{\mathcal{A}} = 0$$
- *finite time stable* for $\mathcal{H}$ if it is stable and finite time attractive for $\mathcal{H}$;
- *globally finite time stable* for $\mathcal{H}$ if it is stable and finite time attractive for $\mathcal{H}$ and $\mathcal{N} = \mathbb{R}^n$.

II. Problem Statement

A. Background on minimum-time control of planar linear systems with bounded inputs

Consider the following planar single input LTI plant
$$\dot{x}_p = A x_p + b u \tag{1}$$
where $x_p \in \mathbb{R}^2$ is the plant's state, $u \in U \coloneqq [-M, M]$ is the control input, where $M \in \mathbb{R}_{\geq 0}$, and $A \in \mathbb{R}^{2\times 2}, b \in \mathbb{R}^2$ are given matrices. Define
$$\mathcal{U} \coloneqq \{u \in \mathcal{L}^{\infty}_{loc}([0,\infty))\colon\; u(t) \in U \quad \forall t \in \operatorname{dom} u\}$$
Given $x_{p0} \in \mathbb{R}^2$, consider the following (minimum-time) optimal control problem:
$$\begin{cases} \min\limits_{u \in \mathcal{U}} \;\mathcal{J}(u) = \displaystyle\int_0^{t_f} dt \\ \text{s.t.} \\ \dot{x}_p(t) = A x_p(t) + b u(t) \qquad \text{for almost all } t \in [0, t_f] \\ x_p(0) = x_{p0}, x_p(t_f) = 0,\; t_f \in \mathbb{R}_{\geq 0} \end{cases} \tag{2}$$

Definition 3: Let $x_{p0} \in \mathbb{R}^2$ and be $(x_p^\star, u^\star)$ a solution pair to (1), with $\operatorname{dom} x_p^\star = [0, t_f^\star]$ and $u^\star \in \mathcal{U}$. We say that the pair $(x_p^\star, u^\star)$ is an *optimal pair* for the optimal control problem (2) if $x_p^\star(0) = x_{p0}$ and

(i) $x_p^\star(t_f^\star) = 0$;

(ii) $\mathcal{J}(u^\star) = \min\limits_{u \in \mathcal{U}} \;\mathcal{J}(u)$.

Moreover, we say that $u^\star \in \mathcal{U}$ is an *optimal control* for (2) if the corresponding solution $x_p^\star$ from x_{p0} to (1) is such that $(x_p^\star, u^\star)$ is an optimal pair for[1] (2).

Consider now the following result, which gathers some important results from [9] and that provides guidelines on how to generate optimal controls for (2), though specialized to case of single input planar LTI plants.

Theorem 1: Let $A \in \mathbb{R}^{2\times 2}$ and $b \in \mathbb{R}^2$ such that (A, b) is controllable, $\sigma(A) \subset \overline{\mathbb{C}}_-$, and let $x_{p0} \in \mathbb{R}^2$. Then, the following properties hold:

(i) there exists $t_f^\star \geq 0$, and a unique optimal control $u^\star \in \mathcal{U}$ with $\operatorname{dom} u^\star = [0, t_f^\star]$ that solves (2);

(ii) the optimal control $u^\star \in \mathcal{U}$ is piecewise constant and such that $\operatorname{rge} u^\star \subset \{-M, M\}$;

(iii) if the eigenvalues of A are real, then the optimal control $u^\star \in \mathcal{U}$ that solves (2) can change sign at most one time.

Remark 1: Having assumed that $\sigma(A) \subset \overline{\mathbb{C}}_-$ rules out the case of exponentially unstable plants, for which problem (2) cannot be solved globally due to U being bounded; see [12]. Moreover, having assumed (A, b) to be controllable ((1) being single input) rules out the existence of singular control in the solution to (2); see [7].

Theorem 1 formally states the well-known *bang-bang principle* for minimum-time optimal control, *i.e.*, the optimal control switches between the two extrema of the admissible input set U. Due to this behavior, it is convenient to define the following objects.

Definition 4: Let $\mathcal{I}$ be a compact interval and $v\colon \mathcal{I} \to \mathbb{R}$ be a piecewise constant function. We denote $n_s(v) \in \mathbb{N}_0$ the number of switchings of v. More precisely, $n_s(v)$ is the smallest nonnegative integer such that
$$v(t) = \sum_{k=0}^{n_s(v)} v_k \chi_{\mathcal{I}_k}(t) \qquad \forall t \in \mathcal{I}$$

[1] Notice that, due to the right-hand side of (1) being linear, and $u^\star$ being Lebesgue-measurable, for each $x_{p0} \in \mathbb{R}^2$ given an optimal control, the corresponding optimal pair is univocally determined.

where $\mathcal{I}_0, \mathcal{I}_1, \ldots, \mathcal{I}_{n_s(v)}$ are some bounded pairwise disjoint intervals such that $\bigcup_{k=0}^{n_s(v)} \mathcal{I}_k = \mathcal{I}$, χ_S is the indicator function of the generic set S, and v_k, for $k = 0, 1, \ldots, n_s(v)$, is a real number.

Definition 5: Let $x_{p0} \in \mathbb{R}^2$ be given, and let $(x_p^\star, u^\star)$ be the corresponding (unique) optimal pair. We denote $\mathcal{L}^\star(x_{p0})$ as the (optimal) number of switchings of $u^\star$, *i.e.*, for each $x_{p0} \in \mathbb{R}^2$, $\mathcal{L}^\star(x_{p0}) = n_s(u^\star)$.

Given the assumptions in Theorem 1, it turns out that the class of planar systems covered by Theorem 1 can be (modulo a linear invertible change of variables) written in the following (reachability) form:

$$\dot{x}_p = \underbrace{\begin{pmatrix} 0 & 1 \\ -a_1 & -a_2 \end{pmatrix}}_{A} x_p + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{b} u \tag{3}$$

with $a_1 a_2 \geq 0$. This class of systems encompasses several systems of relevant interest like, just to cite a few, the double integrator ($a_2 = a_1 = 0$) and the harmonic oscillator with angular speed $\omega > 0$ ($a_2 = 0, a_1 = \omega^2$). Therefore, in the sequel, without loss of generality, we explicitly refer to the class of plants in (3).

