2017 IEEE 56th Annual Conference on Decision and Control (CDC)
December 12-15, 2017, Melbourne, Australia

Tighter Lower Bounds on the Error Variance of Pole and Residue Estimates from Impulse Response Data: an Expository Example

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Abstract— **The estimation of nonrandom pole and residue parameters from impulse-response data is revisited. Specifically, for an expository example (a one-pole discrete-time system), the Hammersley-Chapman-Robbins lower bound (HCRB) on the estimation error variance is derived, and compared with the widely-used Cramer-Rao bound (CRB). The HCRB is found to be significantly tighter than the CRB over a range of parameter values. Simplifications of the HCRB which admit analytical expressions but are guaranteed to outperform the CRB are also derived. The results indicate that CRB-based confidence intervals for pole-residue estimates, which are being used in several mode monitoring applications, need to be examined with caution.**

I. INTRODUCTION AND PROBLEM FORMULATION

Parameter estimation for dynamical systems has been extensively studied. Within this broad literature, one focus has been on the estimation of the poles and residues of linear time-invariant models, which are represented as unknown (nonrandom) parameters, from noisy impulse-response data [1], [2], [3], [4], [5], [6], [7]. Along with the development of algorithms for estimation, bounds on the estimation performance also have been obtained. Particularly, there has been a considerable effort to compute Cramer-Rao lower bound (CRB) on the error variance of pole and residue estimates, for the case where the the observations are subject to additive white Gaussian noise [1], [2], [3]. These bounds are important because they give an indication of the practicality of estimation, regardless of the estimator used. Recently, the CRBs have found specific application in mode (pole/residue) monitoring in complex infrastructures, such as monitoring the time constants of fast power-system dynamics from synchrophasor data, and estimating resonance phenomena in flexible structures [8], [9], [10]. In these applications, the CRBs are being used to give confidence intervals around pole estimates, which can aid infrastructure operators in gauging estimate fidelity in taking corrective actions. The bounds are also being exploited to support sensor placement.

The CRBs on pole and residue estimates from impulse response data are not guaranteed to be tight. In particular, the problem of mode estimation from impulse response data does not generally satisfy the regularity conditions which are needed to guarantee optimality of the maximum-likelihood estimate and tightness of the CRB. The possible gap between the CRB and best possible estimation performance is a significant concern in the mode-monitoring applications of current interest, in that the confidence intervals on the estimates may be misrepresented. Motivated by this concern, in this work we develop the tighter Hammersley-Chapman-Robbins bound (HCRB) on pole and residue estimates from impulse response data [11], [12], for the simplest expository example (a stable discrete-time one-pole system). Using this analysis, the gap between the CRBs and HCRBs is characterized in terms of the parameters of the model (the pole and residue, the noise level, and the number of observations used). It is found that the HCRB significantly improves on the CRB over a wide range of parameter values, indicating the importance of developing tighter lower bounds on mode estimates. Even for the single-pole system, the HCRB computation is rather intricate, and developing tractable generalizations for more sophisticated models is challenging; we thus also pursue a simplification which is still guaranteed to outperform the CRB, but is computationally much more appealing.

A. Relevant Literature

Estimation of nonrandom signal or system parameters from noisy observation data has been a focus of the controls, signal processing, and statistics literatures. The research presented here contributes to the performance analysis of such estimators. Though a number of different analyses and bounds have been developed, a particular focus has been on the Cramer-Rao lower bound (CRB), because it is relatively simple to compute yet has desirable theoretical properties (e.g., tightness guarantees) in some settings. As a starting point, CRBs on parameter estimates for mixtures of exponentials (whether damped or undamped, complex or real) in white noise have been developed in the classical signal processing and communications literature. These analyses have been extended to encompass quasi-polynomial signals, colored noise, and multiplicative noise, among other features [13], [14], [15]. In parallel, the estimation of autoregressive moving-average (ARMA) model parameters from noisy impulse-response data and ambient-noise-driven responses has been studied, and CRBs have been developed [16], [17]. As a part of this effort, CRBs have been developed on particularly on estimates of poles and residues, or alternately poles and zeros [1]. The study [1] also explores the dependence of the CRB on pole and zero locations, while [2] develops graph-structural results for estimation performance for linear dynamics defined on a network. CRBs have also

