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Dominant eigenvalue minimization with trace preserving diagonal perturbation: Subset design problem[☆]

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ABSTRACT

Motivated by network resource allocation needs, we study the problem of minimizing the dominant eigenvalue of an essentially-nonnegative matrix with respect to a trace-preserving or fixed-trace diagonal perturbation, in the case where only a subset of the diagonal entries can be perturbed. Graph-theoretic characterizations of the optimal subset design are obtained: in particular, the design is connected to the structure of a reduced effective graph defined from the essentially-nonnegative matrix. Also, the change in the optimum is studied when additional diagonal entries are constrained to be undesignable, from both an algebraic and graph-theoretic perspective. These results are developed in part using properties of the Perron complement of nonnegative matrices, and the concept of line-sum symmetry. Some results apply to general essentially-nonnegative matrices, while others are specialized for sub-classes (e.g., diagonally-symmetrizable, or having a single node cut).

1. Introduction

The problem of allocating or redistributing limited local control resources to shape a network's dynamics is of interest in several domains, including in the mitigation of network spread processes, management of various compartmental systems, and control of transients in large-scale infrastructures. In many of these application domains, control resources can only be placed or recruited in a limited subset of network locations. The limited control resources thus must be designed to leverage the intrinsic interconnectivity of the network, so as to meet performance criteria. Further, the scale and complexity of the networks often dictate that simple topological rubrics rather than formal methods are needed for resource allocation. Also, in many of these application domains, resource redesign as constraints change is often needed in lieu of or in addition to *ab initio* design.

The purpose of this paper is to study a canonical optimization problem which arises in the design of limited control resources to shape an associated network dynamics. Specifically, a network dynamics defined by an essentially-nonnegative (Metzler) matrix – a matrix whose off-diagonal entries are nonnegative – is considered. Placement of local control resources is abstracted as perturbing diagonal entries of the Metzler matrix (altering local dynamical characteristics). The goal of the design is to optimize this diagonal perturbation, subject to the constraints that (1) only a subset of entries may be perturbed (resource allocations are only permitted at some network locations); and (2) the sum of the perturbed entries is zero (resource re-distribution) or fixed (allocation on a fixed resource budget). The aim of the design is to optimize the dominant eigenvalue of the Metzler matrix, which captures or approximates a dominant propagative dynamics in the network. Succinctly, the problem addressed here is the design of trace-preserving or fixed-trace diagonal perturbations of an essentially nonnegative matrix to minimize a dominant eigenvalue, in the case where only a subset of entries can be designed. We study this *fixed-trace* ***subset design problem***, with a focus on developing graph-theoretic insights into the optimal solution and addressing resource re-design when constraints are changed.

This study extends a research effort in the linear-algebra literature on optimizing the dominant eigenvalue of an essentially-nonnegative matrix over trace-preserving or fixed-trace diagonal perturbations (Johnson, Loewy, Olesky, & Van Den Driessche, 1996; Johnson, Stanford, Dale Olesky, & van den Driessche, 1994), which is part of a broader effort on the fast eigen-decomposition of these matrices (see Johnson, Pitkin, and Stanford (2000), Schneider and Zenios (1990), Zhang, Qi, Luo, and Xu (2013)). These works

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exploit the convexity of the dominant eigenvalue with respect to the diagonal entries along with a similarity transformation to a *line-sum-symmetric* form (where each row sum is equal to the corresponding column sum), to develop computationally-appealing solutions and some structural insights into the optimization (Eaves, Hoffman, Rothblum, & Schneider, 1985; Johnson et al., 1994). This study also contributes to a thrust on resource-constrained control of spread dynamics in the controls community (Enyioha, Preciado, & Pappas, 2013; Preciado, Zargham, Enyioha, Jadbabaie, & Pappas, 2014; Ramirez-Llanos & Martinez, 2014; Ramírez-Llanos & Martínez, 2015; Robertson, Eisenberg, & Tien, 2013; Wan, Roy, & Saberi, 2008), which has addressed parallel optimization problems and generalizations to those considered in the linear-algebra literature, using both structural and numerical approaches (Preciado et al., 2014; Ramírez-Llanos & Martínez, 2015). Of particular relevance, algebraic characterizations of the optimum and numerical optimization algorithms were developed for the subset-design problem in Abad Torres, Roy, and Wan (2017, 2015). The presented research also contributes to a growing effort to characterize the input–output dynamics of sparsely actuated and measured network dynamics (Abad Torres & Roy, 2015b, c; Dhal & Roy, 2013; Liu, Slotine, & Barabási, 2013; Pasqualetti, Zampieri, & Bullo, 2014; Rahmani, Ji, Mesbahi, & Egerstedt, 2009; Roy, Xue, & Das, 2012; Xue, Wang, & Roy, 2014).

Relative to the literature, the main contribution of this study is to (1) develop graph-theoretic insights into the optimal subset design and its performance and (2) systematically address resource re-design as constraints are changed. In particular, we show that the pattern of resource distribution at the optimum is closely tied to the network's graph (the pattern of zero and nonzero entries of the matrix) and the locations of control channels (or designable resources) relative to the graph. Algorithms for resource re-design are also obtained, and the re-allocation is shown to be specially patterned for certain network structures. As a whole, the study shows how resource placements can account for the undesignable structure of a network in shaping response characteristics. We note that some results apply to arbitrary essentially-nonnegative matrices, while are specialized to particular sub-classes (e.g., diagonally symmetrizable, line-sum symmetric, or having a special graph structure).

The article is organized as follows. The design problem is introduced in Section 2. Preliminary algebraic analyses and design algorithms are reviewed in Section 3. Graph-theoretic results on the optimal design are described in Section 4, and the re-design problem is addressed in Section 5. An example is presented (Section 6), and brief conclusions are given (Section 7). Initial results in this direction were given in Abad Torres and Roy (2015a).

2. Problem formulation and notation

An $n \times n$ real essentially-nonnegative (or Metzler) matrix A is considered. The problem of interest is to find a fixed-trace diagonal perturbation matrix $D = \text{diag}(D_1, \dots, D_n)$ such that the dominant eigenvalue of $A + D$ is minimized, subject to the further constraint that some entries of D are restricted to be zero (say, $D_i = 0$ for $i = m+1, \dots, n$, without loss of generality). This problem can be formalized as follows:

$$\begin{aligned} &\underset{D_1,\dots,D_m}{\text{argmin}} \ \lambda_{\max}(A + D) \\ &\qquad \text{s.t. } D_i = 0 \ \ \forall i = m+1, \dots, n, \\ &\qquad \sum_{i=1}^{m} D_i = \Gamma, \end{aligned} \tag{1}$$

where Γ specifies the trace of the imposed perturbation, and $\lambda_{\max}$ refers to the dominant eigenvalue, i.e. the eigenvalue whose real part is largest (most positive). Since $A + D$ is essentially nonnegative, this dominant eigenvalue is real, see Cohen (1981). Some results are focused specifically on the trace-preserving case, where $\Gamma = 0$.

The problem can be interpreted as a resource allocation task, where finite resources D_i are being placed at a subset of network locations to suppress a linear propagative dynamics governed by the state matrix A (with more negative D_i corresponding to higher resource levels). For such network applications, the zero–nonzero pattern of the matrix A specifies the network's topology. Thus, to enable graph-theoretic analysis, we associate with the matrix A a weighted digraph $\mathcal{G} = (V, E : W)$, where the vertices contained in V are labeled $1, \dots, n$, an arc (directed edge) is drawn from vertex i to vertex j ($i \neq j$) if and only if $A_{j,i} \neq 0$, and the arc is assigned a weight $A_{j,i}$.

Some matrix and graph terminology/notation is used in our development. The entries in D that are not constrained to be zero (and corresponding graph vertices) are termed *designable entries (vertices)*; the constrained entries/vertices are called *undesignable*. The diagonal matrix D that minimizes the dominant eigenvalue of $A+D$ is denoted as $\bar{D}$. The dominant eigenvalue and corresponding eigenvectors of $A + \bar{D}$ are denoted as $\bar{\lambda}_{\max}$, $\bar{w}_{\max}$ and $\bar{v}_{\max}$. Further, $w_{\max,i}$ and $v_{\max,i}$ refer to the ith entries of left- and right-eigenvectors associated with the dominant eigenvalue. A couple of standard graph-theoretic terms are also used: a vertex cut set is a set of vertices whose removal results in a disconnected graph, while an (edge) cut set is a set of edges whose removal results in a disconnected graph.

3. Preliminaries: algebraic analysis and algorithms

In Abad Torres et al. (2017), an algebraic analysis was conducted of the spectrum of $A + D$ for the optimal fixed trace subset perturbation design $D = \bar{D}$, and used to develop an algorithm for finding the optimal perturbation. These analyses, which are preliminary to the results developed here, are reviewed (without proof) in the following theorem and lemma.