Although optimal control problems are naturally formalized (and solved) as open-loop control problems, having available a state dependent (closed-loop) expression of the optimal control, as defined next, is of primary importance in practice. In fact, open-loop solutions are unlikely to be robust with respect to mismatches on the plant initial condition or to arbitrarily small (even vanishing in finite time) external perturbations.

Definition 6: The function $\kappa\colon \mathbb{R}^n \to U$ is said to be a state feedback optimal controller for (2) if for each $x_{p0} \in \mathbb{R}^2$, there exists a $t_f \geq 0$, and a unique solution $[0, t_f] \ni t \mapsto \phi^\star(t)$ to

$$\dot{x}_p = A x_p + b\kappa(x_p)$$

such that $(\phi^\star, \kappa \circ \phi^\star)$ is an optimal pair for (2).

Remark 2: As pointed out in Theorem 1, the optimal control takes values only in the set $\{-M, M\}$. Therefore, the optimal feedback κ is necessarily a discontinuous function.

A notable characteristic of minimum-time control of linear LTI plants is that whenever an optimal control exists, provided that the eigenvalues of A are either real or purely imaginary, one can explicitly construct a state-feedback optimal control out of it. In particular the derivation of optimal feedback controllers, for all possible realizations of the plant (3), are thoroughly presented and discussed in [1], [6]. Specifically, from the constructions presented in [1], [6], [3], it turns out that, given a specific realization of the plant (3), and provided that the eigenvalues of A are either real or purely imaginary, then a closed form for a state-feedback optimal controller for (3) exists. In particular, as shown next, the state-feedback optimal controller is univocally identified by a continuous function $s\colon \mathbb{R}^2 \to \mathbb{R}$ that we call the switching surface generator, which is defined as follows

Definition 7 (Switching surface generator): The function s is a switching surface generator if:

(i) there exist continuous functions $\alpha_i\colon \mathbb{R} \to \mathbb{R}$, for $i = 1, 2$, such that s can be written either as

$$s(x) = x_2 + \alpha_1(x_1) \tag{4a}$$

or as

$$s(x) = x_1 + \alpha_2(x_2) \tag{4b}$$

(ii) the functions $\alpha_i\colon \mathbb{R} \to \mathbb{R}$, for $i = 1, 2$, are such that $\alpha_1(0) = \alpha_2(0) = 0$, and for each $p \in \mathbb{R}$, $p\alpha_1(p) \leq 0$ and $p\alpha_2(p) \geq 0$

More specifically, given $s\colon \mathbb{R}^2 \to \mathbb{R}$ satisfying the above properties, if one defines the following nonempty sets

$$\begin{aligned} \mathcal{S} &:= \{x \in \mathbb{R}^2 \colon s(x) = 0\} \\ \mathcal{S}_+ &:= \{x \in \mathbb{R}^2 \colon s(x) > 0\} \\ \mathcal{S}_- &:= \{x \in \mathbb{R}^2 \colon s(x) < 0\} \end{aligned} \tag{5}$$

where $\mathcal{S}$ is called the *switching surface*, then, to generate optimal trajectories from each point of the state space, κ can be defined as follows:[2]

$$\kappa(x_p) = \begin{cases} -M & \text{if } x_p \in \mathcal{S}_+ \\ M & \text{if } x_p \in \mathcal{S}_- \\ -M & \text{if } x_p \in \mathcal{S} \cap (\mathbb{R}_{<0} \times \mathbb{R}) \\ M & \text{if } x_p \in \mathcal{S} \cap (\mathbb{R}_{>0} \times \mathbb{R}) \\ 0 & \text{if } x_p = 0 \end{cases} \tag{6}$$

which univocally determines κ in $\mathbb{R}^2$ due to $\mathcal{S} \cup \mathcal{S}_+ \cup \mathcal{S}_- = \mathbb{R}^2$. For example, in the case of the double integrator, one has that

$$s(x) = x_1 + \underbrace{\frac{1}{2M}|x_2|x_2}_{\alpha_2}$$

where α_2 obviously satisfies all the items in Definition 7; see, [1], [7]. In particular, from the analysis showcased in [1], it turns out that the following fact holds:

Fact 1: Let κ be defined as in (6), $\xi \in \mathbb{R}^2$, and ϕ be the unique maximal solution to

$$\dot{x}_p = A x_p + b\kappa(x_p)$$

with $\phi(0) = \xi$. Then, the following properties hold:

(i) if $\xi \in \mathcal{S}_-$, then there exists $T > 0$, such that $\phi(T) \in \mathcal{S} \cap (\mathbb{R}_{<0} \times \mathbb{R})$ and, for all $t \in [0, T)$, $\phi(t) \in \mathcal{S}_-$;

(ii) if $\xi \in \mathcal{S}_+$, then there exists $T > 0$, such that $\phi(T) \in \mathcal{S} \cap (\mathbb{R}_{>0} \times \mathbb{R})$ and, for all $t \in [0, T)$, $\phi(t) \in \mathcal{S}_+$;

(iii) if $\xi \in \mathcal{S} \cap (\mathbb{R}_{<0} \times \mathbb{R})$, then there exists $T > 0$ such that, for all $t \in [0, T]$, $\phi(t) \in \overline{\mathcal{S}_+} \cap (\mathbb{R}_{<0} \times \mathbb{R})$;

(iv) if $\xi \in \mathcal{S} \cap (\mathbb{R}_{>0} \times \mathbb{R})$, then there exists $T > 0$ such that, for all $t \in [0, T]$, $\phi(t) \in \overline{\mathcal{S}_-} \cap (\mathbb{R}_{>0} \times \mathbb{R})$.

Although the feedback controller (6) provides a viable solution to (2), being a discontinuous controller, it is particularly not robust to the presence of measurement noise, which

[2]As a matter of fact, whenever the matrix A in (3) has real nonzero distinct eigenvalues, *i.e.*, $a_2^2 - 4a_1 > 0$, the definition of κ in (6) holds up to a linear invertible change of coordinates $z_p = \widetilde{T} x_p$. In this case, the optimal feedback κ can be defined for each $x_p \in \mathbb{R}^2$ as $\kappa(x_p) = \tilde{\kappa}(\widetilde{T} x_p)$, where $\tilde{\kappa}\colon \mathbb{R}^2 \to \{-M, M\}$ is defined as in (6). However, to keep the presentation simple, we assume κ to be directly defined as in (6) in the x_p-coordinates.

may result in unwanted behaviors like chattering away from the origin; see [2].

