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Local Control and Estimation Performance in Dynamical Networks: Structural and Graph-Theoretic Results

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***Abstract*— This work examines metrics for *target reachability* and *source observability* in dynamical networks, which are especially relevant in a network security context. Specifically, the energy required to control a target node in a network from a remote input is characterized, and dually the fidelity with which a source state can be estimated from a remote measurement is studied. The work highlights an essential asymmetry between the problems: We show that target reachability is often easy, while source observability is almost always impossible. Several spectral and graph-theoretic results are also presented, which give structural insight into how easy or hard target control and source estimation are.**

I. INTRODUCTION

The integration of cyber-technologies into terrestrial-scale infrastructure networks is providing diverse stakeholders with unprecedented access to the networks' sensors and actuators [4], [13], [18], [28]. The growing access to infrastructure sensors and actuators has stimulated an interdisciplinary research effort on security and resilience of complex cyber-physical infrastructures [7], [8], [12], [20], [24], [26], [30]. As part of this effort, controls engineers have sought to characterize the systemic impacts of local actuation and measurement of network dynamics [17], [21], [23], [25], [29], [33]. Specific research thrusts include characterizing observability and controllability of canonical linear network dynamics, and exploring structural representations of manifest variables in dynamical networks [1], [5], [6], [11], [32], [34], [35]. More recently, graph-theoretic characterizations have also been obtained for the effort required to manipulate the networks dynamics, and dually the fidelity of state estimation or detection from noisy measurements, [9], [21], [29], [33]. These results also connect to an earlier literature on the spectral analysis of Gramian matrices in the controls-engineering and physics literatures [2], [19], [22].

The research described here is also concerned with analyzing the ease with which network dynamics can be monitored and manipulated from local measurements and actuations, from a graph-theoretic perspective. Rather than considering observability and controllability in a global sense, however, we focus here on targeted or local monitoring and management tasks. Such ;ocal estimation and manipulation tasks are often paradigmatic in large-scale infrastructures, given the specialized goals and practical resource and information limits of stakeholders. In this study, the ease or difficulty of local estimation and manipulation of network processes is characterized, in the context of a canonical discrete-time linear dynamics defined on a graph which is actuated at a single node and measured at a single node. The main goal of this work is to develop such structural and graph-theoretic insights into the ease of local estimation and manipulation, which can . The research described here is aligned with a few very recent works on targeted control in networks [10], [15], [31]. Relative to these works, our study contributes by: 1) considering local estimation in addition to control and 2) developing structural results on the estimation/control performance metrics. This study connects to a wide literature on output controllability and partial-state observability [14], [16], but pursues network-theoretic analyses.

A main contribution of this article is to highlight that estimation of local states is fundamentally a more difficult task than localized control. This comparison is further refined to show that, for networks of large size, low-effort targeted control of all individual states may be possible but efficient estimation of all individual states is never possible for symmetric topologies. The article also develops a number of spectral and graph-theoretic results on local estimation and control.

II. PROBLEM FORMULATION

This study is concerned with the energy required for control and the effectiveness of estimation from noisy measurements in dynamical networks. Specifically, we focus on local notions of reachability and observability, which we call *target* reachability and *source* observability. *Target reachability* considers the ability of an input to drive a particular network node's state, i.e. an element in the state vector, to a value[1]. *Source observability* considers the ability to estimate the value of one particular network node from a measurement. Interestingly, our analysis will show that target reachability is often easy (requiring little energy), while source observability is often hard (entailing significant estimation error. At its essence, the asymmetry between the

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[1]We note that target reachability is a specialization of the output reachability/controllability concept, to the case where a single state variable is being driven [3], [16]. However, out interest here is in relating this to the graph.

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concepts arises because ease of is promoted by *reachable* subspaces, while estimation fidelity is governed by *unobservable* subspaces.

Formally, a network with n nodes labeled $i = 1, \dots, n$ is considered, in which each node i has a scalar state $x_i[k]$ that evolves in discrete time. The network's full state $\mathbf{x}[k] = \begin{bmatrix} x_1[k] & \cdots & x_n[k] \end{bmatrix}^T$ is governed by a stable discrete-time linear dynamical model with state matrix A. Further, the dynamics are amenable to actuation at one node s, which we call the source node; and measurement at a second node t, which we call the target node.

