

A DISCRETE UNIFORMIZATION THEOREM FOR POLYHEDRAL SURFACES

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Abstract

A discrete conformality for polyhedral metrics on surfaces is introduced in this paper. It is shown that each polyhedral metric on a compact surface is discrete conformal to a constant curvature polyhedral metric which is unique up to scaling. Furthermore, the constant curvature metric can be found using a finite dimensional variational principle.

1. Introduction

1.1. Statement of results. The Poincaré–Köbe uniformization theorem for Riemann surfaces is a pillar in the last century mathematics. It states that given any Riemannian metric on a connected surface, there exists a complete constant curvature Riemannian metric conformal to the given one. In particular, any simply connected surface with a Riemannian metric is conformally diffeomorphic to the Riemann sphere $\mathbb{S}^2$, the plane $\mathbb{C}$ or the unit disk $\mathbb{D}$. When the Riemann surface is a simply connected domain in the complex plane, the conformal diffeomorphism is the Riemann mapping. There are many effective algorithms and softwares for computing the Riemann mapping. See [40], [26], [39], [38], [20] and others. However, computing the conformal diffeomorphisms, to be called the *uniformization maps*, for surfaces not isometric to planar domains has been a challenging problem. The goal of this paper and its sequel ([29], [30]) is to give a solution to the problem. This is achieved in two steps. In the first step, we establish a discrete counterpart of the uniformization theorem for compact polyhedral surfaces. In the second step, we show that discrete uniformization maps converge to the uniformization map when the triangulations are suitably chosen. The first step is achieved in this paper. In [29], we prove a convergence theorem.

There have been many research activities on establishing various discrete versions of the uniformization theorem and the Riemann mapping

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theorem for polyhedral metric surfaces or topologically triangulated surfaces. See, for instance, [39], [38], [12], [37], [22], [20] and others. A key step in discretization is to define the concept of discrete conformality. Different approaches depend on different notions of discrete conformality. The most prominent one is probably Thurston's circle packing theory ([39], [38]). However, not all polyhedral metrics are of circle packing type. In this paper, we introduce a new discrete conformality for all polyhedral metrics and establish a discrete uniformization theorem for polyhedral metrics (PL metrics) on compact surfaces.

Polyhedral surfaces are ubiquitous in computer graphics and many fields of sciences due to digitization. Organizing polyhedral surfaces according to their conformal classes is a very useful and important principle. Each polyhedral surface carries a natural Riemann surface structure. However, there are no known algorithms to decide if two polyhedral surfaces are conformal in the classical (Riemannian) sense and to compute the uniformization metric. The discrete conformality introduced in this paper is algorithmic.

Given a closed connected surface S and a finite non-empty set $V \subset S$, we call (S, V) a *marked surface*. The objects of our investigation are *polyhedral metrics* (or simply PL metrics) on (S, V) . By definition, a PL metric on (S, V) is a flat cone metric on S whose cone points are contained in V . Geometrically these metrics are obtained by isometric gluing of Euclidean triangles along their edges. For instance, the boundary of a tetrahedron in the 3-space is a PL metric on the 2-sphere with four marked points. The absolute values of holomorphic quadratic differentials on Riemann surfaces are other examples of PL metrics. The *discrete curvature* of a PL metric on (S, V) is the function on V sending a vertex $v \in V$ to 2π less the cone angle at v . A triangulation $\mathcal{T}$ of S with vertex set V is called a *triangulation* of (S, V) . Each PL metric d on (S, V) has a *Delaunay triangulation* $\mathcal{T}(d)$ of (S, V) such that each triangle in $\mathcal{T}(d)$ is Euclidean and the sum of two angles facing each edge is at most π .

Definition 1.1. (Discrete conformality and discrete Riemann surface) Two PL metrics d, d' on (S, V) are discrete conformal if there exist sequences of PL metrics $d_1 = d, \dots, d_m = d'$ on (S, V) and triangulations $\mathcal{T}_1, \dots, \mathcal{T}_m$ of (S, V) satisfying

- (a) (Delaunay condition) each $\mathcal{T}_i$ is Delaunay in d_i ,
- (b) (Vertex scaling condition) if $\mathcal{T}_i = \mathcal{T}_{i+1}$, there exists a function $u : V \rightarrow \mathbb{R}$ so that if e is an edge in $\mathcal{T}_i$ with end points v and v' , then the lengths $l_{d_{i+1}}(e)$ and $l_{d_i}(e)$ of e in d_i and d_{i+1} are related by

$$(1) \quad l_{d_{i+1}}(e) = l_{d_i}(e)e^{u(v)+u(v')},$$

- (c) if $\mathcal{T}_i \neq \mathcal{T}_{i+1}$, then (S, d_i) is isometric to (S, d_{i+1}) by an isometry homotopic to the identity in (S, V) .

The discrete conformal class of a PL metric is called a *discrete Riemann surface*.

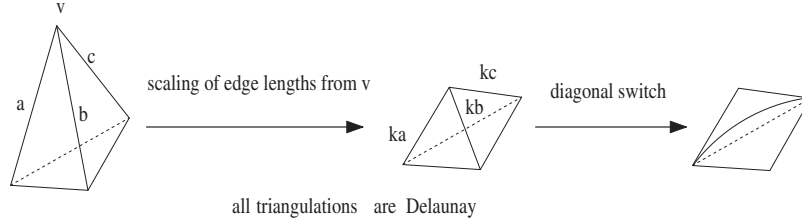


Figure 1. Discrete conformal change of PL metrics.

The vertex scaling condition (1), first introduced in [33] and [27], is a reflection of the following fact from Riemannian geometry (see [29]). Suppose (Σ, g) is a closed surface with a Riemannian metric g whose Riemannian distance is d_g and $\lambda : \Sigma \rightarrow \mathbb{R}_{>0}$ is a smooth function. Then there exists a constant C such that for all points $v, v' \in \Sigma$,

$$|d_{\lambda^4 g}(v, v') - \lambda(v)\lambda(v')d_g(v, v')| \leq C d_g^3(v, v').$$

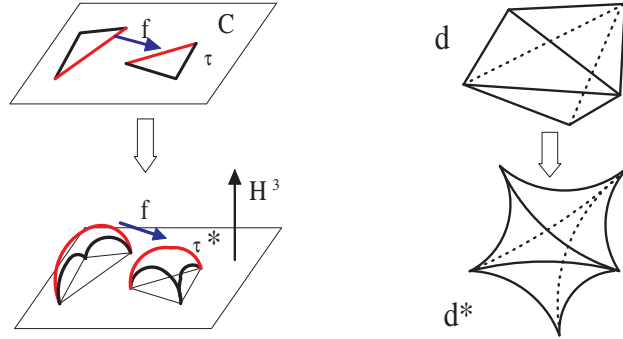


Figure 2. Discrete conformality in terms of hyperbolic geometry.

Theorem 1.2. *Suppose (S, V) is a closed connected marked surface and d is a PL metric on (S, V) . Then for any $K^* : V \rightarrow (-\infty, 2\pi)$ with $\sum_{v \in V} K^*(v) = 2\pi\chi(S)$, there exists a PL metric d' , unique up to scaling and isometry homotopic to the identity on (S, V) , such that d' is discrete conformal to d and the discrete curvature of d' is K^* . Furthermore, the metric d' can be found using a finite dimensional (convex) variational principle.*

In the special case of S being the torus $\mathbb{S}^1 \times \mathbb{S}^1$ and $K^* = 0$, the above result was first proved by Fillastre [16] in a different content. Fillastre

proved an existence and uniqueness of isometric embedding of cusped hyperbolic tori into hyperbolic 3-manifolds in [16]. See also the work of Schlenker [36]. The equivalent of these two results is proved in [29].

For the constant function $K^* = 2\pi\chi(S)/|V|$ in Theorem 1.2, we obtain a constant curvature PL metric d' , unique up to scaling, discrete conformal to d . This is a discrete version of the uniformization theorem. Theorem 1.2 also holds for compact marked surfaces with non-empty boundary. In that case, we double the surface to obtain a closed surface. We omit the details.

1.2. Discrete conformality, hyperbolic geometry, and the Ptolemy identity. The relationship between PL metrics and hyperbolic geometry was first discovered in an important paper by Bobenko–Pinkall–Springborn [6]. Using their work and the Delaunay condition, one sees that the discrete conformality introduced in Definition 1.1 is closely related to convex hull geometry in hyperbolic spaces. Indeed, the following is equivalent to Definition 1.1. Given a PL metric d on (S, V) , let $\mathcal{T}$ be a Delaunay triangulation of (S, V, d) . For each Euclidean triangle $\tau = \triangle ABC$ in $\mathcal{T}$, replace τ by the ideal hyperbolic triangle τ^* in the hyperbolic 3-space $\mathbb{H}^3$ such that τ^* and τ have the same set of vertices $\{A, B, C\}$ in $\mathbb{C}$. Here we consider $\mathbb{C}$ to be in the sphere at infinity of the hyperbolic 3-space $\mathbb{H}^3 = \mathbb{C} \times \mathbb{R}_{>0}$. If τ and σ are two Euclidean triangles in $\mathcal{T}$ glued along their common edge by a Euclidean isometry f , then one glues τ^* and σ^* along their corresponding edges by the *same* isometry f , considered as a rigid motion of $\mathbb{H}^3$. In this way, one produces a complete finite area hyperbolic metric d^* on $S - V$. From the definition of Delaunay triangulation, one sees that d^* is independent of the choices of the Delaunay triangulation $\mathcal{T}$. It will be shown (Corollary 4.8) that two PL metrics d_1 and d_2 on (S, V) are discrete conformal if and only if d_1^* is isometric to d_2^* by an isometry isotopic to the identity in (S, V) .

There are many directions which one can generalize Theorem 1.2. For instance, we can define discrete conformality for spherical, hyperbolic polyhedral surfaces and even polyhedral surfaces obtained by isometric gluing of triangles in the Minkowski plane, or de Sitter space. These will be addressed in our future work.

Our guiding principle for defining discrete conformality is the *Delaunay condition* and the *Ptolemy identity*. The Delaunay condition selects the correct class of triangulations and the Ptolemy identity defines the appropriate “vertex scaling” deformation of the metrics.

Let a, b, a', b' be the edge lengths of a quadrilateral Q labelled cyclically and c, c' be the lengths of the diagonals so that Q is inscribed to a circle. When Q is in the Euclidean plane, then these lengths satisfy the Ptolemy identity

$$aa' + bb' = cc'.$$

Ptolemy's theorem for spherical and hyperbolic quadrilaterals were proved by Darboux, Frobenius and Kubota over one hundred years ago. The results still take the same form $AA' + BB' = CC'$ as the Ptolemy identity. Namely, for Q in the 2-sphere,

$$\sin\left(\frac{a}{2}\right)\sin\left(\frac{a'}{2}\right) + \sin\left(\frac{b}{2}\right)\sin\left(\frac{b'}{2}\right) = \sin\left(\frac{c}{2}\right)\sin\left(\frac{c'}{2}\right),$$

and Q in the hyperbolic plane,

$$\sinh\left(\frac{a}{2}\right)\sinh\left(\frac{a'}{2}\right) + \sinh\left(\frac{b}{2}\right)\sinh\left(\frac{b'}{2}\right) = \sinh\left(\frac{c}{2}\right)\sinh\left(\frac{c'}{2}\right).$$

The above two equations suggest, similar to (1), the correct “vertex scaling” for spherical and hyperbolic polyhedral metrics should be

$$(2) \quad \sin\left(\frac{l_{d_{i+1}}(e)}{2}\right) = \sin\left(\frac{l_{d_i}(e)}{2}\right)e^{u(v)+u(v')},$$

and

$$(3) \quad \sinh\left(\frac{l_{d_{i+1}}(e)}{2}\right) = \sinh\left(\frac{l_{d_i}(e)}{2}\right)e^{u(v)+u(v')}.$$

These two definitions of vertex scaling operation were first introduced in the work of Bobenko–Pinkall–Springborn [6].

Since the Ptolemy identity still holds for quadrilaterals inscribed to a curve of constant curvature in both the Minkowski plane and the de Sitter space, we can define the similar discrete conformality for polyhedral metrics with the Minkowski plane and the de Sitter space as the underlying geometry.

Furthermore, Ptolemy theorem's has been generalized by John Casey in 1866 to the case of four disjoint disks tangent to a circle from one side. Let a, a', b, b', c, c' be the lengths of the exterior common tangent lines between pairs of disks. Then Casey's theorem says that the Ptolemy identity $aa' + bb' = cc'$ still holds. Casey's theorem was generalized by Darboux and Frobenius to the spherical geometry in 1880's and by Kubota [25] to the hyperbolic geometry in 1912. Our recent work [14] shows that Casey's theorem still holds in the Minkowski plane and the de Sitter space. All of the formulas take the same form as the Ptolemy identity $AA' + BB' = CC'$. Therefore, we can define the similar discrete conformality using the Delaunay condition and the “vertex scaling” deformation governed by the Ptolemy identity $aa' + bb' = cc'$ for polyhedral metrics produced in Casey's situation.

1.3. Discrete conformality in spherical and hyperbolic geometries. The counterpart of Theorem 1.2 for hyperbolic polyhedral metrics on (S, V) has been proved in [21] which is a sequel to the current paper. In this case, two hyperbolic polyhedral metrics d, d' on (S, V) are *discrete conformal* if there exist sequences of hyperbolic polyhedral metrics

$d_1 = d, \dots, d_m = d'$ on (S, V) and triangulations $\mathcal{T}_1, \dots, \mathcal{T}_m$ of (S, V) satisfying (a) each $\mathcal{T}_i$ is Delaunay in d_i , and (b) if $\mathcal{T}_i = \mathcal{T}_{i+1}$, there exists a function $u : V \rightarrow \mathbb{R}$ so that if e is an edge in $\mathcal{T}_i$ with end points v and v' , then the lengths $l_{d_{i+1}}(e)$ and $l_{d_i}(e)$ of e in d_i and d_{i+1} are related by (3), and (c) if $\mathcal{T}_i \neq \mathcal{T}_{i+1}$, then (S, d_i) is isometric to (S, d_{i+1}) by an isometry homotopic to the identity in (S, V) .