Theorem 1. *Consider the matrix $A+D$, where $D = diag(D_1, \dots, D_m, 0, \dots, 0)$ and A is a real essentially-nonnegative matrix (which may or may not be irreducible). Consider any $D = \bar{D}$ that minimizes the dominant eigenvalue of $A + D$ subject to $\sum_{i=1}^{m} D_i = \Gamma$. Assume that $A + \bar{D}$ has a real simple dominant eigenvalue. The left and right dominant eigenvectors, $\bar{w}_{\max}$ and $\bar{v}_{\max}$, of $A + \bar{D}$ satisfy one of the following conditions: (1) There exists $\bar{\mu} > 0$ such that $\bar{w}_{\max,i}\bar{v}_{\max,i} = \bar{\mu} \quad \forall i = 1, 2, \dots, m$; (2) $\bar{w}_{\max,i}\bar{v}_{\max,i} = 0 \quad \forall i = 1, 2, \dots, m$. Furthermore, if A is irreducible, then $A+\bar{D}$ has a real simple dominant eigenvalue, and the optimizing $\bar{D}$ and the dominant eigenvectors always satisfy condition 1.*

The algorithm for computing the optimal trace-preserving diagonal perturbation matrix requires some further notation. Specifically, it is useful to partition the *topology matrix* A as $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, where A_{11} is an $m \times m$ matrix. The result also draws on the fact that there always exists a diagonal similarity transformation matrix P such that $PAP^{-1}\vec{1} = P^{-1}A'P\vec{1}$, where $\vec{1}$ is the all ones vector of the appropriate dimension and A' is the transpose of A (see Eaves et al. (1985), Schneider and Zenios (1990) for the computation of the row-sum-symmetrizing transformation P). Here is the algorithm:

Lemma 1. *Consider the matrix $A + D$, where $D = diag(D_1, \dots, D_m, 0, \dots, 0)$, and A is an irreducible essentially-nonnegative matrix. The* *diagonal matrix $D = \bar{D}$ that minimizes the dominant eigenvalue of $A + D$ subject to $\sum_{i=1}^{m} D_i = \Gamma$ can be found as follows:*

1. *Solve the following system of equations for P and $\lambda_{\max}$:*
$$\begin{aligned} PA_r(\lambda_{\max})P^{-1}\vec{1} - P^{-1}A_r(\lambda_{\max})'P\vec{1} &= 0 \\ \vec{1}'PA_r(\lambda_{\max})P^{-1}\vec{1} + \Gamma - \lambda_{\max}\vec{1}'\vec{1} &= 0 \end{aligned} \quad (2)$$
where $A_r(\lambda_{\max}) = A_{11} + A_{12}(\lambda_{\max}I - A_{22})^{-1}A_{21}$, and $P = diag(p_1, \ldots, p_m)$
2. *Calculate $\bar{D}^{(1)} = diag(\lambda_{\max}\vec{1} - PA_r(\lambda_{\max})P^{-1}\vec{1})$ and $\bar{D} = \begin{bmatrix} \bar{D}^{(1)} & 0 \\ 0 & 0 \end{bmatrix}$.*

Remarks. (1) Eqs. (2) admit simple numerical solutions in general, and can be reduced to a single polynomial equation for diagonally symmetrizable A (Abad Torres et al., 2017). (2) Per Theorem 1, the optimal design can be viewed as equalizing the *participation factors* $w_{\max,i}v_{\max,i}$ of the designable channels in the dominant modal dynamics (Pérez-Arriaga, Verghese, & Schweppe, 1982), or equivalently equalizing the sensitivity of the dominant mode to further differential resource placements. (3) The proofs of Theorem 1 and Lemma 1 draw on eigenvalue sensitivity analysis (Wilkinson, Wilkinson, & Wilkinson, 1965), Lagrange multiplier constructs, non-negative matrix analysis, and line-sum symmetrization, see Abad Torres et al. (2017). (4) Theorem 1 slightly generalizes the result given in Abad Torres et al. (2017), in that the irreducible case is explicitly addressed.

4. Graph-theoretic analysis

The subset design algorithm in Lemma 1 can be interpreted as a full trace-preserving diagonal perturbation design for a special "reduced" matrix. This reduced matrix is the Perron complement $A_r(\lambda) = A_{11} + A_{12}(\lambda I - A_{22})^{-1}A_{21}$, evaluated at the particular value λ that achieves the maximum ($\lambda = \bar{\lambda}_{\max}$). The original problem is then translated to the design of a trace-preserving diagonal perturbation, where *all* the diagonal entries can be designed, to minimize the dominant eigenvalue of the *reduced* or *effective* matrix $A_r(\lambda_{\max})$. Consequently, it is natural to associate with the matrix $A_r(\lambda_{\max})$ a weighted digraph $\mathcal{G}_e$, which we call the ***effective graph***. The effective graph is defined as $\mathcal{G}_e = (V_e, E_e : W_e)$, where the vertices contained in V_e are the designable vertices, an arc (directed edge) is drawn from vertex i to vertex j if and only if $A_r(\lambda_{\max})_{j,i} \neq 0$, and the arc is assigned a weight $A_r(\lambda_{\max})_{j,i}$. The effective graph $A_r(\lambda_{\max})$ summarizes interconnections not only within the designable vertices, but also indirect influences through the undesignable part of the graph.

Next, characterizations of the optimal subset design are given in terms of the effective graph $\mathcal{G}_e$ and the original graph $\mathcal{G}$. As a preliminary step, the edge weights in the *original* graph $\mathcal{G}$ and the effective graph $\mathcal{G}_e$ are related:

Theorem 2. *The* effective graph *has a directed edge from i to j ($i = 1, \ldots, m, j = 1, \ldots, m$) if and only if (1) $\mathcal{G}$ has an edge from i to j, or (2) there is a directed path from i to j in $\mathcal{G}$ that passes only through the undesignable vertices (except for i and j). Furthermore, the weight of an edge (i, j) in $\mathcal{G}_e$ is at least the weight of an edge (i, j) in $\mathcal{G}$, i.e. $A_r(\lambda_{\max})_{j,i} \geq A_{j,i}$.*

Proof. The proof is by induction. First, the effective graph is characterized when only one entry of D is undesignable. Then, the effective graph with $k + 1$ undesignable entries is characterized in terms of the case with k designable entries.

The proof requires some notation. Let $A^{(k)}(\lambda_{\max})$ be the $n - k \times n - k$ effective matrix when k vertices are considered undesignable, i.e. at the kth step in the induction. Further, the matrix is partitioned as $A^{(k)}(\lambda_{\max}) = \begin{bmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & A_{22}^{(k)} \end{bmatrix}$ where $A_{11}^{(k)}$ is a $n - k - 1 \times n - k - 1$ matrix and $A_{22}^{(k)}$ is the scalar $A_{n-k,n-k}^{(k)}$. The effective graph matrix associated with $A^{(k)}(\lambda_{\max})$ is $\mathcal{G}^{(k)} = (V^{(k)}, E^{(k)} : W^{(k)})$, where $V^{(k)}$ contains the designable vertices. Further, $A^{(0)} = A$ and $\mathcal{G}^{(0)} = \mathcal{G}$.

Basis:

WLOG, let the nth entry be undesignable. Then, the effective graph matrix is $A^{(1)}(\lambda_{\max}) = A_{11}^{(0)} + A_{12}^{(0)}(\lambda_{\max} - A_{22}^{(0)})^{-1}A_{21}^{(0)}$, where $A_{22}^{(0)}$ is the diagonal entry $A_{n,n}$. Additionally, we see that $Q_a^{(0)} = A_{12}^{(0)}(\lambda_{\max} - A_{22}^{(0)})^{-1}A_{21}^{(0)}$ is a nonnegative matrix, from the definition of A and the fact that the inverse of $\lambda_{\max} - A_{22}^{(0)}$ is positive (see properties of M-matrices (Fiedler, 2008)). In fact, $Q_{a,(i,j)}^{(0)} \neq 0$ if and only if there exists the directed path from j to i that passes through the vertex n. If such a path exists, then we have $A_{i,j}^{(1)} > A_{i,j}^{(0)}$ for $i, j \in \{1, \ldots, n-1\}$. Thus, $\mathcal{G}^{(k)}$ has an edge from j to i, i.e. $A_{i,j}^{(1)} > 0$, if and only if there is an edge from j to i in the graph $\mathcal{G}$, or a path from j to i that passes entirely through the undesignable vertex.

Induction:

Suppose that the undesignable subset has k entries and $A_{i,j}^{(k)} > A_{i,j}^{(k-1)}$ for $i, j \in \{1, \ldots, n-k\}$ if and only if there is a path from j to i that passes through the vertices in the undesignable subset $\{n-k+1, \ldots, n\}$. Additionally, assume that the graph $\mathcal{G}^{(k)}$ has a directed edge (j, i), i.e. $A_{i,j}^{(k)} \neq 0$, if and only if $\mathcal{G}$ has a directed path from j to i that passes entirely through vertices corresponding to the undesignable subset $\{n-k+1, \ldots, n\}$, or it has an edge from j to i.

