Bivariate Dimension Polynomials of Non-Reflexive Prime Difference-Differential Ideals. The Case of One Translation

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ABSTRACT

We use the method of characteristic sets with respect to two term orderings to prove the existence and obtain a method of computation of a bivariate dimension polynomial associated with a non-reflexive difference-differential ideal in the algebra of difference-differential polynomials with several basic derivations and one translation. As a consequence, we obtain a new proof and a method of computation of the dimension polynomial of a non-reflexive prime difference ideal in the algebra of difference polynomials over an ordinary difference field. We also discuss applications of our results to systems of algebraic difference-differential equations.

CCS CONCEPTS

• **Symbolic and Algebraic Manipulation → Algebraic Algorithms**;

KEYWORDS

Difference-differential polynomial, dimension polynomial, reduction, characteristic set

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1 INTRODUCTION

The role of dimension polynomials in differential and difference algebra is similar to the role of Hilbert polynomials in commutative algebra and algebraic geometry. An important feature of such polynomials is that they describe in exact terms the freedom degree of a continuous or discrete dynamic system as well as the number of arbitrary constants in the general solution of a system of partial algebraic differential or difference equations. The notion of a differential dimension polynomial was introduced by E. Kolchin [6] who proved the following fundamental result.

Theorem 1.1. *Let K be a differential field (*Char $K = 0$*), that is, a field considered together with the action of a set $\Delta = \{\delta_1, \dots, \delta_m\}$ of mutually commuting derivations of K into itself. Let Θ denote the free commutative semigroup of all power products of the form $\theta = \delta_1^{k_1} \dots \delta_m^{k_m}$ ($k_i \geq 0$), let* ord $\theta = \sum_{i=1}^{m} k_i$*, and for any $r \geq 0$, let $\Theta(r) = \{\theta \in \Theta \,|\, ord\,\theta \leq r\}$. Furthermore, let $L = K\langle \eta_1, \dots, \eta_n \rangle_\Delta$ be a differential field extension of K generated by a finite set $\eta = \{\eta_1, \dots, \eta_n\}$. (As a field, $L = K(\{\theta\eta_j | \theta \in \Theta, 1 \leq j \leq n\})$.)*

Then there exists a polynomial $\omega_{\eta|K}(t) \in \mathbb{Q}[t]$ such that

(i) *$\omega_{\eta|K}(r) = trdeg_K K(\{\theta\eta_j | \theta \in \Theta(r),\ 1 \leq j \leq n\})$ for all sufficiently large $r \in \mathbb{Z}$;*

(ii) deg *$\omega_{\eta|K} \leq m$ and $\omega_{\eta|K}(t)$ can be written as $\omega_{\eta|K}(t) = \sum_{i=0}^{m} a_i \binom{t+i}{i}$ where $a_0, \dots, a_m \in \mathbb{Z}$;*

(iii) *$d =$* deg *$\omega_{\eta|K}$, a_m and a_d do not depend on the choice of the system of Δ-generators η of the extension L/K (clearly, $a_d \neq a_m$ if and only if $d < m$, that is $a_m = 0$). Moreover, a_m is equal to the differential transcendence degree of L over K, that is, to the maximal number of elements $\xi_1, \dots, \xi_k \in L$ such that the set $\{\theta\xi_i | \theta \in \Theta, 1 \leq i \leq k\}$ is algebraically independent over K.*

The corresponding dimension polynomials of difference and difference-differential field extensions were introduced in [9] and [14]. The importance of these characteristics is determined by at least three factors. First, for a wide class of algebraic differential (respectively, difference or difference-differential) equations, the dimension polynomial of the corresponding field extension expresses the strength of the system of equations in the sense of A. Einstein. In the case of a system of partial differential equations, this concept, introduced in [2] as an important qualitative characteristic of a system, was expressed by a certain differential dimension polynomial in [16]; the corresponding algebraic interpretations of the strength of systems of difference and difference-differential equations were obtained in [8, Sect. 6.4] and [12, Sect. 7.7]. Second, the dimension polynomial associated with a finitely generated differential, difference or difference-differential field extension carries certain birational invariants, that is, numbers that do not change when we switch to another finite system of generators of the extension. These invariants are closely connected with some other important characteristics; for example, one of them is the differential (respectively, difference or difference-differential) transcendence degree of the extension. Finally, properties of dimension polynomials associated with prime differential (respectively, difference or difference-differential) ideals provide a powerful tool in the dimension theory of the corresponding rings (see, for example, [4], [5], [8, Ch.7], and [15]).

In this paper we adjust a generalization of the Ritt-Kolchin method of characteristic sets developed in [13] to the case of (non-inversive) difference-differential polynomials with one translation

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and apply this method to prove the existence and outline a method of computation of a bivariate dimension polynomial associated with a non-reflexive difference-differential polynomial ideal. Our main result (Theorem 4.2) can be viewed as an essential generalization (in the case of one translation) of the existing theorems on bivariate dimension polynomials of difference-differential and difference field extensions, see [10, Theorem 5.4] and [12, Theorems 4.2.16 and 4.2.17]. The latter theorems deal with extensions that arise from factor rings of difference-differential (or difference) polynomial rings by reflexive difference-differential (respectively, difference) prime ideals. Our paper extends these results to the case when the prime ideals are not necessarily reflexive, so the induced translations of the factor rings are not necessarily injective.

We also discuss the relationship between the obtained difference-differential dimension polynomial and the concept of strength of a system of algebraic difference-differential equations in the sense of A. Einstein. Furthermore, as a consequence of our main result, we obtain a new proof and a method of computation of the dimension polynomial of a non-reflexive prime difference ideal in the algebra of difference polynomials over an ordinary difference field. The existence of such a polynomial was first established in [3, Section 4.4], an alternative proof was obtained in [17, Section 5.1]. However, these proofs are not constructive, while our approach leads to an algorithm for computing dimension polynomials.

2 PRELIMINARIES

Throughout the paper $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ denote the sets of all integers, all non-negative integers, all rational numbers and all real numbers, respectively. For any positive integer p, we set $\mathbb{N}_p = \{1, \dots, p\}$. By a ring we always mean an associative ring with unity. Every ring homomorphism is unitary (maps unity onto unity), every subring of a ring contains the unity of the ring, and every algebra over a commutative ring is unitary. Every field considered below is supposed to have zero characteristic.

If $B = A_1 \times \dots \times A_k$ is a Cartesian product of k ordered sets with orders $\leq_1, \dots \leq_k$, respectively ($k \in \mathbb{N}$, $k \geq 1$), then by the product order on B we mean a partial order $\leq_P$ such that $(a_1, \dots, a_k) \leq_P (a'_1, \dots, a'_k)$ if and only if $a_i \leq_i a'_i$ for $i = 1, \dots, k$. In particular, if $a = (a_1, \dots, a_k)$, $a' = (a'_1, \dots, a'_k) \in \mathbb{N}^k$, then $a \leq_P a'$ if and only if $a_i \leq a'_i$ for $i = 1, \dots, k$. We write $a <_P a'$ if $a \leq_P a'$ and $a \neq a'$.

The proof of the following statement can be found in [7, Chapter 0, Lemma 15].

Lemma 2.1. *Let A be an infinite subset of $\mathbb{N}^m \times \mathbb{N}_n$ ($m, n \in \mathbb{N}$, $n \geq 1$). Then there exists an infinite sequence of elements of A, strictly increasing relative to the product order, in which every element has the same projection on $\mathbb{N}_n$.*

NUMERICAL POLYNOMIALS OF SUBSETS OF $\mathbb{N}^m \times \mathbb{Z}$

Definition 2.2. A polynomial $f(t_1, \dots, t_p)$ in p variables $t_1, \dots, t_p$ ($p \in \mathbb{N}$, $p \geq 1$) with rational coefficients is called *numerical* if $f(t_1, \dots, t_p) \in \mathbb{Z}$ for all sufficiently large p-tuples $(t_1, \dots, t_p) \in \mathbb{Z}^p$ (that is, there exist integers $s_1, \dots, s_p$ such that $f(r_1, \dots, r_p) \in \mathbb{Z}$ whenever $(r_1, \dots, r_p) \in \mathbb{Z}$ and $r_i \geq s_i$ for all $i = 1, \dots, p$.)