The first problem of interest is to characterize the effort required to manipulate the state at the target node to a desired value by designing the input at the source node. We call this the target control (TC) problem. The dynamical model considered for the TC problem is:

$$\mathbf{x}[k+1] = A\mathbf{x}[k] + \mathbf{e}_s u[k], \tag{1}$$

where $\mathbf{e}_s$ is a 0–1 indicator vector with entry s nonzero, and the scalar $u[k]$ is the designable input. The network is assumed to be initially relaxed, i.e. $\mathbf{x}[0] = \mathbf{0}$. The input $u[k]$ is to be designed to move the state at the target node to a unit value (without loss of generality) at time $\widehat{k}$, so that $x_t[\widehat{k}] = 1$ or equivalently $\mathbf{e}_t^T \mathbf{x}[\widehat{k}] = 1$ where $\mathbf{e}_t$ is a 0–1 indicator vector with entry t nonzero. Our main interest is to characterize the minimum-energy input, measured in a two-norm sense, required to achieve this goal. That is, the targeted-control effort over a horizon $\widehat{k}$ is computed as $E(\widehat{k}) = min_{u[0],\dots,u[\widehat{k}-1]} \sum_{k=0}^{\widehat{k}-1} u^2[k]$, subject to the the goal state being achieved ($x_t[\widehat{k}] = 1$). We primarily focus on characterizing this minimum energy when ample time is available for control, i.e. we seek to characterize $E = \lim_{\widehat{k}\to\infty} E(\widehat{k})$. This infinite-horizon target-control energy lower bounds the energy required over a finite horizon. It is important to stress that our formulation of the TC problem places no requirements on any network states except the target state.

The second problem of interest is to characterize the fidelity with which the initial state at the source node can be estimated from noisy measurements taken from the target node; we call this the source estimation (SE) problem. The dynamical model considered for the SE problem is:

$$\begin{aligned} \mathbf{x}[k+1] &= A\mathbf{x}[k] \\ y[k] &= \mathbf{e}_t^T \mathbf{x}[k] + N[k], \end{aligned} \tag{2}$$

where $N[k]$ is a zero-mean unit-variance white Gaussian noise signal. The initial state $\mathbf{x}[0]$ is assumed to be a unknown, nonrandom vector. Our interest is in determining the highest fidelity with which the initial state of the source node, i.e. $x_s[0]$ or $\mathbf{e}_s^T \mathbf{x}[0]$, can be estimated from the sequence of observations $y[0], \dots, y[\widehat{k}-1]$. Specifically, the estimation fidelity $F(\widehat{k})$ is computed as the minimum achievable mean-square error in the estimate among unbiased estimators of the source node's state. In analogy with the target control problem, we primarily focus on the setting where ample data is available, i.e. we seek to characterize $F = \lim_{\widehat{k}\to\infty} F(\widehat{k})$.

The effort required for target control, and the fidelity of source estimation, are dependent on the topology of the network. To develop topological results, it is convenient to associate a graph Γ with the state matrix of the dynamical model. Specifically, we define Γ to be a weighted digraph with n vertices labeled $1, \dots, n$, which correspond to the n nodes in the network. An edge is drawn from vertex i to vertex j in the graph (where i and j are not necessarily distinct) if A_{ji} is non-zero. The presence of the edge indicates that the next state of vertex j depends on the current state of vertex i. We note that self-loops (edges from a vertex back to itself) are acknowledged, and also that edge weights may be of either sign.

Three types of analyses are undertaken for the TC and SE problems. First, algebraic expressions for the control effort and estimation fidelity are reviewed, which are basic consequences of the standard linear system theory. These algebraic expressions also immediately yield a simple comparison between the control effort and estimation fidelity – specifically, that SE is more difficult than TC. Second, spectral and graph-theoretic results on the control effort are obtained, which primarily demonstrate that TC is practical for many networks. Third, spectral and graph-theoretic results on the estimation fidelity are obtained, which show that SE is very often impractical in large networks.

