Comments on "Upper and Lower Bounds for Controllable Subspaces of Networks of Diffusively Coupled Agents"

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***Abstract*—A condition for controllability of a network of diffusively coupled linear agents with sparse actuation, which was developed in [1], is examined. An example is presented which demonstrates that the condition is not always sufficient for controllability. The gap in the original proof is clarified, and issues related to developing a necessary and sufficient condition are briefly discussed.**

***Index Terms*—Controllability of dynamical networks, eigenvalue decomposition.**

Several recent studies have considered manipulation of multiagent networks via actuation at a subset of the agents, e.g., [1]–[4]. Theorem 1 proposed in [1], which gives a condition for the controllability of a network of diffusively coupled linear agents where some leaders can be actuated, is an important contribution in this direction. The network controllability question considered in [1] can be expressed formally as the controllability of the pair $(\hat{L}, \hat{M})$, where: $\hat{L} = I_n \otimes A - L \otimes CK$, $\hat{M} = M \otimes B$, n is the number of agents, A is the (common) state matrix of each agent, L is an $n \times n$ symmetric Laplacian matrix that specifies the diffusive coupling topology, CK indicates the (homogeneous) structure of the interagent couplings, M is a matrix whose columns are 0–1 indicators of the leader agents, and B is a local input matrix that specifies how the external input signals actuate the leader agents. Theorem 1 proposed in [1] decomposes controllability of $(\hat{L}, \hat{M})$ into local and network-level controllability conditions. Specifically, it claims that the pair $(\hat{L}, \hat{M})$ is controllable if and only if the following conditions both hold:

i) the pair (L, M) is controllable (network-level condition);
ii) the pairs $(A - \lambda_i CK, B)$ are controllable for $i = 1, \ldots, n$, where λ_i are the eigenvalues of L (local conditions).

The conditions given in Theorem 1 of [1] are incomplete. In particular, while the conditions i) and ii) in the theorem are necessary for controllability of $(\hat{L}, \hat{M})$, they may not be sufficient, as shown by the following example with $n = 3$ agents: $A = \begin{bmatrix} -1.5 & -0.5 \\ -0.5 & -1.5 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $C = I_2$, $K = \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}$, $L = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$, and $M = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. For this example, the pair (L, M) can be readily checked to be controllable. Likewise, the pairs $(A - \lambda_i CK, B)$ for $i = 1, 2, 3$ are controllable. However, $(\hat{L}, \hat{M})$ is not controllable: specifically, the controllability matrix has rank 4 rather than 6.

The proof of sufficiency in [1] relies on a similarity transform which equivalences controllability of $(\hat{L}, \hat{M})$ to that of an alternate pair $(\tilde{L}, \tilde{M})$, where $\tilde{L} = \text{blockdiag}(A - \lambda_i CK)$, $\tilde{M} = \begin{bmatrix} \tilde{M}_1 \\ \vdots \\ \tilde{M}_n \end{bmatrix}$ (with the blocks having commensurate dimension to the diagonal blocks of $\tilde{L}$), $\tilde{M}_i = Q_i \otimes B$, and each Q_i is a function of the eigenvectors of L and the matrix M, see [1] for details. The gap in the proof arises from the fact that controllability of the pairs $(A - \lambda_i CK, \tilde{M}_i)$ is not sufficient for controllability of $(\tilde{L}, \tilde{M})$. Insufficiency may arise, specifically, in the case that the diagonal blocks $A - \lambda_i CK$ of $\tilde{L}$ have repeated (nondefective) eigenvalues across them. Under these circumstances, $\tilde{L}$ may have a left eigenvector with nonzero entries across multiple blocks which is in the null space of $\tilde{M}$, even though none of the left eigenvectors of each diagonal block $A - \lambda_i CK$ are in the null space of $\tilde{M}_i$. In consequence, it is possible that (L, M) and $(A - \lambda_i CK, B)$, $i = 1, \ldots, n$, are all controllable, yet $(\tilde{L}, \tilde{M})$ and hence $(\hat{L}, \hat{M})$ are uncontrollable. Indeed, for the example presented above, the matrix $\tilde{L}$ has eigenvalues at -2 and -4, which are each repeated across two diagonal blocks.

Importantly, sufficiency of the condition in [1] can only be lost when the diagonal blocks $A - \lambda_i CK$ of $\tilde{L}$ corresponding to distinct λ_i share a common eigenvalue. Thus, the necessary and sufficient condition of [1] is valid, when the analysis is restricted to models which do not have shared eigenvalues across the diagonal blocks $A - \lambda_i CK$. It is therefore of interest to develop conditions on A, L, and CK which guarantee that the diagonal blocks $A - \lambda_i CK$ do not share eigenvalues. Alternately, to give a treatment for arbitrary models, the left eigenspaces of $\tilde{L}$ corresponding to the shared eigenvalues need to be characterized. These issues are considered in more depth, as part of a broader study on input–output processes in dynamical networks, in [5].

Broadly, the discussion here exposes that the eigenvector analysis of diffusive network models is incompletely understood, in contrast with the classical eigenvalue analysis for these models [6]. The subtleties in the eigenvector analysis need to be resolved to fully characterize the input–output dynamics of diffusive networks. We believe this to be a fruitful direction of future work.

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