This work was supported by United States National Science Foundation and Lawrence Berkeley National Laboratory

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978-1-5090-2873-3/17/$31.00 ©2017 IEEE

been developed for two-dimensional modal analysis problem [3] and for the multi-observation setting [18]. Relative to these efforts, the main contribution of this study is to evaluate the tightness of the CRB relative to the HCRB in an expository example, and to pursue the development of lower bounds that are tighter yet tractable.

The research presented here is also closely related to the effort on the Hammersley-Chapman-Robbins bound (HCRB), which provides a tighter lower bound on nonrandom parameter estimates than the CRB [11], [12] but has been applied in a more limited way due to computational challenges. Of note, the HCRB has been applied to threshold prediction in direction-of-arrival (DOA) estimation and source localization, since the CRB does not provide tight bounds in the case of low SNR and limited data points in these contexts [19], [20]. The HCRB has also recently been used for estimating sparse non-random vectors in the presence of Gaussian white noise [21], [22]. Likewise, the HCRB is used for estimation of multiple change points in time series, since the change-point location parameters are discrete and the CRB is not applicable [23]. The HCRBs for pole and residue estimates developed here are similarly motivated. Specifically, the regularity condition required for the CRB to be tight are not generally satisfied for the mode estimation problem, hence the gap between the CRB and the optimal estimator performance needs to be characterized, and better bounds are desirable. This is the focus of the paper.

B. Problem Formulation

A single-pole discrete-time system, with transfer function $H(z) = C/(1 - az^{-1})$, is considered. The nonrandom parameters a and C are the pole and residue of the system, respectively. Noisy measurements are made of the system's response at the times $k = 0, \ldots, n$, upon impulsive stimulation at time $k = 0$. Specifically, the measured impulse response is given by

$$y(k) = Ca^k + w(k) \tag{1}$$

for $k = 0, \ldots, n$, where Ca^k is the true impulse response of the system, and $w(k)$ is a zero-mean Gaussian white noise with variance σ^2.

The focus of this study is on estimation of the nonrandom parameters a and C from the measured impulse response. Specifically, lower bounds on the parameter estimation error (error variance) are established and characterized. The following are the main analyses pursued in the article:

- Expressions for the HCRBs on the pole and residue estimates are determined. The bounds are first developed for the cases that the pole or the residue are individually estimated (the other parameter is known), and then the joint estimation of both parameters is considered.
- The HCRB is compared with the CRB, as a function of the pole location, residue value, noise variance, and number of observations.
- The HCRB is computationally intensive to find and does not admit fully analytical characterizations, because it requires searching over the parameter space for extremal values of a function. To overcome these limitations, a simpler lower bound is also extracted from the HCRB computation, which is tighter than the CRB but has an explicit form.

II. REVIEW OF THE HCRB

The single and multiparameter versions of the HCRB are briefly reviewed. The HCRB is a lower bound on the error variance that can be achieved by any unbiased estimator of a nonrandom (deterministic, unknown) parameter. This bound is quite important since it can provide a measure of the efficiency for estimators, and also provide confidence intervals on the reliability of any obtained estimates.

Formally, consider estimation of a single nonrandom parameter θ based on a set of observations $\mathbf{x} = \left[\mathbf{x_1}, \mathbf{x_2}, \cdots, \mathbf{x_n}\right]$, which are random variables generated according to the joint probability density function $f(\mathbf{x_1} = x_1, \mathbf{x_2} = x_2, \cdots, \mathbf{x_n} = x_n; \theta)$ or succinctly $f(\mathbf{x}; \theta)$. An unbiased estimator $T(\mathbf{x})$ of the parameter θ from the observations is considered. The HCRB provides a lower bound on the estimation error variance $Var\{T(\mathbf{x})\} = E\{(T(x) - \theta)^2\}$, that holds for any unbiased estimator.