Obviously, every polynomial with integer coefficients is numerical. As an example of a numerical polynomial in p variables with non-integer coefficients ($p \in \mathbf{N}, p \geq 1$) one can consider a polynomial $\prod_{i=1}^{p} \binom{t_i}{m_i}$ where $m_1, \dots, m_p \in \mathbb{N}$. (As usual, $\binom{t}{k}$ ($k \in \mathbb{Z}$, $k \geq 1$) denotes the polynomial $\frac{t(t-1)\dots(t-k+1)}{k!}$ in one variable t, $\binom{t}{0} = 1$, and $\binom{t}{k} = 0$ if $k < 0$.) It can be shown (see [8, Corollary 2.1.5]) that a numerical polynomial $f(t_1, \dots, t_p)$ in p variables can be expressed as a linear combination of products of the form $\binom{t_1+i_1}{i_1} \dots \binom{t_p+i_p}{i_p}$ with integer coefficients ($i_1, \dots, i_p \in \mathbb{N}$).

In the rest of the section we deal with subsets of $\mathbb{N}^{m+1}$ (m is a positive integer) treated as a Cartesian product $\mathbb{N}^m \times \mathbb{N}$ (so that the last coordinate has a special meaning). If $a = (a_1, \dots, a_{m+1}) \in \mathbb{N}^{m+1}$, we set $\text{ord}_1\, a = \sum_{i=1}^{m} a_i$ and $\text{ord}_2\, a = a_{m+1}$. Furthermore, we treat $\mathbb{N}^{m+1}$ as a partially ordered set with respect to the product order $\leq_P$.

If $A \subseteq \mathbb{N}^{m+1}$, then V_A will denote the set of all elements $v \in \mathbb{N}^{m+1}$ such that there is no $a \in A$ with $a \leq_P v$. Clearly, $v = (v_1, \dots, v_{m+1}) \in V_A$ if and only if for any element $(a_1, \dots, a_{m+1}) \in A$, there exists $i \in \mathbb{N}$, $1 \leq i \leq m+1$, such that $a_i > v_i$. Furthermore, for any $r, s \in \mathbb{N}$, we set

$$A(r,s) = \{x = (x_1, \dots, x_{m+1}) \in A \mid \text{ord}_1\, x \leq r, \text{ord}_2\, x \leq s\}.$$

The following theorem is a direct consequence of the corresponding statement proved in [8, Chapter 2]; it generalizes the well-known Kolchin's result on the univariate numerical polynomials associated with subsets of $\mathbb{N}^m$ (see [7, Chapter 0, Lemma 17]).

Theorem 2.3. *Let A be a subset of $\mathbb{N}^{m+1}$. Then there exists a numerical polynomial $\omega_A(t_1, t_2)$ with the following properties:*

(i) *$\omega_A(r,s) = \text{Card}\, V_A(r,s)$ for all sufficiently large $(r,s) \in \mathbb{N}^2$. (As in Definition 2.2, it means that there exist $r_0, s_0 \in \mathbb{N}$ such that the equality holds for all integers $r \geq r_0$, $s \geq s_0$; as usual,* Card M *denotes the number of elements of a finite set M).*

(ii) *$\deg_{t_1} \omega_A \leq m$ and $\deg_{t_2} \omega_A \leq 1$ (so the total degree $\deg \omega_A$ of the polynomial does not exceed $m+1$).*

(iii) *$\deg\, \omega_A = m+1$ if and only if $A = \emptyset$. In this case $\omega_A(t_1, t_2) = \binom{t_1+m}{m}(t_2+1)$.*

(iv) *ω_A is a zero polynomial if and only if $(0, \dots, 0) \in A$.*

Definition 2.4. The polynomial $\omega_A(t_1, t_2)$ whose existence is stated by Theorem 2.4 is called the dimension polynomial of the set $A \subseteq \mathbb{N}^{m+1}$ associated with the orders ord_1 and ord_2.

A closed-form formula for $\omega_A(t_1, t_2)$ can be found in [8, Proposition 2.2.11].

BASIC NOTATION AND TERMINOLOGY ON DIFFERENCE-DIFFERENTIAL RINGS AND FIELDS

By a *difference-differential ring* we mean a commutative ring R considered together with finite sets $\Delta = \{\delta_1, \dots, \delta_m\}$ and $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ of derivations and injective endomorphisms of R, respectively, such that any two mappings of the set $\Delta \bigcup \Sigma$ commute. In what follows, we consider a special case when the set Σ consists of a single endomorphism σ called a *translation*. The set $\Delta \bigcup \{\sigma\}$

will be referred to as a *basic set* of the difference-differential ring R, which is also called a Δ-σ-ring. If R is a field, it is called a *difference-differential field* or a Δ-σ-field. We will often use prefix Δ-σ- instead of the adjective "difference-differential".

Let T be the free commutative semigroup generated by the set $\Delta \bigcup \{\sigma\}$, that is, the semigroup of all power products

$$\tau = \delta_1^{k_1} \dots \delta_m^{k_m} \sigma^l \qquad (k_i, l \in \mathbb{N}).$$

The numbers $\text{ord}_\Delta \tau = \sum_{i=1}^{m} k_i$ and $\text{ord}_\sigma \tau = l$ are called the *orders* of τ with respect to Δ and σ, respectively. For every $r, s \in \mathbb{N}$, we set

$$T(r, s) = \{\tau \in T \mid \text{ord}_\Delta \tau \le r, \text{ ord}_\sigma \tau \le s\}.$$

Furthermore, Θ will denote the subsemigroup of T generated by Δ, so every element $\tau \in T$ can be written as $\tau = \theta\sigma^l$ where $\theta \in \Theta$, $l \in \mathbb{N}$. If $r \in \mathbb{N}$, we set $\Theta(r) = \{\theta \in \Theta \mid \text{ord}_\Delta \theta \le r\}$.

A subring (ideal) R_0 of a Δ-σ-ring R is called a difference-differential (or Δ-σ-) subring of R (respectively, a difference-differential (or Δ-σ-) ideal of R) if R_0 is closed with respect to the action of any operator of $\Delta \bigcup \sigma$. In this case the restriction of a mapping from $\Delta \bigcup \sigma$ on R_0 is denoted by the same symbol. If a prime ideal P of R is closed with respect to the action of $\Delta \bigcup \sigma$, it is called a *prime difference-differential* (or Δ-σ-) *ideal* of R.

If R is a Δ-σ-field and R_0 a subfield of R which is also a Δ-σ-subring of R, then R_0 is said to be a Δ-σ-subfield of R; R, in turn, is called a difference-differential (or Δ-σ-) field extension or a Δ-σ-overfield of R_0. In this case we also say that we have a Δ-σ-field extension R/R_0.

If R is a Δ-σ-ring and $S \subseteq R$, then the intersection of all Δ-σ-ideals of R containing the set S is, obviously, the smallest Δ-σ-ideal of R containing S. This ideal is denoted by $[S]$; as an ideal, it is generated by all elements $\tau\eta$ where $\tau \in T$, $\eta \in S$. (Here and below we frequently write $\tau\eta$ for $\tau(\eta)$ ($\tau \in T$, $\eta \in R$).) If the set S is finite, $S = \{\eta_1, \dots, \eta_p\}$, we say that the Δ-σ-ideal $I = [S]$ is finitely generated (in this case we write $I = [\eta_1, \dots, \eta_p]$) and call $\eta_1, \dots, \eta_p$ difference-differential (or Δ-σ-) generators of I. A Δ-σ-ideal I of a Δ-σ-ring R is called *reflexive* if the inclusion $\sigma^k(a) \in I$ ($k \in \mathbb{N}$, $a \in R$) implies that $a \in I$. For any Δ-σ-ideal I of R, the set $I^* = \{a \in R \mid \sigma^k(a) \in I \text{ for some } k \in \mathbb{N}\}$ is the smallest reflexive Δ-σ-ideal containing I; it is called the *reflexive closure* of I in R.

If K_0 is a Δ-σ-subfield of a Δ-σ-field K and $S \subseteq K$, then the intersection of all Δ-σ-subfields of K containing K_0 and S is the unique Δ-σ-subfield of K containing K_0 and S and contained in every Δ-σ-subfield of K containing K_0 and S. It is denoted by $K_0\langle S\rangle$. If S is finite, $S = \{\eta_1, \dots, \eta_n\}$, then K is said to be a finitely generated Δ-σ-extension of K_0 with the set of Δ-σ-generators $\{\eta_1, \dots, \eta_n\}$. In this case we write $K = K_0\langle\eta_1, \dots, \eta_n\rangle$. As a field, $K_0\langle\eta_1, \dots, \eta_n\rangle$ coincides with the field $K_0(\{\tau\eta_i | \tau \in T, 1 \le i \le n\})$.