III. Preliminary Algebraic Expressions and Comparison

Algebraic expressions for the target-control effort and source-estimation fidelity follow readily from the standard analysis of linear systems [27]. The following two lemmas give algebraic expressions for the target-control effort and source-estimation fidelity, respectively (we omit the proofs since both results can be easily derived from classic linear system theory):

Lemma 1: The minimum energy required for target control over the horizon $\widehat{k}$ is $E(\widehat{k}) = \frac{1}{\sum_{k=0}^{\widehat{k}-1} (\mathbf{e}_t^T A^k \mathbf{e}_s)^2}$.

Lemma 2: The minimum unbiased-estimator error variance (optimal fidelity) of source estimation over the horizon $\widehat{k}$ is $F(\widehat{k}) = \mathbf{e}_s^T \left(G_o(0, \widehat{k}) \right)^{-1} \mathbf{e}_s$, where the observability Gramian $G_o(0, \widehat{k})$ is given by $G_o(0, \widehat{k}) = \sum_{k=0}^{\widehat{k}-1} (A^T)^k \mathbf{e}_t \mathbf{e}_t^T A^k$.

We notice that the estimation fidelity is the sth diagonal entry of the inverse of the observability Gramian, while the target-control energy is the inverse of sth diagonal entry of the controllability Gramian $G_c(0, \widehat{k}) = \sum_{k=0}^{\widehat{k}-1} A^k \mathbf{e}_s \mathbf{e}_s^T (A^T)^k$.

Although $E(\widehat{k})$ and $F(\widehat{k})$ are two different metrics (one is an input energy and the other is the estimation error variance), they both reflect the difficulty levels of the TC and SE problems, respectively. Therefore, it is instructive to compare the two metrics. This is done in the following theorem:

Theorem 1: For any network, source and target location, and horizon $\widehat{k}$, the target-control energy metric is majorized by the source-estimation fidelity metric, i.e. $E(\widehat{k}) \le F(\widehat{k})$.

The comparison given in Theorem 1 indicates that the canonical target control problem is never harder than the canonical source estimation problem, for a given network and source/target locations. This formal comparison aligns with the conceptual argument that local state estimation is hard compared to local state control, because local control only requires that one easily controllable direction has a projection at the target location while local estimation requires that no hard-to-estimate direction has a projection at the source location. While Theorem 1 establishes a basic comparison, however, it gives no insight into how hard local estimation and control are in an absolute sense, nor into the performance gap between them. The following results provide more detailed answers to these questions, and tie the performance to the network's graph.

IV. Structural and Graph-Theoretic Results on Target Control Effort

The purpose of this section is to develop some structural and graph-theoretic insights into the target control effort, so as to gain an understanding of what network characteristics make target control easy or hard. First, the minimum energy required for target control is expressed in terms of the spectrum (eigenvalues and eigenvectors) of the matrix A. For convenience of presentation, we focus on the case that the eigenvalues of A are not defective, i.e. each eigenvalue is in a Jordan block of size 1; similar results can be obtained for the defective case but are more cumbersome. The following notation is used for the spectrum of A. The eigenvalues are denoted as λ_i for $i = 1, \ldots, n$, and the right eigenvector of A associated with λ_i is denoted as $\mathbf{v}_i$. Meanwhile, the left eigenvector of A associated with λ_i is denoted as $\mathbf{w}_i^T$. We use the notation v_{ij} to indicate the jth entry in the right eigenvector $\mathbf{v}_i$, and similarly use w_{ij} for the jth entry in the left eigenvector $\mathbf{w}_i$.

Using eigenvalue decomposition, the matrix A can be written as $V\Lambda W^T$ where Λ is a diagonal matrix with $\Lambda_{ii} = \lambda_i$ for $i = 1, ..., n$, $V = \begin{bmatrix}\mathbf{v}_1 & \ldots \mathbf{v}_n\end{bmatrix}$, $W = \begin{bmatrix}\mathbf{w}_1 & \cdots & \mathbf{w}_n\end{bmatrix}$, and $W^T = V^{-1}$. Therefore, we can rewrite E from Lemma 1 as: $E = \left(\sum_{k=0}^{\infty}(\mathbf{e}_t^T V\Lambda^k W^T\mathbf{e}_s)^2\right)^{-1}$ Provided that the eigenvalues of A are inside the unit circle (because A is stable), the summation in the denominator converges, and hence the energy E can be rewritten as:

$$E = \left((\mathbf{e}_t^T V \begin{bmatrix} \ddots & & \\ & \frac{1}{1-\lambda_i} & \\ & & \ddots \end{bmatrix} W^T \mathbf{e}_s)^2 \right)^{-1}$$