Theorem 1.3 ([21]). *Suppose d is a hyperbolic polyhedral metric on a closed connected marked surface (S, V) . Then for any $K^* : V \rightarrow (-\infty, 2\pi)$ with $\sum_{v \in V} K^*(v) > 2\pi\chi(S)$, there exists a unique hyperbolic polyhedral metric d' on (S, V) so that d' is discrete conformal to d and the discrete curvature of d' is K^* . Furthermore, the metric d' can be found using a finite dimensional (convex) variational principle. In particular, if $\chi(S) < 0$ and $K^* = 0$, each hyperbolic polyhedral metric on (S, V) is discrete conformal to a unique hyperbolic metric on S .*

We remark that the special case of $K^* = 0$ in Theorem 1.3 was a result of Fillastre [16]. See also [36]. Fillastre proved his theorem in a different content on the isometric embedding of punctured hyperbolic surfaces as the boundaries of convex hulls in Fuchsian hyperbolic 3-manifolds. We thank a referee for pointing out the equivalence of Fillastre's theorem and Theorem 1.3 for $K^* = 0$. See [29] and [21] for more details.

Theorem 1.4 ([21]). *Suppose d and d' are two Euclidean (or hyperbolic or spherical) polyhedral metrics given as isometric gluing of geometric triangles on a closed marked surface (S, V) . There exists an algorithm to decide if d and d' are discrete conformal.*

1.4. Convergence of discrete uniformization maps. The convergence theorem that we prove in [29] is the following. It is motivated by Thurston's conjecture on the convergence of circle packing maps to the Riemann mapping. Thurston's conjecture was solved by B. Rodin and D. Sullivan in [34].

Given a simply connected polygonal disk with a PL metric (D, V, d) and three boundary vertices $p, q, r \in V$, let the metric double of (D, V, d) along the boundary be the polyhedral 2-sphere $(\mathbb{S}^2, V', d')$. Using Theorem 1.2, one produces a new polyhedral surface $(\mathbb{S}^2, V', d^*)$ such that: 1) $(\mathbb{S}^2, V', d^*)$ is discrete conformal to $(\mathbb{S}^2, V', d')$; 2) the discrete curvatures of d^* at p, q, r are $4\pi/3$; 3) the discrete curvatures of d^* at all other vertices are zero; and 4) the area of $(\mathbb{S}^2, V', d^*)$ is $\sqrt{3}/2$. Therefore, $(\mathbb{S}^2, V', d^*)$ is isometric to the metric double $(\mathcal{D}(\Delta ABC), V'', d'')$ of an equilateral triangle ΔABC of edge length 1. Let F be the discrete conformal map from $(\mathcal{D}(\Delta ABC), V'', d'')$ to $(\mathbb{S}^2, V', d')$ such that F sends A, B, C to p, q, r respectively. Due to the uniqueness part of Theorem 1.2, we may assume that $f = F| : \Delta ABC \rightarrow D$ and f sends A, B, C to p, q, r respectively. We call f the *discrete uniformization map* associated $(D, V, d, (p, q, r))$. Given a Jordan domain Ω , Caratheodory's

extension theorem says that the Riemann mapping from the unit disk $\mathbb{D}$ to Ω extends to a homeomorphism from the closure $\overline{\mathbb{D}}$ of $\mathbb{D}$ to the closure $\overline{\Omega}$. In particular, given three boundary points p, q, r of Ω , there exists a unique homeomorphism g from ΔABC to $\overline{\Omega}$ sending A, B, C to p, q, r respectively such that g is conformal in the interior of ΔABC . We call g the extended Riemann mapping for $(\Omega, (p, q, r))$. For any subset X in the plane $\mathbb{C}$, let d_{st} be the restriction of the Euclidean metric on $\mathbb{C}$ to X .

Theorem 1.5 ([29]). *Suppose Ω is a Jordan domain in the complex plane with three distinct points $p, q, r \subset \partial\Omega$. There exists a sequence $(\Omega_n, \mathcal{T}_n, d_{st}, (p_n, q_n, r_n))$ of simply connected triangulated polygonal disks in $\mathbb{C}$ where $\mathcal{T}_n$ are triangulations by equilateral triangles and p_n, q_n, r_n are three boundary vertices such that*

- (a) $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ with $\Omega_n \subset \Omega_{n+1}$ and,
- (b) discrete uniformization maps associated to $(\Omega_n, \mathcal{T}_n, d_{st}, (p_n, q_n, r_n))$ converge uniformly to the extended Riemann mapping associated to $(\Omega, (p, q, r))$.

In [29], a conjectural discrete uniformization for non-compact simply connected surface is proposed. It is related to the existence and uniqueness of convex surfaces in the hyperbolic 3-space and K  be’s circle domain conjecture.

1.5. Previous works on discrete conformal geometry. The vertex scaling condition (1) in Definition 1.1 was introduced by Ro  ek and Williams in physics [33] and by Luo in [27]. Recall that if $u : V \rightarrow \mathbb{R}$ is a function and $x \in \mathbb{R}_{>0}^E$, then the vertex scaling $u * x$ of x is

$$(4) \quad u * x(vv') = x(vv')e^{u(v)+u(v')},$$

for all edges $vv' \in E(\mathcal{T})$. Two PL metrics d, d' on $(S, V, \mathcal{T})$ differ by a vertex scaling if there exists a function $u : V \rightarrow \mathbb{R}$ such that

$$(5) \quad l_{d'} = u * l_d.$$

A convex variational principle (Theorem 5.4) associated to the vertex scaling condition (5) was established by Luo in [27]. There have been many important works on various finite dimensional variational principles closely related this work. The convex function used in [27] is closely related to the hyperbolic volume and the dual volume. Indeed, the hyperbolic geometric meaning of the convex function used in [27] was explained by Bobenko–Pinkall–Springborn in [6] who also obtained an explicit form of it. Other important related works include Schlenker [36], Fillastre [16], Fillastre–Izemetiev [17], Colin de-Verdi  re [11], Br  gger [10], Rivin [35] and others.

The other related notion of discrete conformality comes from circle packings and circle patterns. It is in some sense dual to the vertex scaling (5). See the work of [18], [7] and others.

Note that if $(S, \mathcal{T}, d)$ is a Delaunay PL surface and $(S, \mathcal{T}, d')$ is not Delaunay such that (5) holds, then d and d' are not discrete conformal in the sense of Definition 1.1. In [27], d and d' are defined to be discrete conformal if (5) holds. However, the existence of constant curvature metric within a discrete conformal class is false in this setting.

The main issue is that if we are given two PL triangulated surfaces $(S, \mathcal{T}, l)$ and $(S, \mathcal{T}', l')$ related by a diagonal flip, then the vertex scaled PL metric surface $(S, \mathcal{T}, \lambda * l)$ may not be obtained from $(S, \mathcal{T}', \mu * l')$ by a diagonal flip for any choice of μ .

Our main contribution is to add the Delaunay condition (a) in Definition 1.1. This makes the definition to be triangulation independent. From the computational geometry point of view, Delaunay triangulations are the most natural choices of triangulations. Using of Delaunay condition to define triangulation independent quantities has appeared before. For instance, a triangulation independent discrete Laplace operator on a polyhedral surface was introduced by Bobenko and Springborn in [8].

In the work of Bobenko–Pinkall–Springborn [6], they further extended the work of [27] on the vertex scaling condition (1). In particular, they observed a relationship between vertex scaling condition (1) and Penner’s decorated Teichmüller theory. This observation of [6] is important for the proof of Theorem 4.3. Furthermore, [6] introduced the vertex scaling conditions (2) and (3) for spherical and hyperbolic polyhedral surfaces and established the associated variational principles.

Other different versions of discrete Riemann mapping theorem and discrete conformality have appeared in the work by Thurston [39], Cannon [12], Schramm [37], Glickenstein [18], Herschsky [22], Mercat [31] and others.

A theorem of Troyanov [41] states that the same result of Theorem 1.2 holds if discrete conformality is replaced by the classical Riemannian conformality. The major difference between Troyanov’s work and Theorem 1.2 is that in our case, we discretize the metric and conformality so that metrics are represented as edge length vectors in $\mathbb{R}^N$ and discrete conformality can be decided algorithmically. Theorem 1.2 is also related to the work of Kazdan and Warner [23] and [24] on prescribing Gaussian curvature. It is possible that Theorem 1.2 implies the existence part of Troyanov’s theorem and Kazdan–Warner’s theorem for closed surfaces by approximation.

1.6. Basic ideas of the proof of Theorem 1.2. Given a PL metric d on (S, V) , let $\mathcal{D}(d)$ be the set of all PL metrics discrete conformal to d , considered up to isometries isotopic to the identity (respecting V). The space $\mathcal{D}(d)$ is a discrete analogue of the classical conformal class $\{e^u g | u : \Sigma \rightarrow \mathbb{R}\}$ of all Riemannian metrics conformal to a given Riemannian metric g . The key ingredient in the proof of

Theorem 1.2 is that $\mathcal{D}(d)$ is diffeomorphic to $\mathbb{R}^V$ such that the discrete curvature map $K : \mathcal{D}(d) \rightarrow (-\infty, 2\pi)^V$ is C^1 -smooth. Assuming that $\mathcal{D}(d)$ is C^1 diffeomorphic to $\mathbb{R}^V$, the rest of the proof of Theorem 1.2 follows from the standard continuity method. The basic idea goes as follows. Since a PL metric d' and its scalar multiplied metric $\lambda d'$ have the same discrete curvature, K is defined on the quotient space $\mathcal{D}(d)/\mathbb{R}_{>0}$ where $\mathbb{R}_{>0}$ acts by scaling PL metrics. Using a variational principle established by Luo in [27] (Theorem 5.4), we show that K is the gradient of a C^2 -smooth convex function defined on $\mathbb{R}^V$. This implies that the curvature map $K : \mathcal{D}(d)/\mathbb{R}_{>0} \rightarrow (-\infty, 2\pi)^V$ is injective. By the Gauss–Bonnet theorem, the image of K is contained in the subspace $\{z \in (-\infty, 2\pi)^V \mid \sum_{v \in V} z(v) = 2\pi\chi(S)\}$ whose dimension is the same as that of $\mathcal{D}(d)/\mathbb{R}_{>0}$. Using a result of Akizohsi [1] and degeneration analysis, we show that the image of K is equal to $\{z \in (-\infty, 2\pi)^V \mid \sum_{v \in V} z(v) = 2\pi\chi(S)\}$. This establishes Theorem 1.2.

The space $\mathcal{D}(d)$ is essentially a combinatorial object (e.g., a CW complex) by Definition 1.1 and is covered by cells defined as follows. Each PL metric d' discrete conformal to d admits a Delaunay triangulation $\mathcal{T}$. Let the edge length vector of d' with respect to $\mathcal{T}$ be $l_{d'} \in \mathbb{R}^{E(\mathcal{T})}$ and let $U(\mathcal{T}) \subset \mathbb{R}^V$ be the set of vectors $x \in \mathbb{R}^V$ such that the vertex scaled triangulated PL metric $(S, \mathcal{T}, x * l_{d'})$ is still Delaunay. It can be shown that $U(\mathcal{T})$ is diffeomorphic to a closed convex polytope (Lemma 5.1). Furthermore, by definition, $\mathcal{D}(d)$ is a union of cells $W(\mathcal{T})$ where $W(\mathcal{T}) = \{(S, \mathcal{T}, x * l_{d'}) \mid x \in U(\mathcal{T})\}$. Thus, the space $\mathcal{D}(d)$ is built in a combinatorial way by gluing of cells $W(\mathcal{T})$. In general, it is very difficult to construct a smooth structure on a space using the gluing of cells. We overcome this difficulty by employing Teichmüller theory. Using the works of Penner [32], Bowditch–Epstein [9], Rivin [35], and Bobenko–Pinkall–Springborn [6], we proved that $\mathbb{R}^V = \cup_{\mathcal{T}} U(\mathcal{T})$ and that the corresponding maps $A_{\mathcal{T}} : W(\mathcal{T}) \rightarrow U(\mathcal{T})$ sending $x * l_{d'}$ to x (taken to be chart maps for $\mathcal{D}(d)$) can be glued to produce a global homeomorphism from $\mathcal{D}(d)$ to $\mathbb{R}^V$. Finally, we prove the C^1 -smoothness of the homeomorphism and its inverse by a direct computation (Lemma 4.6). The C^1 -smoothness of natural maps between various moduli spaces of geometric structures have appeared before. See, for instance, the important work of Bonahon [3], [4]. As pointed out by a referee, there should be a more geometric explanation of this C^1 -smoothness phenomenon.

1.7. Notations, conventions and organization of the paper. Triangulations to be used in the paper are defined as follows. Take a finite disjoint union of Euclidean triangles and identify edges in pairs by homeomorphisms. The quotient space is a compact surface together with a *triangulation* $\mathcal{T}$ whose simplices are the quotients of the simplices in the disjoint union. Let $V = V(\mathcal{T})$ and $E = E(\mathcal{T})$ be the sets of vertices and edges in $\mathcal{T}$. If e is an edge in $\mathcal{T}$ adjacent to two distinct triangles t, t' ,

then the *diagonal switch* on $\mathcal{T}$ at e replaces e by the other diagonal in the quadrilateral $t \cup_e t'$ and produces a new triangulation $\mathcal{T}'$ on (S, V) . A PL metric d on (S, V) is obtained as isometric gluing of Euclidean triangles along edges so that the set of cone points is contained in V . It is the same as a flat cone metric on (S, V) whose cone points are contained in V . Given a PL metric d and a triangulation $\mathcal{T}$ on (S, V) , if each triangle in $\mathcal{T}$ (in d metric) is isometric to a Euclidean triangle, we say $\mathcal{T}$ is *geometric* in d . If $\mathcal{T}$ is a triangulation of (S, V) isotopic to a geometric triangulation $\mathcal{T}'$ in a PL metric d , then the *length* of an edge $e \in E(\mathcal{T})$ (or *angle* of a triangle at a vertex in $\mathcal{T}$) is defined to be the length of the corresponding geodesic edge $e' \in E(\mathcal{T}')$ (respectively angle of the corresponding triangle in $\mathcal{T}'$) measured in metric d . The interior of a surface X is denoted by $\text{int}(X)$. If X is a finite set, $|X|$ denotes its cardinality and $\mathbb{R}^X$ denotes the vector space $\{f : X \rightarrow \mathbb{R}\}$. All surfaces are assumed to be connected.