Let R and S be two difference-differential rings with the same basic set $\Delta \bigcup \{\sigma\}$, so that elements of Δ and σ act on each of the rings as derivations and an endomorphism, respectively, and every two mapping of the set $\Delta \bigcup \{\sigma\}$ commute. A ring homomorphism $\phi : R \longrightarrow S$ is called a *difference-differential* (or Δ-σ-) *homomorphism* if $\phi(\alpha a) = \alpha\phi(a)$ for any $\alpha \in \Delta \bigcup \{\sigma\}$, $a \in R$. In this case $\text{Ker}\,\phi$ is a reflexive Δ-σ-ideal of R. Furthermore, if J is a reflexive Δ-σ-ideal of Δ-σ-ring R, then the factor ring R/J has a natural structure of a Δ-σ-ring such that the canonical epimorphism $R \rightarrow R/J$ is a Δ-σ-homomorphism.

If K is a Δ-σ-field and $Y = \{y_1, \dots, y_n\}$ is a finite set of symbols, then one can consider a countable set of symbols $TY = \{\tau y_j | \tau \in T, 1 \le j \le n\}$ and the polynomial ring $R = K[\{\tau y_j | \tau \in T, 1 \le j \le n\}]$ in the set of indeterminates TY over the field K. This polynomial ring is naturally viewed as a Δ-σ-ring where $\alpha(\tau y_j) = (\alpha\tau)y_j$ for any $\alpha \in \Delta \bigcup \{\sigma\}$, $\tau \in T$, $1 \le j \le n$, and the elements of $\Delta \bigcup \{\sigma\}$ act on the coefficients of the polynomials of R as they act in the field K. The ring R is called the *ring of difference-differential* (or Δ-σ-) *polynomials* in the set of difference-differential (Δ-σ-) indeterminates $y_1, \dots, y_n$ over K. This ring is denoted by $K\{y_1, \dots, y_n\}$ and its elements are called difference-differential (or Δ-σ-) polynomials. If $f \in K\{y_1, \dots, y_n\}$ and $\eta = (\eta_1, \dots, \eta_n)$ is an n-dimensional vector with coordinates in some Δ-σ-overfield of K, then $f(\eta)$ (or $f(\eta_1, \dots, \eta_n)$) denotes the result of the replacement of every entry τy_i in f by $\tau\eta_i$ ($\tau \in T$, $1 \le i \le n$).

A Δ-σ-ideal in the ring $K\{y_1, \dots, y_n\}$ is called *linear* if it is generated (as a Δ-σ-ideal) by homogeneous linear Δ-σ-polynomials (i. e., Δ-σ-polynomials of the form $\sum_{i=1}^{d} a_i\tau_i y_{k_i}$ where $a_i \in K$, $\tau_i \in T$, $1 \le k_i \le n$).

Let R be a Δ-σ-ring and $\mathcal{U}$ a family of elements of some Δ-σ-overring of R. We say that $\mathcal{U}$ is Δ-σ-*algebraically) dependent* over R, if the family $T\mathcal{U} = \{\tau u \mid \tau \in T, u \in \mathcal{U}\}$ is algebraically dependent over R (that is, there exist elements $u_1, \dots, u_k \in T\mathcal{U}$ and a nonzero polynomial f in k variables with coefficients in R such that $f(u_1, \dots, u_k) = 0$). Otherwise, the family $\mathcal{U}$ is said to be Δ-σ-*algebraically independent* over R.

If K is a Δ-σ-field and L a Δ-σ-field extension of K, then a set $B \subseteq L$ is said to be a Δ-σ-*transcendence basis* of L over K if B is Δ-σ-algebraically independent over K and every element $a \in L$ is Δ-σ-algebraic over $K\langle B\rangle$ (it means that the set $\{\tau a \mid \tau \in T\}$ is algebraically dependent over the field $K\langle B\rangle$). If L is a finitely generated Δ-σ-field extension of K, then all Δ-σ-transcendence bases of L over K are finite and have the same number of elements (one can easily obtain this result by mimicking the proof of Proposition 4.1.6 of [12]). This number is called the Δ-σ-*transcendence degree* of L over K (or the Δ-σ-transcendence degree of the extension L/K); it is denoted by Δ-σ-tr. $\deg_K L$.

Let K be a Δ-σ-field K and L a finitely generated Δ-σ-extension of K with a set of Δ-σ-generators $\eta = \{\eta_1, \dots, \eta_n\}$, $L = K\langle\eta_1, \dots, \eta_n\rangle$. Then there exists a natural Δ-σ-homomorphism Φ_η of the ring of Δ-σ-polynomials $K\{y_1, \dots, y_n\}$ onto the Δ-σ-subring $K\{\eta_1, \dots, \eta_n\}$ of L such that $\Phi_\eta(a) = a$ for any $a \in K$ and $\Phi_\eta(y_j) = \eta_j$ for $j = 1, \dots, n$. If A is a Δ-σ-polynomial in $K\{y_1, \dots, y_n\}$, then the element $\Phi_\eta(A)$ is called the *value* of A at η and is denoted by $A(\eta)$. Obviously, the kernel P of the Δ-σ-homomorphism Φ_η is a prime reflexive Δ-σ-ideal of $K\{y_1, \dots, y_n\}$. This ideal is called the *defining* ideal of η. If we consider the quotient field Q of the factor ring $K\{y_1, \dots, y_n\}/P$ as a Δ-σ-field (where $\delta(\frac{u}{v}) = \frac{v\delta(u) - u\delta(v)}{v^2}$ and $\sigma(\frac{u}{v}) = \frac{\sigma(u)}{\sigma(v)}$ for any $u, v \in \bar{R}$, $\delta \in \Delta$), then this quotient field is naturally Δ-σ-isomorphic to the field L. The Δ-σ-isomorphism of Q onto L is identical on K and maps the canonical images of the Δ-indeterminates $y_1, \dots, y_n$ in $K\{y_1, \dots, y_n\}/P$ to the elements $\eta_1, \dots, \eta_n$, respectively.

3 REDUCTION OF Δ-σ-POLYNOMIALS. CHARACTERISTIC SETS

Let K be a difference-differential field with a basic set $\Delta\bigcup\{\sigma\}$ ($\Delta = \{\delta_1, \dots, \delta_m\}$ is a set of derivations, σ is an endomorphism of K). Let $R = K\{y_1, \dots, y_n\}$ be the ring of Δ-σ-polynomials in the set of Δ-σ-indeterminates $y_1, \dots, y_n$ over K and let TY denote the set of all elements $\tau y_i \in R$ ($\tau \in T$, $1 \le i \le n$) called *terms*. If $u = \tau y_i \in TY$, then the numbers $\mathrm{ord}_\Delta\,\tau$ and $\mathrm{ord}_\sigma\,\tau$ are called the orders of the term u with respect to Δ and σ, respectively.

We will consider two total orders $<_\Delta$ and $<_\sigma$ on the set of all terms TY defined as follows:

If $u = \delta_1^{k_1} \dots \delta_m^{k_m}\sigma^p y_i$ and $v = \delta_1^{l_1} \dots \delta_m^{l_m}\sigma^q y_j$ ($1 \le i, j \le n$), then $u <_\Delta v$ (respectively, $u <_\sigma v$) if the $(m+3)$-tuple $(\mathrm{ord}_\Delta u, \mathrm{ord}_\sigma u, k_1, \dots, k_m, i)$ is less than the $(m+3)$-tuple $(\mathrm{ord}_\Delta v, \mathrm{ord}_\sigma v, l_1, \dots, l_m, j)$ (respectively, the $(m+3)$-tuple $(\mathrm{ord}_\sigma u, \mathrm{ord}_\Delta u, k_1, \dots, k_m, i)$ is less than the $(m+3)$-tuple $(\mathrm{ord}_\sigma v, \mathrm{ord}_\Delta v, l_1, \dots, l_m, j)$) with respect to the lexicographic order on $\mathbb{N}^{m+3}$. We write $u \le_\Delta v$ if either $u <_\Delta v$ or $u = v$; the relation $\le_\sigma$ is defined in the same way.

An element $\tau \in T$ is said to be divisible by an element $\tau' \in T$ if $\tau = \tau''\tau'$ for some $\tau'' \in T$. In this case we write $\tau' \mid \tau$ and $\tau'' = \dfrac{\tau}{\tau'}$.

The least common multiple of elements $\tau_1, \dots, \tau_p \in T$, where $\tau_i = \delta_1^{k_{i1}} \dots \delta_m^{k_{im}}\sigma^{l_i}$ ($1 \le i \le p$) is defined as $\tau = \delta_1^{d_1} \dots \delta_m^{d_m}\sigma^l$ with $d_j = \max\{k_{1j}, \dots, k_{pj}\}$ ($1 \le j \le m$), $l = \max\{l_1, \dots, l_p\}$; it is denoted by $\mathrm{lcm}\{\tau_1, \dots, \tau_p\}$.

