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PSEUDO-DEVELOPING MAPS FOR IDEAL TRIANGULATIONS II: POSITIVELY ORIENTED IDEAL TRIANGULATIONS OF CONE-MANIFOLDS

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Abstract. We generalise work of Young-Eun Choi to the setting of ideal triangulations with vertex links of arbitrary genus, showing that the set of all (possibly incomplete) hyperbolic cone-manifold structures realised by positively oriented hyperbolic ideal tetrahedra on a given topological ideal triangulation and with prescribed cone angles at all edges is (if non-empty) a smooth complex manifold of dimension the sum of the genera of the vertex links. Moreover, we show that the complex lengths of a collection of peripheral elements give a local holomorphic parameterisation of this manifold.

1. Introduction

The complement N of the vertices in a triangulated orientable 3–dimensional pseudo-manifold P carries a complete hyperbolic cone-manifold structure, where the singular locus is contained in the 1–skeleton and each ideal tetrahedron develops into an ideal hyperbolic tetrahedron. For instance, such a structure is obtained by realising each ideal tetrahedron in N as a regular ideal hyperbolic 3–simplex (see [9] and §2.4). Let $\mathcal{T}$ denote the (topological) ideal triangulation of N and E the set of ideal edges. For any prescribed cone angles $\kappa\colon E \to \mathbb{R}$, let $\mathfrak{D}^+(\mathcal{T}, \kappa)$ be the set of all (possibly incomplete) hyperbolic cone-manifold structures with the ideal tetrahedra in $\mathcal{T}$ realised by positively oriented hyperbolic tetrahedra and with the prescribed cone angles κ. We show that if this set is non-empty, then it is a smooth complex manifold of dimension the sum of the genera of the vertex links (Corollary 3). This generalises the first main theorem of Choi [2]. This result is deduced as a consequence of a more general result (Theorem 2), which is essentially due to Neumann [6]. Moreover, we show that the complex lengths of a collection of non-trivial peripheral elements, g for each vertex link of genus g, give a local holomorphic parametrisation of $\mathfrak{D}^+(\mathcal{T}, \kappa)$ (Corollary 10). This is achieved through a generalisation (Theorem 9) of the second main theorem of Choi [2].

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Our approach is different from Choi's and includes new results on the interplay between tangential angle structures, the boundary map and the study of the volume function. It builds on previous work of Neumann [6] as well as Futer and Guéritaud [3].

The results of this paper show that the examples of pseudo-manifolds M with spherical vertex links in [9] exhibit generic behaviour—for any prescribed cone angles at the edges, the manifold $\mathfrak{D}^+(M,\kappa)$ is either 0–dimensional or empty. We conclude this paper by giving two examples, which both have *two components where one might have expected one.* The first example is a once-cusped hyperbolic 3–manifold of finite volume with the property that the two discrete and faithful characters lie on different components of the $PSL_2(\mathbb{C})$–character variety. It was found via an application of the volume function explained to the authors by Nathan Dunfield. The second example is a manifold with two boundary components—a torus and a genus two surface. In this example, one can find distinct prescribed cone angles κ_1 and κ_2 for the edges of an ideal triangulation with the property that κ_1 and κ_2 differ by 2π at two edges. In contrast, for manifolds with only torus boundary components, a standard Euler characteristic argument shows that at each edge the total cone angle has to be exactly 2π.

2. Preliminaries

2.1. **Conventions for vector spaces.** If X is a finite set, $\mathbb{R}^X$ denotes the real vector space of all functions $X \to \mathbb{R}$ and we assume that $\mathbb{R}^X$ has the standard inner product, so that

$$X^\star = \{x^\star \in \mathbb{R}^X \mid x \in X \text{ and for all } y \in X : x^\star(y) = \delta_{xy}\}$$

is an orthonormal basis of $\mathbb{R}^X$, where Kronecker's notation

$$\delta_{xy} = \begin{cases} 1 & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is used.

2.2. **Pseudo-manifolds and triangulations.** Let $\widetilde{\Delta}$ be a finite union of pairwise disjoint, oriented Euclidean 3–simplices, and Φ be a family of orientation-reversing affine isomorphisms pairing the facets in $\widetilde{\Delta}$, with the properties that $\varphi \in \Phi$ if and only if $\varphi^{-1} \in \Phi$, and every codimension-one facet is the domain of a unique element of Φ. The elements of Φ are termed *face pairings.* The quotient space $P = \widetilde{\Delta}/\Phi$ with the quotient topology is a closed, orientable 3–dimensional pseudo-manifold, and the quotient map is denoted $p\colon \widetilde{\Delta} \to P$. The triple $\mathcal{T} = (\widetilde{\Delta}, \Phi, p)$ is a *(singular) triangulation* of P. The adjective *singular* is usually omitted, and we will not need to distinguish between the cases of a simplicial or a singular triangulation. We will always assume that P is connected. In the case where P is not connected, the results of this paper apply to its connected components.

We will use the following notation:

$$T = \{\sigma_i\} = \widetilde{\Delta}^{(3)}, \qquad E = \{e_j\} = P^{(1)}, \qquad V = \{v_k\} = P^{(0)}.$$

Note that E and V are equivalence classes of 1–simplices and 0–simplices of $\widetilde{\Delta}$ respectively.

The set of non-manifold points of P is contained in the 0–skeleton. Denote this set $V_s \subseteq V = P^{(0)}$. The cases of interest are usually when $V_s = \emptyset$ or $V_s = V$.

In the first case P is a closed 3–manifold. In the second case $\mathcal{T}$ restricts to an *ideal triangulation* of the topologically finite, non-compact 3–manifold $N = P \setminus P^{(0)}$ and P is the *end-compactification* of N.

For each vertex $v \in V$, $\mathrm{Lk}(v)$ is a closed orientable surface of genus $g_v \geq 0$, with a triangulation $\mathcal{T}_v$ induced from $\mathcal{T}$. We will repeatedly make use of the following fact, which follows from a direct Euler characteristic calculation:

Lemma 1. $|T| - |E| + |V| = \sum_{v \in V} g_v$.

2.3. **Quadrilateral index.** Let σ be a 3–simplex. A *quadrilateral type* q in σ is a partition of its set of vertices into two sets of cardinality two. The name alludes to its geometric realisation as a properly embedded quadrilateral disc separating a pair of opposite 1–simplices e_0 and e_1. See [10] for an exposition of this well-known geometric viewpoint, which goes back to Haken [4]. It will be convenient to regard a quadrilateral type as a set containing these two opposite 1–simplices, and we write

$$\{e_0, e_1\} = q < \sigma$$

and say that e_0 and e_1 *face* q.

There are precisely three quadrilateral types in σ. There is a natural action of the symmetric group $\mathrm{Sym}(4)$ on the set of vertices of σ. Choose an orientation of σ. The alternating group $\mathrm{Alt}(4)$ fixes the orientation and permutes the three quadrilateral types. The stabiliser of a quadrilateral type is the Klein four group K. So there is a natural faithful action of the cyclic group $C_3 \cong \mathrm{Alt}(4)/K$ on the set of all quadrilateral types, and hence a natural cyclic order on that set. Throughout, q, q', q'' denote the three quadrilateral types in the 3–simplex σ and the action of C_3 is indicated by the prime mark, so $(q')' = q''$ and $(q'')' = q$ and the natural cyclic order is therefore given by $q \to q' \to q'' \to q$.

Denote by $\square$ the set of all normal quadrilateral types in $\widetilde{\Delta}$. The quotient map $p\colon \widetilde{\Delta} \to P$ induces a natural map $p\colon \widetilde{\Delta}^{(1)} \to P^{(1)} = E$. For any $e \in E$ and $q \in \square$, the number of edges in the preimage $p^{-1}(e) \subset \widetilde{\Delta}^{(1)}$ facing q is:

$$i(q, e) = |q \cap p^{-1}(e)| \in \{0, 1, 2\}.$$

If $i(q, e) > 0$, we say that e faces q.