The paper is organized as follows. In §2, we recall the Teichmüller space $T_{\text{PL}}(S, V)$ of PL metrics and its topology and show that $T_{\text{PL}}(S, V)$ admits a natural cell decomposition from Delaunay triangulations. In §3, we recall Penner's theory of decorated Teichmüller space and its cell decomposition from Delaunay triangulations. In §4, we relate discrete conformality with decorated Teichmüller space and show that there exists a C^1 -smooth diffeomorphism from $T_{\text{PL}}(S, V)$ to decorated Teichmüller space sending discrete conformality to decoration changes. Theorem 1.2 is proved in §5 using Akiyoshi's work and a variational principle developed in [27].

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2. Teichmüller space of PL metrics, its cell decomposition and Delaunay conditions

Suppose (S, V) is a marked connected closed surface. The discrete curvature $K_d : V \rightarrow (-\infty, 2\pi)$ of a PL metric d on (S, V) satisfies the Gauss–Bonnet formula that $\sum_{v \in V} K_d(v) = 2\pi\chi(S)$. Therefore, if $\chi(S - V) \geq 0$, i.e., $(S, V) = (S^2, \{v_1, \dots, v_n\})$ with $n \leq 2$, or $(\mathbb{R}P^2, \{v_1, \dots, v_n\})$ for $n = 1$, the Gauss–Bonnet identity implies there is no PL metric on (S, V) . From now on, we will always assume that the Euler characteristic $\chi(S - V) < 0$.

2.1. Teichmüller space of PL metrics and its length coordinates. The *Teichmüller space of all PL metrics* on (S, V) , denoted by $T_{\text{PL}}(S, V)$ or simply T_{PL} , is the space of all polyhedral metrics on (S, V) ,

considered up to isometry isotopic to the identity map (respecting V). The equivalence class of a PL metric d will be denoted by $[d]$.

Troyanov [41] shows that $T_{\text{PL}}(S, V)$ is homeomorphic to $\mathbb{R}^{-3\chi(S-V)}$. Below, we will use a natural collection of charts on T_{PL} which makes it a real analytic manifold. Suppose $\mathcal{T}$ is a triangulation of (S, V) with the set of edges $E = E(\mathcal{T})$. Let $\mathbb{R}_{\Delta}^{E(\mathcal{T})}$ be the set of $x \in \mathbb{R}^{E(\mathcal{T})}$ such that $x(e) > 0$ for each edge $e \in E$, and such that $x(e_i) + x(e_j) > x(e_k)$ whenever there is a triangle $t \in \mathcal{T}$ with edges e_i, e_j, e_k . By definition, $\mathbb{R}_{\Delta}^{E(\mathcal{T})}$ is non-empty and is an open convex polytope in $\mathbb{R}^E$. For each $x \in \mathbb{R}_{\Delta}^{E(\mathcal{T})}$, one constructs a PL metric d_x on (S, V) by replacing each triangle t with edges e_i, e_j, e_k by a Euclidean triangle of edge lengths $x(e_i), x(e_j), x(e_k)$ and gluing them by isometries along the corresponding edges. This construction produces an injective map

$$\Phi_{\mathcal{T}} : \mathbb{R}_{\Delta}^{E(\mathcal{T})} \rightarrow T_{\text{PL}}(S, V)$$

sending x to $[d_x]$. The image $P(\mathcal{T}) := \Phi_{\mathcal{T}}(\mathbb{R}_{\Delta}^{E(\mathcal{T})})$ is the space of all (isotopy classes of) PL metrics $[d]$ on (S, V) for which $\mathcal{T}$ is isotopic to a geometric triangulation in d . We call x the *length coordinate* of d_x and $\Phi_{\mathcal{T}}(x)$ (with respect to $\mathcal{T}$).

In general $P(\mathcal{T}) \neq T_{\text{PL}}(S, V)$. For instance, let d be the metric double of an obtuse triangle t along its boundary and $\mathcal{T}$ be the natural triangulation whose edges are edges of t . Let $\mathcal{T}'$ be the triangulation obtained by the diagonal switch at the shortest edge of t . Then $\mathcal{T}'$ is not isotopic to any geometric triangulation in d .

Since each PL metric on (S, V) admits a geometric triangulation (for instance, its Delaunay triangulation), we see that $T_{\text{PL}}(S, V) = \cup_{\mathcal{T}} P(\mathcal{T})$ where the union is over all triangulations of (S, V) . The space $T_{\text{PL}}(S, V)$ is a real analytic manifold with coordinate charts $\{(P(\mathcal{T}), \Phi_{\mathcal{T}}^{-1}) | \mathcal{T} \text{ triangulations of } (S, V)\}$. To see that transition functions $\Phi_{\mathcal{T}}^{-1} \Phi_{\mathcal{T}'}$ are real analytic, note that any two triangulations of (S, V) are related by a sequence of diagonal switches. Therefore, it suffices to show the result for $\mathcal{T}$ and $\mathcal{T}'$ which are related by a diagonal switch along an edge e . In this case, the transition function takes the following form:

$$(6) \quad \Phi_{\mathcal{T}}^{-1} \Phi_{\mathcal{T}'}(x_0, x_1, \dots, x_m) = (f(x_0, \dots, x_m), x_1, \dots, x_m),$$

where x_0 is the length of e and f is the length of the diagonal switched edge. See Figure 3(a), (b). Let t, t' be the triangles adjacent to e so that the lengths of edges of t, t' are x_0, x_1, x_2 and x_0, x_3, x_4 . Using the cosine law, we see that f is a real analytic function of $x_0, \dots, x_4$ given by

$$(7) \quad f = \sqrt{x_1^2 + x_4^2 - 2x_1x_4 \cos(\arccos(\frac{x_0^2 + x_1^2 - x_2^2}{2x_0x_1}) + \arccos(\frac{x_0^2 + x_4^2 - x_3^2}{2x_0x_4}))}.$$

In the special case when the quadrilateral $t \cup_e t'$ is inscribed to a circle, we have the Ptolemy identity $x_0 f = x_1 x_3 + x_2 x_4$. Therefore,

Corollary 2.1. *If $\mathcal{T}'$ is obtained from $\mathcal{T}$ by a diagonal switch at the edge e and the quadrilateral $t \cup_e t'$ is inscribed to a circle, then*

$$(8) \quad \Phi_{\mathcal{T}}^{-1} \Phi_{\mathcal{T}'}(x_0, x_1, x_2, \dots, x_m) = \left(\frac{x_1 x_3 + x_2 x_4}{x_0}, x_1, x_2, \dots, x_m \right).$$

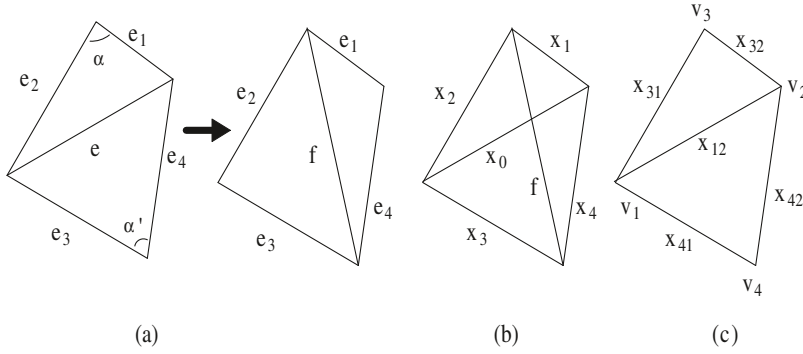


Figure 3. Diagonal switch and lengths of quadrilaterals.

2.2. Delaunay triangulations. Recall that a cell decomposition $\mathcal{C}$ of a marked surface (S, V) is a CW decomposition of S so that V is the set of 0-cells of $\mathcal{C}$. If (S, V, d) is a polyhedral surface, a *geometric* cell decomposition of (S, V, d) is a cell-decomposition of (S, V) so that each open 1-cell is a geodesic arc.

Given a PL metric d on (S, V) , its *Voronoi decomposition* is the collection of 2-cells $\{R(v) | v \in V\}$ where $R(v) = \{x \in S | d(x, v) \leq d(x, v') \text{ for all } v' \in V\}$. The dual cell-decomposition $\mathcal{C}(d)$ of the Voronoi decomposition is called the *Delaunay tessellation* of (S, V, d) ([2], [8]). It is the geometric cell decomposition of (S, V, d) with vertices V and edges corresponding to 1-dimensional connected components of $R(v) \cap R(v')$ for $v, v' \in V$. A *Delaunay triangulation* $\mathcal{T}(d)$ of (S, V, d) is a geometric triangulation of the Delaunay tessellation $\mathcal{C}(d)$ by further triangulating all non-triangular 2-dimensional cells without introducing extra vertices. For a generic PL metric d , $\mathcal{C}(d)$ is a Delaunay triangulation of d . The following was proved by Bobenko–Springborn and Aurenhamer et al.

Lemma 2.2. (See [8], [2]) (1) *Each PL metric d on (S, V) has a Delaunay triangulation.*

(2) *A geometric triangulation $\mathcal{T}$ of (S, V, d) is Delaunay if and only if the sum of opposite angles facing each edge e is at most π . Furthermore, the sum is strictly less than π if and only if the edge e is in $\mathcal{C}(d)$.*

(3) *If $\mathcal{T}$ and $\mathcal{T}'$ are Delaunay triangulations of d , then there exists a sequence of Delaunay triangulations $\mathcal{T}_1 = \mathcal{T}, \mathcal{T}_2, \dots, \mathcal{T}_k = \mathcal{T}'$ of d so that $\mathcal{T}_{i+1}$ is obtained from $\mathcal{T}_i$ by a diagonal switch.*

Definition 2.3. (Delaunay cells in Teichmüller space) For a triangulation $\mathcal{T}$ of (S, V) , the associated Delaunay cell in $T_{\text{PL}}(S, V)$ is defined by

$$D_{\text{PL}}(\mathcal{T}) = \{ [d] \in T_{\text{PL}}(S, V) \mid \mathcal{T} \text{ is isotopic to a Delaunay triangulation of } d \}.$$

Note that $D_{\text{PL}}(\mathcal{T}) \subset P(\mathcal{T})$ and $D_{\text{PL}}(\mathcal{T})$ contains the PL metric such that all edges have length 1.

Lemma 2.4. *Given a finite set of triangulations $\mathcal{T}_1, \dots, \mathcal{T}_m$ of (S, V) , then $\cap_{i=1}^m D_{\text{PL}}(\mathcal{T}_i) \neq \emptyset$ if and only if there exists a cell-decomposition $\mathcal{C}$ of (S, V) so that each $\mathcal{T}_i$ is isotopic to a triangulation of $\mathcal{C}$ with no additional vertices introduced. Furthermore, if $\cap_{i=1}^m D_{\text{PL}}(\mathcal{T}_i) \neq \emptyset$, there exists, unique up to isotopy, a cell-decomposition denoted by $\cap_{i=1}^m \mathcal{T}_i$ which has the maximum number of edges so that $\mathcal{T}_i$ is isotopic to a triangulation of $\cap_{i=1}^m \mathcal{T}_i$.*

Proof. If $\cap_{i=1}^m D_{\text{PL}}(\mathcal{T}_i) \neq \emptyset$, take a point $[d] \in \cap_{i=1}^m D_{\text{PL}}(\mathcal{T}_i)$ and consider the Delaunay tessellation $\mathcal{C}(d)$ associated to d . Then by definition each $\mathcal{T}_i$ is isotopic to a triangulation of $\mathcal{C}(d)$.

Conversely, if $\mathcal{C}$ is a cell-decomposition of (S, V) without 1-gons and bi-gons as 2-cells, produce a flat cone metric d on (S, V) by making all 2-cells in $\mathcal{C}$ regular Euclidean polygons of edge length 1. Since $\mathcal{T}_i$ is isotopic to a geometric triangulation $\mathcal{T}_i'$ of $\mathcal{C}$, by the construction, each $\mathcal{T}_i'$ is Delaunay in d . Therefore, $[d] \in \cap_{i=1}^m D_{\text{PL}}(\mathcal{T}_i)$.

To prove the last statement, if $\cap_{i=1}^m D_{\text{PL}}(\mathcal{T}_i) \neq \emptyset$, let $\mathcal{C}$ be the cell-decomposition of (S, V) whose edges are isotopic to edges in $\mathcal{T}_i$ for each i . Then by the construction, if $\mathcal{C}'$ is a cell-decomposition so that $\mathcal{T}_i$ is a triangulation of $\mathcal{C}'$ for each i , $\mathcal{C}'$ is isotopic to a subcomplex of $\mathcal{C}$. q.e.d.

Assume that $\mathcal{T}$ is geometric in d . One can characterize PL metrics $[d] \in D_{\text{PL}}(\mathcal{T})$ in terms of the length coordinate $x = \Phi_{\mathcal{T}}^{-1}([d])$ as follows. By definition, $\mathcal{T}$ is Delaunay in d if and only if

$$(9) \quad \alpha + \alpha' \leq \pi, \quad \text{i.e.,} \quad \cos(\alpha) + \cos(\alpha') \geq 0, \quad \text{for each edge } e \in E(\mathcal{T})$$

where α, α' are the two angles facing e . See Figure 3(a). Let t and t' be the triangles adjacent to e and e, e_1, e_2 be edges of t and e, e_3, e_4 be the edges of t' . Note that $t' = t$ is allowed. Suppose in the metric d the length of e is x_0 and the length of e_i is x_i , $i = 1, \dots, 4$. By the cosine law, Delaunay condition (9) is the same as

$$(10) \quad \frac{x_1^2 + x_2^2 - x_0^2}{2x_1x_2} + \frac{x_3^2 + x_4^2 - x_0^2}{2x_3x_4} \geq 0, \quad \text{for all edges } e \in E(\mathcal{T}).$$

Inequality (10) shows that $D_{\text{PL}}(\mathcal{T}) \subset T_{\text{PL}}$ is bounded by a finite set of real analytic subvarieties. It turns out $\{D_{\text{PL}}(\mathcal{T}) \mid \mathcal{T}\}$ forms a real analytic cell decomposition of T_{PL} .

Let us recall the basics of real analytic cell decompositions of a real analytic manifold M^n . A closed subspace $C \subset M^n$ is a *real analytic cell* if there is a real analytic diffeomorphism h defined in an open neighborhood U of C into $\mathbb{R}^n$ so that $h(C)$ is a closed (possibly non-compact) convex polytope in $\mathbb{R}^n$. A *face* C' of C is a subset so that $h(C')$ is a face of the polytope $h(C)$. A *real analytic cell decomposition* of M is a locally finite collection of n -dimensional real analytic cells $\{C_i | i \in J\}$ so that $M = \cup_{i \in J} C_i$ and $C_{i_1} \cap \dots \cap C_{i_k}$ is a face of C_{i_j} for all choices of indices.