If $u = \tau_1 y_i$, $v = \tau_2 y_j \in TY$, we say that u divides v and write $u \mid v$ if and only if $i = j$ and $\tau_1 \mid \tau_2$. In this case the ratio $\dfrac{v}{u}$ is defined as $\dfrac{\tau_2}{\tau_1}$. If $u_1 = \tau_1 y_i, \dots, u_p = \tau_p y_i$ are terms with the same Δ-σ-indeterminate y_i, then the least common multiple of these terms, denoted by $\mathrm{lcm}(u_1, \dots, u_p)$, is defined as $\mathrm{lcm}(\tau_1, \dots, \tau_p)y_i$.

The following statement is a consequence of Lemma 2.1.

Lemma 3.1. *Let S be any infinite set of terms in $K\{y_1, \dots, y_n\}$. Then there exists an infinite sequence of terms $u_1, u_2, \dots$ in S such that $u_k \mid u_{k+1}$ for every $k = 1, 2, \dots$.*

If $A \in K\{y_1, \dots, y_n\} \setminus K$, then the highest with respect to the orderings $<_\Delta$ and $<_\sigma$ terms that appear in A are called the *Δ-leader* and the *σ-leader* of A; they are denoted by u_A and v_A, respectively. If A is written as a polynomial in one variable v_A, $A = I_d(v_A)^d + I_{d-1}(v_A)^{d-1} + \dots + I_0$ (Δ-σ-polynomials $I_d, I_{d-1}, \dots, I_0$ do not contain v_A), then I_d is called a *leading coefficient* of A; the partial derivative $\partial A/\partial v_A = dI_d(v_A)^{d-1} + (d-1)I_{d-1}(v_A)^{d-2} + \dots + I_1$ is called a *separant* of A. The leading coefficient and the separant of a Δ-σ-polynomial A are denoted by I_A and S_A, respectively.

Definition 3.2. Let A and B be two Δ-σ-polynomials in the ring $K\{y_1, \dots, y_n\}$. We say that A has lower rank than B and write $\mathrm{rk}\,A < \mathrm{rk}\,B$ if either $A \in K$, $B \notin K$, or $(v_A, \deg_{v_A} A, \mathrm{ord}_\Delta u_A)$ is less than $(v_B, \deg_{v_B} B, \mathrm{ord}_\Delta u_B)$ with respect to the lexicographic order (where the terms v_A and v_B are compared with respect to the order $<_\sigma$ and the other coordinates are compared with respect to the natural order on $\mathbb{N}$). If the two vectors are equal (or $A, B \in K$), we say that the Δ-σ-polynomials A and B are of the same rank and write $\mathrm{rk}\,A = \mathrm{rk}\,B$.

Definition 3.3. If $A, B \in K\{y_1, \dots, y_n\}$, then B is said to be **reduced** with respect to A if

(i) B does not contain terms τv_A such that $\mathrm{ord}_\Delta\,\tau > 0$ and $\mathrm{ord}_\Delta(\tau u_A) \le ord_\Delta u_B$.

(ii) If B contains a term τv_A where $\mathrm{ord}_\Delta\,\tau = 0$, then either $\mathrm{ord}_\Delta\,u_B < \mathrm{ord}_\Delta\,u_A$ or $\mathrm{ord}_\Delta\,u_A \le \mathrm{ord}_\Delta\,u_B$ and $\deg_{\tau v_A} B < \deg_{v_A} A$.

If $B \in K\{y_1, \dots, y_n\}$, then B is said to be reduced with respect to a set $\mathcal{A} \subseteq K\{y_1, \dots, y_n\}$ if B is reduced with respect to every element of $\mathcal{A}$.

Remark 3.4. *It follows from the last definition that a Δ-σ-polynomial B is not reduced with respect to a Δ-σ-polynomial A ($A \notin K$) if either B contains some term τv_A such that $\mathrm{ord}_\Delta\,\tau > 0$ and $\mathrm{ord}_\Delta(\tau u_A) \le \mathrm{ord}_\Delta\,u_B$ or B contains $\sigma^i v_A$ for some $i \in \mathbb{N}$ and in this case $\mathrm{ord}_\Delta\,u_A \le \mathrm{ord}_\Delta\,u_B$ and $\deg_{v_A} A \le \deg_{\sigma^i v_A} B$.*

Definition 3.5. A set of Δ-σ-polynomials $\mathcal{A}$ in $K\{y_1, \dots, y_n\}$ is called **autoreduced** if $\mathcal{A}\bigcap K = \emptyset$ and every element of $\mathcal{A}$ is reduced with respect to any other element of this set.

Proposition 3.6. *Every autoreduced set of Δ-σ-polynomials in the ring $K\{y_1, \dots, y_n\}$ is finite.*

Proof. Suppose that $\mathcal{A}$ is an infinite autoreduced subset of $K\{y_1, \dots, y_n\}$. Then there is an infinite subset $\mathcal{A}'$ of $\mathcal{A}$ such that all Δ-σ-polynomials in $\mathcal{A}'$ have distinct σ-leaders. Indeed, otherwise there exists an infinite set $\mathcal{A}_1 \subseteq \mathcal{A}$ such that all Δ-σ-polynomials in $\mathcal{A}_1$ have the same σ-leader v. It follows that the infinite set $\{\mathrm{ord}_\Delta\,u_A \mid A \in \mathcal{A}_1\}$ contains a nondecreasing infinite sequence $\mathrm{ord}_\Delta\,u_{A_1} \le \mathrm{ord}_\Delta\,u_{A_2} \le \dots$. Since the sequence $\{\deg_v A_i | i = 1, 2, \dots\}$ cannot be strictly decreasing, there exists two indices i and j such that $i < j$ and $\deg_v A_i \le \deg_v A_j$. We obtain that A_j is not reduced with respect to A_i that contradicts the fact that $\mathcal{A}$ is an autoreduced set.

Thus, we can assume that all leaders of our infinite autoreduced set $\mathcal{A}$ are distinct. By Lemma 3.1, there exists an infinite sequence $B_1, B_2, \dots$ of elements of $\mathcal{A}$ such that $v_{B_i} \mid v_{B_{i+1}}$ for all $i = 1, 2, \dots$. (Also, since the leaders of elements of our sequence are distinct, $\dfrac{v_{B_{i+1}}}{v_{B_i}} \neq 1$.)

Let $k_i = \mathrm{ord}_\sigma\,v_{B_i}$ and $l_i = \mathrm{ord}_\Delta\,u_{B_i}$. Since u_{B_i} is the Δ-leader of B_i, $l_i \ge k_i$ ($i = 1, 2, \dots$), so that the infinite set $\{l_i - k_i \mid i \in \mathbb{N},\ i \ge 1\}$ contains a nondecreasing sequence $l_{i_1} - k_{i_1}, l_{i_2} - k_{i_2}, \dots$. Then $\mathrm{ord}_\Delta(\frac{v_{B_{i_2}}}{v_{B_{i_1}}}u_{B_{i_1}}) = k_{i_2} - k_{i_1} + l_{i_1} \le k_{i_2} + l_{i_2} - k_{i_2} = l_{i_2} = \mathrm{ord}_\Delta\,u_{B_{i_2}}$. It follows that B_{i_2} contains a term $\tau v_{B_{i_1}} = v_{B_{i_2}}$ such that $\mathrm{ord}_\Delta\,\tau > 0$ and $\mathrm{ord}_\Delta(\tau u_{B_{i_1}}) \le \mathrm{ord}_\Delta\,u_{B_{i_2}}$. Thus, the Δ-σ-polynomial B_{i_2} is reduced with respect to B_{i_1} that contradicts the fact that $\mathcal{A}$ is an autoreduced set. $\square$

The proof of the following statement is similar to the proof of the reduction theorem for difference-differential polynomials in the case of classical autoreduced sets, see [1].

Proposition 3.7. *Let $\mathcal{A} = \{A_1, \dots, A_p\}$ be an autoreduced set in the ring of Δ-σ-polynomials $K\{y_1, \dots, y_n\}$ and let $B \in K\{y_1, \dots, y_n\}$. Then there exist a Δ-σ-polynomial B_0 and nonnegative integers k_i, l_i*

$(1 \le i \le p)$ such that B_0 is reduced with respect to $\mathcal{A}$, $\operatorname{rk} B_0 \le \operatorname{rk} B$, *and* $\prod_{i=1}^{p} I_{A_i}^{k_i} S_{A_i}^{l_i} B \equiv B_0\, (mod[\mathcal{A}])$.