B. Hybrid robust minimum-time control

We propose a hybrid controller allowing to solve the considered minimum-time control problem robustly. The proposed hybrid controller has state $\eta \in \{-M, M\}$, input $v \in \mathbb{R}^2$, and output $\zeta \in \{-M, M\}$, and is given by

$$\mathcal{H}_K \begin{cases} \dot{\eta} &= f_K(\eta, v) \qquad (v, \eta) \in C_K \\ \eta^+ &= g_K(\eta, v) \qquad (v, \eta) \in D_K \\ \zeta &= \eta \end{cases} \tag{7}$$

where $C_K \subset \mathbb{R}^2 \times \{-M, M\}$, $D_K \subset \mathbb{R}^2 \times \{-M, M\}$, $f_K : \{-M, M\} \times \mathbb{R}^2 \to \mathbb{R}$, $g_K : \{-M, M\} \times \mathbb{R}^2 \to \{-M, M\}$ need to be designed. By interconnecting it to the plant (3) through $v = x_p$ and $u = \zeta$, it leads to the closed-loop system

$$\mathcal{H} \begin{cases} \dot{x} &= f(x) \qquad x \in C_K \\ \eta^+ &= g(x) \qquad x \in D_K \end{cases} \tag{8}$$

where $x = (x_p, \eta)$; for each $x \in C_K$, $f(x) = (Ax_p + b\eta), f_K(x))$; and for each $x \in D_K$, $g(x) = (x_p, g_K(x))$. Defining the set

$$\mathcal{A} = \{0\} \times \{-M, M\} \subset \mathbb{R}^2 \times \{-M, M\} \tag{9}$$

the problem to solve consists of designing the data of $\mathcal{H}$, namely (C_K, f_K, D_K, g_K) such that

1) For each $\xi = (\xi_p, \xi_\eta) \in C_K \cup D_K$, and each $\phi \in \mathcal{S}_\mathcal{H}(\xi)$, there exists $j^\star \in \mathbb{N}_0$ such that $\phi(\mathcal{J}^\star(\xi_p), j^\star) \in \mathcal{A}$;
2) the data of $\mathcal{H}$ satisfies the hybrid basic conditions.

III. Main Results

Consider the following general result that is exploited in this section. For the sake of exposition, we assume completeness of maximal solutions. The general case follows similarly.

Proposition 1: Let $\mathcal{H} = (C, f, D, g)$ be a generic hybrid system with state in $\mathbb{R}^n$ defined as in Section I-A, $\mathcal{A} \subset \mathbb{R}^n$ be compact, $\mathcal{N} \subset \mathbb{R}^n$ be an open neighborhood of $\mathcal{A}$, and $\mathcal{T} : \mathcal{N} \to \mathbb{R}_{\geq 0}$ be locally bounded. Assume that $\mathcal{A}$ is forward invariant and finite-time attractive for $\mathcal{H}$ with settling-time function $\mathcal{T}$ and that maximal solutions to $\mathcal{H}$ are complete. If $\mathcal{H}$ satisfies the hybrid basic conditions, then $\mathcal{A}$ is finite time stable.

Proof: To prove the claim, one only needs to show stability of the set $\mathcal{A}$. To this end, observe that $\mathcal{A}$ being compact and finite time attractive, it follows that every $\phi \in \mathcal{S}_\mathcal{H}(\mathcal{N})$ is bounded. Therefore, according to [4, Proposition 7.5], we show that the set $\mathcal{A}$ is stable by showing that for some positive real scalar μ, $\mathcal{A}$ is uniformly pre-attractive from $\mathcal{A} + \mu\mathbb{B}$[3]. Let

$$\bar{\mu} = \sup\{\mu \in \mathbb{R}_{>0} : \mathcal{A} + \mu\mathbb{B} \subset \mathcal{N}\}$$

[3]A compact set $\mathcal{A}$ is *uniformly pre-attractive* for $\mathcal{H}$ from $\mathcal{F}$ if for each $\varepsilon > 0$, there exists $T > 0$ such that $\phi \in \mathcal{S}_\mathcal{H}(\mathcal{F})$ is bounded, and for each $(t, j) \in \operatorname{dom}\phi$, $t + j \geq T$ implies $|\phi(t, j)|_\mathcal{A} \leq \varepsilon$; see [4].

and for any finite $\mu \in (0, \bar{\mu}]$ define

$$\tau(\mu) = \sup_{\zeta \in \mathcal{A} + \mu\mathbb{B}} \mathcal{T}(\zeta)$$

which is finite due to $\mathcal{T}$ being locally bounded and $\mathcal{A}$ being compact. Pick $\phi \in \mathcal{S}_\mathcal{H}(\mathcal{A} + \mu\mathbb{B})$ and observe that, since $\mathcal{A}$ is finite time attractive and forward invariant, each $(t, j) \in \operatorname{dom}\phi$ with $t + j \geq \tau(\mu)$ implies that $\phi(t, j) \in \mathcal{A}$. Thus, $\mathcal{A}$ is uniformly pre-attractive from $\mathcal{A} + \mu\mathbb{B}$ and this concludes the proof. ■

A. A robust finite-time controller

Given the plant (3), assume that a minimum-time state feedback controller κ is given, and let $s : \mathbb{R}^2 \to \mathbb{R}$ be the corresponding switching surface generator. In particular, for each initial condition $x_{p0} \in \mathbb{R}^2$, we denote $\mathcal{J}^\star(x_{p0})$ as the smallest (minimum) time for the (unique) maximal solution ϕ to $\dot{x}_p = Ax_p + b\kappa(x_p)$ to reach the origin from x_{p0}.