Theorem 2.5. *For a marked closed surface (S, V) with $\chi(S - V) < 0$, the set $\{D_{\text{PL}}(\mathcal{T}) | \mathcal{T} \text{ triangulations of } (S, V)\}$ forms an analytic cell-decomposition of the Teichmüller space $T_{\text{PL}}(S, V)$.*

Proof. To see that $D_{\text{PL}}(\mathcal{T})$ is a real analytic cell of dimension $-3\chi(S - V)$, one takes the open neighborhood of $D_{\text{PL}}(\mathcal{T})$ to be $P(\mathcal{T})$. Recall that $P(\mathcal{T}) = \Phi_{\mathcal{T}}(\mathbb{R}_{\Delta}^{E(\mathcal{T})})$ is the set of all PL metrics $[d]$ for which $\mathcal{T}$ is isotopic to a geometric triangulation in d . Also fixes an edge $e_1 \in E(\mathcal{T})$. Define $h_{\mathcal{T}} : P(\mathcal{T}) \rightarrow \mathbb{R}^E \times \mathbb{R}$ to be the real analytic map sending x to $(\phi_0(x), \ln(x(e_1)))$ where $\phi_0(x)(e) = \pi - (\alpha + \alpha')$ with α and α' being angles facing e . The fact that $D_{\text{PL}}(\mathcal{T})$ is a real analytic cell follows from,

Theorem 2.6. (Rivin, [35]) *The map $h_{\mathcal{T}} : P(\mathcal{T}) \rightarrow \mathbb{R}^E \times \mathbb{R}$ is a real analytic embedding whose image is an open set in the affine subspace $P^* = \{(y, t) \in \mathbb{R}^E \times \mathbb{R} | \sum_{e \in E} y(e) = \pi(|V| - \chi(S))\}$. Furthermore, $h_{\mathcal{T}}(D_{\text{PL}}(\mathcal{T})) = P^* \cap (\mathbb{R}_{\geq 0}^E \times \mathbb{R})$ is a closed convex polytope whose codimension-1 faces are defined by $\alpha + \alpha' = \pi$ for some edges e .*

To show that $\{D_{\text{PL}}(\mathcal{T}) | \mathcal{T}\}$ forms a cell decomposition, consider $W = D_{\text{PL}}(\mathcal{T}_1) \cap \dots \cap D_{\text{PL}}(\mathcal{T}_m) \neq \emptyset$. We will show that $h_{\mathcal{T}_i}(W)$ is a face of the convex polytope $h_{\mathcal{T}_i}(D_{\text{PL}}(\mathcal{T}_i))$ for each i . By Lemma 2.4, consider the cell-decomposition $\cap_{i=1}^m \mathcal{T}_i$ of (S, V) . Let U_i be the face of $h_{\mathcal{T}_i}(D_{\text{PL}}(\mathcal{T}_i))$ defined by the set of linear equalities: $\alpha + \alpha' = \pi$ for all edges e in $\mathcal{T}_i$ which is not isotopic to an edge of $\cap_{j=1}^m \mathcal{T}_j$. We claim $h_{\mathcal{T}_i}(W) = U_i$.

To see that $h_{\mathcal{T}_i}(W) \subset U_i$, take $[d] \in W$ and an edge e of $\mathcal{T}_i$ which is not isotopic to an edge of $\cap_{j=1}^m \mathcal{T}_j$. Then by Lemma 2.4, the Delaunay tessellation $\mathcal{C}(d)$ is isotopic to a subcomplex of $\cap_{j=1}^m \mathcal{T}_j$. Therefore, e is not isotopic to an edge of $\mathcal{C}(d)$. By Lemma 2.2(2), this implies $\alpha + \alpha' = \pi$ where α and α' are angles opposite to e in $\mathcal{T}_i$, i.e., $h_{\mathcal{T}_i}(W) \subset U_i$.

Conversely, suppose $[d]$ satisfies $h_{\mathcal{T}_i}([d]) \in U_i$. By definition $[d] \in D_{\text{PL}}(\mathcal{T}_i)$ and $\alpha + \alpha' = \pi$ for all edges e of $\mathcal{T}_i$ which are not isotopic to edges of $\cap_{j=1}^m \mathcal{T}_j$. Let $\mathcal{C}(d)$ be the Delaunay tessellation associated to d and e be an edge of $\mathcal{C}(d)$. Then by Lemma 2.2(2), $\alpha + \alpha' < \pi$ for the two angles α and α' facing e . Therefore, e is isotopic to an edge in $\cap_{j=1}^m \mathcal{T}_j$. This shows that $\mathcal{C}(d)$ is isotopic to a subcomplex of $\cap_{j=1}^m \mathcal{T}_j$.

Since $\mathcal{T}_j$ is isotopic to a triangulation of $\cap_{j=1}^m \mathcal{T}_j$, it follows that $\mathcal{T}_j$ is isotopic to a triangulation of $\mathcal{C}(d)$. Therefore, $[d] \in D_{\text{PL}}(\mathcal{T}_j)$ for all j and $U_i \subset h_{\mathcal{T}_i}(W)$. q.e.d.

Note that the cell decomposition $T_{\text{PL}}(S, V) = \cup_{[\mathcal{T}]} D_{\text{PL}}(\mathcal{T})$ of the Teichmüller space is invariant under the action of the mapping class group.

3. Penner's work on decorated Teichmüller spaces

One of the main tools used in the proof of Theorem 1.2 is the decorated Teichmüller space theory developed by R. Penner [32]. We will recall a natural cell structure on the decorated Teichmüller space discovered by Penner and Bowditch–Epstein [9] and the work of Akiyoshi [1].

3.1. Decorated triangles. Let $\mathbb{H}^2$ be the 2-dimensional hyperbolic plane. An *ideal triangle* is a hyperbolic triangle in $\mathbb{H}^2$ with three vertices v_1, v_2, v_3 at the circle at infinity of $\mathbb{H}^2$. Any two ideal triangles are isometric. A *decorated ideal triangle* τ is an ideal triangle so that each vertex v_i is assigned a horoball H_i centered at v_i . Let e_i be the complete geodesic edge of τ opposite to the vertex v_i . The inner *angle* a_i of τ is the length of the portion of the horocycle ∂H_i between e_j and e_k , $\{i, j, k\} = \{1, 2, 3\}$. The *length* $l_i \in \mathbb{R}$ of the edge e_i in τ is the signed distance between H_j and H_k ($j, k \neq i$). To be more precise, if $H_j \cap H_k = \emptyset$, then $l_i > 0$ is the distance between H_k and H_j . If $H_j \cap H_k \neq \emptyset$, then $-l_i$ is the distance between two end points of $\partial(e_i \cap H_j \cap H_k)$. Penner calls $L_i = e^{l_i/2}$ the λ -length of e_i .

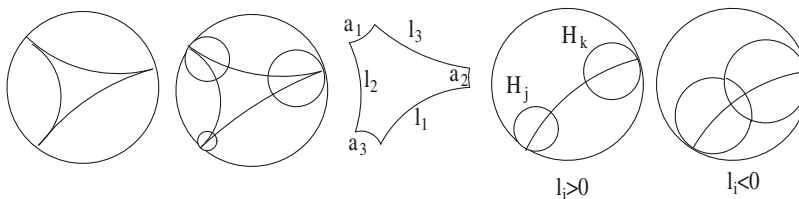


Figure 4. Decorated ideal triangles and their edge lengths.

It is known that for any $l_1, l_2, l_3 \in \mathbb{R}$, there exists a unique decorated ideal triangle of edge lengths l_1, l_2, l_3 . The relationship between the lengths l_i and angles a_j 's is the following *cosine law* proved by Penner: (11)

$$a_i = e^{\frac{1}{2}(l_i - l_j - l_k)} = \frac{L_i}{L_j L_k}, \quad \ln(a_i) + \ln(a_j) = -l_k, \quad \{i, j, k\} = \{1, 2, 3\}.$$

Let S be a closed connected surface and $V = \{v_1, \dots, v_n\} \subset S$. We assume $n \geq 1$ and $\chi(S - V) < 0$. Following Penner, a *decorated hyperbolic*

metric on $S - V$ is a complete finite area hyperbolic metric d on $S - V$ together with a horoball H_i centered at the i -th cusp corresponding to v_i . We can parameterize it as (d, w) where $w = (w_1, \dots, w_n) \in \mathbb{R}_{>0}^n$ and w_i is the length of the horocycle ∂H_i . Two decorated hyperbolic metrics on $S - V$ are *equivalent* if there is an isometry h between them so that h is homotopic to the identity and h preserves the horoballs. The space of all equivalence classes of decorated hyperbolic metrics on $S - V$ is defined to be the *decorated Teichmüller space* $T_D(S - V)$. If we use $T(S - V)$ to denote the usual Teichmüller space of complete hyperbolic metrics of finite area on $S - V$, then there is a natural homeomorphism from $T_D(S - V)$ to $T(S - V) \times \mathbb{R}_{>0}^n$ by sending $[(d, w)]$ to $([d], w)$. The projection $T_D(S - V) \rightarrow T(S - V)$ sending $[(d, w)]$ to $[d]$ records the underlying hyperbolic metric.

Now suppose $\mathcal{T}$ is a triangulation of (S, V) with $E = E(\mathcal{T})$. Then Penner introduced a homeomorphism map $\Psi_{\mathcal{T}} : \mathbb{R}_{>0}^E \rightarrow T_D(S - V)$ called the λ -length coordinate as follows. For each $x \in \mathbb{R}_{>0}^E$, i.e., $x : E \rightarrow \mathbb{R}_{>0}$, $\Psi_{\mathcal{T}}(x)$ is the equivalence class of the decorated hyperbolic metric (d, w) on $S - V$ obtained as follows. If t is a triangle in $\mathcal{T}$ with three edges e_i, e_j, e_k , one replaces t by the decorated ideal triangle of edge lengths $2 \ln x(e_i), 2 \ln x(e_j)$ and $2 \ln x(e_k)$ and glues these decorated ideal triangles isometrically along the corresponding edges preserving decoration. One obtains a decorated hyperbolic metric (d, w) on $S - V$. The horoballs are the gluing of the corresponding portions of horoballs associated to ideal triangles. In particular, w_i is the sum of all angles of the decorated ideal triangles at v_i . Penner proved, using his Ptolemy identity, that $\Psi_{\mathcal{T}}^{-1} \Psi_{\mathcal{T}'}$ is real analytic for any two triangulations $\mathcal{T}$ and $\mathcal{T}'$. Here Ptolemy identity for decorated ideal quadrilaterals states that $AA' + BB' = CC'$ where A, A', B, B' are the λ -lengths of the edges of a quadrilateral labelled cyclically and C, C' are the λ -lengths of the diagonals. See Figure 5. In particular, $\{\Psi_{\mathcal{T}}|\mathcal{T}\}$ forms real analytic charts for $T_D(S - V)$.

The following lemma is well known. We omit the proof.

Lemma 3.1. *Suppose C is an embedded horocycle of length w_i centered at a cusp in a complete hyperbolic surface and C' is another embedded horocycle of smaller length w'_i centered at the same cusp. Then the $w_i = w'_i e^t$ where $t = d(C, C')$ is the distance between C and C' .*

By the lemma and definition, if $\Psi_{\mathcal{T}}(x) = [(d, w)]$ then for any $k > 0$, $\Psi_{\mathcal{T}}(kx) = [(d, \frac{1}{k}w)]$. Thus, for any decorated metric (d, w) , by choosing k large, one may assume the associated horoballs are disjoint and embedded in $(d, \frac{1}{k}w)$.

3.2. Delaunay triangulations. Given a decorated hyperbolic metric (d, w) on $S - V$, there is a natural *Delaunay triangulation* $\mathcal{T}$ associated to (d, w) . The geometric definition of $\mathcal{T}$ goes as follows. First assume

that the associated horoballs $H_1(w), \dots, H_n(w)$ are embedded and disjoint in $S - V$. Consider the Voronoi cell decomposition of the compact surface $X_w = S - V - \cup_{i=1}^n \text{int}(H_i(w))$ so that the 2-cell $R_i(w)$ associated to v_i is $\{x \in X_w \mid d(x, \partial H_i(w)) \leq d(x, \partial H_j(w)), \text{ all } j\}$. Recall that an *orthogeodesic arc* in X_w is a geodesic segment from ∂X_w to ∂X_w perpendicular to ∂X_w . The dual of the Voronoi decomposition is a decomposition $\mathcal{C}(d, w)$ of X_w by a collection of disjoint embedded orthogeodesic arcs $\{s'\}$ constructed as follows. If $s \subset R_i(w) \cap R_j(w)$ is a 1-dimensional connected component, take a point $p \in s$ and consider the two shortest geodesic paths b_i and b_j in $R_i(w)$ and $R_j(w)$ respectively from p to $\partial H_i(w)$ and $\partial H_j(w)$. The shortest orthogeodesic arc s' in X_w homotopic to $b_i^{-1} * b_j$ is in $\mathcal{C}(d, w)$ and is dual to s . A *Delaunay triangulation* of X_w is a subdivision of $\mathcal{C}(d, w)$ by decomposing (using orthogeodesic arcs) all non-hexagonal 2-cells into hexagonal 2-cells. Since each orthogeodesic arc extends to a complete geodesic from cusp to cusp, one obtains a *Delaunay triangulation* $\mathcal{T}(d, w)$ of the decorated metric (d, w) on $S - V$ by extending each orthogeodesic arc s' to a complete geodesic. For a generic metric (d, w) , a Delaunay triangulation is the dual to the Voronoi decomposition. By the definition of Voronoi cells and Lemma 3.1, Delaunay triangulations of (d, w) and $(d, w/k)$ are the same when $k > 1$. Due to this, for a general decorated metric (d, w) , we define a Delaunay triangulation of (d, w) to be that of $(d, w/k)$ for k large.

Given triangulation $\mathcal{T}$ of (S, V) , let $D_D(\mathcal{T})$ be the set of all equivalence classes of decorated hyperbolic metrics (d, w) in $T_D(S - V)$ so that $\mathcal{T}$ is isotopic to a Delaunay triangulation of (d, w) . Penner proved the following important theorem in [32]. See also [9]. Details on the real analytic diffeomorphism part of the decomposition can be found in [19].