Let us add one more component to the undesignable subset, say component $n - k$. The effective network matrix is $A^{(k+1)}(\lambda_{\max}) = A_{11}^{(k)} + A_{12}^{(k)}(\lambda_{\max} - A_{22}^{(k)})^{-1}A_{21}^{(k)}$, where $A_{22}^{(k)}$ is the diagonal entry $A_{n-k,n-k}^{(k)}$. It is clear that $Q_a^{(k+1)} = A_{12}^{(k)}(\lambda_{\max} - A_{22}^{(k)})^{-1}A_{21}^{(k)}$ is a nonnegative matrix. In fact, an entry i, j of $Q_a^{(k+1)}$ is nonzero, $\{Q_a^{(k+1)}\}_{i,j} \neq 0$, if and only if there is a directed path from j to i that passes through the component $n - k$ in $\mathcal{G}^{(k)}$. This implies that either the path $j \to n-k \to i$ exists in $\mathcal{G}^{(0)}$ or there is a path from j to $n - k$ that passes through the vertices $\{n-k+1, \ldots, n\}$. If such a path exists, then $A_{i,j}^{(k+1)} > A_{i,j}^{(k)}$ for $i, j \in \{1, \ldots, n-k-1\}$. Also, it immediately follows that $\mathcal{G}^{(k+1)}$ has a directed edge (j, i), i.e. $A_{i,j}^{(k+1)} \neq 0$, if and only if $\mathcal{G}$ has a directed path from j to i that passes entirely through vertices corresponding to the undesignable subset $\{n-k, \ldots, n\}$, or it has an edge from j to i. This proves the induction.

Consider applying the induction until $n - m$ vertices are undesignable, i.e. the designable vertices are $1, \ldots, m$. In this case, we have that $A_r(\lambda_{\max}) = A^{(n-m)}$ and $\mathcal{G}_e = \mathcal{G}^{(n-m)}$. Thus, from the induction, the theorem statement follows immediately.

Fig. 1 shows the original graph $\mathcal{G}$ and the effective graph $\mathcal{G}_e$, for a small example. The reduced graph is seen to encapsulate direct links among the designable vertices and indirect connections through the undesignable graph.

Theorem 2 provides a graph-theoretic interpretation for the subset-design problem. This is valuable because operators of many networks (e.g., critical infrastructures) design resource placements based on simple graph-theoretic insights rather than formal optimizations. The graph-theoretic approach is appealing because it can (1) provide simple-to-implement and intuitive rubrics for design and (2) allow robust solutions that work reasonably well even if models are incomplete/uncertain. The above theorem shows that resource design is closely connected to a network's graph even when only some locations are permitted resources, however the design should be done based on a modified effective graph.

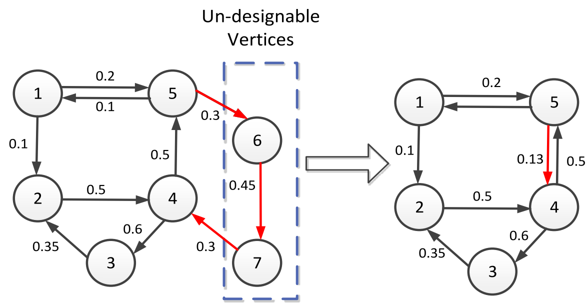

Fig. 1. Graph $\mathcal{G}$ (left), and effective graph $\mathcal{G}_e$ (right).

The following theorem describes a simpler computation of the optimal diagonal trace preserving matrix for the special case that the undesignable and designable vertices in $\mathcal{G}$ are separated by a single-vertex-cut (see Chartrand (2006)). This case is reflective of circumstances where one authority has the wherewithal to enact resource changes in a network partition, while the remainder of the network is undesignable. For this case, the line-sum transformation matrix can be directly computed using only A_{11}, thus simplifying solution of Eq. (2). This analysis follows Johnson et al. (1994), which addresses the full trace-preserving design in the single-vertex-cut case.

Theorem 3. *Suppose the digraph $\mathcal{G}$ associated with the essentially nonnegative matrix A has a single-vertex cut-set. Without loss of generality, let us label the cut-vertex as k, and label the vertices in one partition formed by removal of this vertex by $1, \ldots, k-1$ and in the other by $k+1, \ldots, n$. We assume that the vertices in $V_1 = 1, \ldots, k$ are designable, while those in $V_2 = k+1, \ldots, n$ are not. Then the optimal trace preserving diagonal perturbation $\bar{D}$ can be computed as follows:*

1. *Find a diagonal matrix P such that $PA_{11}P^{-1}\vec{1} = P^{-1}A'_{11}P\vec{1}$.*
2. *Solve for $\lambda_{\max}$: $\vec{1}'PA_r(\lambda_{\max})P^{-1}\vec{1} - \lambda_{\max}\vec{1}'\vec{1} = 0$*
3. *Compute $\bar{D}^{(1)} = diag(PA_r(\lambda_{\max})P^{-1}\vec{1} - \lambda_{\max}\vec{1})$ and $\bar{D} = \begin{bmatrix} \bar{D}^{(1)} & 0 \\ 0 & 0 \end{bmatrix}$.*

Proof. According to Theorem 2, the effective graph $\mathcal{G}_e$ has identical structure to the induced subgraph of $\mathcal{G}$ on V_1. Further, $A_r(\lambda_{\max})_{(i,j)} = A_{11(i,j)}$, for $i, j = 1, \ldots, k$, with the exception that $A_r(\lambda_{\max})_{(k,k)} > A_{11(k,k)}$. Since the only entry in $A_r(\lambda_{\max})$ and A_{11} that is different is a diagonal one, the same matrix P transforms $A_r(\lambda_{\max})$ and A_{11} to line-sum symmetric matrices. The remainder of the proof follows directly from Lemma 1.

The simplification in Theorem 3 depends on the fact that A_{11} and A_r admit the same transformation to line-sum symmetry. This simplification may also arise for other graph structures, for instance if the undesignable part of the graph is symmetric with respect to the designable part (i.e. A_{22} is symmetric and $A_{12} = A_{21}^T$). Similar simplified algorithms for finding the optimal diagonal perturbation can be developed in these cases. Unfortunately, Theorem 3 does not directly generalize to the multi-vertex-cut case, because the modification to A_{11} that yields A_r is no longer diagonal. However, the design obtained from Theorem 2 is simplified even for this case, in the sense that the perturbation is sparse.

The optimal designs developed here are connected to graph-theoretic studies of grounded-Laplacian matrices, see Fitch and Leonard (2013), Pirani, Shahrivar, and Sundaram (2015) and Dorfler and Bullo (2013). These studies characterize performance measures of the network process (e.g., coherence and convergence measures), and some of the papers consider selection of grounding locations to shape the performance measures is also pursued. Relative to this literature, the main contributions of this work are to (1) address the optimal design of constrained resources at a subset of network nodes, (2) show that the designs are fundamentally tied to the Kron reduction, and (3) develop graph-theoretic analyses for the broader class of essentially-nonnegative matrices which correspond to directed graphs.

5. Redesign for changed subsets

We next study how the entries of the optimal diagonal trace-preserving perturbation change when an originally designable entry is constrained to a fixed value. This analysis is useful for redesigning resources when new restrictions on resource allocations come into play, or conversely are alleviated. The results also permit comparison of optimal designs for different designable subsets, and give insight into the design when the design variables are constrained to be within a range (see e.g. Abad Torres et al. (2015), Ramírez-Llanos and Martínez (2015)). The results are developed as follows: first some spectral characterizations and equivalences of the further-constrained design are obtained, then limited-computation algorithms for redesign are presented, and finally some insights into the redesigned solution are developed.

Throughout this development, we primarily compare the optimal trace-preserving design for the original subset design problem, and for a modified problem where a single additional entry has been set to 0. Formally, we consider the matrix $A + D$ where $D = \text{diag}(D_1, \ldots, D_m, 0, \ldots, 0)$, and A is an irreducible essentially-nonnegative matrix. The notation $D = \bar{D}$ is used for the trace-preserving diagonal matrix that minimizes the dominant eigenvalue of $A + D$, i.e. for the solution to the original problem (1). Meanwhile, the notation $\hat{D}$ is used for the trace-preserving diagonal matrix D that minimizes the dominant eigenvalue of $A+D$, when additionally the qth entry of D ($q \in 1, \ldots, m$) has been set to zero. The bar ($\bar{}$) and hat ($\hat{}$) notations are also used to distinguish other characteristics of the two optima, e.g. the optimized dominant eigenvalue, corresponding eigenvectors, etc. The two optima are referred to as the original and *further-constrained* or redesigned solutions. In a couple of results, we consider the case where multiple diagonal entries are set to zero in the further-constrained solution; these cases are made explicit in the theorem statements.