2.4. **Cone-deformation variety.** The *cone-deformation variety* $\mathfrak{D}(\mathcal{T}; \star)$ is the set of all $(z, \xi) \in \mathbb{C}^{\square} \times (S^1)^E$ satisfying:

i) for each edge $e \in E$,

$$\prod_{q \in \square} z(q)^{i(q,e)} = \xi(e), \tag{1}$$

ii) for each $q \in \square$,

$$z(q') = \frac{1}{1 - z(q)}. \tag{2}$$

Equation (2) is the *parameter relation* for q. Applying the cyclic ordering gives:

$$z(q'') = 1 - \frac{1}{z(q)} = \frac{z(q) - 1}{z(q)} \qquad \text{and} \qquad z(q)z(q')z(q'') = -1.$$

Equation (1) is the *cone-hyperbolic gluing equation* for e. Multiplying all of these equations gives the identity

$$\prod_{e\in E} \xi(e) = 1.$$

If $\xi(e) = 1$, then equation (1) is the usual *hyperbolic gluing equation* of e.

We have the projections onto the factors

$$s\colon \mathfrak{D}(\mathcal{T};\star) \to \mathbb{C}^{\square} \qquad \text{and} \qquad c\colon \mathfrak{D}(\mathcal{T};\star) \to (S^1)^E,$$

where s gives the *shapes* of the tetrahedra and c gives the *curvature* at the edges.

Denote the upper half plane in $\mathbb{C}$ by $\mathbb{H}$ and for each $\xi \in (S^1)^E$ define

$$\mathfrak{D}^+(\mathcal{T};\xi) = \mathfrak{D}(\mathcal{T};\star) \cap (\mathbb{H}^{\square} \times \{\xi\}).$$

It was observed in [9] that the cone-deformation variety is non-empty for any triangulation since one may choose each $z(q) = \frac{1}{2}(1+\sqrt{-3})$ to be the shape of the regular hyperbolic ideal 3–simplex, and this turns N into a complete hyperbolic cone-manifold. Hence there always exists $\xi \in (S^1)^E$ such that $\mathfrak{D}^+(\mathcal{T};\xi) \neq \emptyset$.

One can study the (topological) connected components of $\mathfrak{D}^+(\mathcal{T};\xi)$ via the set of all $\kappa \in \mathbb{R}^E$ with the property that $z \in \mathbb{H}^{\square}$ satisfies the parameter relations and for each edge e, we have:

$$\sum_{q\in\square} i(q,e)\ \log(z(q)) = \kappa(e), \tag{3}$$

and

$$\xi(e) = \exp(i\kappa(e)). \tag{4}$$

Throughout this paper, log is the standard branch on $\mathbb{C} \setminus (-\infty, 0]$ unless stated otherwise. It follows from analytic continuation that on each connected component of $\mathfrak{D}^+(\mathcal{T};\xi)$, the left-hand side of (3) is constant.

Below Corollaries 3 and 10 imply that if $\mathfrak{D}^+(\mathcal{T};\xi)$ is non-empty, then it is a smooth complex manifold of dimension the sum of the genera of the vertex links, and each of its components has a holomorphic parametrisation by the holonomies of peripheral elements, one for each genus. This will be proved using the more general *complex-curvature* and *log-curvature* maps defined in §2.5, which respectively generalise the left-hand sides of (1) and (3).

2.5. **The log-curvature map.** Let $\mathbb{H}$ be the upper half plane and $Z = \{z \in \mathbb{H}^{\square} | z(q') = \frac{1}{1-z(q)}\}$. The *log-curvature map* $G\colon Z \to \mathbb{C}^E$ is defined by:

$$G(z)(e) = \sum_{q\in\square} i(q,e)\log(z(q)).$$

The log-curvature map is, of course, closely related to the complex-curvature map $c\colon Z \to \mathbb{C}^E$,

$$c(z)(e) = \prod_{q\in\square} z(q)^{i(q,e)}.$$

For instance, a well-known Euler characteristic argument shows that if the link of each vertex is a torus, then $G^{-1}(2\pi i, \dots, 2\pi i) = c^{-1}(1, \dots, 1)$. In general, $c^{-1}(u)$ is a countable union of sets of the form $G^{-1}(u_k)$, and these sets are pairwise disjoint by analytic continuation. But since $c^{-1}(u)$ is an affine algebraic set, at most finitely many of these sets will be non-empty. Hence each level set $G^{-1}(u_k)$ of the log-curvature function is also an affine algebraic set.

3. Statements and proofs of the main results

Our central result is Theorem 9, generalising Choi's main technical result [2, Theorem 4.13] to pseudo-manifolds with positively oriented triangulations and prescribed log-curvature.

3.1. **Rank of the log-curvature map.** Fixing a preferred quadrilateral type $q < \sigma$ for every 3–simplex, there is a natural identification between Z and $\mathbb{H}^T$ via the projection map $\pi\colon\ Z \to \mathbb{H}^T$, where

$$\pi(z)(\sigma) = z(q).$$

Hence the log-curvature map can and will be viewed as a map $G : \mathbb{H}^T \to \mathbb{C}^E$. The following result is a corollary of [6, Theorem 4.1]:

Theorem 2 (Neumann). *$\mathrm{d}\,G$ has constant rank $|T| - \sum_{v\in V} g_v$.*

Proof. For each tetrahedron $\sigma_i \in T$, fix a quadrilateral type $q_i < \sigma_i$. Using the quadrilateral index, we let $B = (b_{ij})$ be the $2|T| \times |E|$ matrix, where for all even i:

$$\begin{cases} b_{ij} = i(q_i, e_j) - i(q_i', e_j), \\ b_{i+1j} = i(q_i', e_j) - i(q_i'', e_j). \end{cases}$$

It is well known that $\mathrm{d}\,G$ has the same rank as B (see [8]), hence it is enough to show that $\operatorname{rank} B = |T| - \sum_{v\in V} g_v$. This is the content of [6, Theorem, 4.1]. Neumann defines a linear map $\beta : C_1 \to J$, where C_1 is the free $\mathbb{Z}$–module generated by E and J is the free $\mathbb{Z}$–module generated by T^2, and shows that

$$\dim(\operatorname{Ker}\beta^* / \operatorname{Im}\beta) = \dim \bigoplus_{v\in V} \mathrm{H}_1(\mathrm{Lk}(v)) = \sum_{v\in V} 2g_v,$$

where β^* is the dual map. Moreover, Neumann shows $\operatorname{Ker}\beta^* = 2|T| - |E| + |V|$, and therefore

$$\dim(\operatorname{Im}\beta) = \dim(\operatorname{Ker}\beta^*) - \dim \bigoplus_{v\in V} \mathrm{H}_1(\mathrm{Lk}(v)) = 2|T| - |E| + |V| - \sum_{v\in V} 2g_v.$$

Using particular bases of C_0 and J, B is the matrix associated to β, giving

$$\operatorname{rank} B = \dim(\operatorname{Im}\beta) = 2|T| - |E| + |V| - \sum_{v\in V} 2g_v = T - \sum_{v\in V} g_v,$$

where the last equality follows from Lemma 1. □

Theorem 2 and the implicit function theorem imply the following result.

Corollary 3. *For all $u \in \mathbb{C}^E$, the complex variety $G^{-1}(u)$ is either empty or a smooth complex manifold of dimension $\sum_{v\in V} g_v$.*

3.2. **Boundary map.** Let α be an oriented closed normal curve on $\mathrm{Lk}(v)$, representing a non–trivial element in $H_1(\mathrm{Lk}(v))$ and let $t \in \mathcal{T}_v^{(2)}$ be a normal triangle contained in tetrahedron $\sigma \in T$. A segment of α with respect to t is an oriented connected component of $\alpha \cap t$. Let S_α^t be the set of all segments of α with respect to t, and $S_\alpha = \bigcup_{v\in V} \bigcup_{t\in\mathcal{T}_v^{(2)}} S_\alpha^t$ be the set of all segments of α.