Theorem 3.2. (Penner) *The decorated Teichmüller space $T_D(S - V)$ has a real analytic cell decomposition by $\{D_D(\mathcal{T}) \mid \mathcal{T}\}$ and*

$$T_D(S - V) = \cup_{[\mathcal{T}]} D_D(\mathcal{T}),$$

where the union is over all isotopy classes of triangulations. The decomposition is invariant under the action of the mapping class group.

3.3. Finite set of Delaunay triangulations. The following theorem of Akiyoshi [1] holds for decorated finite volume hyperbolic manifolds of any dimension.

Theorem 3.3. (Akiyoshi) *For any finite area complete hyperbolic metric d on $S - V$, there are only finitely many isotopy classes of triangulations $\mathcal{T}$ so that $([d] \times \mathbb{R}_{>0}^n) \cap D_D(\mathcal{T}) \neq \emptyset$. In particular, there exist triangulations $\mathcal{T}_1, \dots, \mathcal{T}_k$ so that for any $w \in \mathbb{R}_{>0}^n$, any Delaunay triangulation of (d, w) is isotopic to one of $\mathcal{T}_i$.*

We thank B. Springborn for informing us the above result was known before and was a theorem of Akiyoshi. However, our proof is different and short. For completeness, we present our proof in the appendix.

4. Euclidean polyhedral metrics and decorated hyperbolic metrics

The goal of this section is to associate a complete finite area hyperbolic metric d^* on $S - V$ to each PL metric d on (S, V) such that two PL metrics d_1 and d_2 are discrete conformal if and only if d_1^* and d_2^* are isometric by an isometry isotopic to the identity (respecting V). This is a discrete version of the classical theorem that each Riemannian metric g on $S - V$ corresponds to a complete hyperbolic metric g^* (the Poincaré metric) such that $g_1^* = g_2^*$ if and only if g_1 and g_2 are conformal. We achieve this by using Penner's decorated hyperbolic metrics.

Let us begin with an important observation of Bobenko–Pinkall–Springborn [6] which relates Euclidean polyhedral metrics to decorated hyperbolic metrics. Suppose $(S, \mathcal{T}, l)$ is a triangulated polyhedral surface with edge length vector $l \in \mathbb{R}^{E(\mathcal{T})}$. Let $\mathcal{T} - V$ be the associated ideal triangulation of the punctured surface $S - V$. One constructs a decorated hyperbolic metric $d_{l, \mathcal{T}}$ on the ideal triangulated surface $(S - V, \mathcal{T} - V)$ by replacing each triangle $\tau \in \mathcal{T}$ by a decorated ideal triangle τ^* whose λ -length at an edge e is the Euclidean length $l(e)$. By the construction and definition, one sees that edge length vectors l and $u * l$ correspond to two decorated hyperbolic metrics having the same the underlying hyperbolic metric, i.e., if $d_{l, \mathcal{T}} = (p, w)$, then $d_{u * l, \mathcal{T}} = (p, w')$. As remarked in §1.5, due to the lacking of Delaunay condition, this construction does not fully capture the discrete conformality.

We now use this construction for Delaunay triangulations to prove that the space $\mathcal{D}(d)$ of all PL metrics discrete conformal to a given metric d is homeomorphic to $\mathbb{R}^V$. Fix a triangulation $\mathcal{T}$ of (S, V) , we have two coordinate maps $\Phi_{\mathcal{T}}^{-1} : P(\mathcal{T}) \rightarrow \mathbb{R}^{E(\mathcal{T})}$ and $\Psi_{\mathcal{T}} : \mathbb{R}^{E(\mathcal{T})} \rightarrow T_{\mathbb{D}}(S - V)$. Consider the injective map $A_{\mathcal{T}} : P(\mathcal{T}) \rightarrow T_{\mathbb{D}}(S - V)$ defined by $\Psi_{\mathcal{T}} \circ \Phi_{\mathcal{T}}^{-1}$ which is the Bobenko–Pinkall–Springborn construction associated to the triangulation $\mathcal{T}$. By definitions and Lemma 3.1, one sees that two vectors $x, y \in \mathbb{R}_{\Delta}^{E(\mathcal{T})}$ are related by a vertex scaling $y = u * x$ for some $u \in \mathbb{R}^V$, if and only if $\Psi_{\mathcal{T}}(x)$ and $\Psi_{\mathcal{T}}(y)$ have the same underlying hyperbolic structure.

Proposition 4.1. $A_{\mathcal{T}}|_{D_{\text{PL}}(\mathcal{T})}$ is a real analytic diffeomorphism from $D_{\text{PL}}(\mathcal{T})$ onto $D_{\mathbb{D}}(\mathcal{T})$.

Proof. Since $A_{\mathcal{T}}$ is injective and real analytic, it suffices to show that $\Phi_{\mathcal{T}}^{-1}(D_{\text{PL}}(\mathcal{T})) = \Psi_{\mathcal{T}}^{-1}(D_{\mathbb{D}}(\mathcal{T}))$.

Recall that the characterization of a PL metric d which is Delaunay in $\mathcal{T}$ in terms of $x = \Phi_{\mathcal{T}}^{-1}(d)$ is as follows. Take an edge $e \in E(\mathcal{T})$ and

let t and t' be the triangles adjacent to e so that e, e_1, e_2 are edges of t and e, e_3, e_4 are the edge of t' . Suppose α, α' are the angles (measured in d) in t and t' facing e . Then the Delaunay condition is equivalent to

$$(12) \quad \alpha + \alpha' \leq \pi, \quad \text{i.e.,} \quad \cos(\alpha) + \cos(\alpha') \geq 0, \quad \text{for all edges } e \in E(\mathcal{T}).$$

Suppose the length of e (in d) is x_0 and the length of e_i is x_i , $i = 1, \dots, 4$. By the cosine law, the condition (12) is the same as

$$(13) \quad \frac{x_1^2 + x_2^2 - x_0^2}{2x_1x_2} + \frac{x_3^2 + x_4^2 - x_0^2}{2x_3x_4} \geq 0, \quad \text{for all edges } e \in E(\mathcal{T}).$$

This shows that $\Phi_{\mathcal{T}}^{-1}(D_{\text{PL}}(\mathcal{T}))$ is the set of $x \in \mathbb{R}_{>0}^{E(\mathcal{T})}$ such that (13) and (14) hold where

$$(14) \quad x(e_i) + x(e_j) > x(e_k), \quad e_i, e_j, e_k \text{ form edges of a triangle in } \mathcal{T}.$$

We thank a referee for informing us that the lemma below was proved by Penner as Lemma 5.27 in [32]. We provide the proof for completeness.

Lemma 4.2. (Penner) $\Phi_{\mathcal{T}}^{-1}(D_{\text{PL}}(\mathcal{T})) = \{x \in \mathbb{R}_{>0}^{E(\mathcal{T})} \mid (13) \text{ holds for each edge } e \in E(\mathcal{T})\}.$

Proof. This is the same as showing that if (13) holds for all edges, then (14) holds for all triangles. Suppose otherwise, there exists $x \in \mathbb{R}_{>0}^E$ so that (13) holds but there is a triangle with edges e_i, e_j, e_k so that

$$(15) \quad x(e_i) \geq x(e_j) + x(e_k).$$

In this case, we say e_i is a “bad” edge. Let e be a “bad” edge of the largest x value, i.e., $x(e) = \max\{x(e_i) \mid (15) \text{ holds}\}$. Let t, t' be the triangles adjacent to e and the edges of t and t' be $\{e, e_1, e_2\}$ and $\{e, e_3, e_4\}$. Note that $t' = t$ is allowed if e is adjacent to only one triangle. Let $x_0 = x(e)$, $x_i = x(e_i)$ for $i = 1, 2, 3, 4$. Without loss of generality we may assume that

$$(16) \quad x_1 + x_2 \leq x_0.$$

Since e is a “bad” edge of the largest x value, we have $x_3 < x_0 + x_4$ and $x_4 < x_0 + x_3$, i.e.,

$$(17) \quad |x_3 - x_4| < x_0.$$

On the other hand, inequality (13) holds for $x_0, x_1, \dots, x_4$. It is the same as

$$(18) \quad \frac{x_0^2 - (x_1 + x_2)^2}{2x_1x_2} \leq \frac{(x_3 - x_4)^2 - x_0^2}{2x_3x_4}.$$

Inequality (16) says the left-hand-side of (18) is at least 0 and inequality (17) says the right-hand-side of (18) is negative. This is a contradiction. q.e.d.

The space $\Psi_{\mathcal{T}}^{-1}(D_D(\mathcal{T}))$ can be characterized as follows. Suppose that the λ -length vector for $(d', w) \in D_D(\mathcal{T})$ is $x = \Psi_{\mathcal{T}}^{-1}(d', w)$. For each edge e in $(S, \mathcal{T}, d')$, let a, a' be the two angles facing e and b, b', c, c' be the angles adjacent to the edge e . Then $\mathcal{T}$ is Delaunay for (d', w) if and only if for each edge $e \in E(\mathcal{T})$ (see [32] or [19]),

$$(19) \quad a + a' \leq b + b' + c + c'.$$

Let t and t' be the triangles adjacent to e and e, e_1, e_2 be edges of t and e, e_3, e_4 be the edges of t' . Let the λ -length of e be x_0 and the λ -length of e_i be x_i . Then using the cosine law (11), one sees that (19) is equivalent to

$$(20) \quad \frac{x_0^2}{x_1 x_2} + \frac{x_0^2}{x_3 x_4} \leq \frac{x_1}{x_2} + \frac{x_2}{x_1} + \frac{x_3}{x_4} + \frac{x_4}{x_3}, \quad \text{for each } e \in E(\mathcal{T}).$$

Inequality (20) is equivalent to

$$(21) \quad 0 \leq \frac{x_1^2 + x_2^2 - x_0^2}{2x_1 x_2} + \frac{x_3^2 + x_4^2 - x_0^2}{2x_3 x_4}, \quad \text{for each } e \in E(\mathcal{T}).$$

Therefore,

$$\Psi_{\mathcal{T}}^{-1}(D_D(\mathcal{T})) = \{x \in \mathbb{R}_{>0}^E \mid (21) \text{ holds at each edge } e \in E(\mathcal{T})\}.$$

However, inequality (21) is the same as (13). This shows $\Phi_{\mathcal{T}}^{-1}(D_{PL}(\mathcal{T})) = \Psi_{\mathcal{T}}^{-1}(D_D(\mathcal{T}))$.

Finally, since both $\Phi_{\mathcal{T}}$ and $\Psi_{\mathcal{T}}$ are real analytic diffeomorphisms and $A_{\mathcal{T}} = \Psi_{\mathcal{T}} \circ \Phi_{\mathcal{T}}^{-1}$ and $A_{\mathcal{T}}^{-1} = \Phi_{\mathcal{T}} \circ \Psi_{\mathcal{T}}^{-1}$, we see that $A_{\mathcal{T}}$ is a real analytic diffeomorphism. q.e.d.

This concludes the proof of Proposition 4.1.

4.1. Globally defined map, diagonal switch and Ptolemy relation.

Theorem 4.3. *Suppose $\mathcal{T}$ and $\mathcal{T}'$ are two triangulations of (S, V) so that $D_{PL}(\mathcal{T}) \cap D_{PL}(\mathcal{T}') \neq \emptyset$. Then*

$$(22) \quad A_{\mathcal{T}}|_{D_{PL}(\mathcal{T}) \cap D_{PL}(\mathcal{T}')} = A_{\mathcal{T}'}|_{D_{PL}(\mathcal{T}) \cap D_{PL}(\mathcal{T}')}.$$

In particular, the gluing of these $\mathbf{A}_{\mathcal{T}}|_{D_{PL}(\mathcal{T})}$ mappings produces a homeomorphism $\mathbf{A} = \cup_{\mathcal{T}} \mathbf{A}_{\mathcal{T}}|_{D_{PL}(\mathcal{T})} : T_{PL}(S, V) \rightarrow T_D(S - V)$ such that $A([d])$ and $A([d'])$ have the same underlying hyperbolic structure if and only if d and d' are discrete conformal.

Proof. Suppose $[d] \in D_{PL}(\mathcal{T}) \cap D_{PL}(\mathcal{T}')$, i.e., both $\mathcal{T}$ and $\mathcal{T}'$ are Delaunay in the PL metric d . Then there exists a sequence of triangulations $\mathcal{T}_1 = \mathcal{T}, \mathcal{T}_2, \dots, \mathcal{T}_k = \mathcal{T}'$ on (S, V) so that each $\mathcal{T}_i$ is Delaunay in d and $\mathcal{T}_{i+1}$ is obtained from $\mathcal{T}_i$ by a diagonal switch. In particular, $A_{\mathcal{T}}([d]) = A_{\mathcal{T}'}([d])$ follows from $A_{\mathcal{T}_i}([d]) = A_{\mathcal{T}_{i+1}}([d])$ for $i = 1, 2, \dots, k-1$. Thus, it suffices to show $A_{\mathcal{T}}([d]) = A_{\mathcal{T}'}([d])$ when $\mathcal{T}'$ is obtained from $\mathcal{T}$ by a diagonal switch along an edge e . In this

case the transition functions $\Phi_{\mathcal{T}}^{-1}\Phi_{\mathcal{T}'}$ and $\Psi_{\mathcal{T}}^{-1}\Psi_{\mathcal{T}'}$ are the same. Indeed, $\Phi_{\mathcal{T}}^{-1}\Phi_{\mathcal{T}'}$ was calculated in (6). Penner proved that the λ -lengths satisfy the Ptolemy identity for decorated ideal quadrilaterals. See [32] and Figure 5. This result, translated into the language of length coordinates, says that $\Psi_{\mathcal{T}}^{-1}\Psi_{\mathcal{T}'}(x)$ takes the same form as in (8). Thus, (22) holds. Taking the inverse, we obtain

$$(23) \quad A_{\mathcal{T}}^{-1}|_{D_D(\mathcal{T}) \cap D_D(\mathcal{T}')} = A_{\mathcal{T}'}^{-1}|_{D_D(\mathcal{T}) \cap D_D(\mathcal{T}')}.$$

Lemma 4.4. (a) $D_{\text{PL}}(\mathcal{T}) \cap D_{\text{PL}}(\mathcal{T}') \neq \emptyset$ if and only if $D_D(\mathcal{T}) \cap D_D(\mathcal{T}') \neq \emptyset$.

(b) The gluing map $\mathbf{A} = \cup_{\mathcal{T}} \mathbf{A}_{\mathcal{T}}|_{D_{\text{PL}}(\mathcal{T})} : T_{\text{PL}} \rightarrow T_D$ is a homeomorphism invariant under the action of the mapping class group.