To define the data of the controller $\mathcal{H}_K$, we mimic the bang-bang working principle of the feedback optimal controller κ. Specifically, we enforce the state η of the controller $\mathcal{H}_K$ to be constant during flows and toggle its value whenever a jump occurs. This leads to the following definitions for the flow map and for the jump map of $\mathcal{H}_K$

$$\begin{aligned} f_K(\eta, x_p) &= 0 \qquad \forall (\eta, x_p) \in C_K \\ g_K(\eta, x_p) &= -\eta \qquad \forall (\eta, x_p) \in D_K \end{aligned} \tag{10}$$

With the aim of defining the flow set C_K and the jump set D_K of $\mathcal{H}_K$, let us define the following sets:

$$\begin{aligned} \mathcal{C}_M &= \kappa^{-1}(M) := \{x \in \mathbb{R}^2 : \kappa(x) = M\} \\ \mathcal{C}_{-M} &= \kappa^{-1}(-M) := \{x \in \mathbb{R}^2 : \kappa(x) = -M\} \end{aligned} \tag{11}$$

In particular, from the definition of κ in (6)

$$\begin{aligned} \mathcal{C}_M &= \mathcal{S}_- \cup (\mathcal{S} \cap (\mathbb{R}_{\geq 0} \times \mathbb{R})) \\ \mathcal{C}_{-M} &= \mathcal{S}_+ \cup (\mathcal{S} \cap (\mathbb{R}_{\leq 0} \times \mathbb{R})) \end{aligned}$$

Moreover, still from the definition of κ and Fact 1, it follows that the optimal feedback switches from $-M$ to M whenever the switching surface is crossed in the first and fourth quadrant, and it switches from M to $-M$ whenever the switching surface is crossed in the second an the third quadrant. Therefore, we define D_K to enforce a jump whenever the following condition holds:

$$x \in ((\underbrace{\mathcal{S} \cap \mathcal{C}_M}_{\mathcal{D}^s_{-M}}) \times \{-M\}) \cup ((\underbrace{\mathcal{S} \cap \mathcal{C}_{-M}}_{\mathcal{D}^s_M}) \times \{M\}) =: D^s_K \tag{12}$$

In particular, it follows that

$$\begin{aligned} \mathcal{D}^s_{-M} &= \mathcal{S} \cap (\mathbb{R}_{\geq 0} \times \mathbb{R}) \\ \mathcal{D}^s_M &= \mathcal{S} \cap (\mathbb{R}_{\leq 0} \times \mathbb{R}) \end{aligned}$$

Moreover, to fully capture the mechanism of the static minimum-time feedback, one needs to define D_K so to also include points in $(\mathcal{S}_- \times \{-M\}) \cup (\mathcal{S}_+ \times \{M\})$. However, by directly defining D_K as $D^s_K \cup (\mathcal{S}_- \times \{-M\}) \cup (\mathcal{S}_+ \times \{M\})$, $\mathcal{S}_-$ and $\mathcal{S}_+$ being open, would prevent from the possibility

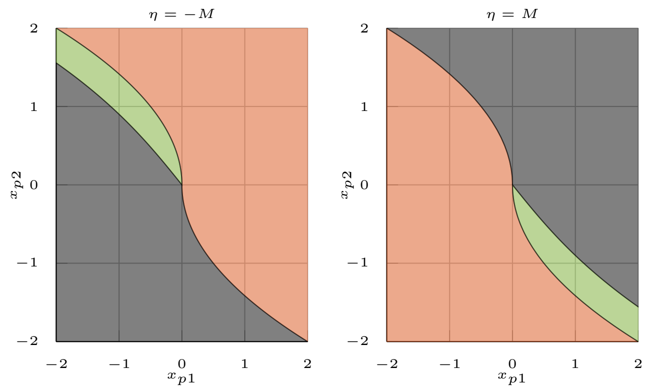

Fig. 1: The sets $\overline{\mathcal{C}_{-\mathcal{M}}}$ (left, orange), $\mathcal{I}_{-M}$ (left, green), $\overline{\mathcal{C}_{\mathcal{M}}}$ (right, orange), $\mathcal{I}_M$ (right, green), $\mathcal{D}_{-\mathcal{M}}$ (left, black), and $\mathcal{D}_{\mathcal{M}}$ (right, black).

of obtaining a closed-loop system satisfying the hybrid basic conditions. To overcome this problem, define

$$D_z := \underbrace{(\mathcal{S} \cap (\mathbb{R}_{\leq 0} \times \mathbb{R}) \times \{-M\})}_{D_z^{-M}} \cup \underbrace{(\mathcal{S} \cap (\mathbb{R}_{\geq 0} \times \mathbb{R}) \times \{M\})}_{D_z^{M}}$$

and let $\mathcal{I}_{-M}$ and $\mathcal{I}_M$ be two closed subsets, respectively, of $\overline{\mathcal{S}_-} \cap (\mathbb{R}_{\leq 0} \times \mathbb{R})$ and $\overline{\mathcal{S}_+} \cap (\mathbb{R}_{\geq 0} \times \mathbb{R})$ such that

$$\overline{((\mathcal{S}_- \setminus \mathcal{I}_{-M}) \times \{-M\}) \cup ((\mathcal{S}_+ \setminus \mathcal{I}_M) \times \{M\})} \cap D_z = \mathcal{A} \tag{13}$$

where $\mathcal{A}$ is defined in (9). Then, we define the jump set of $\mathcal{H}_K$ as follows

$$D_K := \overline{((\underbrace{\mathcal{S}_- \setminus \mathcal{I}_{-M}}_{\mathcal{D}_{-M}}) \times \{-M\})} \cup \overline{((\underbrace{\mathcal{S}_+ \setminus \mathcal{I}_M}_{\mathcal{D}_M}) \times \{M\})} \tag{14}$$

which, due to the assumptions on $\mathcal{I}_{-M}$ and $\mathcal{I}_M$, is such that $D_K \supset D_K^s$.

Remark 3: Observe that due to the definition of the data of $\mathcal{H}$ given in (10), (15), (14), and of $\mathcal{A}$ in (9); one has that

$$g(D_K \setminus \mathcal{A}) \subset C_K \setminus D_K$$

which prevents from the existence of purely discrete solutions to $\mathcal{H}$ from points in $(C_K \cup D_K) \setminus \mathcal{A}$. Such a property directly follows from the definition of the set D_K in (14).