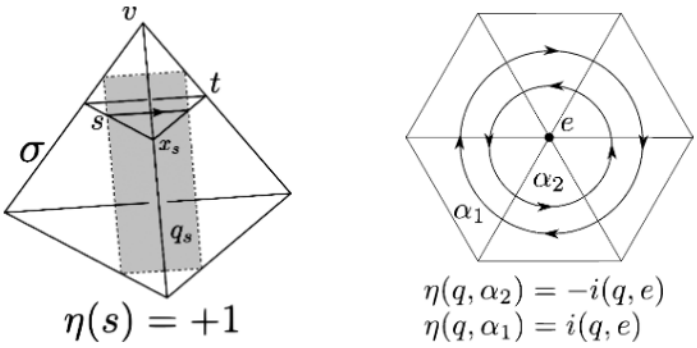

FIGURE 1. On the left: Each $s \in S_\alpha$ determines a quadrilateral type q_s and isolates a vertex x_s. For an observer sitting in the cusp, x_s here lies on the right side of s. On the right: α_1, α_2 encircle the endpoint of an edge $e \in E$.

Each $s \in S_\alpha$ uniquely determines a quadrilateral type $q \in \square$ such that $s \subset q$, which will be denoted by q_s. If $s \in S_\alpha^t$, s divides t in two regions and isolates one of the three vertices, say x_s (Figure 1). With respect to the orientation of s and the induced orientation on $\mathrm{Lk}(v)$, it makes sense to say that x_s lies on the right side or on the left side of s when viewed from the cusp. We therefore define

$$\eta(s) = \begin{cases} +1 & \text{if } x_s \text{ lies on the right side of } s, \\ -1 & \text{if } x_s \text{ lies on the left side of } s, \end{cases}$$

$$\eta(q, s) = \begin{cases} \eta(s) & \text{if } q = q_s, \\ 0 & \text{otherwise,} \end{cases}$$

and finally

$$\eta(q, \alpha) = \sum_{s \in S_\alpha} \eta(q, s).$$

In particular, when α encircles one endpoint of an edge $e \in E$, then $\eta(q, \alpha) = \pm i(q, e)$ for all $q \in \square$, where the sign depends on whether α is oriented anticlockwise or clockwise with respect to the cusp (Figure 1).

Hence we can write the log holonomy of α as $\mathrm{h}_\alpha : Z \to \mathbb{C}$,

$$\mathrm{h}_\alpha(z) = \sum_{q \in \square} \eta(q, \alpha) \log z(q).$$

Notice that if α_1 is normally isotopic to α_2, then $\mathrm{h}_{\alpha_1}(z) = \mathrm{h}_{\alpha_2}(z)$, and if α_1 is the union of n normal curves all normally isotopic to α_2, then $\mathrm{h}_{\alpha_1}(z) = n\,\mathrm{h}_{\alpha_2}(z)$. We therefore have a natural extension of the log holonomy to $\mathcal{N}_\mathbb{R}$, the $\mathbb{R}$–vector space with basis the normal isotopy classes of curves on $\mathrm{Lk}(v)$. For all $a, b \in \mathbb{R}$ and $\alpha, \beta \in \mathcal{N}_\mathbb{R}$ define

$$\mathrm{h}_{a\alpha + b\beta}(z) := a\,\mathrm{h}_\alpha(z) + b\,\mathrm{h}_\beta(z).$$

Note that if α^{-1} is the closed normal curve α with opposite orientation, then $\mathrm{h}_{\alpha + \alpha^{-1}}(z) = 0$.

If $G(z) \neq (2\pi i, \ldots, 2\pi i)$, then the *value* of $\mathrm{h}_\alpha(z)$ may not be an invariant of the homology class of α, but depends on the choice of normal representative since an isotopy pushing α over a vertex in $\mathrm{Lk}(v)$ may change its value.

For each link $\mathrm{Lk}(v_i)$, let g_i denote its genus and choose a canonical homotopy group generating set $\mathcal{L}_i \cup \mathcal{M}_i$, where $\mathcal{L}_i = \{\lambda_1^i, \dots, \lambda_{g_i}^i\}$ is the *set of longitudes* and $\mathcal{M}_i = \{\mu_1^i, \dots, \mu_{g_i}^i\}$ the *set of meridians*, indexed and oriented such that the algebraic intersection number satisfies $\iota(\lambda_j^i, \mu_j^i) = 1$ for all j and $\iota(\alpha, \beta) = 0$ for all other pairs of elements $\alpha, \beta \in \mathcal{L}_i \cup \mathcal{M}_i$. Moreover, we assume that each element of $\mathcal{L}_i \cup \mathcal{M}_i$ is an oriented normal curve on $\mathrm{Lk}(v_i)$.

Let $\mathcal{L} = \bigcup_i \mathcal{L}_i$ and define the *boundary map* $H_{\mathcal{L}} : Z \to \mathbb{C}^{\mathcal{L}}$ by

$$H_{\mathcal{L}}(z)(\lambda) = \mathrm{h}_\lambda(z) \qquad \text{for all } \lambda \in \mathcal{L}.$$

Similarly, we let $\mathcal{M} = \bigcup_i \mathcal{M}_i$ and $\mathrm{Lk}(V) = \bigsqcup_i \mathrm{Lk}(v_i)$. Hence $\mathcal{L} \cup \mathcal{M}$ is a generating set for $H_1(\mathrm{Lk}(V))$.

3.3. **Tangential angle structures.** Our study of derivatives naturally leads us (at least implicitly) to the tangent space of the space of all angle structures on $\mathcal{T}$. Following [5], the space of all *tangential angle structures* $\mathrm{TAS} = \mathrm{TAS}(\mathcal{T})$ is the set of all $\alpha \in \mathbb{R}^{\square}$ such that

- $\sum_{q<\sigma} \alpha(q) = 0$, $\forall \sigma \in T$, and
- $\sum_{q \in \square} i(q, e)\alpha(q) = 0$, $\forall e \in E$.

The endgame of the proof of Theorem 9 uses a specific spanning set of

$$\mathrm{TAS}_{\mathcal{L}} = \mathrm{TAS}_{\mathcal{L}}(\mathcal{T}) = \{w \in \mathrm{TAS} \mid \sum_{q \in \square} \eta(q, \lambda) w(q) = 0 \ \forall \lambda \in \mathcal{L}\} \leq \mathrm{TAS},$$

which we will now determine. For every edge $e \in E$ and every normal curve γ on a vertex link, let $Q_e, Q_\gamma \in \mathbb{R}^{\square}$ be defined by

$$Q_e = \sum_{q \in \square} i(q, e) \left((q')^* - (q'')^* \right),$$
$$Q_\gamma = \sum_{q \in \square} \eta(q, \gamma) \left((q')^* - (q'')^* \right).$$

In their study of angle structure on cusped manifolds, Futer and Guéritaud [3, Section 4] introduced Q_e and Q_γ under the names of *leading–trailing deformation around* e and *leading–trailing deformation along* γ respectively. In fact, for every edge e, one may choose a normal closed curve ρ about one endpoint of e and an orientation of ρ such that $Q_e = Q_\rho$. Hence Q_e can be thought of as a leading–trailing deformation along ρ. Some of the results of [3] extend directly to the more general setting of an oriented pseudo-manifold. As in [3, Lemma 4.5], we have $Q_e, Q_\gamma \in \mathrm{TAS}$.

Lemma 4 ([3, Lemma 4.4])**.** *Let α, β be oriented closed normal curves on $\mathrm{Lk}(V)$ that intersect transversely, if at all. Then*

$$\frac{\partial}{\partial Q_\beta} \mathrm{im}(\mathrm{h}_\alpha) = \sum_{q \in \square} \eta(q, \alpha) Q_\beta(q) = 2\, \iota(\alpha, \beta).$$

Proof. The proof in [3, Lemma 4.4] does not use the fact that ∂M is a union of tori (and in particular that in this case h_α is independent of the choice of normal curve in a homology class) and applies verbatim. □

Lemma 5. *The set $\{Q_e\}_{e\in E}$ spans a subspace of dimension $|T| - \sum_i g_i$.*

Proof. For each tetrahedron $\sigma_j \in T$ fix a normal quadrilateral type $q_j < \sigma_j$. Let B be the $(|E| \times 3|T|)$ matrix whose i–th row is the vector Q_{e_i}, namely

$$\left(Q_{e_i}(q_1), Q_{e_i}(q_1'), Q_{e_i}(q_1''), \ldots, Q_{e_i}(q_{|T|}), Q_{e_i}(q_{|T|}'), Q_{e_i}(q_{|T|}'')\right),$$

where $e_i \in E$. For every vertex $v \in V$, we define a row vector $r_v \in \mathbb{R}^{|E|}$, whose i–th entry is the number of endpoints that the i–th edge e_i has at the vertex v.