With the notation of the last proposition, we say that the Δ-σ-polynomial B *reduces to* B_0 modulo $\mathcal{A}$.

Throughout the rest of the paper, while considering an autoreduced set $\mathcal{A} = \{A_1, \dots, A_p\}$ in the ring $K\{y_1, \dots, y_n\}$ we always assume that its elements are arranged in order of increasing rank, $rk\, A_1 < \dots < rk\, A_p$.

Definition 3.8. If $\mathcal{A} = \{A_1, \dots, A_p\}$, $\mathcal{B} = \{B_1, \dots, B_q\}$ are two autoreduced sets of Δ-σ-polynomials $K\{y_1, \dots, y_n\}$, we say that $\mathcal{A}$ has lower rank than $\mathcal{B}$ if one of the following two cases holds:

(1) There exists $k \in \mathbb{N}$ such that $k \le \min\{p, q\}$, $\operatorname{rk} A_i = \operatorname{rk} B_i$ for $i = 1, \dots, k-1$ and $\operatorname{rk} A_k < \operatorname{rk} B_k$.

(2) $p > q$ and $\operatorname{rk} A_i = \operatorname{rk} B_i$ for $i = 1, \dots, q$.

If $p = q$ and $\operatorname{rk} A_i = \operatorname{rk} B_i$ for $i = 1, \dots, p$, then $\mathcal{A}$ is said to have the same rank as $\mathcal{B}$. In this case we write $\operatorname{rk} \mathcal{A} = \operatorname{rk} \mathcal{B}$.

Repeating the arguments of the proof of the corresponding result for autoreduced sets of differential polynomials (see [7, Chapter I, Proposition 3]) we obtain the following statement.

Proposition 3.9. *In every nonempty family of autoreduced sets of differential polynomials there exists an autoreduced set of lowest rank.*

This statement shows that if J is a Δ-σ-ideal (or even a subset) of the ring of Δ-σ-polynomials $K\{y_1, \dots, y_n\}$, then J contains an autoreduced subset of lowest rank.(Clearly, the set of all autoreduced subsets of J is not empty: if $A \in J$, then $\{A\}$ is an autoreduced subset of J.)

Definition 3.10. If J is a subset (in particular, a Δ-σ-ideal) of the ring of Δ-σ-polynomials $K\{y_1, \dots, y_n\}$, then an autoreduced subset of J of lowest rank is called a **characteristic set** of J.

Proposition 3.11. *Let $\mathcal{A} = \{A_1, \dots, A_p\}$ be a characteristic set of a nonempty subset J of the ring of Δ-σ-polynomials $R = K\{y_1, \dots, y_n\}$. Then an element $B \in J$ is reduced with respect to $\mathcal{A}$ if and only if $B = 0$.*

Proof. First of all, note that if $B \ne 0$ and $\operatorname{rk} B < \operatorname{rk} A_1$, then $\{B\}$ is an autoreduced set and $\operatorname{rk}\{B\} < \operatorname{rk} \mathcal{A}$ that contradicts the fact that $\mathcal{A}$ is a characteristic set of J. Let $\operatorname{rk} B > \operatorname{rk} A_1$ and let $A_1, \dots, A_j$ $(1 \le j \le p)$ be all elements of $\mathcal{A}$ whose rank is lower that the rank of B. Then the set $\mathcal{A}' = \{A_1, \dots, A_j, B\}$ is autoreduced. Indeed, the Δ-σ-polynomials $A_1, \dots, A_j$ are reduced with respect to each other and B is reduced with respect to the set $\{A_1, \dots, A_j\}$, since B is reduced with respect to $\mathcal{A}$. Furthermore, each A_i $(1 \le i \le j)$ is reduced with respect to B because $\operatorname{rk} A_i < \operatorname{rk} B$. By the choice of B, if $j < p$, then $\operatorname{rk} B < \operatorname{rk} A_{j+1}$, so $\operatorname{rk} \mathcal{A}' < \operatorname{rk} \mathcal{A}$; if $j = p$, then we still have the inequality $\operatorname{rk} \mathcal{A}' < \operatorname{rk} \mathcal{A}$ by the second part of Definition 3.8. It follows that $\mathcal{A}$ is not a characteristic set of J, contrary to our assumption. Thus, $B = 0$. □

Definition 3.12. Let $\mathcal{A} = \{A_1, \dots, A_p\}$ be an autoreduced set in the ring $K\{y_1, \dots, y_n\}$ such that all Δ-σ-polynomials A_i $(1 \le i \le p)$ are linear. Then the set $\mathcal{A}$ is said to be coherent if it satisfies the following two conditions.

(i) τA_i reduces to zero modulo $\mathcal{A}$ for any $\tau \in T$, $1 \le i \le p$.

(ii) For every A_i, $A_j \in \mathcal{A}$, $1 \le i < j \le p$, let $w = \operatorname{lcm}\{v_{A_i}, v_{A_j}\}$ and $\tau' = \dfrac{w}{v_{A_i}}$, $\tau'' = \dfrac{w}{v_{A_j}}$. Then $(\tau'' I_{A_j})(\tau' A_i) - (\tau' I_{A_i})(\tau'' A_j)$ reduces to zero modulo $\mathcal{A}$.

The proof of the following statement can be obtained by mimicking the proof of the corresponding result for autoreduced sets of difference polynomials, see [8, Theorem 6.5.3].

Proposition 3.13. *Every characteristic set of a linear Δ-σ-ideal in the ring of Δ-σ-polynomials $K\{y_1, \dots, y_n\}$ is a coherent autoreduced set. Conversely, if $\mathcal{A}$ is a coherent autoreduced set in $K\{y_1, \dots, y_n\}$ consisting of linear Δ-σ-polynomials, then $\mathcal{A}$ is a characteristic set of the linear Δ-σ-ideal $[\mathcal{A}]$.*

4 DIMENSION POLYNOMIALS. THE MAIN THEOREM

Let K be a Δ-σ-field (as before $\Delta = \{\delta_1, \dots, \delta_m\}$ is a set of mutually commuting derivations of K and σ is an endomorphism of K that commutes with every δ_i). Let $R = K\{y_1, \dots, y_n\}$ be the ring of Δ-σ-polynomials over K and P a prime Δ-σ-ideal of R. Let P^* denote the reflexive closure of P in R (as we have mentioned, P^* is also a prime Δ-σ-ideal of R) and for every $r, s \in \mathbb{N}$, let $R_{rs} = K[\{\tau y_i \mid \tau \in T(r,s), 1 \le i \le n\}]$. In other words, R_{rs} is a polynomial ring over K in indeterminates τy_i such that $\operatorname{ord}_\Delta \tau \le r$ and $\operatorname{ord}_\sigma \tau \le s$. Let $P_{rs} = P \bigcap R_{rs}$, $P^*_{rs} = P^* \bigcap R_{rs}$, and let L, L^*, L_{rs} and L^*_{rs} denote the quotient fields of the integral domains R/P, R/P^*, R_{rs}/P_{rs} and R_{rs}/P^*_{rs}, respectively. If η_i denotes the canonical image of y_i in R_{rs}/P^*_{rs}, then L^* is a Δ-σ-field extension of K, $L^* = K\langle \eta_1, \dots, \eta_n \rangle$, and $L^*_{rs} = K(\{\tau \eta_i \mid \tau \in T(r,s),\ 1 \le i \le n\})$.

The following statement is an analog of the theorem on the dimension polynomial of an inversive difference-differential field extension proved in [13].

Theorem 4.1. *With the above notation, there exists a numerical polynomial $\phi_{P^*}(t_1, t_2) \in \mathbb{Q}[t_1, t_2]$ such that*

(i) *$\phi_{P^*}(r, s) = \operatorname{tr.deg}_K L^*_{rs}$ for all sufficiently large pairs $(r, s) \in \mathbb{N}^2$.*

(ii) *The polynomial $\phi_{P^*}(t_1, t_2)$ is linear with respect to t_2 and $\deg_{t_1} \phi_{P^*} \le m$, so this polynomial can be written as*

$$\phi_{P^*}(t_1, t_2) = \phi^{(1)}_{P^*}(t_1) t_2 + \phi^{(2)}_{P^*}(t_1)$$

where $\phi^{(1)}_{P^*}(t_1)$ and $\phi^{(2)}_{P^*}(t_1)$ are numerical polynomials in one variable that, in turn, can be written as

$$\phi^{(1)}_{P^*}(t_1) = \sum_{i=0}^{m} a_i \binom{t_1 + i}{i} \quad and \quad \phi^{(2)}_{P^*}(t_1) = \sum_{i=0}^{m} b_i \binom{t_1 + i}{i}$$

with a_i, $b_i \in \mathbb{Z}$ $(1 \le i \le m)$. Furthermore, $a_m = \Delta$-σ-$\operatorname{tr.deg}_K L^*$.