Proof. By (22) and (23), the maps $\mathbf{A} = \cup_{\mathcal{T}} \mathbf{A}_{\mathcal{T}}|_{D_{\text{PL}}(\mathcal{T})} : T_{\text{PL}} \rightarrow T_D$ and $\mathbf{B} = \cup_{\mathcal{T}} \mathbf{A}_{\mathcal{T}}^{-1}|_{D_D(\mathcal{T})} : T_D \rightarrow T_{\text{PL}}$ are well defined and continuous. Since $\mathbf{A}(D_{\text{PL}}(\mathcal{T}) \cap D_{\text{PL}}(\mathcal{T}')) \subset D_D(\mathcal{T}) \cap D_D(\mathcal{T}')$ and $\mathbf{B}(D_D(\mathcal{T}) \cap D_D(\mathcal{T}')) \subset D_{\text{PL}}(\mathcal{T}) \cap D_{\text{PL}}(\mathcal{T}')$, part (a) follows. To see part (b), since $T_D = \cup_{\mathcal{T}} D_D(\mathcal{T})$, the map $\mathbf{A}$ is onto. To see that $\mathbf{A}$ is injective, suppose $x_1 \in D_{\text{PL}}(\mathcal{T}_1), x_2 \in D_{\text{PL}}(\mathcal{T}_2)$ so that $\mathbf{A}(x_1) = \mathbf{A}(x_2) \in D_D(\mathcal{T}_1) \cap D_D(\mathcal{T}_2)$. Apply (23) to $\mathbf{A}_{\mathcal{T}_1}^{-1}|, \mathbf{A}_{\mathcal{T}_2}^{-1}|$ on the set $D_D(\mathcal{T}_1) \cap D_D(\mathcal{T}_2)$ at the point $\mathbf{A}(x_1)$, we conclude that $x_1 = x_2$. This shows that $\mathbf{A}$ is a bijection with inverse $\mathbf{B}$. Since both $\mathbf{A}$ and $\mathbf{B}$ are continuous, $\mathbf{A}$ is a homeomorphism. q.e.d.

Now if d and d' are two discrete conformally equivalent PL metrics, then $\mathbf{A}([d])$ and $\mathbf{A}([d'])$ are of the form (p, w) and (p, w') by the definition of $\Psi_{\mathcal{T}}^{-1}\Phi_{\mathcal{T}}$. If d, d' are two PL metrics such that $\mathbf{A}([d])$ and $\mathbf{A}([d'])$ are of the form (p, w) and (p, w') , we prove that d and d' are discrete conformal as follows. Consider a generic smooth path $\gamma(t) = (p, w(t)), t \in [0, 1]$, in $T_D(S - V)$ from (p, w) to (p, w') such that $\gamma(t)$ intersects the cells $D_D(\mathcal{T})$'s transversely. This is possible since $\{D_D(\mathcal{T})|\mathcal{T}\}$ forms a real analytic cell decomposition of $T_D(S - V)$ and there are only finitely many $D_D(\mathcal{T})$ which intersects $\{p\} \times \mathbb{R}_{>0}^V$. Therefore, the path γ passes through a finite set of cells $D_D(\mathcal{T}_i)$ and $\mathcal{T}_j$ and $\mathcal{T}_{j+1}$ are related by a diagonal switch. Let $t_0 = 0 < \dots < t_m = 1$ be a partition of $[0, 1]$ so that $\gamma([t_i, t_{i+1}]) \subset D_D(\mathcal{T}_i)$. Say d_i is the PL metric so that $\mathbf{A}([d_i]) = \gamma(t_i) \in D_D(\mathcal{T}_i) \cap D_D(\mathcal{T}_{i+1})$, $d_1 = d$ and $d_m = d'$. Then by definition, the sequences $\{d_1, \dots, d_m\}$ and the associated Delaunay triangulations $\{\mathcal{T}_1, \dots, \mathcal{T}_m\}$ satisfy the definition of discrete conformality for d, d' . q.e.d.

Theorem 4.5. The homeomorphism $\mathbf{A} : T_{\text{PL}}(S, V) \rightarrow T_D(S - V)$ is a C^1 -diffeomorphism.

Proof. It suffices to show that for a point $[d] \in D_{\text{PL}}(\mathcal{T}) \cap D_{\text{PL}}(\mathcal{T}')$, the derivatives $DA_{\mathcal{T}}[d]$ and $DA_{\mathcal{T}'}[d]$ are the same. Since both $\mathcal{T}$ and $\mathcal{T}'$ are Delaunay in d and are related by a sequence of Delaunay

triangulations (in d) $\mathcal{T}_1 = \mathcal{T}, \mathcal{T}_2, \dots, \mathcal{T}_k = \mathcal{T}'$, $DA_{\mathcal{T}}([d]) = DA_{\mathcal{T}'}([d])$ follows from $DA_{\mathcal{T}_i}([d]) = DA_{\mathcal{T}_{i+1}}([d])$ for $i = 1, 2, \dots, k-1$. Therefore, it suffices to show $DA_{\mathcal{T}}([d]) = DA_{\mathcal{T}'}([d])$ when $\mathcal{T}$ and $\mathcal{T}'$ are related by a diagonal switch at an edge e . In the coordinates $\Phi_{\mathcal{T}}$ and $\Psi_{\mathcal{T}}$, the fact that $DA_{\mathcal{T}}([d]) = DA_{\mathcal{T}'}([d])$ is equivalent to the following smoothness question on the diagonal lengths.

Lemma 4.6. *Let Q be a convex Euclidean quadrilateral whose four edges lengths are x, y, z, w labelled cyclically and the length of a diagonal be a . Let $A(x, y, z, w, a)$ be the length of second diagonal and $B(x, y, z, w, a) = \frac{xz+yw}{a}$. If a point (x, y, z, w, a) satisfies $A(x, y, z, w, a) = B(x, y, z, w, a)$, i.e., Q is inscribed to a circle, then $DA(x, y, z, w, a) = DB(x, y, z, w, a)$ where DA is the derivative of A . (See Figure 5).*

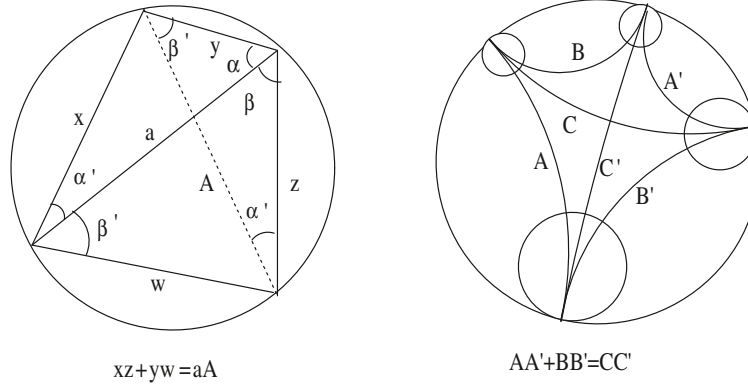


Figure 5. Euclidean and hyperbolic Ptolemy.

Proof. Since the roles of x, y, z, w are symmetric with respect to A , it suffices to show that $\frac{\partial A}{\partial a} = \frac{\partial B}{\partial a}$ and $\frac{\partial A}{\partial x} = \frac{\partial B}{\partial x}$. First, we have $\frac{\partial B}{\partial x} = \frac{z}{a}$ and $\frac{\partial B}{\partial a} = -\frac{B}{a}$.

Now let $\alpha, \alpha', \beta, \beta'$ be the angles formed by the pairs of edges $\{y, a\}$, $\{a, x\}$, $\{a, z\}$ and $\{a, w\}$. By the cosine law, we have

$$A^2 = y^2 + z^2 - 2yz \cos(\alpha + \beta).$$

Take partial x derivative of it. We obtain

$$2A \frac{\partial A}{\partial x} = 2yz \sin(\alpha + \beta) \frac{\partial \alpha}{\partial x}.$$

But it is well known (see, for instance, [28]) that in the triangle of lengths x, y, a ,

$$(24) \quad \frac{\partial \alpha}{\partial x} = \frac{x}{ay \sin(\alpha)}.$$

Therefore,

$$\frac{\partial A}{\partial x} = \frac{xz \sin(\alpha + \beta)}{aA \sin(\alpha)}.$$

Now at the point where $A(x, y, z, w, a) = B(x, y, z, w, a)$, the quadrilateral is inscribed to the circle. Therefore, $\frac{\sin(\alpha+\beta)}{\sin(\alpha)} = \frac{A}{x}$. By putting these together, we see that $\frac{\partial A}{\partial x} = \frac{xzA}{aAx} = \frac{z}{a} = \frac{\partial B}{\partial x}$.

Next, we calculate $\frac{\partial A}{\partial a}$. By the formula above, we obtain $2A \frac{\partial A}{\partial a} = 2yz \sin(\alpha + \beta) (\frac{\partial \alpha}{\partial a} + \frac{\partial \beta}{\partial a})$. Now by the derivative cosine law ([13]), we have $\frac{\partial \alpha}{\partial a} = -\frac{\partial \alpha}{\partial x} \cos(\alpha')$ which in turn is $-\frac{x \cos(\alpha')}{ay \sin(\alpha)}$ by (24). Similarly, we have $\frac{\partial \beta}{\partial a} = -\frac{w \cos(\beta')}{az \sin(\beta)}$. Putting these together, we obtain,

$$\frac{\partial A}{\partial a} = -\frac{yz \sin(\alpha + \beta)}{aA} \left(\frac{x \cos(\alpha')}{y \sin(\alpha)} + \frac{w \cos(\beta')}{z \sin(\beta)} \right).$$

Now since $A = B$, the quadrilateral is inscribed in a circle, therefore, $\frac{\sin(\alpha+\beta)}{\sin(\alpha)} = \frac{A}{x}$ and $\frac{\sin(\alpha+\beta)}{\sin(\beta)} = \frac{A}{w}$. Therefore, $\frac{\partial A}{\partial a} = -\frac{1}{a} (z \cos(\alpha') + y \cos(\beta')) = -\frac{A}{a} = -\frac{B}{a} = \frac{\partial B}{\partial a}$ where the identity $A = z \cos(\alpha') + y \cos(\beta')$ comes from the triangle of lengths y, z, A and the fact that Q is inscribed in a circle. q.e.d.

This concludes the proof of Theorem 4.5.

Suppose d is a PL metric on (S, V) whose associated decorated hyperbolic metric is $\mathbf{A}([d]) = (p, w)$ where $p \in T(S - V)$ is the underlying hyperbolic metric.

Corollary 4.7. *The space $\mathcal{D}(d) \subset T_{\text{PL}}(S, V)$ of all equivalence classes of PL metrics discrete conformal to d is C^1 -diffeomorphic to $\{p\} \times \mathbb{R}_{>0}^V$ under the diffeomorphism $\mathbf{A}$.*

The underlying (isotopy class of) hyperbolic metric p on $S - V$ can be constructed geometrically as described in §1.2. Let $\mathcal{T}$ be a Delaunay triangulation of (S, V, d) . For each Euclidean triangle τ in $\mathcal{T}$ (τ is considered as a subset of $\mathbb{C}$), let τ^* be the ideal hyperbolic triangle in $\mathbb{H}^3$ having the same set of vertices as that of τ . Here $\mathbb{C}$ is considered to be in the sphere at the infinity of the hyperbolic 3-space $\mathbb{H}^3 = \mathbb{C} \times \mathbb{R}_{>0}$. If $\tau, \sigma \in \mathcal{T}$ are two Euclidean triangles in $\mathcal{T}$ glued along their common edge by a Euclidean isometry f , then one glues τ^* and σ^* along their corresponding edges by the *same* isometry f , considered as a hyperbolic motion of $\mathbb{H}^3$. In this way, one produces a complete finite area hyperbolic metric d^* on $S - V$.

Corollary 4.8. *The underlying hyperbolic metric p on $S - V$ is given by $[d^*]$.*

We prove the corollary by checking that the shear coordinates of p and d^* are the same at each edge e of $\mathcal{T}$. For details on shear coordinate

see [32] or [4]. Let τ, σ be two Euclidean triangles in $\mathcal{T}$ adjacent to e and let the quadrilateral $\tau \cup_e \sigma$ be isometrically embedded into $\mathbb{C}$. For simplicity, we may assume without loss of generality that $\tau, \sigma \subset \mathbb{C}$ such that $\tau \cap \sigma = e$. Let the vertices of e be v_1, v_3 and the vertices of τ, σ be v_1, v_2, v_3 and v_1, v_3, v_4 respectively. We use l_{ij} to denote the Euclidean distance between v_i and v_j . Penner [32] showed that the shear coordinate of p at e is $\ln(\frac{l_{23}l_{14}}{l_{12}l_{34}})$. On the other hand, the shear coordinate of the union $\tau^* \cup \sigma^* \subset \mathbb{H}^3$ at the common geodesic edge e^* can be calculated using the complex cross ratio of the four vertices v_1, v_2, v_3, v_4 as $\ln(|\frac{(v_1-v_4)(v_3-v_2)}{(v_1-v_2)(v_3-v_4)}|)$. Since $\ln(|\frac{(v_1-v_4)(v_3-v_2)}{(v_1-v_2)(v_3-v_4)}|) = \ln(\frac{l_{23}l_{14}}{l_{12}l_{34}})$, the corollary follows. q.e.d.

5. A proof of the main Theorem 1.2

Recall that S is a closed connected surface and $V = \{v_1, \dots, v_n\} \subset S$. We will identify $\mathbb{R}^V$ with $\mathbb{R}^n$ by sending x to $(x_1, \dots, x_n)$ where $x_i = x(v_i)$ in this section. The discrete curvature K is a map from $T_{\text{PL}}(S, V)$ to $(-\infty, 2\pi)^V \cap GB$ where $GB = \{z \in \mathbb{R}^n \mid \sum_{i=1}^n z_i = 2\pi\chi(S)\}$ is defined by the Gauss–Bonnet identity. Due to $K_{\lambda d} = K_d$ for any PL metric d and $\lambda > 0$, the discrete curvature is defined on the quotient space $T_{\text{PL}}(S, V)/\mathbb{R}_{>0}$ where the action of $\mathbb{R}_{>0}$ on the Teichmüller space is by scaling PL metrics.