In [3, Lemma 3.3], a matrix A and vectors r_c were constructed in a similar fashion, and they are related to B and r_v as follows:

- Let $\sigma \in S_3$ be the permutation (123) and A_i, B_i be the i-th columns of A, B respectively. Then for all $0 \le k \le |T|$ and $1 \le i \le 3$, B_{3k+i} is the column vector made up of the last $|E|$ entries of $A_{3k+\sigma^{-1}(i)} - A_{3k+\sigma(i)}$;
- r_v is the row vector made of the last $|E|$ entries of r_c.

Following the proof of [3, Lemma 3.3], one checks that the vectors r_v form a basis for the row null space of B and therefore $\operatorname{rank}(B) = |E| - |V|$. The conclusion follows from Lemma 1. □

The following result generalises [3, Proposition 4.6]. To simplify notation, we write $\{Q_\lambda\} = \{Q_\lambda\}_{\lambda\in\mathcal{L}}$, etc.

Lemma 6. *The set $\{Q_\lambda\}\cup\{Q_\mu\}$ is linearly independent, and* $\operatorname{Span}(\{Q_\lambda\} \cup \{Q_\mu\}) \cap \operatorname{Span}\{Q_e\} = \{0\}$. *In particular,*

i) $\dim \operatorname{Span}\{Q_\mu\} = \sum_i g_i$,
ii) $\dim \operatorname{Span}(\{Q_e\} \cup \{Q_\lambda\}) = |T|$,
iii) $\operatorname{Span}(\{Q_e\} \cup \{Q_\lambda\} \cup \{Q_\mu\}) = \operatorname{TAS}(\mathcal{T})$.

Proof. For every edge e, choose a normal closed curve ρ about one endpoint of e. Then we can choose an orientation on ρ such that $Q_e = Q_\rho$ and think of Q_e as a leading–trailing deformation along ρ.

Let $I : \operatorname{Span}(\{Q_\lambda\} \cup \{Q_\mu\} \cup \{Q_\rho\}) \longrightarrow \mathbb{R}^{|\mathcal{L}|+|\mathcal{M}|}$ be the map defined as follows. If $X = \sum a_i Q_{\lambda_i} + \sum b_j Q_{\mu_j} + \sum c_k Q_{\rho_k}$, set $\xi = \sum a_i\lambda_i + \sum b_j\mu_j + \sum c_k\rho_k$ and define

$$I(X) = \left(\frac{\partial}{\partial Q_{\lambda_1}} \operatorname{im}(\mathrm{h}_\xi), \ldots, \frac{\partial}{\partial Q_{\lambda_{|\mathcal{L}|}}} \operatorname{im}(\mathrm{h}_\xi), \frac{\partial}{\partial Q_{\mu_1}} \operatorname{im}(\mathrm{h}_\xi), \ldots, \frac{\partial}{\partial Q_{\mu_{|\mathcal{M}|}}} \operatorname{im}(\mathrm{h}_\xi)\right).$$

The linearity of h_ξ implies that I is linear, and by Lemma 4, we have

- $I(Q_\rho) = (0, \ldots, 0)$, i.e. $\operatorname{Span}\{Q_e\} \subset \operatorname{Ker} I$;
- $I(Q_{\lambda_i})$ has 2 in the $(|\mathcal{L}| + i)$–th entry and 0 everywhere else, i.e. maps to twice the $(|\mathcal{L}| + i)$–th standard basis vector.
- $I(Q_{\mu_j})$ has -2 in the j–th entry and 0 everywhere else, i.e. maps to twice the negative j–th standard basis vector.

This implies $\operatorname{Span}(\{Q_\lambda\} \cup \{Q_\mu\}) \cong \operatorname{Im} I = \mathbb{R}^{|\mathcal{L}|+|\mathcal{M}|}$, so $\{Q_\lambda\}_{\lambda\in\mathcal{L}} \cup \{Q_\mu\}_{\mu\in\mathcal{M}}$ is linearly independent, and $\operatorname{Span}(\{Q_\lambda\} \cup \{Q_\mu\}) \cap \operatorname{Span}\{Q_e\} = \{0\}$. This implies (i), and together with Lemma 5, it implies (ii). By [5, Corollary 2.3], we have

$$\dim \operatorname{TAS} = |V| - |E| + 2|T| = |T| + \sum_i g_i.$$

It now follows from Lemma 5 that $\dim \operatorname{Span}(\{Q_e\} \cup \{Q_\lambda\} \cup \{Q_\mu\}) = \dim \operatorname{TAS}$, which implies (iii). We also note that this shows $\operatorname{Span}\{Q_e\} = \operatorname{Ker} I$. □

Theorem 7. *The set* $\{Q_e\} \cup \{Q_\lambda\}$ *is a subset of* $\mathrm{TAS}_{\mathcal{L}}$*, whereas* $\mathrm{Span}(\{Q_\mu\}) \cap \mathrm{TAS}_{\mathcal{L}} = \{0\}$*. Therefore* $\mathrm{Span}\left(\{Q_e\} \cup \{Q_\lambda\}\right) = \mathrm{TAS}_{\mathcal{L}}$.

Proof. Since $\{Q_e\} \cup \{Q_\lambda\}$ is a subset of TAS, the first part follows from checking that for all $\lambda, \lambda' \in \mathcal{L}$ and $e \in E$,

$$\sum_{\overline{q} \in \square} \eta(\overline{q}, \lambda') Q_e(\overline{q}) = 0, \tag{5}$$

$$\sum_{\overline{q} \in \square} \eta(\overline{q}, \lambda') Q_\lambda(\overline{q}) = 0. \tag{6}$$

Observe that for all $q, \overline{q} \in \square$,

$$(q')^*(\overline{q}) - (q'')^*(\overline{q}) = -\left((\overline{q}')^*(q) - (\overline{q}'')^*(q)\right),$$

hence

$$\begin{aligned}
\sum_{\overline{q} \in \square} \eta(\overline{q}, \lambda') Q_e(\overline{q}) &= \sum_{\overline{q} \in \square} \eta(\overline{q}, \lambda') \sum_{q \in \square} i(q, e) \left((q')^*(\overline{q}) - (q'')^*(\overline{q})\right) \\
&= \sum_{q \in \square} i(q, e) \sum_{\overline{q} \in \square} \eta(\overline{q}, \lambda') \left((q')^*(\overline{q}) - (q'')^*(\overline{q})\right) \\
&= -\sum_{q \in \square} i(q, e) \sum_{\overline{q} \in \square} \eta(\overline{q}, \lambda') \left((\overline{q}')^*(q) - (\overline{q}'')^*(q)\right) \\
&= -\sum_{q \in \square} i(q, e) Q_{\lambda'}(q) = 0,
\end{aligned}$$

as $Q_{\lambda'} \in \mathrm{TAS}$. This shows $Q_e \in \mathrm{TAS}_{\mathcal{L}}$.

Now suppose $\lambda, \lambda' \in \mathcal{L}$; then $\iota(\lambda', \lambda) = 0$ as they are disjoint by assumption. By Lemma 4, this shows that $\{Q_\lambda\}$, and in particular $\{Q_e\} \cup \{Q_\lambda\}$, is a subset of $\mathrm{TAS}_{\mathcal{L}}$, and by Lemma 6$(ii)$,

$$\dim \mathrm{TAS}_{\mathcal{L}} \geq |T|.$$

For the next part, let $\sum_{i,j} a^i_j Q_{\mu^i_j}$ be an element in $\mathrm{Span}(\{Q_\mu\})$ and suppose by contradiction $\sum_{i,j} a^i_j Q_{\mu^i_j} \in \mathrm{TAS}_{\mathcal{L}}$. Then for all l and for all $\lambda^l_k \in \mathcal{L}_l$, by Lemma 4,

$$0 = \sum_{q \in \square} \eta(q, \lambda^l_k) \left(\sum_{i,j} a^i_j Q_{\mu^i_j}(q) \right) = \sum_{i,j} a^i_j \, \iota(\lambda^l_k, \mu^i_j).$$