Proof. Let $\mathcal{A} = \{A_1, \dots, A_p\}$ be a characteristic set of the Δ-σ-ideal P^* and for any $r, s \in \mathbb{N}$, let

$U_{rs} = \{u \in TY \mid \operatorname{ord}_\Delta u \le r$, $\operatorname{ord}_\sigma u \le s$ and either u is not a multiple of any v_{A_i} or u is a multiple of some σ-leader of an element of $\mathcal{A}$ and for every $\tau \in T, A \in \mathcal{A}$ such that $u = \tau v_A$, one has $\operatorname{ord}_\Delta(\tau u_A) > r\}$.

Using our concept of an autoreduced and the arguments of the proof of Theorem 6 in [7, Chapter II], we obtain that the set $U_{rs}(\eta) = \{u(\eta) \mid u \in U_{rs}\}$ is a transcendence basis of L^*_{rs} over K. In order to evaluate the number of elements of U_{rs} (and therefore, tr. $\deg_K L^*_{rs}$), let us consider the sets $U'_{rs} = \{u \in TY \mid \text{ord}_\Delta u \le r, \text{ord}_\sigma u \le s$ and u is not a multiple of any $v_{A_i}\}$ and $U''_{r,s} = \{u \in TY \mid \text{ord}_\Delta u \le r, \text{ord}_\sigma u \le s$ and there exist $A \in \mathcal{A}$ such that $u = \tau v_A$ and $\text{ord}_\Delta(\tau u_A) > r\}$. Clearly, $U'_{rs} \bigcap U''_{r,s} = \emptyset$ and $U_{r,s} = U'_{r,s} \bigcup U''_{r,s}$.

By Theorem 2.3, there exists a numerical polynomial in two variables $\phi^{(1)}(t_1, t_2)$ such that $\phi^{(1)}(r, s) = \text{Card}\, U'_{rs}$ for all sufficiently large $(r, s) \in \mathbb{N}^2$, $\deg_{t_1} \phi^{(1)} \le m$, and $\deg_{t_2} \phi^{(1)} \le 1$. Furthermore, repeating the arguments of the proof of Theorem 4.2 of [11] (considered in the case of one translation) we obtain that there exists a bivariate numerical polynomial $\phi^{(2)}(t_1, t_2)$ such that $\phi^{(2)}(r, s) = \text{Card}\, U''_{rs}$ for all sufficiently large $(r, s) \in \mathbb{N}^2$ and $\phi^{(2)}(t_1, t_2)$ is an alternating sum of bivariate numerical polynomials of subsets of $\mathbb{N}^{m+1}$ described in section 2. Each such a polynomial can be represented in the form (2), so $\deg_{t_1} \phi^{(2)} \le m$ and $\deg_{t_2} \phi^{(2)} \le 1$. Clearly the polynomial $\phi_{P^*}(t_1, t_2) = \phi^{(1)}(t_1, t_2) + \phi^{(2)}(t_1, t_2)$ satisfies all conditions of the theorem. The fact that $a_m = \Delta$-σ-tr. $\deg_K L^*$ can be established in the same way as in the last part of the proof of Theorem 3.1 in [13]. □

Note that in the case when σ is an automorphism of K, the statement of the last theorem was proved in [10] with the use of a theorem on the multivariate dimension polynomial of a difference-differential module and properties of modules of Kähler differentials.

The following theorem is the main result of the paper.

Theorem 4.2. *With the notation introduced at the beginning of this section, there exists a bivariate numerical polynomial $\psi_P(t_1, t_2)$ such that*

(i) $\psi_P(r, s) =$ tr. $\deg_K L_{rs}$ *for all sufficiently large pairs* $(r, s) \in \mathbb{N}^2$.

(ii) *The polynomial $\psi_P(t_1, t_2)$ is linear with respect to t_2 and $\deg_{t_1} \psi_P \le m$, so it can be written as*

$$\psi_P(t_1, t_2) = \psi_P^{(1)}(t_1)t_2 + \psi_P^{(2)}(t_1)$$

where $\psi_P^{(1)}(t_1)$ and $\psi_P^{(2)}(t_1)$ are numerical polynomials in one variable.

Proof. We start with the proof for the case $\Delta = \emptyset$. In this case we will use the above notation and conventions just replacing the prefix Δ-σ- by σ- (and "difference-differential" by "difference"). Let $\mathcal{A} = \{A_1, \dots, A_p\}$ be a characteristic set of the σ-ideal P^* (the reflexive closure of the prime σ-ideal P of the ring of σ-polynomials $R = K\{y_1, \dots, y_n\}$) and let v_j denote the σ-leader of A_j ($j = 1, \dots, p$). Let $\eta_i = y_i + P$ ($1 \le i \le n$), $L = K(\{\sigma^k \eta_i \mid k \in \mathbb{N}, 1 \le i \le n\})$ (the quotient field of R/P) and $L_s = K(\{\sigma^k \eta_i \mid 0 \le k \le s, 1 \le i \le n\})$.

For every $j = 1, \dots, p$, let s_j be the smallest nonnegative integer such that $\sigma^{s_j}(A_j) \in P$. Furthermore, let

$$V = \{u \in TY \mid u \neq \sigma^i v_j \text{ for any } i \in \mathbb{N}, 1 \le j \le p\},$$

$V_r = \{v \in V \mid \text{ord}_\sigma v \le r\}$ ($r \in \mathbb{N}$), $V(\eta) = \{u(\eta) \mid u \in V\}$, $W = \{\sigma^k v_j \mid 1 \le j \le p, 0 \le k \le s_j - 1\}$, and $W(\eta) = \{u(\eta) \mid u \in W\}$.

It is easy to see that the set $V(\eta)$ is algebraically independent over K. Indeed, suppose there exist $v_1, \dots, v_l \in V$ and a polynomial $f(X_1, \dots, X_l)$ in l variables with coefficients in the field K such that $f(v_1(\eta), \dots, v_l(\eta)) = 0$. Then $f(v_1, \dots, v_l) \in P \subseteq P^*$ and $f(v_1, \dots, v_l)$ is reduced with respect to the characteristic set $\mathcal{A}$ (this σ-polynomial does not contain any transforms of the leaders of elements of $\mathcal{A}$). Therefore, $f = 0$, so the set $V(\eta)$ is algebraically independent over K.

Now we notice that every element of the field L is algebraic over its subfield $K(V(\eta) \bigcup W(\eta))$.

Indeed, since $L = K(V(\eta) \bigcup W(\eta) \bigcup \{\sigma^k v_j(\eta) \mid 1 \le j \le p, k \ge s_i\})$, it is sufficient to prove that every element $\sigma^k v_j(\eta)$ with $k \ge s_j$ ($1 \le i \le p$) is algebraic over $K(V(\eta) \bigcup W(\eta))$.

Since $\sigma^{s_j} A_j \in P$, we have $\sigma^{s_j} A_j(\eta) = 0$, hence $\sigma^k A_j(\eta) = 0$ for all $k \ge s_j$. If one writes A_j as a polynomial in v_j,

$$A_j = I_{q_j}^{(j)} v_j^{q_j} + I_{q_j-1}^{(j)} v_j^{q_j-1} + \dots + I_0^{(j)}$$

($I_{q_j}^{(j)}, \dots, I_0^{(j)}$ do not contain v_j) and $k \ge s_j$, then

$$\sigma^k A_j(\eta) = \left(\sigma^k I_{q_j}^{(j)}(\eta)\right) v_j(\eta)^{q_j} + \left(\sigma^k I_{q_j-1}^{(j)}(\eta)\right) v_j(\eta)^{q_j-1} + \dots + \sigma^k I_0^{(j)}(\eta) = 0.$$

Note that $I_{q_j}^{(j)} \notin P^*$, since $I_{q_j}^{(j)}$ is an initial of an element of the characteristic set of P^* and therefore is reduced with respect to this set (by Proposition 3.11, the inclusion $I_{q_j}^{(j)} \in P^*$ would imply $I_{q_j}^{(j)} = 0$). Since the σ-ideal P^* is reflexive, $\sigma^k I_{q_j}^{(j)} \notin P^*$, hence $\sigma^k I_{q_j}^{(j)}(\eta) \neq 0$. It follows that $\sigma^k v_j(\eta)$ is algebraic over the field $K\left(V(\eta) \bigcup W(\eta) \bigcup \{u(\eta) \mid u <_\sigma \sigma^k u_i\}\right)$ (the term ordering $<_\sigma$ was defined at the beginning of section 3). By induction on the well-ordered (with respect to $<_\sigma$) set of terms TY we obtain that all elements $\sigma^k v_j(\eta)$ ($1 \le j \le p, k \in \mathbb{N}$) are algebraic over $K(V(\eta) \bigcup W(\eta))$, so L is algebraic over this field as well.