Fix a PL metric d on (S, V) and let $\mathcal{D}(d)$ be the set of all PL metrics on (S, V) discrete conformal to d modulo isometries isotopic to the identity on (S, V) . By definition $\mathcal{D}(d) \subset T_{\text{PL}}(S, V)$ and is invariant under scalar multiplication. Theorem 1.2 is equivalent to the statement that the restriction map $K| : \mathcal{D}(d)/\mathbb{R}_{>0} \rightarrow (-\infty, 2\pi)^V \cap GB$ is a bijection.

Using the diffeomorphism $\mathbf{A} : T_{\text{PL}}(S, V) \rightarrow T_{\text{D}}(S - V)$ and Theorem 4.3, we see that $\mathcal{D}(d)$ is C^1 -diffeomorphic to $\{p\} \times \mathbb{R}_{>0}^n \subset T_{\text{D}}(S - V)$ where p the projection of $\mathbf{A}([d]) \in T(S - V) \times \mathbb{R}_{>0}^n$ to $T(S - V)$. Let $u = (u_1, \dots, u_n) \in \mathbb{R}^n$, $w_i = e^{u_i}$, and $w = (w_1, \dots, w_n) \in \mathbb{R}_{>0}^n$. Then $w = w(u)$ is a diffeomorphism from $\mathbb{R}^n$ to $\mathbb{R}_{>0}^n$. Define the curvature map $\mathbf{F} : \mathbb{R}^n \rightarrow (-\infty, 2\pi)^n$ to be

$$(25) \quad \mathbf{F}(u) = K_{\mathbf{A}^{-1}(p, w(u))}.$$

Due to $K_{\lambda d} = K_d$, the map satisfies that $\mathbf{F}(v + k(1, 1, \dots, 1)) = \mathbf{F}(v)$. Let $P = \{z \in \mathbb{R}^n \mid \sum_{i=1}^n z_i = 0\}$ be the affine plane, $Q = (-\infty, 2\pi)^n \cap GB$ and consider the restriction map $F := \mathbf{F}|_P : P \rightarrow Q$. Theorem 1.2 is equivalent to the statement that $F : P \rightarrow Q$ is a bijection. In this section, we will show a stronger result that F is a C^1 -diffeomorphism using a variational principle.

5.1. Injectivity of F . The proof uses a variational principle developed in [27]. Since $\mathbf{A}$ is a C^1 -diffeomorphism and the discrete curvature $K : T_{\text{PL}}(S, V) \rightarrow \mathbb{R}^V$ is real analytic, the map $\mathbf{F}$ defined by (25) is

C^1 -smooth. By Theorem 3.3, let $\mathcal{T}_i$, $i = 1, \dots, m$, be the set of all triangulations so that $(\{p\} \times \mathbb{R}_{>0}^n) \cap D_D(\mathcal{T}_i) \neq \emptyset$ and $\{p\} \times \mathbb{R}_{>0}^n \subset \cup_{i=1}^m D_D(\mathcal{T}_i)$.

Lemma 5.1. *Let $\phi : \mathbb{R}^n \rightarrow \{p\} \times \mathbb{R}_{>0}^n$ be $\phi(u_1, \dots, u_n) = (p, e^{u_1}, \dots, e^{u_n})$, $U_i = \phi^{-1}((\{p\} \times \mathbb{R}_{>0}^n) \cap D_D(\mathcal{T}_i)) \subset \mathbb{R}^n$ and $J = \{i | U_i \text{ has non-empty interior in } \mathbb{R}^n\}$. Then $\mathbb{R}^n = \cup_{i \in J} U_i$ and U_i is real analytically diffeomorphic to a closed convex polytope in $\mathbb{R}^n$.*

Proof. By definition, both $\{p\} \times \mathbb{R}_{>0}^n$ and $D_D(\mathcal{T}_i)$ are closed and semi-algebraic in $T_D(S - V)$. Therefore, U_i is closed and semi-real analytic. Now by definition, $X := \cup_{i \in J} U_i$ is a closed subset of $\mathbb{R}^n$ since U_i is closed. If $X \neq \mathbb{R}^n$, then the complement $\mathbb{R}^n - X$ is a non-empty open set which is a finite union of real analytic sets of dimension less than n . This is impossible.

Next we show that for any triangulation $\mathcal{T}$ of (S, V) and $p \in T(S - V)$, the intersection $U = \phi^{-1}((\{p\} \times \mathbb{R}_{>0}^n) \cap D_D(\mathcal{T}))$ is real analytically diffeomorphic to a convex polytope in a Euclidean space. In fact, $\Psi_{\mathcal{T}}^{-1}(U) \subset \mathbb{R}^{E(\mathcal{T})}$ is real analytically diffeomorphic to a convex polytope. To this end, let $b = \Psi_{\mathcal{T}}^{-1}(p, (1, 1, \dots, 1))$. By definition, $\Psi_{\mathcal{T}}^{-1}(U)$ consists of vectors $x \in \mathbb{R}_{>0}^{E(\mathcal{T})}$ such that (20) holds and $x = \ln(\lambda) * b$ for some $\lambda \in \mathbb{R}_{>0}^V$. Here $\ln(\lambda) = (\ln(\lambda_1), \dots, \ln(\lambda_n))$ for $\lambda = (\lambda_1, \dots, \lambda_n)$. We claim that the Delaunay condition (20) consists of linear inequalities in the variable $\delta : V \rightarrow \mathbb{R}_{>0}$ where $\delta(v) = \lambda(v)^{-2}$. Indeed, suppose the two triangles adjacent to the edge e have vertices v_1, v_2, v_3 and v_1, v_2, v_4 as shown in Figure 3(c). Let x_{ij} (respectively b_{ij}) be the value of x (respectively b) at the edge joining v_i, v_j , and $\lambda_i = \lambda(v_i)$. By definition, $x_{ij} = b_{ij} \lambda_i \lambda_j$. The Delaunay condition (20) at the edge e says that

$$(26) \quad \frac{x_{12}^2}{x_{31}x_{32}} + \frac{x_{12}^2}{x_{41}x_{42}} \leq \frac{x_{31}}{x_{32}} + \frac{x_{32}}{x_{31}} + \frac{x_{41}}{x_{42}} + \frac{x_{42}}{x_{41}}.$$

It is the same as, using $x_{ij} = b_{ij} \lambda_i \lambda_j$,

$$c_3 \frac{\lambda_1 \lambda_2}{\lambda_3^2} + c_4 \frac{\lambda_1 \lambda_2}{\lambda_4^2} \leq c_1 \frac{\lambda_2}{\lambda_1} + c_2 \frac{\lambda_1}{\lambda_2},$$

where c_i is some constant depending only on b_{jk} 's. Dividing above inequality by $\lambda_1 \lambda_2$ and using $\delta_i = \lambda_i^{-2}$, we obtain

$$(27) \quad c_3 \delta_3 + c_4 \delta_4 \leq c_1 \delta_1 + c_2 \delta_2,$$

at each edge $e \in E(\mathcal{T})$. This shows for b fixed, the set of all possible values of δ form a convex polytope $\hat{P}$ defined by (27) at all edges and $\delta(v) > 0$ at all $v \in V$. On the other hand, by definition, the map from $\hat{P}$ to $\Psi_{\mathcal{T}}^{-1}(U)$ sending δ to $x = x(\delta)$ given by $x(vv') = \frac{b(vv')}{\sqrt{\delta(v)\delta(v')}} is a real analytic diffeomorphism. Thus, the result follows. q.e.d.$

Recall that the curvature map $\mathbf{F} = (F_1, \dots, F_n) : \mathbb{R}^n \rightarrow Q$ is C^1 -smooth. The value $F_i(u)$ is the discrete curvature of the PL metric $\mathbf{A}^{-1}(p, w(u))$ at the i -th vertex v_i . We will prove,

Proposition 5.2. *There exists a C^2 -smooth convex function $W : \mathbb{R}^n \rightarrow \mathbb{R}$ so that its gradient ∇W is $\mathbf{F}$ and the restriction $W|_{\{u \in \mathbb{R}^n | \sum_{i=1}^n u_i = 0\}} \rightarrow \mathbb{R}$ is strictly convex.*

On the other hand, the following is a well known fact from analysis,

Lemma 5.3. *If $W : \Omega \rightarrow \mathbb{R}$ is a C^1 -smooth strictly convex function on an open convex set $\Omega \subset \mathbb{R}^m$, then its gradient $\nabla W : \Omega \rightarrow \mathbb{R}^m$ is an embedding.*

Combining Proposition 5.2 and the lemma, we conclude the map $F := \mathbf{F} : P \rightarrow Q$ is injective.

It remains to prove Proposition 5.2.

Proof. The proposition follows by showing that (i) $\frac{\partial F_i}{\partial u_j} = \frac{\partial F_j}{\partial u_i}$ and (ii) the matrix $[\frac{\partial F_i}{\partial u_j}]_{n \times n}$ is positive semi-definite so that its null vectors are $\{\lambda(1, \dots, 1) | \lambda \in \mathbb{R}\}$. Indeed, in this case the function $W(u) = \int_0^u \sum_{i=1}^n F_i(u) du_i$. Since $\mathbb{R}^n = \cup_{i \in J} U_i$, it suffices to verify assertions (i) and (ii) on each subset U_k .

To this end, take a point $u \in \mathbb{R}^n$. Since $\mathbb{R}^n = \cup_{i \in J} U_i$, there exists U_k which contains u such that $\text{int}(U_k) \neq \emptyset$ in $\mathbb{R}^n$. This means $(p, w(u)) \in D_D(\mathcal{T}_k)$ and $\text{int}(D_D(\mathcal{T}_k)) \cap \mathbb{R}^n \neq \emptyset$. Let $l = \Psi_{\mathcal{T}_k}^{-1}(p, (1, \dots, 1)) \in \mathbb{R}_{>0}^{E(\mathcal{T}_k)}$. Then by definition, the PL metric on $(S, V, \mathcal{T}_k)$ whose edge length is $l(e)e^{u_a+u_b}$ at each edge e with $\partial e = \{v_a, v_b\}$ represents the point $\mathbf{A}^{-1}(p, w(u))$ in $T_{\text{PL}}(S, V)$.

Recall the main theorem proved in [27] (see Theorems 1.2, 2.1, and Corollary 2.3) shows,

Theorem 5.4. ([27]) *Suppose a PL metric on $(S, V, \mathcal{T})$ has edge length vector $l \in \mathbb{R}^{E(\mathcal{T})}$ and $u \in \mathbb{R}^V$ satisfies*

$$(28) \quad l(e_i)e^{u_j+u_k} + l(e_j)e^{u_i+u_k} > l(e_k)e^{u_k+u_i},$$

for all triangles of edges $e_i = (v_j, v_k), e_j = (v_k, v_i), e_k = (v_i, v_j)$ where $u_m = u(v_m)$ and $v_m \in V$. Then the discrete curvature $K = (K_1, \dots, K_n)$ of the vertex scaled PL metric on (S, V) of edge lengths $l(e)e^{u_i+u_j}$ with $\partial e = \{v_i, v_j\}$ satisfies $\frac{\partial K_i}{\partial u_j} = \frac{\partial K_j}{\partial u_i}$ and the matrix $[\frac{\partial K_i}{\partial u_j}]_{n \times n}$ is positive semi-definite so that its null space is $\{\lambda(1, \dots, 1) | \lambda \in \mathbb{R}\}$ on the open set in $\mathbb{R}^V$ defined by (28).

By Theorem 5.4, one concludes that assertions (i) and (ii) hold on each U_k and, therefore, they hold on $\mathbb{R}^n$. Therefore, the C^1 -smooth 1-form $\eta = \sum_i F_i(u) du_i$ on $\mathbb{R}^n$ satisfies $d\eta = 0$ on each $U_k, k \in J$. Due to $\mathbb{R}^n = \cup_{k \in J} U_k$, we obtain $d\eta = 0$ in $\mathbb{R}^n$. Hence, the integral $W(u) = \int_0^u \eta$

is a well defined C^2 -smooth function on $\mathbb{R}^n$ so that its Hessian matrix is positive semi-definite. Therefore, W is convex in $\mathbb{R}^n$ so that its gradient $\nabla W = \mathbf{F}$. Furthermore, since the kernel of the Hessian of W consists of diagonal vectors $\lambda(1, 1, \dots, 1)$, the Hessian of the function $W|_P$ is positive definite. Hence, $W|_P$ is strictly convex. q.e.d.

5.2. Surjectivity of F . Since both P and Q are connected manifolds of dimension $n - 1$ and F is injective and continuous, the invariance of domain theorem implies that $F(P)$ is open in Q . To show that F is onto, it suffices to prove that $F(P)$ is closed in Q .

To this end, take a sequence $\{u^{(m)}\}$ in P which leaves every compact set in P . We will show that $\{F(u^{(m)})\}$ leaves each compact set in Q . By taking subsequences, we may assume that for each index $i = 1, 2, \dots, n$, the limit $\lim_m u_i^{(m)} = t_i$ exists in $[-\infty, \infty]$. Furthermore, using Akiyoshi's Theorem 3.3 that the space $\{p\} \times \mathbb{R}_{>0}^n$ is in the union of a finite set of Delaunay cells $D_D(\mathcal{T})$, we may assume, after taking another subsequence, that the corresponding PL metrics $d_m = \mathbf{A}^{-1}(p, w(u^{(m)}))$ are Delaunay in one triangulation $\mathcal{T}$. All calculations below use the length coordinate $\Phi_{\mathcal{T}}$.

Due to the normalization that $\sum_i u_i^{(m)} = 0$ and that $u^{(m)}$ does not converge to any vector in P , there exist $t_i = \infty$ and $t_j = -\infty$. Let us label vertices $v \in V$ by *black* and *white* as follows. The vertex v_i is black if and only if $t_i = -\infty$ and all other vertices are white.

Lemma 5.5. (a) *There does not exist a triangle $\tau \in \mathcal{T}$ with exactly two white vertices.*

(b) *If $\Delta v_1 v_2 v_3$ is a triangle in $\mathcal{T}$ with exactly one white vertex at v_1 , then the inner angle of the triangle at v_1 converges to 0 as $m \rightarrow \infty$ in the metrics d_m .*

Proof. To see (a), suppose otherwise, say the triangle in $\mathcal{T}$ with vertices v_1, v_2, v_3 has exactly two white vertices at v_2, v_3 . Let the edge lengths of $\Delta v_1 v_2 v_3$ be $a_i e^{u_j^{(m)} + u_k^{(m)}}$, $\{i, j, k\} = \{1, 2, 3\}$, where $\lim_m u_i^{(m)} > -\infty$ for $i = 2, 3$ and $\lim_m u_1^{(m)} = -\infty$. By the triangle inequality, we have

$$a_2 e^{u_1^{(m)} + u_3^{(m)}} + a_3 e^{u_1^{(m)} + u_2^{(m)}} > a_1 e^{u_2^{(m)} + u_3^{(m)}}.$$

This is the same as

$$a_2 e^{-u_2^{(m)}} + a_3 e^{-u_3^{(m)}} > a_1 e^{-u_1^{(m)}}.$$

However, by the assumption, the right-hand-side tends to ∞ and the left-hand-side is bounded. The contradiction shows that (a) holds.