But $\iota(\lambda^l_k, \mu^i_j) \neq 0$ if and only if $i = l$ and $k = j$, therefore $a^l_k = 0$ for all l, k and

$$\mathrm{Span}(\{Q_\mu\}) \cap \mathrm{TAS}_{\mathcal{L}} = \{0\}.$$

It follows from Lemma 6(i) that $\dim \mathrm{TAS}_{\mathcal{L}} \leq |T|$. Hence $\dim \mathrm{TAS}_{\mathcal{L}} = |T|$ and so

$$\mathrm{Span}\left(\{Q_e\} \cup \{Q_\lambda\}\right) = \mathrm{TAS}_{\mathcal{L}}.$$ □

3.4. **A parametrisation of $G^{-1}(u)$.** Using the identification $Z = \mathbb{H}^T$, we write $H_{\mathcal{L}} : \mathbb{H}^T \to \mathbb{C}^{\mathcal{L}}$, giving the map

$$\begin{aligned}
(G, H_{\mathcal{L}}) : \mathbb{H}^T \to &\quad \mathbb{C}^E \times \mathbb{C}^{\mathcal{L}} \\
z \quad \mapsto &\quad \left(G(z), H_{\mathcal{L}}(z)\right).
\end{aligned}$$

We already remarked that when $G(z) \neq (2\pi i, \ldots, 2\pi i)$, the *value* of the boundary map $H_{\mathcal{L}}$ depends on the choice of normal curves representing homology classes of

peripheral curves. The following lemma shows that the *injectivity* of the differential of $(G, H_{\mathcal{L}})$ is independent of this choice.

Lemma 8. *For each vertex $v_i \in V$, let g_i be the genus of $\mathrm{Lk}(v_i)$. Let $A_i = \{\alpha_1^i, \dots, \alpha_{g_i}^i\}$ and $B_i = \{\beta_1^i, \dots, \beta_{g_i}^i\}$ be two sets of longitudes on $\mathrm{Lk}(v_i)$, such that α_j^i and β_j^i are representatives of the same element in $H_1(\mathrm{Lk}(v_i))$, and set $A = \bigcup_i A_i$ and $B = \bigcup_i B_i$. Then*

$$\operatorname{rank} \mathrm{d}(G, H_A) = \operatorname{rank} \mathrm{d}(G, H_B).$$

In particular $\mathrm{d}(G, H_A)$ is injective if and only if $\mathrm{d}(G, H_B)$ is injective.

Proof. As α_j^i and β_j^i are in the same homotopy class,

$$\mathrm{h}_{\alpha_j^i}(z) = \mathrm{h}_{\beta_j^i}(z) + \sum_{e \in E} a_e G(z)(e) \qquad \text{for some } a_e \in \mathbb{R}.$$

Hence

$$\nabla \mathrm{h}_{\alpha_j^i}(z) = \nabla \mathrm{h}_{\beta_j^i}(z) + \sum_{e \in E} a_e \nabla G(z)(e),$$

i.e. each row of $\mathrm{d}(H_A)$ is a linear combination of the corresponding row of $\mathrm{d}(H_B)$ and rows of $\mathrm{d}(G)$. It follows that $\mathrm{d}(G, H_A)$ is related to $\mathrm{d}(G, H_B)$ by elementary row operations, hence they have the same rank. □

Theorem 9. *The derivative $\mathrm{d}(G, H_{\mathcal{L}}) : T_z(\mathbb{H}^T) \to \mathbb{C}^E \times \mathbb{C}^{\mathcal{L}}$ is injective for any $z \in \mathbb{H}^T$.*

Proof. Let $z \in Z$. For every $q \in \square$, $z(q) \in \mathbb{H}$ uniquely determines three angles α, β and γ of the triangle with vertices $(0, 1, z(q))$ such that $z(q) = \frac{\sin(\beta)}{\sin(\gamma)} e^{i\alpha}$ and

$$z(q') = \frac{\sin(\gamma)}{\sin(\alpha)} e^{i\beta}, \qquad z(q'') = \frac{\sin(\alpha)}{\sin(\beta)} e^{i\gamma}.$$

Hence we can identify Z with the set $X = \{x \in \mathbb{R}^{\square}_{>0} | \sum_{q<\sigma} x(q) = \pi \ \forall \sigma \in T\}$ via the map $\phi^T : X \to Z$, where

$$\phi^T(x)(q) = \frac{\sin(x(q'))}{\sin(x(q''))} e^{ix(q)}.$$

Under this identification, we can write $(G, H_{\mathcal{L}})$ as a map $(G, H_{\mathcal{L}}) : X \to \mathbb{C}^E \times \mathbb{C}^{\mathcal{L}} = (\mathbb{R} \times \mathbb{R})^{E \cup \mathcal{L}}$ where, for each edge $e \in E$ and curve $\lambda \in \mathcal{L}$,

$$\begin{aligned} G(x)(e) &= \sum_{q \in \square} i(q, e) \log\left(\frac{\sin(x(q'))}{\sin(x(q''))} e^{ix(q)}\right) \\ &= \left(\sum_{q \in \square} i(q, e)\Big(\log(\sin(x(q'))) - \log(\sin(x(q''))\Big), \sum_{q \in \square} i(q, e) x(q)\right), \\ H_{\mathcal{L}}(x)(\lambda) &= \sum_{q \in \square} \eta(q, \lambda) \log\left(\frac{\sin(x(q'))}{\sin(x(q''))} e^{ix(q)}\right) \\ &= \left(\sum_{q \in \square} \eta(q, \lambda)\Big(\log(\sin(x(q'))) - \log(\sin(x(q''))\Big), \sum_{q \in \square} \eta(q, \lambda) x(q)\right). \end{aligned}$$

Now our goal is to show that $\mathrm{d}(G, H_{\mathcal{L}}) : T_xX \to (\mathbb{R}\times\mathbb{R})^{E\cup\mathcal{L}}$ is injective for all $x \in X$. Let $w \in T_xX = \{w \in \mathbb{R}^{\square} | \sum_{q<\sigma} w(q) = 0 \ \forall \sigma \in T\}$ be a tangent vector such that $\mathrm{d}(G, H_{\mathcal{L}})(x)(w) = 0$.

We recall the elementary fact that $\frac{\mathrm{d}}{\mathrm{d}\,t}\log(\sin t) = \cot t$. For every edge e,

$$\sum_{q\in\square} i(q,e)\left(\cot(x(q'))w(q') - \cot(x(q''))w(q'')\right) = 0, \tag{7}$$

$$\sum_{q\in\square} i(q,e)w(q) = 0, \tag{8}$$

and for every curve λ,

$$\sum_{q\in\square} \eta(q,\lambda)\left(\cot(x(q'))w(q') - \cot(x(q''))w(q'')\right) = 0, \tag{9}$$

$$\sum_{q\in\square} \eta(q,\lambda)w(q) = 0. \tag{10}$$

Observe that (8) and (10) imply $w \in \mathrm{TAS}_{\mathcal{L}} \leq \mathrm{TAS} \subset T_xX$.

Let $F : \mathbb{R}^{\square} \to \mathbb{R}$ be the volume function

$$F(x) = \sum_{q\in\square} \Lambda(x(q)),$$

where $\Lambda(x) = -\int_0^x \log|2\sin(u)|du$ is the Lobachevsky function. The Hessian of F is the diagonal matrix with diagonal entries $-\cot(x(q))$, and it is negative definite at each point $x \in X$ (see, for instance, [3, pg. 175]). Recall that for $e \in E$ and $\lambda \in \mathcal{L}$, we have ,

$$Q_e = \sum_{q\in\square} i(q,e)\left(\ (q')^* - (q'')^*\ \right),$$
$$Q_\lambda = \sum_{q\in\square} \eta(q,\lambda)\left((q')^* - (q'')^*\right).$$

Then by (7) and (9)

$$Q_e \cdot \mathrm{Hess}_x(F)w = 0, \qquad \forall e \in E,$$
$$Q_\lambda \cdot \mathrm{Hess}_x(F)w = 0, \qquad \forall \lambda \in \mathcal{L}.$$

Theorem 7 shows that $\mathrm{TAS}_{\mathcal{L}}$ is spanned by $\{Q_e\}_{e\in E} \cup \{Q_\lambda\}_{\lambda\in\mathcal{L}}$. Hence w can be written as a linear combination of the Q_e and Q_λ, and so

$$w \cdot \mathrm{Hess}_x(F)w = 0.$$

But since $\mathrm{Hess}_x(F)$ is negative definite, we have $w = 0$ and hence $\mathrm{d}(G, H_{\mathcal{L}})$ is injective. □

Theorem 2 shows that the rank of $\mathrm{d}\,G$ is constant $|T| - \sum_{v\in V} g_v$, hence Theorem 9 implies the following result.