To define the flow set, since our goal is to guarantee that $C_K \cup D_K = \mathbb{R}^2 \times \{-M, M\}$, we select

$$\begin{aligned} C_K := \overline{\mathbb{R}^2 \times \{-M, M\} \setminus D_K} =& (\overline{\mathcal{C}_{-M}} \cup \mathcal{I}_{-M}) \times \{-M\} \cup \\ & (\overline{\mathcal{C}_M} \cup \mathcal{I}_M) \times \{M\} \end{aligned} \tag{15}$$

See Fig. 1 for a pictorial representation of the above defined sets for the case of the double integrator, see also Section IV.

Now we are in a position to state a first result characterizing some properties of the hybrid system $\mathcal{H}$ defined in (8) when restrained to a certain subset of the state space. Before that, we define the following notion.

Definition 8: Let $\xi \in C \cup D$. We say that ξ is viable for $\mathcal{H}$ if there exists $\epsilon > 0$, and an absolutely continuous function $z\colon [0, \epsilon] \to \mathbb{R}^2 \times \{-M, M\}$ such that $z(0) = \xi$ and

$$z(t) \in C, \quad \dot{z}(t) = f(z(t)) \qquad \text{for almost all } t \in [0, \epsilon]$$

Whenever the above property does not hold, we say that ξ is not viable for $\mathcal{H}$ meaning that no flow is allowed from ξ.

Remark 4: Notice that, as pointed out by (13), $D_K \supset \mathcal{A}$. Therefore, since $g(\mathcal{A}) \subset \mathcal{A}$, if points in $\mathcal{A}$ are not viable for $\mathcal{H}$, then $\mathcal{A}$ is strongly forward invariant for $\mathcal{H}$, since the only solution from $\mathcal{A}$ is purely discrete, complete, and stays in $\mathcal{A}$.

Lemma 1: Let $\mathcal{H} = (C_K, f, D_K, g)$ with state $x = (x_p, \eta) \in \mathbb{R}^2 \times \{-M, M\}$ and

$$\begin{aligned} f(x) &= (Ax_p + b\eta, 0) \quad && \forall x \in C_K \\ g(x) &= (x_p, -\eta) && \forall x \in D_K \end{aligned}$$

where C_K and D_K are defined, respectively, in (15) and (14). Assume that each $\xi \in C_K \cap D_K$ is not viable for $\mathcal{H}$ and define $C_K^s = (\overline{C}_{-M} \times \{-M\}) \cup (\overline{C}_M \times \{M\})$. Then, the following properties hold:

(i) Each $\phi := (\phi_p, \phi_\eta) \in \mathcal{S}_{\mathcal{H}}(\mathcal{A})$ is unique, complete, and purely discrete. In particular:

$$\phi(t, j) = (0, (-1)^j \phi_\eta(0, 0)) \qquad \operatorname{dom} \phi = \{0\} \times \mathbb{N}_0$$

(ii) Let $\xi = (\xi_p, \xi_\eta) \in C_K^s \cup D_K$ and $\phi \in \mathcal{S}_{\mathcal{H}}(\xi)$. Then, ϕ is complete, unique, eventually discrete, and in particular for each

$$j \geq \mathcal{L}^\star(\xi_p) + \chi_{\mathcal{D}_{-M} \cup \mathcal{D}_M}(\xi_p)$$

it satisfies

$$(t^\star, j) \in \operatorname{dom} \phi, \quad \phi(t^\star, j) \in \mathcal{A}$$

where $t^\star := \mathcal{J}^\star(\xi_p)$ and $\mathcal{A}$ is given in (9).

Proof: To prove the above result, first observe that the set $C_K^s \cup D_K$ is strongly forward invariant for $\mathcal{H}$. Indeed, solutions from $C_K^s \cap D_K \subset C_K \cap D_K$ can only jump due to points in $C_K \cap D_K$ not viable for $\mathcal{H}$ and by construction $g(D_K) \subset C_K^s \cup D_K$. Moreover, solutions to $\mathcal{H}$ from $C_K^s \setminus D_K$, according to Fact 1, do not leave C_K^s and reach D_K in finite time. Therefore, maximal solutions to $\mathcal{H}$ from $C_K^s \cup D_K$ coincide with maximal solutions to $\mathcal{H}^s = (C_K^s, f, D_K, g)$. Hence, in the remainder of the proof, for simplicity, we will make use of $\mathcal{H}^s$ to prove our statement.

Item (i) follows from the fact that $\xi \in C_K \cap D_K$ is not viable for $\mathcal{H}^s$, $\mathcal{A} \subset D_K$, and $g(\mathcal{A}) \subset \mathcal{A}$, showing that $\mathcal{A}$ is forward invariant for $\mathcal{H}$.

To show item (ii), first observe that, since each $\xi \in C_K^s \cap D_K$ is not viable for $\mathcal{H}^s$, thanks to [4, Proposition 2.11], for every $\xi \in C_K^s \cup D_K$, there exists a unique maximal solution ϕ to $\mathcal{H}^s$ with $\phi(0, 0) = \xi$. Let $\xi = (\xi_p, \xi_\eta) \in (\overline{\mathcal{S}_+} \times \{-M\}) \cup (\overline{\mathcal{S}_-} \times \{M\})$ with $\xi_p \neq 0$, and consider the unique maximal solution ϕ_c to

$$\dot{x}_p = Ax_p + b\kappa(x_p)$$

with $\phi_c(0) = \xi_p$, and for which $\operatorname{dom} \phi_c = [0, \mathcal{J}^\star(\xi_p)]$ and $\phi_c(\mathcal{J}^\star(\xi_p)) = 0$. Let $\{t_j\}_{j=1}^{\mathcal{L}^\star(\xi_p)}$ be the sequence (possibly empty) of switching times of $[0, \mathcal{J}^\star(\xi_p)] \ni t \mapsto \kappa(\phi_c(t))$, and whenever $\mathcal{L}^\star(\xi_p) = 0$, define $t_1 = \mathcal{J}^\star(\xi_p)$. For simplicity, assume that $\kappa(\xi_p) = \xi_\eta$, analogous considerations hold for the general case. Define the following hybrid time domain (see [4] for a definition of hybrid time domain)