The HCRB on the estimation error variance denoted $HCRB_\theta$ is given by [11]

$$Var\{T(\mathbf{x})\} \geq \sup_h \left(\frac{1}{E_\theta\{J_\theta\}}\right) = HCRB_\theta, \tag{2}$$

where

$$J_\theta = \frac{1}{h^2}\{[\frac{f(\mathbf{x}; \theta + h)}{f(\mathbf{x}; \theta)}]^2 - 1\} \tag{3}$$

Here, the notation $E_\theta\{\cdot\}$ means the expectation is taken with respect to $\mathbf{x}$, with the result parameterized by θ.

Next, consider an unbiased estimator $\mathbf{T}(\mathbf{x})$ for a nonrandom parameter vector $\boldsymbol{\theta} = [\theta_1 \ \theta_2 \cdots \theta_k]^T \in \mathbb{R}^k$ based on a set of observations $\mathbf{x} = \left[\mathbf{x_1}, \mathbf{x_2}, \cdots, \mathbf{x_n}\right]$. The observations are modeled as random variables generated according to the joint probability density function $f(\mathbf{x_1} = x_1, \mathbf{x_2} = x_2, \cdots, \mathbf{x_n} = x_n; \boldsymbol{\theta})$ or succinctly $f(\mathbf{x}; \boldsymbol{\theta})$. The HCRB provides a lower bound on the covariance matrix $COV\{\mathbf{T}(\mathbf{x})\} = E\{[\mathbf{T}(\mathbf{x}) - \boldsymbol{\theta}][\mathbf{T}(\mathbf{x}) - \boldsymbol{\theta}]^T\}$, where the diagonal entries of the covariance matrix are the estimation error variances for each parameter $\theta_i, \ i = 1, 2, \cdots, k$.

The multiparameter version of unconstrained HCRB on the covariance matrix denoted $HCRB_{\boldsymbol{\theta}}$ is given by [24]

$$COV\{\mathbf{T}(\mathbf{x})\} \geq \sup\left(\mathbf{I}_{HCRB}{}^\dagger\right) = HCRB_{\boldsymbol{\theta}} \tag{4}$$

where

$$\mathbf{I}_{HCRB} = [\boldsymbol{V}] \ E_{\boldsymbol{\theta}}\left\{[\frac{\delta f_{\boldsymbol{\theta}}}{f_{\boldsymbol{\theta}}}]^T \ [\frac{\delta f_{\boldsymbol{\theta}}}{f_{\boldsymbol{\theta}}}]\right\}[\boldsymbol{V}]^T \tag{5}$$

Here $[\boldsymbol{V}]$ is the concatenation of the direction vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots \mathbf{v}_k \in \mathbb{R}^k$, and $\delta f_{\boldsymbol{\theta}}$ is the concatenation of the finite differences of density functions due to changes in the parameter $\boldsymbol{\theta} \in \mathbb{R}^k$ in the directions of $\mathbf{v}_1, \mathbf{v}_2, \cdots \mathbf{v}_k \in \mathbb{R}^k$. Specifically,

$$[\boldsymbol{V}] = [\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \cdots \ \ \mathbf{v}_k] \tag{6}$$

$$\delta f_{\boldsymbol{\theta}} = [\delta_1 f_{\boldsymbol{\theta}} \;\; \delta_2 f_{\boldsymbol{\theta}} \;\; \cdots \;\; \delta_k f_{\boldsymbol{\theta}}] \tag{7}$$

$$\delta_i f_{\boldsymbol{\theta}} = \frac{f_{\boldsymbol{\theta}+h_i \mathbf{v}_i} - f_{\boldsymbol{\theta}}}{h_i}, \qquad i = 1, 2, \cdots, k \tag{8}$$

Here $h_1, h_2, \cdots, h_k$ are scalars. Note that, in (4) $\{\cdot\}^\dagger$ denotes Moore-Penrose pseudo inverse, and the supremum is taken over all possible direction vectors and respective magnitude scalars.