With a little algebra effort, we can show that the minimum energy required for target control over the infinite horizon is

$$E \triangleq \lim_{\widehat{k}\to\infty} E(\widehat{k}) = \frac{1}{\sum_{i=1}^{n}(v_{it}w_{is}\frac{1}{1-\lambda_i})^2} \tag{3}$$

The above expression shows the finite-energy target control is possible if and only if at least one eigenvalue of the state matrix A is both controllable from the source location ($w_{is} \neq 0$) and observable from the target location ($v_{it} \neq 0$).

The spectral expression indicates that target control is easy (requires little energy) if the state matrix has an eigenvalue close to 1 whose left eigenvector has a large entry corresponding to the source location, and whose right eigenvector has a large entry corresponding to the target location. In fact, the required control energy is limited if there is any eigenvalue λ_i such that $v_{it}w_{is}$ is large, regardless of whether this eigenvalue is close to 1; this is because the magnitude of $1-\lambda_i$ is bounded by 2 no matter where in the unit circle λ_i is, which implies that the energy will be relatively small if $v_{it}w_{is}$ is large. This result confirms the intuition, developed in the pedagogical example, that target control only requires the ability to manipulate the target state via one controllable eigenvector.

In the following lemmas, the infinite-horizon target-control energy metric is characterized in terms of the network's graph, and the locations of the source and target relative to the graph. As a starting point, conditions for finite-energy control are given. In particular, the following lemma demonstrates that a directed path from the source to the target is necessary for finite-energy target control. Further, the condition is also sufficient if the state matrix A is nonnegative or essentially nonnegative. Here is the result:

Lemma 3: Finite-energy target control is possible only if the network graph Γ has a directed path from the source vertex to the target vertex. In the case that the state matrix A is nonnegative or Metzler, the condition is necessary and sufficient.

A similar result to the above theorem, which also addresses the case of multiple targets, was shown in [10].

Remark: If the network's graph has a unique directed path of minimum length from the source vertex to the target vertex, then finite-energy target control can be guaranteed even if the matrix A is not Metzler, since the product of edge weights along the unique directed path is necessarily nonzero.

In the next lemma, the effect on the target-control energy of changing an edge weight in the network graph is examined, for the case where the state matrix is nonnegative.

Lemma 4: Consider a network with nonnegative state matrix A. If the weight of any edge in the graph Γ is increased, or a new edge is added to graph, the minimum energy for target control $E(k)$ decreases or does not increase, for any horizon k.

Remark: If there is no path from the source node to the target node which includes the edge with the changed weight, the value of E remains unaltered after increasing the weight of that edge.

Finally, for the case of nonnegative A, an upper bound for the minimum required energy E can be derived in terms of the edge weights of paths between the source and target in the network graph Γ. This upper bound is introduced in the following lemma:

Lemma 5: Consider a network with nonnegative state matrix A, and suppose that there is at least one directed path from the source vertex to the target vertex in the network graph. Let us use the notation $(s, q_1, q_2, ..., q_r, t)$ for the path whose product of edge weights $A_{q_1 s}A_{q_2 q_1}...A_{tq_r}$ is largest among all paths between s and t. Then the energy required for target control over the infinite horizon is upper bounded as $E \leq \frac{1}{(A_{q_1 s}A_{q_2 q_1}...A_{tq_r})^2}$

The bound could alternatively be stated in terms of edge-weight products on any path between the source and target, however the largest-product path gives the strongest bound. We notice that the bound is simplistic: much tighter bounds could be obtained by considering edge-weight products on multiple independent paths. Also, lower bounds on the energy can be obtained be considering the edge weights on cuts separating the source and target. Due to space constraints, these results will not be presented here.

From Lemmas 4 and 5, it can be concluded that for networks with dense graph structure, the minimum energy required for target control is small for any source and target pair. Although these theorems provide simple comparisons and bounds, which are far from tight, they verify the intuition that network-wide target control is possible even in very large networks provided that there are strong paths between the network's nodes. Lemma 5 also indicates that source-target pairs that are close in the graph Γ are particularly amenable to target control.