$$E = \bigcup_{j=0}^{\mathcal{L}^\star(\xi_p)} [t_j, t_{j+1}] \times \{j\}$$

and notice that

$$\sup E = (T^\star, \mathcal{L}^\star(\xi_p)) \in E$$

and for each $j \in \{1, 2, \ldots, \mathcal{L}^\star(\xi_p)\}$, $[t_j, t_{j+1}]$ is nonempty and with positive length, and $\kappa(\phi_c(t))$ is constant for each $[t_j, t_{j+1})$, with $j \in \{1, 2, \ldots, \mathcal{L}^\star(\xi_p)\}$. Now, for each $(t, j) \in E$, define the following hybrid arc

$$\phi(t,j) = \begin{cases} (\phi_c(t), \lim_{s\uparrow t} \kappa(\phi_c(s))) & \text{if } \{(t,j),(t,j+1)\} \subset E \\ (\phi_c(t), \kappa(\phi_c(t))) & \text{elsewhere} \end{cases}$$

for which one has $\phi(\mathcal{J}^\star(\xi_p), \mathcal{L}^\star(\xi_p)) \in \mathcal{A}$. Then, in light of the definition of $\mathcal{H}$, it is straightforward to check that ϕ is the unique solution to $\mathcal{H}^s$ from ξ. Let ψ be the unique maximal solution to $\mathcal{H}^s$ from $\phi(\mathcal{J}^\star(\xi_p), \mathcal{L}^\star(\xi_p)) \in \mathcal{A}$ which, as shown in item (i), is purely discrete, for each $\mathbb{N}_0 \ni j > \sup_j E$, define $\widetilde{\psi}(j) = \psi(0, j - \sup_j E)$ and $\widehat{E} := \{(t,j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}_0 \colon t = \sup_t E, j > \sup_j E\}$ Then, the following hybrid arc

$$\varphi(t,j) = \begin{cases} \phi(t,j) & \forall (t,j) \in E \\ \widetilde{\psi}(j) & \forall (t,j) \in \widehat{E} \end{cases}$$

with

$$\operatorname{dom} \varphi = E \cup \widehat{E} = E \cup \bigcup_{\mathbb{N}_0 \ni j > \sup_j E} (\sup_t E, j)$$

is the unique maximal solution to $\mathcal{H}^s$ from ξ, and this completes the proof. ■

Remark 5: The applicability of the above result requires points in $C_K \cap D_K$ not being viable for $\mathcal{H}$. On the other hand, such an assumption can be directly verified by inspection of the phase portrait obtained with $u = \pm M$ for the possible realizations of (3) considered in this paper. In particular, such a property is easy to check for points in $\mathcal{A}$.

Lemma 1 shows that, under some mild assumptions, solutions to the closed-loop system $\mathcal{H}$ from the set $C_K^s \cup D_K$ converge to the set $\mathcal{A}$ in minimum ordinary time $t^\star$. The next result illustrates key properties for the closed-loop system (8) and characterizes its behavior from the whole state space $C_K \cup D_K = \mathbb{R}^2 \times \{-M, M\}$.

Proposition 2: Let $\mathcal{H} = (C_K, f, D_K, g)$ with state $x = (x_p, \eta) \in \mathbb{R}^2 \times \{-M, M\}$ and

$$\begin{aligned} f(x) &= (A x_p + b\eta, 0) & \forall x \in C_K \\ g(x) &= (x_p, -\eta) & \forall x \in D_K \end{aligned}$$

where C_K and D_K are defined, respectively, in (15) and (14), and define $C_K^s = (\overline{C}_{-M} \times \{-M\}) \cup (\overline{C}_M \times \{M\})$. Assume that each for each $\phi \in \mathcal{S}_\mathcal{H}(C_K \setminus C_K^s)$, there exists $(T, J) \in \operatorname{dom} \phi$ such that $\phi(T, J) \in D_K$. Moreover, assume that each point $\xi \in C_K \cap D_K$ is not viable for $\mathcal{H}$. Then, the following properties hold:

(i) For each $\xi \in C_K \cup D_K$, there exists a nontrivial solution ϕ such that $\phi(0,0) = \xi$. Moreover, each $\phi \in \mathcal{S}_\mathcal{H}(C_K \cup D_K)$ is complete;

(ii) Let $\xi \in C_K \setminus C_K^s$ and $\phi := (\phi_p, \phi_\eta) \in \mathcal{S}_\mathcal{H}(\xi)$. Then, ϕ is eventually discrete. In particular, there exists $\hat{T} > 0$ such that $[0, \hat{T}] \times \{0\} \in \operatorname{dom} \phi$ and for each

$$j \geq 1 + \mathcal{L}^\star(\phi_p(\hat{T}, 0)))$$

one has $(\hat{T} + \mathcal{J}^\star(\phi_p(\hat{T}, 0)), j) \in \operatorname{dom} \phi$ implies

$$\phi(\hat{T} + \mathcal{J}^\star(\phi_p(\hat{T}, 0)), j) = 0$$

(iii) The set $\mathcal{A}$ in (9) is globally finite time stable for $\mathcal{H}$.

Proof: To prove item (i), notice that, since by assumption maximal solutions to $\mathcal{H}$ from $C_K \setminus C_K^s$ converge to D_K in finite time, and by construction $g(D_K) \subset C_K^s \cup D_K$, existence and completeness of maximal solutions to $\mathcal{H}$ follow from item (ii) in Lemma 1.

To show item (ii), it suffices to observe that by assumption maximal solutions to $\mathcal{H}$ from $C_K \setminus C_K^s$ converge in finite time to D_K, and solutions to $\mathcal{H}$ from $D_K \setminus \mathcal{A}$ converge in one jump in $C_K^s \setminus D_K$. Therefore, one has tails of solutions to $\mathcal{H}$ from $C_K \cup D_K$ are solutions to $\mathcal{H}$ from $C_K^s \setminus D_K$. Hence, thanks to Lemma 1, item (ii) is proven.