A brief comparison of the HCRB with the widely-used CRB is worthwhile. The HCRB does not require any of the regularity assumptions of the CRB, but does require the weak condition that the support of $f(\mathbf{x};\boldsymbol{\theta} + h_i\mathbf{v}_i)$ is subset of the support of $f(\mathbf{x};\boldsymbol{\theta})$ for $i = 1, 2, \cdots, k$. Furthermore, the HCRB is as tight as the CRB, with the two bounds coinciding when the supremum is achieved at $h_i \to 0$ for all i with the direction vectors taken as the unit vectors on $\mathbb{R}^k$ [11].

III. HCRB Analysis for the Single-Pole System

HCRBs are obtained on pole and residue estimates from impulse response data given by (1) and compared with the CRBs to get insight into the gap between the bounds. Simplified bounds are also developed. The analysis is undertaken first for the cases where only the pole or only the residue needs to be estimated (Sections III-A and III-B), and then the joint estimation of both parameters is considered. Due to space constraints, proofs of the theorems have been removed, see [25].

A. HCRB Analysis for Pole Estimation

First the HCRB is characterized in the case that only the pole needs to be estimated (i.e., the residue is known), which we refer to as the *pole-only estimation problem*. The following theorem gives an expression for the HCRB:

Theorem 1 *For the pole-only estimation problem, the HCRB on the pole estimate is given by $HCRB_a = \sup_{h_1}(Z)$ where*

$$Z = \frac{1}{\frac{1}{h_1^2}[\exp(\frac{C^2}{\sigma^2}S_1) - 1]}, \tag{9}$$

$$S_1 = \sum_{k=0}^{n}(p^{2k} - 2q^k + a^{2k}) = \frac{1-p^{2n+2}}{1-p^2} - \frac{2(1-q^{n+1})}{1-q} + \frac{1-a^{2n+2}}{1-a^2}, \tag{10}$$

and $p = (a + h_1)$, $q = a(a + h_1)$. □

Several remarks about the result are worthwhile. First, it is worth noting that Z serves as a lower bound on the estimator error variance for any value of h_1, and the HCRB is the greatest of all such lower bounds over h_1. Unfortunately, the optimization to find the HCRB does not admit an analytical treatment. However, the minimum value of $\frac{1}{h_1^2}[\exp(\frac{C^2}{\sigma^2}S_1) - 1]$, which corresponds to the supremum of Z, can readily be found numerically. One easy way to do so is to take the logarithm of this term and differentiate it with respect to h_1, which gives

$$\phi_1(h_1) = \frac{-2}{h_1} + \frac{e^{\frac{C^2}{\sigma^2}S_1}}{\exp(\frac{C^2}{\sigma^2}S_1) - 1}\frac{C^2}{\sigma^2}(S_2 - aS_3) \tag{11}$$

where $S_2 = \sum_{k=0}^{n} kp^{2k-1} = \frac{p-p^{2n+1}+np^{2n+3}-np^{2n+1}}{(1-p^2)^2}$ and $S_3 = \sum_{k=0}^{n} kq^{k-1} = \frac{1-q^n-nq^n+nq^{n+1}}{(1-q)^2}$. Solving the equation $\phi_1(h_1) = 0$ yields the value of h_1 which minimizes $\frac{1}{h_1^2}[e^{\frac{C^2}{\sigma^2}S_3} - 1]$, and hence supremizes Z in the HCRB expression. Alternately, one can also sweep over h_1 to find the supremum value.