First, the dominant eigenvectors for the original and redesigned optima are compared:

Lemma 2. *The entries of the dominant left and right eigenvectors of $A + \bar{D}$ (the original optimum) $A + \hat{D}$ (the redesigned optimum) are related as follows:*

- *$\hat{w}_{\max,i}\hat{v}_{\max,i} = \hat{\mu} > \hat{w}_{\max,q}\hat{v}_{\max,q}$ for all $i = 1, \ldots, q-1, q+1, \ldots, m$ if $\bar{D}_q > 0$;*
- *$\hat{w}_{\max,i}\hat{v}_{\max,i} = \hat{\mu} < \hat{w}_{\max,q}\hat{v}_{\max,q}$ for all $i = 1, \ldots, q-1, q+1, \ldots, m$ if $\bar{D}_q < 0$;*

where the eigenvectors are assumed scaled to unit length.

Proof. This lemma is proved using a sensitivity analysis of the dominant eigenvalue together with the fact that the dominant eigenvalue of an essentially-nonnegative matrix is a convex function of its diagonal entries (Cohen, 1981). Let $\hat{D} - \bar{D} = \Delta$, where $\Delta = \text{diag}(\delta_1, \ldots, \delta_m, 0, \ldots, 0)$, be the difference between the further-constrained and original optima. Notice that $\sum_{i=1}^m \delta_i = 0$. Suppose that the qth entry in the optimal solution $\bar{D}$ satisfies $\bar{D}_q = \delta > 0$. Then we have that $\delta_q = -\delta$, since the qth entry of $\hat{D}$ is zero. Further, since the trace of Δ is zero, it follows that δ_i, for $i \neq q$, are functionally dependent on δ. The sensitivity of the dominant eigenvalue with respect to δ, which impacts the diagonal entries of $A+\hat{D}$, can be found using the standard eigenvalue sensitivity formula, and then applying the chain rule. Doing this, we find $\partial\lambda_{\max}(A+\hat{D})/\partial\delta = -\hat{w}_{\max,q}\hat{v}_{\max,q} + \sum_{i=1,i\neq q}^{m}\hat{w}_{\max,i}\hat{v}_{\max,i}\partial\delta_i(\delta)/\partial\delta = -\hat{w}_{\max,q}\hat{v}_{\max,q} + \hat{\mu}$, where we have used that $\hat{w}_{\max,i}\hat{v}_{\max,i}$ are identically $\hat{\mu}$ for $i = 1,\ldots,q-1, q+1,\ldots,m$ according to Theorem 1, and have also used the fact that the sum of δ_i, where $i = 1,\ldots,m \forall i \neq q$, is δ.

However, since $\lambda_{\max}(A+\hat{D})$ is the optimal solution, it is known that $\partial\lambda_{\max}(A+\hat{D})/\partial\delta \geq 0$, and further $\hat{w}_{\max,q}\hat{v}_{\max,q} \neq \hat{\mu}$. We thus have that $\hat{D}$ minimizes $\lambda_{\max}(A+D)$ after setting $\bar{D}_q > 0$ to zero, if and only if $\hat{w}_{\max,i}\hat{v}_{\max,i} > \hat{w}_{\max,q}\hat{v}_{\max,q}$.

The second condition follows from an identical analysis.

The lemma can be interpreted as follows, from a network dynamics perspective. If the control resource allocation at a node is decreased due to a constraint (moved from a negative value to 0) and the system is re-optimized, the participation and sensitivity of that node in the dominant modal dynamics increases compared to the other designable nodes. That is, there is more to be gained by replacing resources at that location, if it were still permitted, than to place resources at other designable locations.

The remainder of the section is concerned with developing algorithms for finding the further-constrained optimal solution, and gaining insight into the change. In addition to the spectral result, this analysis requires a classical lemma that expresses the dominant eigenvalue of a nonnegative matrix as a minimization, see Eaves et al. (1985) for the proof:

Lemma 3. *Consider a nonnegative matrix Q in $\mathbb{R}^{n\times n}$ let x and y positive vectors in $\mathbb{R}^n$, and let $g(y,x) = y'Qx$ be a real function defined on $H = \{(x,y) : x, y \in \mathbb{R}^n, x, y > 0, x_iy_i = \mu_i \forall i = 1,\ldots,n\}$. The following two conditions are equivalent: (1) The pair $(\tilde{y},\tilde{x})$ minimizes the function $g(y,x)$ over H. (2) The matrix $\text{diag}(\tilde{y})Q\,\text{diag}(\tilde{x})$ is line-sum symmetric.*

It is clear that if we define $\mu_i = w_{\max,i}v_{\max,i}$, where $w_{\max}$ and $v_{\max}$ are the left- and right-eigenvectors associated with the dominant eigenvalue of Q, the pair $(\tilde{y},\tilde{x})$ minimizing the function $g(y,x)$ over H are $w_{\max}$ and $v_{\max}$.

Next, we show that, as an alternative to finding the redesigned solution $\hat{D}$ directly using Lemma 1, $\hat{D}$ can be found as a further perturbation of the known optimal solution when m entries are designable (i.e., as a further perturbation of $\bar{D}$). Specifically, the following lemma shows that this further perturbation can be found by solving a subset design problem on $m-1$ entries where the perturbation matrix has a fixed trace. The performance gap between the original and the additionally-constrained solutions is also characterized.

Lemma 4. *The further-constrained subset design problem can be solved instead by solving the following optimization problem for $\Delta_1,\ldots,\Delta_{q-1},\Delta_{q+1},\ldots,\Delta_m$:*

$$\begin{aligned}&\underset{\substack{\Delta_i\forall i\neq q\\ i=1,\ldots,m}}{\operatorname{argmin}}\ \lambda_{\max}(A+\bar{D}-\Gamma e_q e_q'+\Delta)\\ &\quad s.t.\ \Delta_i = 0\ \ \forall i = q, m+1, m+2,\ldots,n,\\ &\qquad \sum_{i=1,i\neq q}^{m}\Delta_i = \Gamma,\end{aligned} \tag{3}$$

where $e_q \in \mathbb{R}^n$ is a $0-1$ indicator vector whose qth is 1 and $\Gamma = \bar{D}_q$. The solution then is $\hat{D} = \bar{D}+\Delta-\Gamma e_q e_q'$.

The minimum dominant eigenvalues $\bar{\lambda}_{\max}$ and $\hat{\lambda}_{\max}$ achieved by the designs without and with the additional constraint, respectively, are related by $\bar{\lambda}_{\max} \leq \hat{\lambda}_{\max} \leq \bar{\lambda}_{\max} + \Gamma\left(\hat{\mu}-\mu\right)/\hat{w}_{\max}'\hat{v}_{\max}$, where $\hat{w}_{\max}$ and $\hat{v}_{\max}$ are the left and right eigenvectors associated with the eigenvalue $\hat{\lambda}_{\max}$, $\mu = \hat{w}_{\max,q}\hat{v}_{\max,q}$ and $\hat{\mu} = \hat{w}_{\max,i}\hat{v}_{\max,i}$ where i indicates any other designable entry.

Proof. The equivalence follows immediately from the change of variables $\Delta = D - \bar{D} + \Gamma e_q e_q'$, where we note that only $\Delta_1,\ldots,\Delta_{q-1},\Delta_{q+1},\ldots,\Delta_m$ are nonzero.

To prove the bound on the dominant eigenvalue, we use the new formulation together with properties of essentially-nonnegative matrices and Lemma 3. To simplify the proof, without loss of generality, we suppose that $q = m$. Additionally, we define $Z_0 = A+\bar{D}$, and notice that the dominant eigenvalue of Z_0 is $\bar{\lambda}_{\max}$. Also, we let $Z_1 = Z_0 - \Gamma e_m e_m' + \Delta$, where $\Delta = \text{diag}(\Delta_1,\ldots,\Delta_{m-1},0,0,\ldots,0)$ and $\sum_{i=1}^{m-1}\Delta_i = \Gamma$. The dominant eigenvalue of Z_1 is $\hat{\lambda}_{\max}$. Additionally, we let $\hat{w}_{\max}$, $\hat{v}_{\max}$, $\bar{w}_{\max}$ and $\bar{v}_{\max}$ be the left- and right-eigenvectors associated with the dominant eigenvalues $\hat{\lambda}_{\max}$ and $\bar{\lambda}_{\max}$, respectively.