Corollary 10. *For all $u \in \mathbb{C}^E$ such that $G^{-1}(u) \neq \emptyset$, the restriction map*

$$H_{\mathcal{L}}\big|_{G^{-1}(u)} : G^{-1}(u) \longrightarrow \mathbb{C}^{\mathcal{L}}$$

is a local diffeomorphism onto its image.

4. Two Dehn-surgery components

Our first example of a one-cusped hyperbolic 3–manifold of finite volume illustrates Choi's original result. It also has the property that the two discrete and faithful characters lie on different components of the $PSL_2(\mathbb{C})$–character variety. It was found via the following construction due to Nathan Dunfield. Let N be a 2–cusped manifold with strong geometric isolation (see [7] for a definition). Let A be the collection of all hyperbolic Dehn fillings on the first cusp in N; let M_1, M_2, M_3,... be distinct manifolds in A. Fix an orientation of N, and let χ_0 be the character of an associated holonomy representation. Let χ_n be the character of the holonomy representation of M_n that is in a small open neighbourhood of χ_0, and let X_n be the component of the character variety of M_n which contains χ_n. By strong geometric isolation, the image of X_n in the character variety of the second cusp is independent of n, call it C. Now the differential of the volume function on X_n pulls back from a 1–form on C. Hence, the quantity

$$\max\{\mathrm{vol}(\chi) \mid \chi \in X_n\} - \min\{\mathrm{vol}(\chi) \mid \chi \in X_n\} \tag{11}$$

is a constant V independent of n. By volume rigidity, $V \leq 2\min(\mathrm{vol}(M_n))$. So if M_n has non-minimal volume (and of course $\mathrm{vol}(M_n) \to \mathrm{vol}(N)$ so there are many such), it follows that χ_n and its complex conjugate must lie on different components of the character variety of M_n.

The following explicit example was found by looking in the census due to Callahan, Hildebrand and Weeks (as shipped with `Regina` [1]) for an example given by Neumann and Reid [7]. Let M be the manifold $t12046$ in this census, and let N be the manifold obtained from M by $(2,1)$-Dehn filling on the first cusp. We fix the following ideal triangulation $\mathcal{T}$ on N:

Table 1. The ideal triangulation $\mathcal{T}$ on N.

Tetrahedron	Face 012	Face 013	Face 023	Face 123
0	4 (203)	2 (321)	6 (032)	5 (120)
1	3 (312)	2 (012)	4 (013)	6 (031)
2	1 (013)	4 (213)	5 (123)	0 (310)
3	6 (312)	4 (012)	5 (203)	1 (120)
4	3 (013)	1 (023)	0 (102)	2 (103)
5	0 (312)	6 (012)	3 (203)	2 (023)
6	5 (013)	1 (132)	0 (032)	3 (120)

Let z_i be a shape parameter of the tetrahedron T_i, with respect to the edge (01), with orientation such that z_i' is the parameter at (02) and z_i'' is at (03). A standard computation provides the two solutions

$$z^{(0)} = \left(i,\ \frac{1+i}{2},\ i, \frac{2+i}{2},\ \frac{3+i}{5},\ i\ , \frac{-1+i}{2}\right) \in \mathbb{C}^7$$

and its complex conjugate $\overline{z^{(0)}}$ corresponding to discrete and faithful characters.

Notice that $z^{(0)}$ is positively oriented. The 1-dimensional components C_0 and C_0' containing them have the following rational parametrisations:

$$\varphi_0(u) = \begin{pmatrix} \frac{-1-i+u}{-1+(1-i)u} \\ \frac{1+i}{(2+2i)-2u} \\ (1+i)-\frac{i}{u} \\ \frac{(1-i)-2u+2u^2}{2(-1+u)u} \\ \frac{(-1+u)(-i+(1+i)u)}{1-(1+i)u+(1+i)u^2} \\ u \\ \frac{(1-i)+2iu}{(2+2i)u-2u^2} \end{pmatrix} \quad \text{and} \quad \varphi_0'(u) = \begin{pmatrix} \frac{-1+i+u}{-1+(1+i)u} \\ \frac{1-i}{(2-2i)-2u} \\ (1-i)+\frac{i}{u} \\ \frac{(1+i)-2u+2u^2}{2(-1+u)u} \\ \frac{(-1+u)(i+(1-i)u)}{1-(1-i)u+(1-i)u^2} \\ u \\ \frac{(1+i)-2iu}{(2-2i)u-2u^2} \end{pmatrix}.$$

In particular, notice that $\varphi_0(u) = \overline{\varphi_0'(\overline{u})}$.

Henceforth, we will refer to the j-th components of $\varphi_0(u)$ and $\varphi_0'(u)$ by $(\varphi_0)_j(u)$ and $(\varphi_0')_j(u)$ respectively. The natural domain of φ_0 is $\mathbb{C} \setminus X$, where X is the finite set of poles of $(\varphi_0)_j$. A direct computation shows that X contains precisely 6 elements, whose image via φ_0 are all ideal points of C_0, therefore

$$\overline{\mathrm{Im}(\varphi_0)} = \mathrm{Im}(\varphi_0).$$

Together with the image of $u = \infty$, they sum up to a total of 7 ideal points, hence the image of φ_0 is a 7–punctured sphere. Moreover, a direct calculation reveals that the images of φ_0 and φ_0' are disjoint.

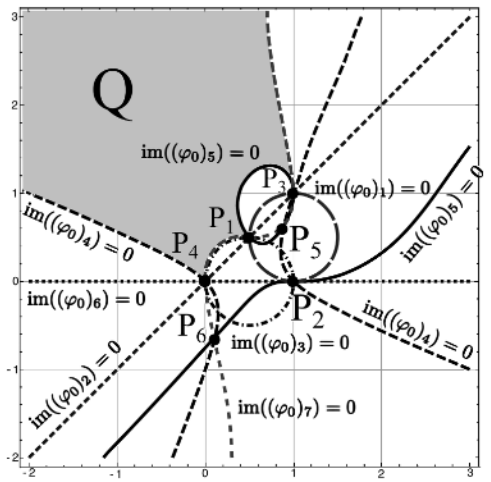

FIGURE 2. Q is an ideal square.

Denoting by $\mathrm{im}(w)$ the imaginary part of a complex number $w \in \mathbb{C}$, we recall that the complete solution $\varphi_0(i) = z^{(0)}$ is geometric in the sense that

$$\mathrm{im}\left((\varphi_0)_j(i)\right) = \mathrm{im}(z_j^{(0)}) > 0, \qquad \forall j \in \{1, \ldots, 7\}.$$

The set of all positively oriented solutions is

$$Q = \bigcap_{j=1}^{7} \{u \in \mathbb{C} \mid \mathrm{im}\left((\varphi_0)_j(u)\right) > 0\}.$$

For $u = x + iy$, $\{\mathrm{im}\left((\varphi_0)_j(u)\right) = 0\}$ is a 1-dimensional subvariety of $\mathbb{R}^2$, which can be explicitly computed from φ_0,

$$\begin{aligned}
\mathrm{im}\left((\varphi_0)_1\right) &= (x-1)^2 + (y - \frac{1}{2})^2 - \frac{1}{4},\\
\mathrm{im}\left((\varphi_0)_2\right) &= y - x,\\
\mathrm{im}\left((\varphi_0)_3\right) &= x^2 + y^2 - x,\\
\mathrm{im}\left((\varphi_0)_4\right) &= (y + \frac{1}{2})^2 - (x\sqrt{3} - \frac{1}{2\sqrt{3}})^2 - \frac{1}{6},\\
\mathrm{im}\left((\varphi_0)_5\right) &= -1 + 3x - 3x^2 + x^3 - x^2 y + y^2 + xy^2 - y^3,\\
\mathrm{im}\left((\varphi_0)_6\right) &= y,\\
\mathrm{im}\left((\varphi_0)_7\right) &= 2x - 3x^2 - 2x^3 + 2xy + y^2 - 2xy^2.
\end{aligned}$$

As shown in Figure 2, Q is a simply connected open set of the plane and

$$\partial Q \subset V\left(\mathrm{im}\left((\varphi_0)_4(u)\right), \mathrm{im}\left((\varphi_0)_5(u)\right), \mathrm{im}\left((\varphi_0)_7(u)\right)\right).$$

We deduce that $\varphi_0(Q)$ is an ideal square contained in $\pi'(C_0)$.