Let $\{w_1, \dots, w_q\}$ be a maximal subset of W such that the set $\{w_1(\eta), \dots, w_q(\eta)\}$ is algebraically independent over $K(V(\eta))$. Then $V(\eta) \bigcup \{w_1(\eta), \dots, w_q(\eta)\}$ is a transcendence basis of the field L over K. Furthermore, since the set $W(\eta)$ is finite, there exists $r_0 \in \mathbb{N}$ such that

(i) $w_1, \dots, w_q \in R_{r_0}$;
(ii) $r_0 \ge \max\{\text{ord}_\sigma v_j + s_j \mid 1 \le j \le p\}$;
(iii) Every element of $W(\eta)$ is algebraic over the field $K\left(V_{r_0}(\eta) \bigcup \{w_1(\eta), \dots, w_q(\eta)\}\right)$.

Let $r \ge r_0$, $R_r = K[\{\sigma^k y_i \mid 1 \le i \le n, 0 \le k \le r\}]$, and $P_r = P \bigcap R_r$. Let L_r denote the quotient field of the integral domain R_r/P_r and $\zeta_i^{(r)} = y_i + P_r \in R_r/P_r \subseteq L_r$ ($1 \le i \le n$). Furthermore, let $\zeta^{(r)} = \{\zeta_1^{(1)}, \dots, \zeta_n^{(1)}\}$, and $V_r(\zeta^{(r)}) = \{v(\zeta^{(r)}) \mid v \in V_r\}$. We are going to prove that

$$B_r = V_r(\zeta^{(r)}) \bigcup \{w_1(\zeta^{(r)}), \dots, w_q(\zeta^{(r)})\}$$

is a transcendence basis of L_r over K.

Repeating the arguments of the proof of Theorem 4.1 (applied to $V_r(\zeta^{(r)})$ instead of $V(\eta)$) we obtain that $V_r(\zeta^{(r)})$ is algebraically independent over K. Let us show that the elements $w_1(\zeta^{(r)}), \dots, w_q(\zeta^{(r)})$ are algebraically independent over the field $K(V_r(\zeta^{(r)}))$. Suppose that $g(w_1(\zeta^{(r)}), \dots, w_q(\zeta^{(r)})) = 0$ for some polynomial g in q

indeterminates with coefficients in $K\left(V_r(\zeta^{(r)})\right)$. Then there exist elements $z_1, \dots, z_d \in V_r$ such that all coefficients of f lie in $K(z_1(\zeta^{(r)}), \dots, z_d(\zeta^{(r)}))$. Multiplying f by the common denominator of these coefficients we obtain a nonzero polynomial g in $d+q$ indeterminates such that

$$g(z_1(\zeta^{(r)}), \dots, z_d(\zeta^{(r)}), w_1(\zeta^{(r)}), \dots, w_q(\zeta^{(r)})) = 0,$$

hence $g(z_1, \dots, z_d, w_1, \dots, w_q) \in P_r \subseteq P$.

Considering the image of g under the natural homomorphism $R \to R/P \subseteq L$ we obtain that $g(z_1(\eta), \dots, z_d(\eta), w_1(\eta), \dots, w_q(\eta)) = 0$ where $z_i(\eta) \in K(V(\eta))$, $1 \le i \le d$. Since $V(\eta) \bigcup \{w_1(\eta), \dots, w_q(\eta)\}$ is a transcendence basis of L/K, $g = 0$, a contradiction. Therefore, the elements $w_1(\zeta^{(r)}), \dots, w_q(\zeta^{(r)})$ are algebraically independent over $K\left(V_r(\zeta^{(r)})\right)$, so that the set B_r is algebraically independent over K.

Now let $r \ge r_0$ and let $u = \sigma^l y_i$ where $0 \le l \le r$ $(1 \le i \le n)$. If u is not a transform of any v_k $(1 \le k \le p)$, then $u \in V_r$ and $u(\zeta) \in V_r(\zeta)$.

If $u = \sigma^j v_k$ where $0 \le j \le s_k - 1$ (in this case $\mathrm{ord}_\sigma u < r$), then $u \in W$, hence the element $u(\eta)$ is algebraic over the field $K\left(V_r(\eta) \bigcup \{w_1(\eta), \dots, w_q(\eta)\}\right)$. As above, we obtain the existence of a nonzero polynomial h in $d+q+1$ variables ($d \in \mathbb{N}$) and elements $z_1, \dots, z_d \in V_r$ such that $h(z_1, \dots, z_d, w_1, \dots, w_q, u) \in P_r$. Then $h(z_1(\zeta^{(r)}), \dots, z_d(\zeta^{(r)}), w_1(\zeta^{(r)}), \dots, w_q(\zeta^{(r)}), u(\zeta^{(r)})) = 0$ hence $u(\zeta^{(r)})$ is algebraic over the field $K(B_r)$.

Suppose that $u = \sigma^j v_k$ where $s_k \le j \le r - \mathrm{ord}\, v_k$ $(1 \le k \le p)$. Then $\sigma^j A_k \in P_r$, hence $\sigma^j A_k(\zeta^{(r)}) = 0$. If one writes A_k as a polynomial of v_k,

$$A_k = I_{kd_k} v_k^{d_k} + \dots + I_{k1} v_k + I_{k0}$$

(I_{ij} do not contain v_k and all terms in I_{ij} are lower than v_k with respect to $<_\sigma$), then $\sigma^j I_{kd_k} \notin P^*$, since I_{kd_k} is the initial of an element of a characteristic set of P^* and the ideal P^* is reflexive. Therefore, $\sigma^j I_{kd_k}(\zeta^{(r)}) \ne 0$, so the equality $\sigma^j A_k(\zeta^{(r)}) = 0$ shows that the element $u(\zeta^{(r)}) = \sigma^j v_k(\zeta^{(r)})$ is algebraic over the field $K\left(V_r(\zeta^{(r)}) \bigcup \{v(\zeta^{(r)}) \,|\, v \in TY,\ v <_\sigma u\}\right)$.

Using the induction on the well-ordered (with respect to the order $<_\sigma$) set TY we obtain that $u(\zeta^{(r)})$ is algebraic over the field $K\left(V_r(\zeta^{(r)}) \bigcup \{\sigma^j v_k(\zeta^{(r)}) \,|\, 1 \le k \le d,\ 0 \le j \le s_k - 1\}\right)$, which, as we have seen, is algebraic over $K(B_r)$. It follows that $u(\zeta^{(r)})$ is algebraic over $K(B_r)$ for every term u with $\mathrm{ord}_\sigma u \le r$. Therefore, B_r is a transcendence basis of L_r over K.

Now we are going to complete the proof of the theorem considering the case when $\mathrm{Card}\,\Delta = m > 0$. In this case, the field L_{rs} can be treated as the subfield $K(\{\theta\sigma^j\xi_i \,|\, \theta \in \Theta(r), 0 \le j \le s, 1 \le i \le n\})$ of the differential (Δ-) overfield $K\langle\{\sigma^j\xi_i \,|\, 0 \le j \le s, 1 \le i \le n\}\rangle_\Delta$ of K. (ξ_i is the canonical image of y_i in R_{rs}/P_{rs}; the index Δ indicates that we consider a differential, not a difference-differential, field extension.)

By the Kolchin's theorem (Theorem 1.1), for any $s \in \mathbb{N}$, there exists a numerical polynomial $\chi_s(t) = \sum_{i=0}^{m} a_i(s)\binom{t+i}{i}$ in one variable t such that $\chi_s(r) = \mathrm{tr.deg}_K L_{rs}$ for all sufficiently large $r \in \mathbb{N}$ and $a_i(s) \in \mathbb{Z}$ $(0 \le i \le m)$.