Because the HCRB does not admit an explicit analytical expression, it is appealing to develop simpler bounds that are less tight than the HCRB, but outperform the CRB. Such bounds make computation easier, give insight into the dependence of estimator performance on the location of the pole, and perhaps provide a route toward developing improved lower bounds for more general systems. One way to develop simpler lower bounds is to evaluate Z for particular values of h_1. It turns out that choosing $h_1 = -a$ yields a lower bound [1] that: 1) is guaranteed to be tighter than the CRB over a range of parameter values, and 2) performs well in practice in that it is close to the HCRB (see simulations section). The following theorem formalizes that this bound improves on the CRB:

Theorem 2 *Consider the lower bound Q on the estimation error variance obtained by evaluating Z as h_1 approaches $-a$, $Q = \lim_{h_1 \to -a} Z$. If either a or C is sufficiently small in magnitude, it follows that Q is larger than the CRB on the pole estimate, i.e. $Q > CRB_a$.* □

Since the CRB and the lower bound Q in Theorem 2 are both easy to compute, a simplified bound that is always tighter than the CRB can be readily defined. Specifically, we define the simplified HCRB (SHCRB) as

$$SHCRB_a = max(Q, CRB_a) \tag{12}$$

From the definition, the SHCRB is clearly a lower bound on the estimation error variance, and is at least as tight as the CRB (and is strictly tighter if either the pole or the residue is sufficiently small, per Theorem 2). The simulation results presented later show that the bound performs well over a wide range of parameter values.

B. HCRB analysis for residue-only estimation

The HCRB on the residue estimate in the case that the pole is known, which we call *residue-only estimation*, is considered. In this case, the CRB and HCRB are identical. Further, the bounds are tight in the sense that the maximum-likelihood estimator of the pole achieves the bound. Verifying this simply requires noting that the measurements $y[k]$ are a linear function of the parameter to be estimated (the residue C), subject to additive Gaussian noise with fixed variance. Drawing on standard results for the CRB in the linear Gaussian case, it follows that the CRB is achieved by the maximum liklihood estimate. Since the HCRB is guaranteed to be tighter than the CRB, it follows also that the two bounds are identical. For the sake of completeness, it is helpful to

[1] Since S_1 becomes undefined when $h_1 = -a$ according to (10), so we take $h_1 \to -a$ instead.

present the expression for the CRB/HCRB in this case, as is done in the following lemma:

Lemma: *For the residue-only estimation problem, the HCRB and CRB on the estimation error variance are identically given by:* $CRB_C = HCRB_C = \frac{\sigma^2}{S_6}$, *where* $S_6 = \sum_{k=0}^{n} a^{2k} = \frac{1-a^{2n+2}}{1-a^2}$. □

C. HCRB Analysis for Joint Pole and Residue Estimation

Joint estimation of the pole and residue assuming that both parameters are unknown is considered. The following theorem gives an expression for the HCRB, for the joint estimation problem.

Theorem 3 *For the joint estimation problem, the HCRB is* $HCRB_{a,C} = \sup_{h_1,h_2} \left(\mathbf{I}_{HCRB}{}^{\dagger} \right)$ *where* $\mathbf{I}_{HCRB}$ *is a* 2×2 *matrix with the following entries:*

$$I_{11} = \frac{1}{h_1^2}[\exp(\frac{C^2}{\sigma^2}S_1) - 1] \quad (13)$$

$$I_{22} = \frac{1}{h_2^2}[\exp(\frac{h_2^2}{\sigma^2}S_6) - 1] \quad (14)$$

$$I_{12} = I_{21} = \frac{1}{h_1 h_2}[\exp(\frac{Ch_2}{\sigma^2}S_7) - 1] \quad (15)$$

where S_1 *is given by (10) and*

$$S_6 = \sum_{k=0}^{n} a^{2k} = \frac{1-a^{2n+2}}{1-a^2}, \quad (16)$$

$$S_7 = \sum_{k=0}^{n}(q^k - a^{2k}) = \frac{1-q^{n+1}}{1-q} - \frac{1-a^{2n+2}}{1-a^2}, \quad (17)$$