To conclude the proof, observe that item (iii) follows directly from item (ii) thanks to Proposition 1, $\mathcal{A}$ being strongly forward invariant and globally finite time attractive for $\mathcal{H}$ with locally bounded settling time function, and maximal solutions to $\mathcal{H}$ being complete. ■

Remark 6: Similarly to Remark 5, the applicability of the above result requires points in $C_K \cap D_K$ not being viable for $\mathcal{H}$, and that maximal solutions from $C_K \setminus C_K^s$ reach the set D_K in finite time. Assumptions can be directly verified by inspection for the possible realizations of (3) considered in this paper.

Remark 7: The above result states that maximal solutions to $\mathcal{H}$ from $\mathbb{R}^2 \times \{-M, M\}$ converge to the set $\mathcal{A}$ in finite-time and points out that $\mathcal{A}$ is globally finite time stable for the closed-loop system. However, maximal solutions to $\mathcal{H}$ from $C_K \setminus C_K^s$ converge to the set $\mathcal{A}$ in non-minimum-time. Nevertheless, notice that

$$C_K \setminus C_K^s = \mathcal{I}_{-M} \times \{-M\} \cup \mathcal{I}_M \times \{M\}$$

where the sets $\mathcal{I}_{-M}$ and $\mathcal{I}_M$ represent a degree of freedom in the design of the controller (7); in particular such sets can be made arbitrarily small. Therefore, the set from which optimality is lost can be determined by choosing the parameters in a convenient way. For this reason, the proposed controller can be seen as an "almost optimal controller". More insights on these aspects are given through numerical examples in the next section.

IV. Case studies

We illustrate the effectiveness of the proposed construction in two specific examples.

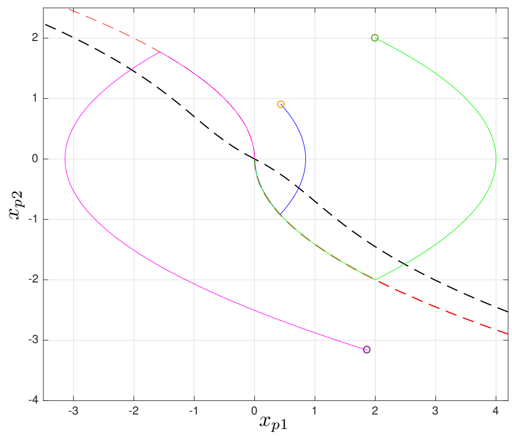

Fig. 2: Evolution of the closed-loop system $\mathcal{H}$ projected onto (x_{p1}, x_{p2})-plane from different initial conditions, the switching surface (dashed-red), and the zero-level set of the function $x \mapsto s(x) + \tau\eta(x)$ (dashed-black).

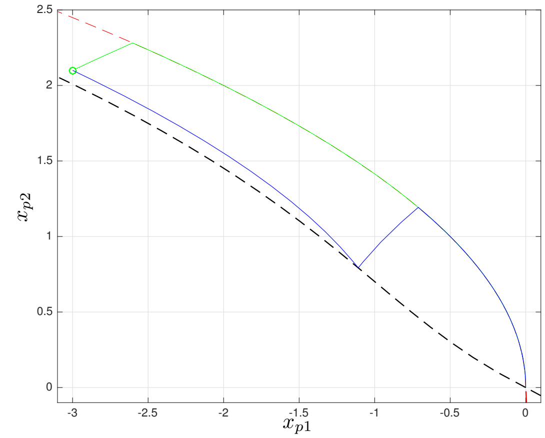

Fig. 3: Evolution of the closed-loop system $\mathcal{H}$ projected onto (x_{p1}, x_{p2})-plane from different initial conditions: $x_0 = (-3, 2.1, 1)$ (green) and $x_0 = (-3, 2.1, -1)$ (blue).

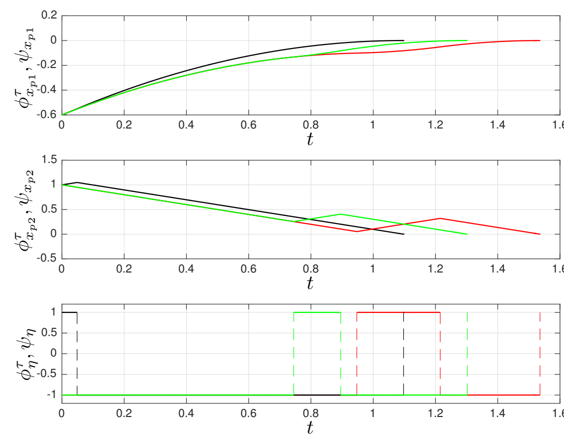

Fig. 4: Solutions ψ (black) and ϕ^τ with $\tau = 1$ (red) and $\tau = 0.25$ (green) projected onto ordinary time.

A. Double Integrator

The optimal control problem (2) is solved for the double integrator by following the methodology presented in Section III-A. In this case, as mentioned earlier, by selecting $M = 1$, the switching surface generator is defined as follows $\mathbb{R}^2 \ni x \mapsto s(x) = x_1 + \frac{1}{2}|x_2|x_2$. To generate the sets C_K and D_K in (15) and (14), respectively, define the following continuous function

$$\mathbb{R}^2 \ni x \mapsto \eta(x) := \begin{cases} (1 - e^{-2x_2}) & \text{if } x_2 \geq 0 \\ -(1 - e^{2x_2}) & \text{if } x_2 < 0 \end{cases}$$

Then, we select

$$\begin{aligned} \mathcal{I}_{-M} &= \{x \in \mathbb{R}_{\leq 0} \times \mathbb{R} \colon s(x) \leq 0, s(x) + \tau\eta(x) \geq 0\} \\ \mathcal{I}_M &= \{x \in \mathbb{R}_{\geq 0} \times \mathbb{R} \colon s(x) \geq 0, s(x) + \tau\eta(x) \leq 0\} \end{aligned} \tag{16}$$

where $\tau > 0$ is a tuning parameter that can be selected to shrink the size of the sets $\mathcal{I}_{-M}$ and $\mathcal{I}_M$, enlarging the set of initial conditions for which minimum-time convergence is guaranteed. On the other hand, notice that by shrinking the sets $\mathcal{I}_{-M}$ and $\mathcal{I}_M$, the response of the resulting closed-loop system approaches the one of the discontinuous closed-loop optimal feedback, which may lead to behavior overly sensitive to measurement noise. Fig. 2 shows some solutions to the closed-loop system $\mathcal{H}$ projected onto (x_{p1}, x_{p2})-plane whenever $\tau = 1$. To underline the impact on the closed-loop response of the initialization of the controller state η, in Fig. 3, the response of the closed-loop system from $(3, 2.1, 1)$ and $(-3, 2.1, -1)$ are reported. As shown in the picture, whenever the controller state η is initialized to the "wrong" value, *i.e.*, -1, the trajectory of the plant state deviates from the optimal solution, but finite time convergence to the origin is guaranteed.