To see (b), let the length $l_i^{(m)}$ of the edge $v_j v_k$ in metric d_m be $a_i e^{u_j^{(m)} + u_k^{(m)}}$, $\{i, j, k\} = \{1, 2, 3\}$. Let $\alpha_1(m)$ be the inner angle at v_1 .

Note that the triangle is similar to the triangle of lengths $\{a_i e^{-u_i^{(m)}}\}$. Since $\lim_m a_i e^{-u_i^{(m)}}$ is ∞ when $i = 2, 3$ and is finite for $i = 1$, the angle α_1 tends to 0. q.e.d.

We now finish the proof of $F(P) = Q$ as follows. Since the surface S is connected and $\sum_i u_i^{(m)} = 0$, there exists an edge $e \in E(\mathcal{T})$ whose end points v, v_1 have different colors. Assume v is white and v_1 is black. Let $v_1, \dots, v_k$ be the set of all vertices adjacent to v so that v, v_i, v_{i+1} form vertices of a triangle and let $v_{k+1} = v_1$. Now apply above lemma to triangle $\Delta v v_1 v_2$ with v white and v_1 black, we conclude that v_2 must be black. Repeating this to $\Delta v v_2 v_3$ with v white and v_2 black, we conclude v_3 is black. Inductively, we conclude that all v_i 's, for $i = 1, 2, \dots, k$, are black. By part (b) of the above lemma that the angle at v of each triangle $\Delta v_i v_{i+1} v$ tends to 0, we conclude that the curvature of d_m at v tends to 2π . This shows that $F(u^{(m)})$ tends to infinity of Q . Therefore, $F(P) = Q$.

5.3. Finding the solution using a variational principle. The proof above shows if $K^* \in Q$, then the solution $u^* \in P$ to $F(u^*) = K^*$ is the unique critical point of the convex function $\int_0^u \sum_{i=1}^n (F_i(u) - K_i^*) du_i$ on the hyper-plane $\{u \in \mathbb{R}^n \mid \sum_{i=1}^n u_i = 0\}$, i.e., the solution can be found by a convex variational principle. We define the discrete Yamabe flow with surgery to be the gradient flow of the convex function $\int_0^u \sum_{i=1}^n (F_i(u) - K_i^*) du_i$. The flow is defined on the space $\mathcal{D}(d)$ of PL metrics discrete conformal to a given metric d . It takes the form $\frac{du_i(t)}{dt} = K_i(u) - K_i^*$ and $u(0) = 0$ when one uses the parametrization of $\mathcal{D}(d)$ by $u \in \mathbb{R}^V$. The exponential convergence of the flow to the solution u^* was established in Theorem 1.4 of [27]. Note that the flow equation $\frac{du_i(t)}{dt} = K_i(u) - K_i^*$ depends on the triangulation $\mathcal{T}$. A solution $u(t)$, $t \in [0, \infty)$ to the equation may go through several different charts U_i 's. In this case, due to the triangulation change, the flow equation $\frac{du_i(t)}{dt} = K_i(u) - K_i^*$ takes different expressions with respect to different triangulations. We call these “surgery change” of the flow.

Appendix: A proof of Akiyoshi's theorem

For completeness, we present our proof in this appendix. The theorem and the proof hold for decorated finite volume hyperbolic manifolds of any dimension. We state the 2-dimensional case for simplicity.

Theorem 5.6. (Akiyoshi [1]) *For a finite area complete hyperbolic metric d on $S - V$, there exist triangulations $\mathcal{T}_1, \dots, \mathcal{T}_k$ so that for any $w \in R_{>0}^n$, any Delaunay triangulation of (d, w) is isotopic $\mathcal{T}_i$, $i \in \{1, 2, \dots, k\}$.*

Proof. We begin by study the shortest geodesics in a complete finite area hyperbolic surface $(S - V, d)$. Recall the Shimizu lemma [5] which

implies that if $w \in (0, 1)^n$, then the associated horoballs $H_i(w)$ in the decorated metric (d, w) are embedded and pairwise disjoint. Let us assume without loss of generality that $w \in (0, 1)^n$. A geodesic α from cusp v_i to v_j in $(S - V, d)$ is called a *shortest geodesic* from v_i to v_j if there exists a $w \in (0, 1)^n$ so that $\alpha \cap X_w$ is a shortest path among all homotopically non-trivial paths in X_w joining $\partial H_i(w)$ to $\partial H_j(w)$. The shortest property implies that $\alpha \cap X_w$ is an orthogeodesic. Furthermore, by Lemma 3.1, if α is a shortest geodesic, then for any $w' \in (0, 1)^n$, $\alpha \cap X_{w'}$ is again a shortest geodesic in $X_{w'}$ from $\partial H_i(w')$ to $\partial H_j(w')$, i.e., being a shortest geodesic from v_i to v_j is independent of the choice of decorations. Indeed, for any geodesic β from cusp v_i to v_j , we have

$$(29) \quad l(\beta \cap X_{w'}) = l(\beta \cap X_w) - \ln(w'_i) - \ln(w'_j) + \ln(w_i) + \ln(w_j).$$

Lemma 5.7. *Suppose $(S - V, d)$ is a finite area complete hyperbolic surface. Then*

- (a) *there are only finitely many shortest geodesics from v_i to v_j .*
- (b) *there is $\delta_{ij} = \delta_{ij}(S - V, d) > 0$ so that if α is a shortest geodesic from v_i to v_j and β is another geodesic from v_i to v_j with $|l(\beta \cap X_w) - l(\alpha \cap X_w)| \leq \delta_{ij}$, then β is a shortest geodesic.*
- (c) *given v_i , if α is a shortest orthogeodesic geodesics among all orthogeodesics in X_w from ∂H_i to ∂X_w , then α^* , the complete geodesic containing α , is an edge of a Delaunay triangulation of the decorated metric (d, w) and the mid-point of α is in $R_j(w)$.*

Proof. The first part follows from the simple fact that on any compact surface X_w , for any constant C , there are only finitely many orthogeodesics of length at most C . Part (b) follows from (a) and equality (29). Part (c) follows from the definition of Voronoi cells and its dual. Note that in general, if β is a shortest orthogeodesic in X_w between $\partial H_i(w)$ and $\partial H_j(w)$, β^* may not be an edge in any Delaunay triangulation of (d, w) . q.e.d.

Now we prove the theorem by contradiction. Suppose otherwise, there exists a sequence of decorated metrics $(d, w^{(m)})$ where $w^{(m)} = (w_1^{(m)}, \dots, w_n^{(m)}) \in \mathbb{R}^n$ so that the associated Delaunay triangulations $\mathcal{T}_m = \mathcal{T}(d, w^{(m)})$ are pairwise distinct in $(S - V, d)$. After normalizing $w^{(m)}$ by scaling, relabel the vertices $v_1, \dots, v_n$ and taking subsequences, we may assume

- (i) $w_1^{(m)} = \max\{w_i^{(m)} | i = 1, 2, \dots, n\} = 1/2$;
- (ii) for each $i = 1, 2, \dots, n$, the limit $\lim_m w_i^{(m)} = t_i \in [0, 1/2]$ exists;
- (iii) $t_1, \dots, t_k > 0$ and $t_{k+1} = \dots = t_n = 0$.

For simplicity, we use $E_{ij}(\mathcal{T})$ to denote the subset of all edges of $\mathcal{T}$ joining v_i to v_j . We will derive a contradiction by showing that $\cup_m E_{ij}(\mathcal{T}_m)$ is a finite set.

Lemma 5.8. *There exists a constant $C > 0$ so that for all $i, j \leq k$, and all $e \in E_{ij}(\mathcal{T}_m)$, the length*

$$l(e \cap X_{w^{(m)}}) \leq C.$$

In particular, $\cup_m E_{ij}(\mathcal{T}_m)$ is a finite set.

Proof. For any $\delta \in (0, 1/2)$, let $u^{(m)}(\delta) = (w_1^{(m)}, \dots, w_k^{(m)}, \delta, \dots, \delta) \in \mathbb{R}^n$. Fix a δ , since $\lim_m w_j^{(m)} = 0$ for $j > k$, for m large, each point $x \in X_{u^{(m)}(\delta)}$ is in some Voronoi cell $R_i(w^{(m)})$ for some $i \leq k$. Therefore, there is a small $\delta > 0$ so that for all $i, j = 1, 2, \dots, k$, all large m , and all $e \in E_{ij}(\mathcal{T}_m)$, $e \cap X_{w^{(m)}} \subset X_{u^{(m)}(\delta)}$. By the assumption that $t_1, \dots, t_k > 0$ and by choosing δ smaller than $\min\{t_1, \dots, t_k\}$, we see that the surface $X_{u^{(m)}(\delta)}$ is a subset of the compact surface $X_{(\delta, \dots, \delta)}$. Therefore, there is a constant $C > 0$ so that $\text{diam}(X_{u^{(m)}(\delta)}) \leq C/2$ for all m . Note if $e \in E(\mathcal{T}(d, w))$ is an edge, then the length of the orthogeodesic $e \cap X_w$ in metric d satisfies,

$$(30) \quad l(e \cap X_w) \leq 2\text{diam}(X_w),$$

where $\text{diam}(Y)$ is the diameter of a metric space Y . Indeed, $l(e \cap X_w) \leq \text{diam}(R_i(w)) + \text{diam}(R_j(w)) \leq 2\text{diam}(X_w)$. This shows, by (30), that

$$l(e \cap X_{w^{(m)}}) \leq l(e \cap X_{u^{(m)}(\delta)}) \leq 2\text{diam}(X_{u^{(m)}(\delta)}) \leq C.$$

Finally, since for any constant C , there are only finitely many orthogeodesics in $X_{(\delta, \dots, \delta)}$ of lengths at most C , it follows that $\cup_m E_{ij}(\mathcal{T}_m)$ is finite. q.e.d.

Now for m large, each point in $X_{u^{(m)}(1/2)}$ is in $\cup_{i=1}^k R_i(w^{(m)})$. Therefore, for large m , if $i, j > k$, then $E_{ij}(\mathcal{T}_m) = \emptyset$ since an edge $e \in E_{ij}(\mathcal{T}_m)$ must intersect $X_{u^{(m)}(1/2)}$. Hence, if $E_{jh}(\mathcal{T}_m) \neq \emptyset$, then $h \leq k$.

Lemma 5.9. *There is n_0 so that if $m \geq n_0$, $j > k$ and $e \in E_{ij}(\mathcal{T}_m)$, then e is a shortest geodesic from v_i to v_j . In particular, for $j > k$ and $i \leq k$, the set $\cup_m E_{ij}(\mathcal{T}_m)$ is finite.*

Proof. We need to study the Voronoi cell $R_j(w^{(m)})$. Since $\lim_m w_j^{(m)} = 0$ and $t_i > 0$, for large m , the Voronoi cell $R_j(w^{(m)}) \subset H_j(u^{(m)}(1/2))$. Let $\partial_0 R_j(w^{(m)})$ be the piecewise geodesic boundary component $\partial R_j(w^{(m)}) - \partial H_j(w^{(m)})$.

Claim. *For any two edges a_m, b_m in $\partial_0 R_j(w^{(m)})$,*

$$(31) \quad \lim_m |\text{dist}(a_m, H_j(w^{(m)})) - \text{dist}(b_m, H_j(w^{(m)}))| = 0.$$

Assuming the claim, we finish the proof of the lemma as follows. Let ϵ_m be a shortest orthogeodesic in X_w from $\partial H_j(w^{(m)})$ to ∂X_w and $e'_m = \epsilon_m^*$ be the complete geodesic containing ϵ_m . Then by Lemma 5.7, $e'_m \in \cup_{i=1}^n E_{ij}(\mathcal{T}_m)$. Let the dual of e'_m be the edge a_m of $\partial_0 R_j(w^{(m)})$.

For any edge $e_m \in E_{ij}(w^{(m)})$ dual to an edge b_m of $\partial_0 R_j(w^{(m)})$, we have $l(e_m \cap X_{w^{(m)}}) = 2 \operatorname{dist}(b_m, H_j(w^{(m)}))$ by the definition of Delaunay. Therefore, by (31)

$$\lim_m |l(e_m \cap X_{w^{(m)}}) - l(e'_m \cap X_{w^{(m)}})| = 0.$$

By Lemma 5.7, since e'_m is a shortest geodesic, e_m is also a shortest geodesic for m large.

To see the claim (31), recall that a simple geodesic loop on $(S - V, d)$ is a smooth map $\alpha : [0, 1] \rightarrow S - V$ so that $\alpha(0) = \alpha(1)$, $\alpha|_{(0,1)}$ is a geodesic and $\alpha|_{[0,1]}$ is injective. Now for each $i \leq k$ and for m large, the equidistance curve $\alpha_{i,j}(m)$ between $H_i(w^{(m)})$ and $H_j(w^{(m)})$ is a simple geodesic loop in the cusp region $H_j(s_m(1, 1, \dots, 1))$ where $\lim_m s_m = 0$. This is due to the fact that $w_j^{(m)} \rightarrow 0$ and $w_i^{(m)} \rightarrow t_i > 0$. It is well known that if α is a simple geodesic loop in a cusp region $H_j(w)$, then the length of α is less than w_j . Therefore, $l(\alpha_{i,j}(m)) \leq s_m$ and $\lim_m l(\alpha_{i,j}(m)) = 0$. By definition, the boundary $\partial_0 R_j(w^{(m)}) \subset \cup_i \alpha_{i,j}(m)$. If a_m, b_m are two edges $\partial_0 R_j(w^{(m)})$, then by definition $|\operatorname{dist}(a_m, H_j(w^{(m)})) - \operatorname{dist}(b_m, H_j(w^{(m)}))| \leq \sum_{i=1}^k l(\alpha_{i,j}(m))$. Therefore, (31) follows from $\lim_m l(\alpha_{i,j}(m)) = 0$. q.e.d.

q.e.d.

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