5. A reducible complex-curvature level set

Let M be the oriented pseudo–manifold given by the ideal triangulation $\mathcal{T}$ shown in Figure 3. M has two vertices v_1 and v_2, whose links are closed orientable surfaces of genus $g_1 = 1$ and $g_2 = 2$ respectively. The induced triangulations of the links $\mathrm{lk}(v_1)$ and $\mathrm{lk}(v_2)$ are shown in the bottom right corner of Figure 3 and in Figure 4, respectively.

Table 2. The ideal triangulation $\mathcal{T}$ on N.

Tetrahedron	Face 012	Face 013	Face 023	Face 123
0	2 (032)	4 (012)	2 (123)	2 (120)
1	2 (013)	1 (213)	3 (013)	1 (103)
2	0 (312)	1 (012)	0 (021)	0 (023)
3	4 (013)	1 (023)	4 (312)	4 (230)
4	0 (013)	3 (012)	3 (312)	3 (230)

We use the same notation as in the previous example, so z_i is the shape parameter of tetrahedron T_i with respect to edge (12), and $z = (z_0, z_1, z_2, z_3, z_4)$. From the face pairings of $\mathcal{T}$ we deduce the following complex-curvature and log-curvature maps.

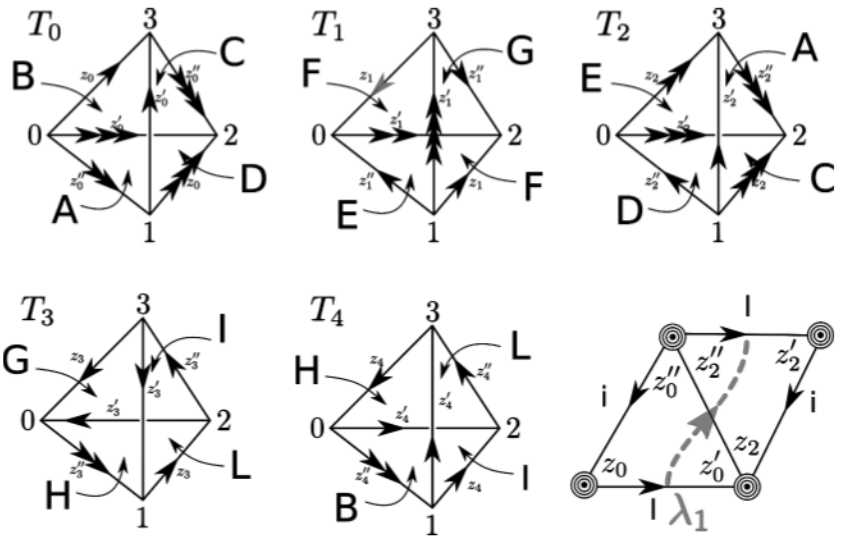

FIGURE 3. Shown are the five tetrahedra in the triangulation of M and, in the bottom right corner, the induced triangulation of the vertex link of v_1 (viewed from the cusp).

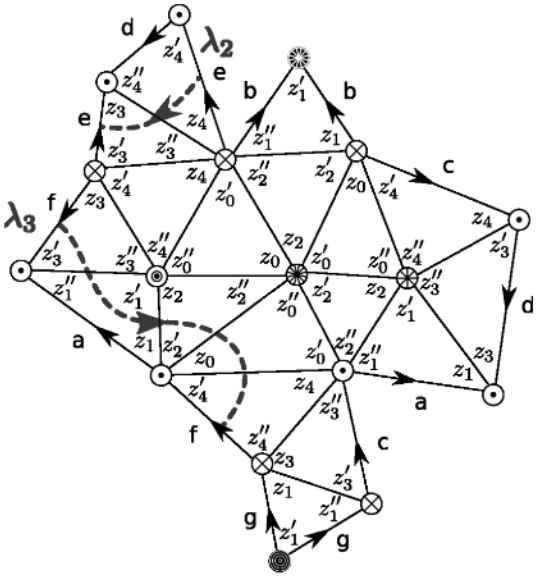

FIGURE 4. Triangulation induced on v_2 by $\mathcal{T}$.

(12)
$$c(z) = \begin{pmatrix} z_1' \\ z_0'' z_1' z_2 z_3'' z_4'' \\ (z_0 z_0' z_0'') (z_2 z_2' z_2'') \\ z_0 z_0' (z_1 z_1'')^2 z_2' z_2'' (z_3 z_3')^2 z_3'' (z_4 z_4')^2 z_4'' \end{pmatrix} = \begin{pmatrix} z_1' \\ z_0'' z_1' z_2 z_3'' z_4'' \\ 1 \\ z_0 z_0' (z_1 z_1'')^2 z_2' z_2'' z_3 z_3' z_4 z_4' \end{pmatrix};$$

(13)
$$G(z) = \begin{pmatrix} \log(z_1') \\ \log(z_0'') + \log(z_1') + \log(z_2) + \log(z_3'') + \log(z_4'') \\ \log(z_0 z_0' z_0'') + \log(z_2 z_2' z_2'') \\ \log(z_0 z_0') + 2\log(z_1 z_1'') + \log(z_2' z_2'') + \log((z_3 z_3')^2 z_3'') + \log((z_4 z_4')^2 z_4'') \end{pmatrix}.$$

Notice that $\log(z-1) = \log(1-z) \pm \pi i$ where the sign ambiguity depends on the argument of z. However, applying the combinatorial Gauss–Bonnet theorem to the vertex, we deduce that

$$G(z)(e_2) = 2\pi i,$$
$$G(z)(e_0) + G(z)(e_1) + G(z)(e_3) = 8\pi i.$$

Hence

$$G(z) = \begin{pmatrix} -\log(1-z_1) \\ \log(z_0-1) - \log(z_0) - \log(1-z_1) + \log(z_2) + \log(z_3-1) - \log(z_3) + \log(z_4-1) - \log(z_4) \\ 2\pi i \\ \log(z_0) - \log(z_0-1) + 2\log(1-z_1) - \log(z_2) + \log(z_3) - \log(z_3-1) + \log(z_4) - \log(z_4-1) + 8\pi i \end{pmatrix}.$$

We observe that $\mathrm{d}\, G$ has constant rank $|T| - g_1 - g_2 = 5 - 1 - 2 = 2$,

$$\mathrm{d}\, G(z) = \begin{pmatrix} 0 & \frac{-1}{z_1-1} & 0 & 0 & 0 \\ \frac{1}{z_0(z_0-1)} & \frac{-1}{z_1-1} & \frac{1}{z_2} & \frac{1}{z_3(z_3-1)} & \frac{1}{z_4(z_4-1)} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{-1}{z_0(z_0-1)} & \frac{2}{z_1-1} & \frac{-1}{z_2} & \frac{-1}{z_3(z_3-1)} & \frac{-1}{z_4(z_4-1)} \end{pmatrix}.$$

Moreover, for all $u = (u_0, u_1, u_2, u_3) \in \operatorname{Im} G$, $G^{-1}(u)$ is the 3–dimensional variety generated by the ideal

$$I = \big(z_1 + e^{-u_0} - 1, (z_0-1)z_2(z_3-1)(z_4-1) - e^{u_1} z_0(1-z_1)z_3 z_4\big).$$