To conclude with this example, we show want to show the influence of the parameter τ in the definition of the sets $\mathcal{I}_{-M}$ and $\mathcal{I}_M$ defined in (16) on the convergence time of the closed-loop system. In particular, in Fig. 4, we compare solutions $\phi^\tau = (\phi^\tau_{x_p}, \phi^\tau_\eta)$ to $\mathcal{H}$ from $(-0.6, 1, -1) \in C_K \setminus C^s_K$ obtained with different value of τ, with the solution $\psi = (\psi_{x_p}, \psi_\eta)$ (τ-independent) to $\mathcal{H}$ from $(-0.6, 1, 1) \in C^s_K$, which gives rise to minimum-time convergence. As shown by the figure, the smaller τ the smaller the convergence time. Specifically, numerical experiments show that for $\tau = 0.15$ minimum-time convergence is practically recovered.

B. Harmonic Oscillator

The optimal control problem (2) is solved for the harmonic oscillator with unitary angular speed, *i.e.*,

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

by following the methodology presented in Section III-A. In this case, as shown in [3], by selecting $M = 1$, the switching

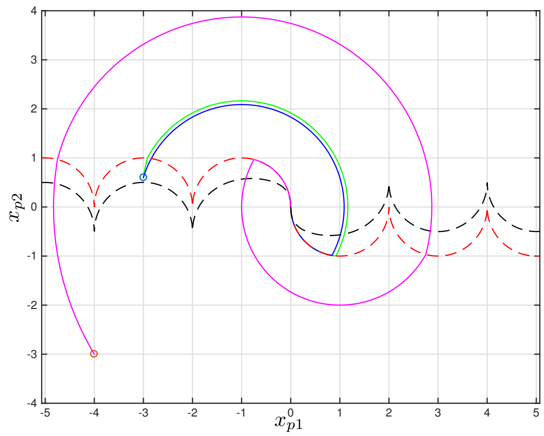

Fig. 5: Evolution of the closed-loop plant state from different initial conditions: $(-4, -3, 1)$ (magenta), $(-3, -0.6, 1)$ (green), $(-3, -0.6, -1)$ (blue), the switching surface (dashed-red), and zero-level set of the function $x \mapsto s(x) + \eta_h(s)$ (dashed-black).

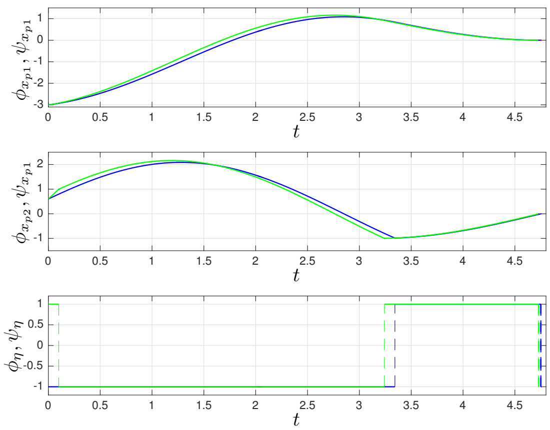

Fig. 6: Solutions ψ (green) and ϕ (blue) projected onto ordinary time.

surface generator is defined as follows

$$s(x) = x_2 + \text{sign}(x_1)\sqrt{1 - \left(x_1 - 2\left\lfloor \frac{1}{2}x_1 \right\rfloor - 1\right)^2}$$

To generate the sets C_K and D_K in (15) and (14), respectively, in a similar fashion as in the previous example, let us define the following continuous function

$$\mathbb{R}^2 \ni x \mapsto \eta_h(x) := \begin{cases} -(1 - e^{-2x_1}) & \text{if } x_1 \geq 0 \\ (1 - e^{2x_1}) & \text{if } x_1 < 0 \end{cases}$$

Then, we select

$$\mathcal{I}_{-M} = \{x \in \mathbb{R}_{\leq 0} \times \mathbb{R} \colon s(x) \leq 0, s(x) + \tau\eta_h(x) \geq 0\}$$
$$\mathcal{I}_M = \{x \in \mathbb{R}_{\geq 0} \times \mathbb{R} \colon s(x) \geq 0, s(x) + \tau\eta_h(x) \leq 0\}$$

Fig. 5 shows the evolution of the closed-loop plant state from different initial conditions whenever $\tau = 0.5$. The figure points out that whenever the controller state is initialized to the "wrong" value, although the resulting trajectory (blue line) deviates from the optimal one (green line), the difference in the evolution is somehow restrained, which reflects on the resulting convergence times. In particular, for such an initial condition the optimal convergence time is ≈ 4.7324, while for the non-optimal solution it is ≈ 4.7527, a mismatch of about 0.429%. These considerations are made evident in Fig. 6, where we compare the solution $\phi := (\phi_{x_p}, \phi_\eta)$ to $\mathcal{H}$ from $(-3, -0.6, -1)$ with the solution $\psi := (\psi_{x_p}, \psi_\eta)$ to $\mathcal{H}$ from $(-3, -0.6, 1)$, which leads to minimum-time convergence.

V. Conclusion

This paper proposed a hybrid controller to solve robustly, and "almost optimally" the minimum-time control problem for a class of planar systems for which a discontinuous state-feedback optimal controller is available. The design of the controller is performed to achieve global finite time stability of a compact set wherein the plant state is zero. Such a property is relevant since it is practically semiglobally (asymptotically) maintained in the presence of small perturbations. The resulting controller provides time optimal response from initial conditions in a certain subset of the state space, and finite time convergence elsewhere.

Future research directions include the extension of the proposed methodology to more general plants, as well as to minimum fuel control problem.

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