It is immediate that $\hat{\lambda}_{\max} \geq \bar{\lambda}_{\max}$, since the optimal solution for the further-constrained problem is a feasible solution for the original problem. To prove the second inequality, let $H = \{(x,y) : x, y \in \mathbb{R}^n, x, y > 0, x_iy_i = \hat{w}_{\max,i}\hat{v}_{\max,i}\forall i = 1,\ldots,n\}$. Additionally, $\hat{w}_{\max,i}\hat{v}_{\max,i} = \hat{\mu}$ for $i = 1,\ldots,m-1$, $\hat{w}_{\max,m}\hat{v}_{\max,m} = \mu$ since Z_1 is the matrix that minimizes the dominant eigenvalue after D_m is set to zero. Let us choose the pair $(v_0, s) \in H$ such that $v_0' = \begin{bmatrix}\vec{1}_m' & \bar{v}_0'\end{bmatrix}$ and $s' = \begin{bmatrix}\hat{\mu}\vec{1}_{m-1}' & \mu & \vec{q}'\end{bmatrix}$, where each entry of the vector $\vec{q}$ is $q_i = \frac{\hat{w}_{\max,i+m}\hat{v}_{\max,i+m}}{\bar{v}_{0i}}$. We note that $\bar{v}_0$ contains the last $n-m$ components of the right eigenvector of Z_0. Now, we write $\hat{\lambda}$ as the minimization: $\hat{\lambda}_{\max}\hat{w}_{\max}'\hat{v}_{\max} = \min_{x,y\in H} y'Z_1x = \min_{x,y\in H} y'(Z_0 - \Gamma e_m e_m' + \Delta)x = \min_{x,y\in H} y'Z_0x - \Gamma\mu + \Gamma\hat{\mu}$. Further, $\min_{x,y\in H} y'Z_0x \leq s'Z_0v_0 = \bar{\lambda}_{\max}s'v_0 = \bar{\lambda}_{\max}\hat{w}_{\max}'\hat{v}_{\max}$. Consequently, $\bar{\lambda}_{\max} \leq \hat{\lambda}_{\max} \leq \bar{\lambda}_{\max} + (\Gamma\hat{\mu} - \Gamma\mu)/\hat{w}_{\max}'\hat{v}_{\max}$.

The inequality in the above lemma shows that withdrawal of control resources at one network location does not cause much degradation in the performance of the optimal resource allocation, if either few resources were allocated originally (Γ is small), or the participation/sensitivity of the dominant mode to the resource withdrawal is limited.

Lemma 4 is a starting point for comparing the optimal solutions for the original and further-constrained problems. These comparisons are of interest in control applications since they show how design resources should be re-allocated, if control capabilities become unavailable (or, conversely, available) at certain network locations. They also provide a means for understanding design if resources at some network locations are subject to constraint (Abad Torres et al., 2015). Our first result in this direction focuses on diagonally symmetrizable matrices, and shows that reducing the optimal entries $\bar{D}_i > 0$ to zero, or any other fixed value, will only increase the other entries D_i in the new optimal solution:

Theorem 4. *Assume that A is an irreducible diagonal symmetrizable matrix. Consider the index set $\mathcal{I}_+ = \{i : \bar{D}_i > 0\}$. Suppose that the diagonal entries of D whose indices are in $\mathcal{I}_C \subset \mathcal{I}_+$ are also constrained, i.e. $D_j = 0 \quad \forall j \in \mathcal{I}_C$. Then the entries of the optimal solution $\hat{D}$ with the additional constraints satisfy $\hat{D}_i \geq \bar{D}_i$ for $i \neq j \in \mathcal{I}_C$.*

Proof. Without loss of generality suppose that $\mathcal{I}_C = \{k, k+1,\ldots,m\}$, for $k > 1$ and A is a symmetric matrix. We prove this theorem by induction.

Basis:

Suppose we set $D_m = 0$; noting that $\bar{D}_m = \Gamma$. Let $\bar{\lambda}_{\max}$, $\bar{v}_{\max}$, $\hat{\lambda}_{\max}$, and $\hat{v}_{\max}$ be the dominant eigenvalues and its respective right eigenvector of $A+\bar{D}$ and $A+\hat{D}$, respectively. From Lemma 2, it is clear that the optimal solution for the first $m-1$ entries will satisfy

$\hat{v}^2_{\max,i} > \hat{v}^2_{\max,m}$ for $i = 1, \dots, m-1$. Further, $\hat{\lambda}_{\max} > \bar{\lambda}_{\max}$ since $\bar{\lambda}_{\max}$ is the optimal solution over the first m entries.

Let $Z_0 = A_r(\bar{\lambda}) + \bar{D}$, and $Z_1 = Z_0 - \Gamma e_m e'_m + \begin{bmatrix} \Delta^{(1)} & \\ & 0 \end{bmatrix}$. We want to prove that the entries of the diagonal matrix $\Delta^{(1)}$ are positive.

The Perron complements of the Z_0 and Z_1 are $Z_{0,r}(\bar{\lambda}_{\max}) = Z_{0,11} + (\bar{\lambda}_{\max} - Z_{0,22})^{-1} Z_{0,12} Z_{0,21}$, where $Z_{0,22} = \{Z_0\}_{m,m}$, and $Z_{1,r}(\hat{\lambda}_{\max}) = Z_{0,11} + \Delta^{(1)} + (\hat{\lambda}_{\max} + \Gamma - Z_{0,22})^{-1} Z_{0,12} Z_{0,21}$, respectively. Noting that $Z_{0,r}$, and $Z_{1,r}$ are also symmetric matrix whose dominant eigenvector is a scaling of $\vec{1}$, the all one vector of appropriate dimension.

Let $\beta_0 = \bar{\lambda}_{\max} - Z_{0,22}$ and $\beta_1 = \hat{\lambda}_{\max} + \Gamma - Z_{0,22} = \bar{\lambda}_{\max} + \delta + \Gamma - Z_{0,22}$ with $\delta > 0$. Then, $0 < \beta_0 < \beta_1$ and $0 < \beta_1^{-1} < \beta_0^{-1}$. Additionally, the Perron complement of Z_1 can be written as $Z_{1,r}(\hat{\lambda}_{\max}) = Z_{0,r}(\bar{\lambda}_{\max}) + \Delta^{(1)} - \gamma Z_{0,12} Z_{0,21}$, where $\gamma > 0$. The eigenvector equation of $Z_{1,r}$ can be written as follows: $Z_{1,r}(\hat{\lambda}_{\max})\vec{1} = Z_{0,r}(\bar{\lambda}_{\max})\vec{1} + \Delta^{(1)}\vec{1} - \gamma Z_{0,12} Z_{0,21}\vec{1}$ where $\hat{\lambda}_{\max}\vec{1} = \bar{\lambda}_{\max}\vec{1} + \Delta^{(1)}\vec{1} - \gamma Z_{0,12} Z_{0,21}\vec{1}$. Therefore, $\Delta^{(1)}\vec{1} = \delta\vec{1} + \gamma Z_{0,12} Z_{0,21}\vec{1} > 0$ since $Z_{0,12} Z_{0,21}$ is a nonnegative matrix and $\delta = \hat{\lambda}_{\max} - \bar{\lambda}_{\max}$.

Inductive Step:

Suppose we have set l entries of $\mathcal{I}_C$ to zero. Let us also suppose that $\hat{v}_{\max,i} > \hat{v}_{\max,j}$ for $j = m, m-1, \dots, m-l-1$ and $i = 1, \dots, m-l$ (optimal condition). Additionally, suppose that the entries D_i for $i = k, k+1, \dots, m-l$ are at least 0. We would like to prove that after setting another entry of D, say D_{m-l}, to zero, we still have the appropriate eigenvector pattern (Lemma 2) and satisfy the condition on the entries of D. Let $\lambda^{(l)}_{\max}$ be the optimal dominant eigenvalue after setting l entries of $\mathcal{I}_C$ to zero and $\lambda^{(l+1)}_{\max}$ the dominant eigenvalue after setting one more entry of $\mathcal{I}_C$ to zero. Since $\lambda^{(l)}_{\max}$ is the optimal dominant eigenvalue when one can design $m-l+1$ entries of D, it is clear that $\lambda^{(l+1)}_{\max} > \lambda^{(l)}_{\max}$. Following similar analysis than the one used in the basis step, we see that the first $m-l-1$ entries of the diagonal perturbation have increase. Consequently, we can set all the entries in $\mathcal{I}_C$ to zero and the designable entries not in $\mathcal{I}_C$ will increase.

If A is a diagonally symmetrizable matrix, one can apply a diagonal similar transformation P such that $\tilde{A} = PAP^{-1}$ is symmetric. It follows that the optimal trace-preserving diagonal perturbation $\bar{D}$ minimizes the dominant eigenvalue of both $A + D$ and $\tilde{A} + D$. Hence, the result also holds for the diagonally-symmetrizable case.

The following theorem studies the case when the qth entry is set to zero, where the original optimal satisfies $\bar{D}_q < 0$. In this case, the new optimal on $m-1$ entries can increase or decrease depending on the specifics of the graph topology. Specifically, the change depends on how the vertex q is connected to other vertices in the effective graph.