For instance, let

$$z^0 = \big(e^{\frac{\pi}{3}i}, e^{\frac{\pi}{3}i}, e^{\frac{\pi}{3}i}, e^{\frac{\pi}{3}i}, e^{\frac{\pi}{3}i}\big);$$

then $u^0 = G(z^0) = (\frac{\pi}{3}i, \frac{5}{3}\pi i, 2\pi i, 6\pi i)$ and

$$G^{-1}(u^0) = \Big\{(z_0, z_1, z_2, z_3, z_4) \in \mathbb{H}^5 \mid z_1 = e^{\frac{\pi}{3}i}, z_2 = \frac{z_0 z_3 z_4 e^{\frac{4}{3}\pi i}}{(z_0-1)(z_3-1)(z_4-1)}\Big\}.$$

For

$$z^1 = \left(\frac{1}{1-e^{\frac{5}{6}\pi i}}, e^{\frac{\pi}{3}i}, \frac{1}{1-e^{\frac{5}{6}\pi i}}, \frac{1}{1-e^{\frac{5}{6}\pi i}}, \frac{1}{1-e^{\frac{5}{6}\pi i}}\right),$$

$u^1 = G(z^1) = \big(\frac{\pi}{3}i, \frac{11}{3}\pi i, 2\pi i, 4\pi i\big)$, so

$$G(z^0) \neq G(z^1) \quad \text{but} \quad c(z^0) = c(z^1).$$

By analytic continuation $G^{-1}(u^0)$ and $G^{-1}(u^1)$ are disjoint, hence the above argument shows that $G^{-1}(u^0)$ and $G^{-1}(u^1)$ are disjoint complex varieties contained

in the complex variety $c^{-1}(c(z^0))$. Moreover, because all shape parameters are assumed to have a positive imaginary part, it follows from (13) that $0 < G(z)(e_1) < 5\pi$ and hence

$$c^{-1}(c(z^0)) = G^{-1}(u^0) \bigcup G^{-1}(u^1).$$

Let $\mathcal{L} = \{\lambda_1, \lambda_2, \lambda_3\}$ be the set of three simple closed curves shown in Figures 3 and 4. The boundary map with respect to $\mathcal{L}$ and its differential are

$$e^{H_{\mathcal{L}}}(z) = \begin{pmatrix} \frac{z_0'}{z_2''} \\ \\ \frac{z_3}{z_4} \\ \\ \frac{z_0 z_2' z_3' z_4'}{z_1'} \end{pmatrix}, \; H_{\mathcal{L}}(z) = \begin{pmatrix} -\log(1-z_0)+\log(z_2)-\log(z_2-1) \\ \\ \log(z_3)-\log(z_4) \\ \\ \log(z_0)+\log(1-z_1)-\log(1-z_2)-\log(1-z_3)-\log(1-z_4) \end{pmatrix},$$

and

$$\mathrm{d}\, H_{\mathcal{L}}(z) = \begin{pmatrix} \frac{1}{1-z_0} & 0 & \frac{-1}{z_2(1-z_2)} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{z_3} & \frac{-1}{z_4} \\ \frac{1}{z_0} & \frac{-1}{1-z_1} & \frac{1}{1-z_2} & \frac{1}{1-z_3} & \frac{1}{1-z_4} \end{pmatrix}.$$

Together with the differential of the log-curvature map we obtain:

$$\mathrm{d}(G,H)(z) = \begin{pmatrix} 0 & \frac{-1}{z_1-1} & 0 & 0 & 0 \\ \frac{1}{z_0(z_0-1)} & \frac{-1}{z_1-1} & \frac{1}{z_2} & \frac{1}{z_3(z_3-1)} & \frac{1}{z_4(z_4-1)} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{-1}{z_0(z_0-1)} & \frac{2}{z_1-1} & \frac{-1}{z_2} & \frac{-1}{z_3(z_3-1)} & \frac{-1}{z_4(z_4-1)} \\ \frac{1}{1-z_0} & 0 & \frac{-1}{z_2(1-z_2)} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{z_3} & \frac{-1}{z_4} \\ \frac{1}{z_0} & \frac{-1}{1-z_1} & \frac{1}{1-z_2} & \frac{1}{1-z_3} & \frac{1}{1-z_4} \end{pmatrix}.$$

The determinant of the minor obtained by removing the second and third row from $\mathrm{d}(G,H)(z)$ is

$$-\frac{2(-1+z_0+z_3+z_4+z_0z_2-z_0z_2z_3-z_0z_2z_4-z_3z_4-z_0z_3z_4+z_0z_2z_3z_4)}{z_0z_1z_2z_3z_4(1-z_0)(1-z_1)(1-z_2)(1-z_3)(1-z_4)}.$$

For all $u \in \mathrm{Im}(G)$, the restriction map

$$H_{\mathcal{L}}\big|_{G^{-1}(u)} : G^{-1}(u) \longrightarrow \mathbb{C}^3$$

is a local diffeomorphism onto its image. In particular we can parameterise $G^{-1}(u)$ through $H_{\mathcal{L}}$. For u^0 as above, we get the following local parameterisation around z^0,

$$\begin{cases} z_0 = 1 - z_4 k, \\ z_1 = e^{\frac{\pi}{3}}, \\ z_2 = \frac{-z_4 k'}{1-z_4 k'}, \\ z_3 = z_4 e^{t_2}, \\ z_4 \text{ is the solution of the equation } \frac{kk'}{1-z_4k'} = \frac{(1-z_4k)e^{t_2+\frac{4}{3}\pi i}}{(z_4e^{t_2}-1)(z_4-1)}, \end{cases}$$

where $k = e^{\frac{5}{6}\pi i + \frac{t_2+t_3-t_1}{2}}$, $k' = e^{\frac{5}{6}\pi i + \frac{t_2+t_3+t_1}{2}}$ and for all $(t_1,t_2,t_3) \in \mathbb{C}^3$ close to

$$H_{\mathcal{L}}(e^{\frac{\pi}{3}i}, e^{\frac{\pi}{3}i}, e^{\frac{\pi}{3}i}, e^{\frac{\pi}{3}i}, e^{\frac{\pi}{3}i}) = (0,0,\pi i).$$

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References

[1] Benjamin A. Burton and Ryan Budney and William Pettersson and others, *Regina: Software for 3-manifold topology and normal surface theory*, `http://regina.sourceforge.net/`, 1999–2014.

[2] Young-Eun Choi, *Positively oriented ideal triangulations on hyperbolic three-manifolds*, Topology **43** (2004), no. 6, 1345–1371, DOI 10.1016/j.top.2004.02.002. MR2081429

[3] David Futer and François Guéritaud, *From angled triangulations to hyperbolic structures*, Interactions between hyperbolic geometry, quantum topology and number theory, Contemp. Math., vol. 541, Amer. Math. Soc., Providence, RI, 2011, pp. 159–182, DOI 10.1090/conm/541/10683. MR2796632

[4] Wolfgang Haken, *Theorie der Normalflächen* (German), Acta Math. **105** (1961), 245–375. MR0141106

[5] Feng Luo, *Volume optimization, normal surfaces, and Thurston's equation on triangulated 3-manifolds*, J. Differential Geom. **93** (2013), no. 2, 299–326. MR3024308

[6] Walter D. Neumann, *Combinatorics of triangulations and the Chern-Simons invariant for hyperbolic 3-manifolds*, Topology '90 (Columbus, OH, 1990), Ohio State Univ. Math. Res. Inst. Publ., vol. 1, de Gruyter, Berlin, 1992, pp. 243–271. MR1184415

[7] Walter D. Neumann and Alan W. Reid, *Rigidity of cusps in deformations of hyperbolic 3-orbifolds*, Math. Ann. **295** (1993), no. 2, 223–237, DOI 10.1007/BF01444885. MR1202390

[8] Walter D. Neumann and Don Zagier, *Volumes of hyperbolic three-manifolds*, Topology **24** (1985), no. 3, 307–332, DOI 10.1016/0040-9383(85)90004-7. MR815482

[9] Henry Segerman and Stephan Tillmann, *Pseudo-developing maps for ideal triangulations I: essential edges and generalised hyperbolic gluing equations*, Topology and geometry in dimension three, Contemp. Math., vol. 560, Amer. Math. Soc., Providence, RI, 2011, pp. 85–102, DOI 10.1090/conm/560/11093. MR2866925

[10] Stephan Tillmann, *Normal surfaces in topologically finite 3-manifolds*, Enseign. Math. (2) **54** (2008), no. 3-4, 329–380. MR2478091

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