and, $p = (a + h_1)$, $q = a(a + h_1)$. □

Individual HCRBs for pole and residue estimation errors can be found by taking the inverse of $\mathbf{I}_{HCRB}$ and considering the diagonal entries. This yields $HCRB_{a;a,C} = \sup_{h_1,h_2}\left(\frac{I_{22}}{I_{11}I_{22}-I_{12}^2}\right)$ and $HCRB_{C;a,C} = \sup_{h_1,h_2}\left(\frac{I_{11}}{I_{11}I_{22}-I_{12}^2}\right)$. We notice that $Z_1 = \left(\frac{I_{22}}{I_{11}I_{22}-I_{12}^2}\right)$ and $Z_2 = \left(\frac{I_{11}}{I_{11}I_{22}-I_{12}^2}\right)$ serve as lower bounds for pole and residue estimation errors, respectively, for any values of h_1 and h_2. The HCRBs are the greatest of these lower bounds over h_1 and h_2.

As before, the HCRBs do not admit analytical solutions, and finding the supremum over the two variables h_1 and h_2 may also be computationally expensive. Hence, it is again appealing to develop simpler bounds that are less tight than the HCRB, but outperform the CRB. One way to develop simpler lower bounds is to evaluate Z_1 and Z_2 for particular values of h_1 and h_2. In this case, choosing $h_1 = -a$ and $h_2 = 0$ yields lower bounds [2] that are guaranteed to be tighter than the CRBs over a range of parameter values, and perform well in practice in that they are close to the HCRBs (see simulations section). The following theorem formalizes that these bounds improve on the CRBs:

[2] Actually we take the limits $h_1 \to -a$ and $h_2 \to 0$ instead of choosing $h_1 = -a$ and $h_2 = 0$, to avoid obtaining undefined expressions.

Theorem 4 *For the joint estimation problem, consider the lower bounds on the estimation error variances of pole and residue obtained by evaluating* Z_1 *and* Z_2*, respectively, as* h_1 *approaches* $-a$ *and* h_2 *approaches* 0*, i.e.* $Q_a = \lim_{h_1\to -a,\ h_2\to 0} Z_1$ *and* $Q_C = \lim_{h_1\to -a,\ h_2\to 0} Z_2$. *If either* a *or* C *is sufficiently small in magnitude, it follows that* Q_a *and* Q_C *are larger than their respective CRBs, i.e.* $Q_a > CRB_{a;a,C}$ *and* $Q_C > CRB_{C;a,C}$ □

Since the CRBs and the lower bounds Q_a and Q_C in Theorem 4 are easy to compute, a simplified bound that is always tighter than the CRBs can be readily defined. Specifically, we define the simplified HCRBs (SHCRBs) for joint estimation problem as $SHCRB_{a;a,C} = max(Q_a, CRB_{a;a,C})$ and $SHCRB_{C;a,C} = max(Q_C, CRB_{C;a,C})$.

From the definition, the SHCRBs are clearly lower bounds on the estimation error variances, and are at least as tight as the CRBs (and are strictly tighter if either the pole or the residue is sufficiently small, per Theorem 4). The simulation results presented later show that the bounds perform well over a wide range of parameter values. It is worthwhile to note that the improvement provided by $SHCRB_{a;a,C}$ will be larger than that of $SHCRB_{C;a,C}$, if either the residue or pole is small.

IV. Numerical Computations and Discussion

Numerical computations of the CRB, HCRB, and SHCRB are undertaken, to gain further insight into the gaps between the bounds and their dependencies on the pole and residue locations. The bound computations are undertaken for poles in the range $0 < a < 1$ and residues in the range $0 < C < \infty$; we observe that the bounds are symmetric for negative values of a and C. For the simulations, a long time horizon ($n = 10000$) was used, and the noise level was assumed to be $\sigma = 0.5$ if not stated otherwise.

Bounds for the pole-only estimation problem are shown as a function of the pole for three different residue values, in Figures 1-3. The figures show that the bounds decrease monotonically with the increase of the pole and also the residue. Also, they become almost equal when the pole is close to 0 or 1. For other values of the pole, there is a significant gap between HCRB and CRB for small residue values, but the gap decreases as the residue increases. The proposed simplified bound (SHCRB) improves significantly the CRB (equivalently, is close to the HCRB) when either the pole or the residue is small. From the formal expressions, we note that the CRB exhibits an inverse-square dependence on the residue; the HCRB has a more complicated dependence, which approaches the inverse-square dependence for larger residues. Meanwhile, the expressions show that the bounds increase with the noise variance, with the CRB having a proportional dependence while the HCRB shows a more complex relationship.