Theorem 5. *Consider an irreducible diagonal symmetrizable matrix A. Consider the index set $\mathcal{I}_- = \{i : \bar{D}_i < 0\}$. Suppose an entry $q \in \mathcal{I}_C$, where $\mathcal{I}_C \subset \mathcal{I}_-$, is set to zero. Then the entries of the further-constrained optimal solution $\hat{D}$ satisfy:*

- *$\hat{D}_i > \bar{D}_i$ for all $i \in \mathcal{I}_-$ such that there is no directed path $i \to q \to k$ to a vertex $k \in \mathcal{I}_-$ in the effective graph $\mathcal{G}_e$.*
- *$\hat{D}_i < \bar{D}_i$ for some $i \in \mathcal{I}_-$ such that there is directed path $i \to q \to k$ to a vertex $k \in \mathcal{I}_-$ in the effective graph $\mathcal{G}_e$.*

Proof. Without loss of generality suppose that A is symmetric and $m \in \mathcal{I}_-$ is set to zero, i.e. $m = q$. Let $\bar{D}_m = \Gamma = -\alpha$ for $\alpha > 0$. Let $\bar{\lambda}_{\max}$, $\bar{v}_{\max}$, $\hat{\lambda}_{\max}$, and $\hat{v}_{\max}$ be the dominant eigenvalues and its respective right eigenvector of $A + \bar{D}$ and $A + \hat{D}$, respectively. From Lemma 4, it is clear that the optimal solution for the first $m-1$ entries will satisfy $\hat{v}^2_{\max,i} < \hat{v}^2_{\max,m}$ for $i = 1, \dots, m-1$. Further, $\hat{\lambda}_{\max} > \bar{\lambda}_{\max}$ since $\bar{\lambda}_{\max}$ is the optimal solution over the first m entries.

Let $Z_0 = A_r(\bar{\lambda}_{\max}) + \bar{D}$, and $Z_1 = Z_0 - \Gamma e_m e'_m + \begin{bmatrix} \Delta^{(1)} & \\ & 0 \end{bmatrix}$. First, we want to prove that the entries of the diagonal $\Delta^{(1)}_i$ are positive if there is no directed path $i \to m \to k$.

The Perron complements of the Z_0 and Z_1 are $Z_{0,r}(\bar{\lambda}_{\max}) = Z_{0,11} + (\bar{\lambda}_{\max} - Z_{0,22})^{-1} Z_{0,12} Z_{0,21}$, where $Z_{0,22} = \{Z_0\}_{m,m}$, and $Z_{1,r}(\hat{\lambda}_{\max}) = Z_{0,11} + \Delta^{(1)} + (\hat{\lambda}_{\max} + \Gamma - Z_{0,22})^{-1} Z_{0,12} Z_{0,21}$, respectively. Noting that $Z_{0,r}$ and $Z_{1,r}$ are also symmetric matrix whose dominant eigenvector is a scaling of $\vec{1}$, the all one vector of appropriate dimension.

Let $\beta_0 = \bar{\lambda}_{\max} - Z_{0,22}$ and $\beta_1 = \hat{\lambda}_{\max} + \Gamma - Z_{0,22} = \bar{\lambda}_{\max} + \delta + \Gamma - Z_{0,22}$ with $\delta > 0$. Further, $\delta = \frac{\partial \lambda_{\max}(Z_1)}{\partial \alpha} = \frac{\alpha(\hat{v}^2_{\max,m} - \hat{\mu})}{\hat{v}'_{\max}\hat{v}_{\max}}$, and $\delta - \alpha = \delta + \Gamma = \frac{\alpha(\hat{v}^2_{\max,m} - \hat{\mu} - \hat{v}'_{\max}\hat{v}_{\max})}{\hat{v}'_{\max}\hat{v}_{\max}} < 0$ since the product $\hat{v}^2_{\max,m} < \hat{v}'_{\max}\hat{v}_{\max}$. Consequently, $0 < \beta_1 < \beta_0$, or equivalently $0 < \beta_0^{-1} < \beta_1^{-1}$.

The Perron complement of Z_1 can be written as $Z_{1,r}(\hat{\lambda}_{\max}) = Z_{0,r}(\bar{\lambda}_{\max}) + \Delta^{(1)} + \gamma Z_{0,12} Z_{0,21}$, where $\gamma > 0$. The eigenvector equation of $Z_{1,r}$ can be written as $Z_{1,r}(\hat{\lambda}_{\max})\vec{1} = Z_{0,r}(\bar{\lambda}_{\max})\vec{1} + \Delta^{(1)}\vec{1} + \gamma Z_{0,12} Z_{0,21}\vec{1}$, where $\hat{\lambda}_{\max}\vec{1} = \bar{\lambda}_{\max}\vec{1} + \Delta^{(1)}\vec{1} + \gamma Z_{0,12} Z_{0,21}\vec{1}$. Therefore, $\Delta^{(1)}\vec{1} = \delta\vec{1} - \gamma Z_{0,12} Z_{0,21}\vec{1}$. We note that $Q = Z_{0,12} Z_{0,21}$ is a nonnegative matrix. Further, the entries $Q_{k,i}$ and $Q_{i,k}$ are strictly positive if and only if the path $i \to m \to k$ and $k \to m \to i$ exist. Hence, $\Delta^{(1)}_i$ and $\Delta^{(1)}_k$ are positive if such paths does not exist. If such paths exist then $\Delta^{(1)}_i$ and $\Delta^{(1)}_k$ may be positive, but since the trace of $\Delta^{(1)}$ is negative at least one of these entries should be nonpositive.

The previous two theorems show an interesting dichotomy in the re-distribution of resources in a dynamical network. If the resource level at a node is fixed above its original level (i.e., D_q is changed from a positive value to 0), then resource allocations are reduced at all other designable locations in achieving the new optimum. However, if the resource level is fixed below its original level, then the resource allocations at individual designable locations may either increase or decrease.

Next, the original and further-constrained optima are compared when the effective graph has a single-edge cut. Interestingly, when an end of the single-edge-cut is the further-constrained vertex, the change in the optimal solution has a particular spatial pattern:

Theorem 6. *Consider an irreducible essentially-nonnegative matrix A, and suppose the effective graph $\mathcal{G}_e$ has a single-edge-cut, where p and q are the vertices on the two ends of the cut. Further, assume that the only arcs from/to vertex q in the effective graph originate/terminate at vertex p. Suppose the entry associated with vertex q is further constrained to be zero. Let $\bar{D}_q = \Gamma$ be the qth entry of the optimal solution $\bar{D}$. Then the entries of the further-constrained optimal solution $\hat{D}$ satisfy the following:*

- *$\hat{D}_i \geq \bar{D}_i$ for $i \neq q$ if $\Gamma > 0$*
- *$\hat{D}_i \geq \bar{D}_i$ for $i \neq p, q$ if $\Gamma < 0$.*

Proof. Let $\bar{\lambda}_{\max}$, $\bar{v}_{\max}$, $\hat{\lambda}_{\max}$, and $\hat{v}_{\max}$ be the dominant eigenvalues and its respective right eigenvector of $A + \bar{D}$ and $A + \hat{D}$, respectively.

First, consider that $\Gamma > 0$. Let $Z_0 = PA_r(\bar{\lambda}_{\max})P^{-1} + \bar{D}$, i.e. the matrix with the optimal dominant eigenvalue when there are m designable entries. Without loss of generality consider the $q = m$ and $p = m-1$. Few remarks about Z_0: Z_0 is line-sum symmetric and the left- and right-eigenvector are $\vec{1}$, or a scaling of this vector. Let also $Z_{0,r}(\bar{\lambda}_{\max}) = Z_{0,11} + (\bar{\lambda}_{\max} - Z_{0,22})^{-1} Z_{0,12} Z_{0,21}$, where $Z_{0,22} = \{Z_0\}_{m,m}$. $Z_{0,r}$ is the Perron complement of Z_0 and hence has the same dominant eigenvalue than Z_0 and the eigenvectors of $Z_{0,r}$ are the $m-1$ entries of the dominant eigenvector of Z_0.

Let $Z_1 = Z_0 - \Gamma e_m e_m' + \begin{bmatrix} \Delta^{(1)} & \\ & 0 \end{bmatrix}$, i.e. the new optimal for the first $m-1$ entries. The Perron complement of Z_1 is $Z_{1,r}(\hat{\lambda}_{\max}) = Z_{0,11} + \Delta^{(1)} + (\hat{\lambda}_{\max} + \Gamma - Z_{0,22})^{-1} Z_{0,12} Z_{0,21}$. $\hat{\lambda}_{\max} = \bar{\lambda}_{\max} + \delta$ and $\delta > 0$. Let $\beta_0 = \bar{\lambda}_{\max} - Z_{0,22}$ and $\beta_1 = \bar{\lambda}_{\max} + \delta + \Gamma - Z_{0,22}$. We note that $0 < \beta_0 < \beta_1$, or equivalently $0 < \beta_1^{-1} < \beta_0^{-1}$.

We can re-write $Z_{1,r}(\hat{\lambda}_{\max})$ as $Z_{1,r}(\hat{\lambda}_{\max}) = Z_{0,r}(\bar{\lambda}_{\max}) + \Delta^{(1)} - \gamma Z_{0,12} Z_{0,21}$, where $\gamma > 0$. Noting that the only entries of $Z_{0,r}(\bar{\lambda}_{\max})$ that change to form $Z_{1,r}(\hat{\lambda}_{\max})$ are the diagonal ones. Consequently, $Z_{1,r}$ is line-sum symmetric and its dominant right- and left-eigenvalue is a scaling of $\vec{1}$ (see Lemma 3 and Eaves et al. (1985)). We have that $Z_{1,r}(\hat{\lambda}_{\max})\vec{1} = Z_{0,r}(\bar{\lambda}_{\max})\vec{1} + \Delta^{(1)}\vec{1} - \gamma Z_{0,12} Z_{0,21}\vec{1}$ and $\hat{\lambda}_{\max}\vec{1} = \bar{\lambda}_{\max}\vec{1} + \Delta^{(1)}\vec{1} - \gamma Z_{0,12} Z_{0,21}\vec{1}$. Therefore, $\Delta^{(1)}\vec{1} = \delta\vec{1} + \gamma Z_{0,12} Z_{0,21}\vec{1} > 0$.