On the other hand, the first part of the proof (with the use of the finite set of σ-indeterminates $\{\Theta(r)y_i \,|\, \theta \in \Theta(r),\ 1 \le i \le n\}$ instead of $\{y_1, \dots, y_n\}$) shows that $\mathrm{tr.deg}_K L_{rs} = \mathrm{Card}\, V_{rs} + \lambda(r)$ where $V_{rs} = \{u = \tau y_i \in TY \,|\, \tau \in T(r,s)$ and $u \ne \tau' v_j$ for any $\tau' \in T,\ 1 \le j \le p\}$. ($v_j$ denotes the σ-leader of the element A_j of a characteristic set $\mathcal{A} = \{A_1, \dots, A_p\}$ of the reflexive closure P^* of P.) Since the set W in the first part of the proof is finite and depends only on the σ-orders of terms of A_j, $1 \le j \le p$, the number of elements of the corresponding set in the general case depends only on r; we have denoted it by $\lambda(r)$.

By Theorem 2.3, there exist $r_0, s_0 \in \mathbb{N}$ and a bivariate numerical polynomial $\omega(t_1, t_2)$ such that $\omega(r,s) = \mathrm{Card}\, V_{rs}$ for all $r \ge r_0$, $s \ge s_0$, $\deg_{t_1} \omega \le m$ and $\deg_{t_2} \omega \le 1$. Thus, $\mathrm{tr.deg}_K L_{rs} = \omega(r,s) + \lambda(r)$ for all $r \ge r_0$, $s \ge s_0$. At the same time, we have seen that $\mathrm{tr.deg}_K L_{rs_0} = \chi_{s_0}(r) = \sum_{i=0}^{m} a_i(s_0)\begin{pmatrix} r+i \\ i \end{pmatrix}$ for all sufficiently large $r \in \mathbb{N}$ ($a_i(s_0) \in \mathbb{Z}$). It follows that $\lambda(r)$ is a polynomial of r for all sufficiently large $r \in \mathbb{N}$, say, for all $r \ge r_1$. Therefore, for any $s \ge s_0$, $r \ge \max\{r_0, r_1\}$, $\mathrm{tr.deg}_K L_{rs} = \omega(r,s) + \lambda(r)$ is expressed as a bivariate numerical polynomial in r and s. □

Definition 4.3. The numerical polynomial $\psi_P(t_1, t_2)$ whose existence is established by Theorem 4.2 is called the Δ-σ-dimension polynomial of the Δ-σ-ideal P.

The proof of the last theorem (as well as the proof of Theorem 4.1) shows that the main step in the computation of a Δ-σ-dimension polynomial is the construction of a characteristic set in the sense of section 3. It can be realized by the corresponding generalization of the Ritt-Kolchin algorithm described in [8, Section 5.5], but the development and implementation of such a generalization is the subject of future research.

The following illustrating example uses the notation of the proofs of Theorems 4.1 and 4.2.

Example 4.4. Let K be a difference-differential (Δ-σ-) field with two basic derivations, $\Delta = \{\delta_1, \delta_2\}$, and one basic endomorphism σ. Let $K\{y\}$ be the ring of Δ-σ-polynomials in one Δ-σ-indeterminate y over K and let P be a linear (and therefore prime) Δ-σ-ideal of $K\{y\}$ generated by the Δ-σ-polynomial $A = \sigma^2 y + \sigma\delta_1^2 y + \sigma\delta_2^2 y$ (that is, $P = [A]$). Then $P^* = [B]$, where $B = \sigma y + \delta_1^2 y + \delta_2^2 y$, and Proposition 3.13 shows that $\{B\}$ is a characteristic set of the Δ-σ-ideal P^*. With the notation of the proof of Theorem 4.1, we have $U'_{rs} = \{u \in TY \,|\, \mathrm{ord}_\Delta u \le r,\ \mathrm{ord}_\sigma u \le s$ and u is not a multiple of $\sigma y\}$ and $U''_{rs} = \{u \in TY \,|\, \mathrm{ord}_\Delta u \le r,\ \mathrm{ord}_\sigma u \le s$ and there is $\tau \in T$ such that $u = \tau(\sigma y)$ and $\mathrm{ord}_\Delta(\tau\delta_1^2) > r\}$. Then

$$\mathrm{Card}\, U'_{rs} = \mathrm{Card}\{\delta_1^i\delta_2^j y \,|\, i+j \le r\} = \begin{pmatrix} r+2 \\ 2 \end{pmatrix} \text{ and}$$

$$\mathrm{Card}\, U''_{rs} = \mathrm{Card}\{\sigma^i\delta_1^j\delta_2^k y \,|\, 1 \le i \le s,\ r-2 < j+k \le r\} =$$

$$s\left(\begin{pmatrix} r+2 \\ 2 \end{pmatrix} - \begin{pmatrix} r+2-2 \\ 2 \end{pmatrix}\right) = (2r+1)s.$$

Since $\sigma B \in P$, the proof of Theorem 4.2 shows that if $\psi_P(t_1, t_2)$ is the Δ-σ-dimension polynomial of the Δ-σ-ideal P, then

$$\psi(r,s) = \mathrm{Card}\, U'_{rs} + \mathrm{Card}\, U''_{rs} + \mathrm{Card}\{\sigma\delta_1^i\delta_2^j y \,|\, i+j \le r-2\}$$

for all sufficiently large $(r,s) \in \mathbb{N}^2$. *It follows that* $\psi_P(t_1,t_2) = (2t_1+1)t_2 + \binom{t_1+2}{2} + \binom{t_1}{2}$, *that is*

$$\psi_P(t_1,t_2) = (2t_1+1)t_2 + t_1^2 + t_1 + 1.$$

We conclude with a brief discussion of the connection between the Δ-σ-dimension polynomial and the concept of strength of a system of difference-differential equations in the sense of A. Einstein.

Consider a system of difference-differential equations

$$A_i(f_1,\dots,f_n) = 0 \qquad (i = 1,\dots,q) \tag{1}$$

with m basic partial derivations and one translation σ over a field K of functions of m real variables $x_1,\dots,x_m$ treated as a difference-differential field with basic set of derivations $\Delta = \{\delta_1,\dots,\delta_m\}$ and one translation σ where $\delta_i = \partial/\partial x_i$ $(1 \le i \le m)$ and $\sigma : f(\overline{x}) \mapsto f(\overline{x}+\overline{h})$ is a shift of the argument $\overline{x} = (x_1,\dots,x_m)$ by some vector $\overline{h}$ in $\mathbb{R}^m$. ($f_1,\dots,f_n$ are unknown functions of $x_1,\dots,x_m$). We assume that system (3) is algebraic, that is, all $A_i(y_1,\dots,y_n)$ are elements of a ring of Δ-σ-polynomials $K\{y_1,\dots,y_n\}$ over the functional Δ-σ-field K.

Let us consider a sequence of nodes in $\mathbb{R}^m$ that begins at some initial node $\mathcal{P}$ and goes in the direction of the vector $\overline{h}$ with step $|\overline{h}|$. We say that *a node* Q *has* σ*-order* i (with respect to $\mathcal{P}$) if the distance between Q and $\mathcal{P}$ is $i|\overline{h}|$.

Let us consider the values of the unknown functions $f_1,\dots,$ f_n and their partial derivatives of order at most r at the nodes of σ-order at most s (r and s are positive integers). With the notation of section 2, we can say that we consider the values $\tau f_i(\mathcal{P})$ where $\tau \in T$, $\mathrm{ord}_\Delta \tau \le r$ and $\mathrm{ord}_\sigma \tau \le s$.

If $f_1,\dots,f_n$ should not satisfy any system of equations (or any other condition), these values can be chosen arbitrarily. Because of the system (and equations obtained from the equations of the system by partial differentiations and translations in the direction $\overline{h}$, the number of independent values of the functions $f_1,\dots,f_n$ and their partial derivatives whose order does not exceed r at the nodes of σ-order at most s decreases. This number, which is a function of two variables, r and s, is the "measure of strength" of the system in the sense of A. Einstein. We denote it by S_{rs}. Suppose that the Δ-σ-ideal P generated in $K\{y_1,\dots,y_n\}$ by the Δ-σ-polynomials $A_1,\dots,A_q$ is prime (e. g., the polynomials are linear). Then we say that the system of difference-differential equations (1) is prime. In this case, the Δ-σ-dimension polynomial $\psi_P(t_1,t_2)$ has the property that $\psi_P(r,s) = S_{rs}$ for all sufficiently large $(r,s) \in \mathbb{N}^2$, so this dimension polynomial is the measure of strength of the system of difference-differential equations (1) in the sense of A. Einstein.

An important perspective for the use of the obtained results is the computation of dimension polynomials (and therefore the Einstein's strength) of differential equations with delay that arise in applications. Examples of the corresponding computation in the differential and inverse difference cases can be found in [8, Chapters 6 and 9]. Computations of the same kind in the non-inverse case (based on the results of this paper) is a subject for future work.

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