For the residue-only estimation problem, the HCRB and CRB are identical (and achieved by the ML estimator). The bound decreases with the pole location, as shown in Figure 4. The bound does not depend on the residue, but is proportional to the noise variance.

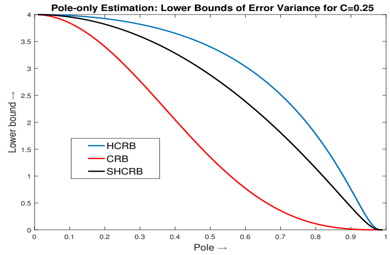

Fig. 1. Pole-only estimation: lower bounds as a function of the pole, for residue $C = 0.25$.

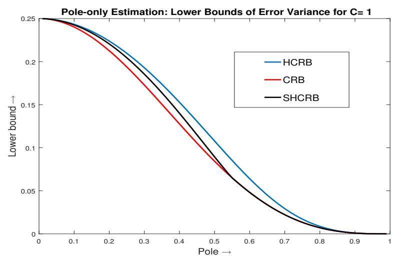

Fig. 2. Pole-only estimation: lower bounds as a function of the pole, for residue $C = 1$.

For the joint estimation case, the characteristics of the bounds for pole and residue estimates are coherent with the pole-only and residue-only cases, see Figures 5 and 6. The figures show that the bounds decrease with the increase of the pole and also with the residue. The bounds are nearly equal when the pole is nearly 0 or nearly 1. For other pole locations, there is a significant gap between HCRB and the CRB if the residue is small, but the gap becomes smaller for larger residues. The residue estimate shows a smaller gap between HCRB and CRB than the pole estimate, but the gap is non-zero. Meanwhile, the SHCRB significantly improves on the CRB for both pole and residue estimation, for the small values of poles or residues. Increasing noise power increases the lower bound of variances, as expected. The bounds are higher for the pole-only or residue-only estimation for the same system, as expected.

Figures 7 and 8 show the dependency of the bounds for very small set of observations ($n = 5$, $\sigma = 0.5$) and at high noise ($n = 10000$, $\sigma = 5$) respectively for $C = 0.25$. As we see from comparing Figures 5 and 8 for high noise, the gap between CRBs and HCRBs gets significantly larger and SHCRBs improve on CRBs for a wider range of pole values. And from comparing Figures 5 and 7 we see for very small set of observations, the gap and improvement are significant for the pole values around 1 and in fact unlike the asymptotic case or very large set of observations, CRBs and HCRBs are significantly different when the pole is nearly 1.

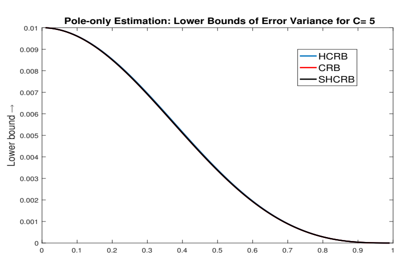

Fig. 3. Pole-only estimation: lower bounds as a function of the pole, for residue $C = 5$.

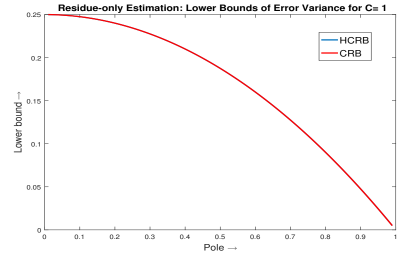

Fig. 4. Residue-only estimation: lower bounds as a function of the pole, for residue $C = 1$.