Now suppose that $\Gamma < 0$. For simplicity let $\Gamma = -\alpha$, where $\alpha > 0$. Then, $Z_1 = Z_0 + \alpha e_m e_m' + \begin{bmatrix} \Delta^{(1)} & \\ & 0 \end{bmatrix}$. The Perron complements of the Z_0 and Z_1 are $Z_{0,r}(\bar{\lambda}_{\max}) = Z_{0,11} + (\bar{\lambda}_{\max} - Z_{0,22})^{-1} Z_{0,12} Z_{0,21}$, where $Z_{0,22} = \{Z_0\}_{m,m}$, and $Z_{1,r}(\hat{\lambda}_{\max}) = Z_{0,11} + \Delta^{(1)} + (\hat{\lambda}_{\max} + \Gamma - Z_{0,22})^{-1} Z_{0,12} Z_{0,21}$, respectively. Noting that $Z_{0,r}$, and $Z_{1,r}$ are also line-sum symmetric matrices whose dominant eigenvector is a scaling of $\vec{1}$.

Let $\beta_0 = \bar{\lambda}_{\max} - Z_{0,22}$ and $\beta_1 = \hat{\lambda}_{\max} + \Gamma - Z_{0,22} = \bar{\lambda}_{\max} + \delta + \Gamma - Z_{0,22}$ with $\delta > 0$. Further, $\delta = \frac{\partial \lambda_{\max}(Z_1)}{\partial \alpha} = \frac{\alpha(\hat{w}_{\max,m}\hat{v}_{\max,m} - \hat{\mu})}{\hat{w}_{\max}'\hat{v}_{\max}}$, and $\delta + \Gamma = \frac{\alpha(\hat{w}_{\max,m}\hat{v}_{\max,m} - \hat{\mu} - \hat{w}_{\max}'\hat{v}_{\max})}{\hat{w}_{\max}'\hat{v}_{\max}} < 0$ since the product $\hat{w}_{\max,m}\hat{v}_{\max,m} < \hat{w}_{\max}'\hat{v}_{\max}$. Consequently, $0 < \beta_1 < \beta_0$, or equivalently $0 < \beta_0^{-1} < \beta_1^{-1}$.

The Perron complement of Z_1 can be written as $Z_{1,r}(\hat{\lambda}_{\max}) = Z_{0,r}(\bar{\lambda}) + \Delta^{(1)} + \gamma Z_{0,12} Z_{0,21}$, where $\gamma > 0$. The eigenvector equation of $Z_{1,r}$ can be written as $Z_{1,r}(\hat{\lambda}_{\max})\vec{1} = Z_{0,r}(\bar{\lambda}_{\max})\vec{1} + \Delta^{(1)}\vec{1} + \gamma Z_{0,12} Z_{0,21}\vec{1}$, where $\hat{\lambda}_{\max}\vec{1} = \bar{\lambda}_{\max}\vec{1} + \Delta^{(1)}\vec{1} + \gamma Z_{0,12} Z_{0,21}\vec{1}$. Therefore, $\Delta^{(1)}\vec{1} = \delta\vec{1} - \gamma Z_{0,12} Z_{0,21}\vec{1}$. Since vertex m is only connected to vertex $m-1$, the entries $\Delta_i^{(1)} > 0$ for $i = 1, \ldots, m-2$. However, the entry $\Delta_{m-1}^{(1)} = -\alpha - (m-2)\delta < 0$ since the trace of $\Delta^{(1)}$ is negative.

6. Example

The effective-graph concept and re-design results are illustrated in an example, which is focused on optimal infection-spread control using multi-group or contact-network models (see Abad Torres et al., (2017)). Here, a multi-group model for spread is considered, which tracks infection prevalences in interconnected subpopulations or groups within a community. Control resources (e.g., quarantine or treatment) can be directed to a subset of these groups, but are expensive and hence budget-constrained. We are interested in designing or redesigning the limited control resources to minimize the dominant time constant of the spread process, so as to achieve the fastest possible elimination of the infection. This design problem can be phrased as the subset design problem considered here, see Abad Torres et al. (2017, 2015).

Specifically, a network with 35 nodes or groups or subpopulations is considered, see Fig. 2. A subset of 20 nodes is amenable to control (Node Set 1). To illustrate the redesign results (Theorems 4 and 5), we also consider that five more nodes may be subject to constraint and become undesignable. We stress that the network matrix is a grounded-Laplacian matrix associated with a weighted digraph.

The optimized dominant eigenvalue and *participation factors* are shown in Table 1, as nodes are sequentially made undesignable (from 20 designable nodes down to 15 designable nodes). The table

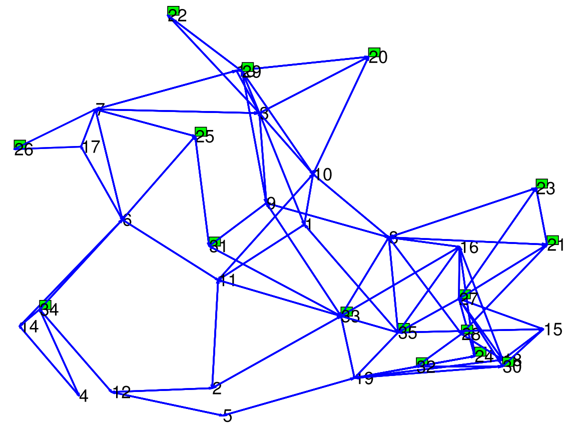

Fig. 2. Network graph.

also shows the optimal resource allocation for the node that was made undesignable. One can observe that the participation factors for each node set move according to Lemma 2. Also, the change in the eigenvalue is correlated with the change in the participation factor and the original value of the diagonal entry that is set to zero, per Lemma 4.

The optimized designs for the six node sets are shown in Fig. 3. The changes in the optimal distributions for the node sets show a pattern, which follows Theorems 4 and 5. For instance, we notice that the optimal resource allocation for Node Set 5 (i.e., values of D_i) are all slightly greater than for Node Set 4. This matches the result of Theorem 4, since the additionally-constrained resource in Node Set 5 (D_{17}) is positive. We note that the optimized designs involve both positive and negative resource allocations at nodes: such designs are acceptable for some spread control problems while others have a sign constraint, see Abad Torres et al. (2107) for further discussion.

Fig. 4 compares the effective graphs when 20 and 15 nodes are designable (Node Set 1 and Node Set 6, respectively). The figure highlights new directed edges (gray edges) that encapsulate the interactions among the designable and the undesignable nodes (Theorem 2). For Node Set 6, the node 15 is seen to have many incident gray edges, since it starts/finishes many paths through the undesignable nodes.

As a whole, the example shows that additional constraints on the designable locations only cause a limited degradation in the performance, with the optimal eigenvalue only increasing from -0.59 to -0.49 between the case with 20 designable nodes and the case with 15 designable nodes.

7. Conclusions

The problem of designing a fixed-trace additive diagonal perturbation to minimize the dominant eigenvalue of an essentially-nonnegative matrix has been studied, in the case where only a subset of the diagonal entries can be designed. The study resulted in two primary contributions: (1) graph-theoretic interpretations of the optimal design and its performance, and (2) insights into re-optimization when additional constraints are placed on the diagonal entries. In particular, the subset-design problem was equivalenced to a full-diagonal-perturbation design problem for a reduced matrix and associated effective graph. This equivalence was then used to give some insight into the optimal resource-distribution pattern. Additionally, redesign of the optimum when additional diagonal entries were constrained to be zero was studied, yielding: (1) comparisons of the dominant eigenvector

Table 1
Optimal dominant eigenvalue.

	Designable nodes	Constrained D_i	$\lambda_{\max}$	$w_{\max,i}v_{\max,i}$
Node Set 1	1-20		−0.59129	0.00791
Node Set 2	1-19	$D_{20} = -0.02111$	−0.59129	0.00791
Node Set 3	1-18	$D_{19} = -1.45565$	−0.56084	0.00607
Node Set 4	1-17	$D_{18} = -1.70717$	−0.52312	0.00391
Node Set 5	1-16	$D_{17} = 0.17046$	−0.52298	0.00393
Node Set 6	1-15	$D_{16} = -2.32504$	−0.49302	0.00263

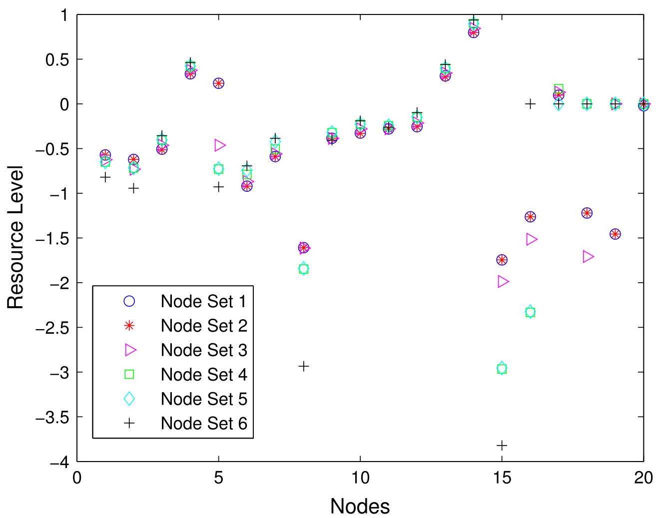

Fig. 3. Resource distribution among the designable nodes.

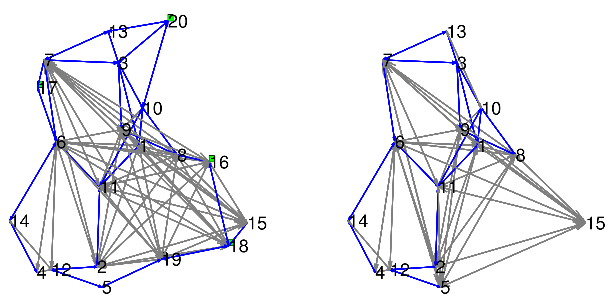

Fig. 4. Effective graph with 20 (left) and 15 (right) designable nodes.

entries, (2) performance bounds between the original and further-constrained design problems, and (3) graph-theoretic results on the resource re-distribution. As a whole, the analyses show that the optimal subset design is closely tied to the structure of the essentially-nonnegative matrix or, equivalently, the topology of the associated graph. These graph-theoretic results are appealing in the context of network resource allocation problems, in that they may enable simple strategies for resource design or re-design to shape performance metrics. Two particular applications of interest are in (1) control of infection spread (e.g., Abad Torres et al. (2017)) and (2) design of leader behavior in network synchronization processes that are subject to noise inputs, see Fitch and Leonard (2013) and Pirani et al. (2015). These applications also motivate similar optimization problems with alternate cost metrics (e.g. the trace of the matrix inverse) and design variables (e.g., multiplicative perturbations, off-diagonal design elements). We anticipate that these alternate problems may admit similar graph-theoretic analyses, but leave the details to further work.

References

Abad Torres, J., & Roy, S. (2015a). Dominant eigenvalue minimization with trace preserving diagonal perturbation: Subset design problem. In *2015 IEEE 54th annual conference on decision and control*. (CDC), (pp. 4208–4213). IEEE.

Abad Torres, J., & Roy, S. (2015b). Graph-theoretic analysis of network input–output processes: Zero structure and its implications on remote feedback control. *Automatica*, *61*, 73–79.

Abad Torres, J., & Roy, S. (2015c). A two-layer transformation for characterizing the zeros of a network input–output dynamics. In *2015 IEEE 54th annual conference on decision and control*. (CDC), (pp. 902–907). IEEE.

Abad Torres, J., Roy, S., & Wan, Y. (2017). Sparse resource allocation for linear network spread dynamics. *IEEE Transactions on Automatic Control*, *62*(4), 1714–1728.

Abad Torres, J., Roy, S., & Wan, Y. (2015). Sparse allocation of resources in dynamical networks with application to spread control. In *American control conference (ACC), Vol. 2015* (pp. 1873–1878). IEEE.

Chartrand, G. (2006). *Introduction to graph theory*. Tata McGraw-Hill Education.

Cohen, J. E. (1981). Convexity of the dominant eigenvalue of an essentially nonnegative matrix. *Proceedings of the Americal Mathematical Society*, *81*(4), 657–658.

Dhal, R., & Roy, S. (2013). Vulnerability of continuous-time network synchronization processes: A minimum energy perspective. In *2013 IEEE 52nd annual conference on decision and control*. (CDC), (pp. 823–828). IEEE.

Dorfler, F., & Bullo, F. (2013). Kron reduction of graphs with applications to electrical networks. *IEEE Transactions on Circuits and Systems. I. Regular Papers*, *60*(1), 150–163.

Eaves, B. C., Hoffman, A. J., Rothblum, U. G., & Schneider, H. (1985). Line-sum-symmetric scalings of square nonnegative matrices. In *Mathematical programming essays in honor of George B. Dantzig Part II* (pp. 124–141). Springer.

Enyioha, C., Preciado, V., & Pappas, G. (2013). Bio-inspired strategy for control of viral spreading in networks. In *Proceedings of the 2nd ACM international conference on high confidence networked systems* (pp. 33–40). ACM.

Fiedler, M. (2008). *Special matrices and their applications in numerical mathematics*. Courier Dover Publications.

Fitch, K., & Leonard, N. E. (2013). Information centrality and optimal leader selection in noisy networks. In *2013 IEEE 52nd annual conference on decision and control*. (CDC), (pp. 7510–7515). IEEE.

Johnson, C. R., Loewy, R., Olesky, D., & Van Den Driessche, P. (1996). Maximizing the spectral radius of fixed trace diagonal perturbations of nonnegative matrices. *Linear Algebra and its Applications*, *241*, 635–654.

Johnson, C. R., Pitkin, J., & Stanford, D. P. (2000). Line-sum symmetry via the domeig algorithm. *Computational Optimization and Applications*, *17*(1), 5–10.

Johnson, C. R., Stanford, D. P., Dale Olesky, D., & van den Driessche, P. (1994). Dominant eigenvalues under trace-preserving diagonal perturbations. *Linear Algebra and its Applications*, *212*, 415–435.

Liu, Y.-Y., Slotine, J.-J., & Barabási, A.-L. (2013). Observability of complex systems. *Proceedings of the National Academy of Sciences*, *110*(7), 2460–2465.

Pasqualetti, F., Zampieri, S., & Bullo, F. (2014). Controllability metrics, limitations and algorithms for complex networks. *IEEE Transactions on Control of Network Systems*, *1*(1), 40–52.

Pérez-Arriaga, I. J., Verghese, G. C., & Schweppe, F. C. (1982). Selective modal analysis with applications to electric power systems, part i: Heuristic introduction. *IEEE Transactions on Power Apparatus and Systems*,(9), 3117–3125.

Pirani, M., Shahrivar, E. M., & Sundaram, S. (2015). Coherence and convergence rate in networked dynamical systems. In *2015 IEEE 54th annual conference on decision and control*. (CDC), (pp. 968–973). IEEE.

Preciado, V. M., Zargham, M., Enyioha, C., Jadbabaie, A., & Pappas, G. J. (2014). Optimal resource allocation for network protection against spreading processes. *IEEE Transactions on Control of Network Systems*, *1*(1), 99–108.

Rahmani, A., Ji, M., Mesbahi, M., & Egerstedt, M. (2009). Controllability of multi-agent systems from a graph-theoretic perspective. *SIAM Journal on Control and Optimization*, *48*(1), 162–186.

Ramirez-Llanos, E., & Martinez, S. (2014). A distributed algorithm for virus spread minimization. In *American control conference (ACC), Vol. 2014* (pp. 184–189). IEEE.

Ramírez-Llanos, E., & Martínez, S. (2015). Distributed and robust resource allocation algorithms for multi-agent systems via discrete-time iterations. In *2015 IEEE 54th annual conference on decision and control.* (CDC), (pp. 1390–1395). IEEE.

Robertson, S. L., Eisenberg, M. C., & Tien, J. H. (2013). Heterogeneity in multiple transmission pathways: modelling the spread of cholera and other waterborne disease in networks with a common water source. *Journal of Biological Dynamics*, *7*(1), 254–275.

Roy, S., Xue, M., & Das, S. K. (2012). Security and discoverability of spread dynamics in cyber-physical networks. *IEEE Transactions on Parallel and Distributed Systems*, *23*(9), 1694–1707.

Schneider, M. H., & Zenios, S. A. (1990). A comparative study of algorithms for matrix balancing. *Operations Research*, *38*(3), 439–455.

Wan, Y., Roy, S., & Saberi, A. (2008). Designing spatially heterogeneous strategies for control of virus spread. *IET Systems Biology*, *2*(4), 184–201.

Wilkinson, J. H., Wilkinson, J. H., & Wilkinson, J. H. (1965). *The algebraic eigenvalue problem. Vol. 87*. Oxford: Clarendon Press.

Xue, M., Wang, W., & Roy, S. (2014). Security concepts for the dynamics of autonomous vehicle networks. *Automatica*, *50*(3), 852–857.

Zhang, L., Qi, L., Luo, Z., & Xu, Y. (2013). The dominant eigenvalue of an essentially nonnegative tensor. *Numerical Linear Algebra with Applications*, *20*(6), 929–941.

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