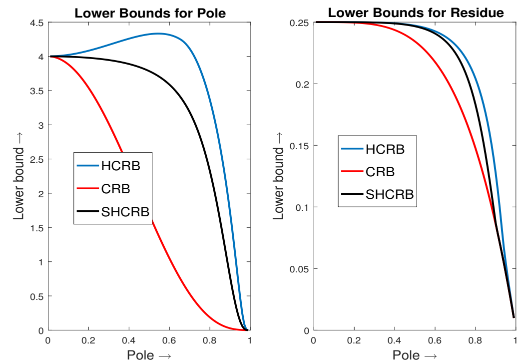

Fig. 5. Joint estimation: lower bounds on the pole and residue estimates as a function of the pole, for $C = 0.25$.

V. CONCLUSIONS

Parameter estimation for a one-pole discrete-time linear system from noisy impule-response data has been considered. The Hammersley-Chapman-Robbins lower bound (HCRB) has been derived, and compared with the Cramer-

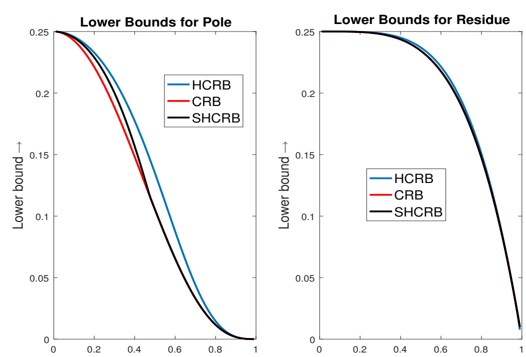

Fig. 6. Joint estimation: lower bounds on the pole and residue estimates as a function of the pole, for $C = 1$.

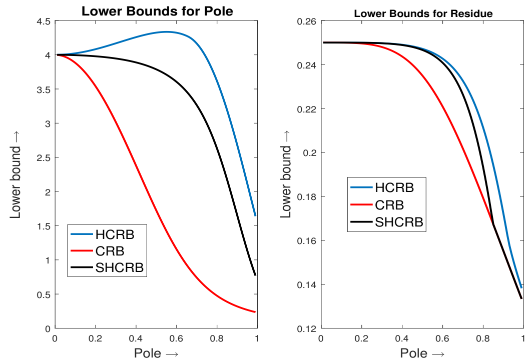

Fig. 7. Joint estimation: lower bounds on the pole and residue estimates for very low set of observations ($n = 5$)

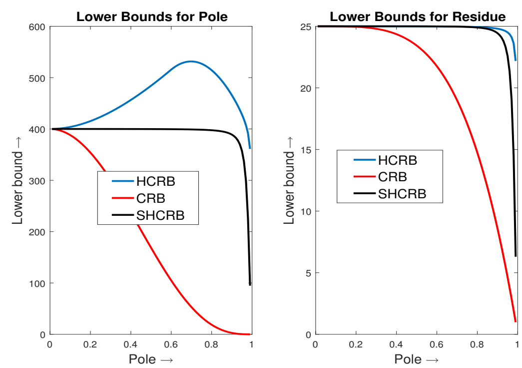

Fig. 8. Joint estimation: lower bounds on the pole and residue estimates for high noise ($\sigma = 5$)

Rao lower bound (CRB) which is often used to present confidence intervals on pole/residue estimates. A simplification of the HCRB which works well over a range of parameter values has also been developed. As a whole, the analyses and associated numerical simulations demonstrate that there is a significant gap between the HCRB and CRB, for a wide range of parameter values. These results suggest that caution is required in using the CRB to present confidence intervals on pole/residue estimates, or design sensing schemes to achieve small bounds. In particular, estimates of dominant (slow) poles may be significantly worse than indicated by the CRBs. Given the gap between the bounds, there is a strong motivation to develop improved bounds for general linear systems, and also to ascertain whether the HCRB is in fact tight; these are possible directions of future work.

Acknowledgements: The authors gratefully acknowledge the support of the United States National Science Foundation under grants CNS-1545104 and CMMI-1635184, and of Lawrence Berkeley National Laboratory.

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