The next graphical result compares the target control energy for different possible target locations, for the special case where A is a nonnegative matrix with row sums less than or equal to 1 (i.e. diffusive matrix). Specifically, let $E(\widehat{k})$ be the target control energy for a pair of source and target nodes. Now consider a node cutset separating the source node from this target node on the graph Γ, and without loss of generality let $i = 1, \cdots, m$ be the corresponding labels of these cutset nodes (assume that these i's are not equal to s or t). For at least one node i on the cutset, the target control energy between the original source and this cutset node, denoted as $E_i(\widehat{k})$, is smaller than $E(\widehat{k})$ (see Figure 1). A couple of other useful notations are also introduced

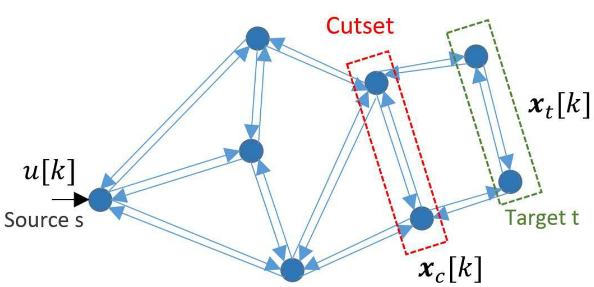

Fig. 1: Illustration of partitions by a node cutset.

here: let $\mathbf{x}_c[k]$ be a vector containing the states of the nodes from the cutset (i.e. $\mathbf{x}_c[k] = \begin{bmatrix} x_1[k] & \cdots & x_m[k] \end{bmatrix}^T$) and $\mathbf{x}_t[k]$ be a vector containing the states of all the nodes that are separated from the source node by the cutset. Note that $\mathbf{x}_t[k]$ also contains the target state $x_t[k]$.

Let us first present a lemma which is the key to prove the above comparison result.

Lemma 6: Consider a target control problem where A is a nonnegative and diffusive matrix. Now consider a node cutset that separates the target from the source. If an optimal input sequence at the source node is able to to drive the target state from zero to one at time $\widehat{k}$, then there exists at least one node on the cutset whose state will reach a value greater than or equal to one before time $\widehat{k}$.

The result in Lemma 6 is not surprising since the control information flows from the source node through the cutset first. It also provides a key step to our next result on comparing target control energies at different target locations.

Theorem 2: Consider the target control problem where the state matrix A is a nonnegative diffusive matrix. Also consider a node cutset the separates the target from the source. Then there exists at least one node on the cutset, such that the minimum input energy required to drive the state of this cutset node from zero to one is no larger than the minimum energy required to drive the target state from zero to one.

V. Structural and Graph-Theoretic Insights into Source Estimation Fidelity

An algebraic expression for the source estimation fidelity, i.e. the error variance for the minimum variance unbiased estimate of the source, has been presented in Lemma 2. Our purpose in this section is to develop spectral and graph-theoretic results on source estimation fidelity.

The source estimation fidelity metric requires characterizing both the observability Gramian and its inverse. The Gramian and its inverse are expressed in terms of the spectrum of A, in the case that its eigenvalues are not defective. The infinite-horizon observability Grammian $G_o \triangleq G_o(0,\infty) = \sum_{k=0}^{\infty}(A^T)^k \mathbf{e}_t\mathbf{e}_t^T A^k$ can be written as

$$G_o = (V^{-1})^T \begin{bmatrix} \frac{V_{1t}^2}{1-\lambda_1^2} & \frac{V_{1t}V_{2t}}{1-\lambda_1\lambda_2} & \cdots & \frac{V_{1t}V_{nt}}{1-\lambda_1\lambda_n} \\ \frac{V_{2t}V_{1t}}{1-\lambda_2\lambda_1} & \frac{V_{2t}^2}{1-\lambda_2^2} & \cdots & \frac{V_{2t}V_{nt}}{1-\lambda_2\lambda_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{V_{nt}V_{1t}}{1-\lambda_n\lambda_1} & \frac{V_{nt}V_{2t}}{1-\lambda_n\lambda_2} & \cdots & \frac{V_{nt}^2}{1-\lambda_n^2} \end{bmatrix} V^{-1},$$

where λ_i is the ith eigenvalue of A, V is the right eigenvector matrix of A, and V_{it} is the tth entry in the right eigenvector $\mathbf{v}_i$. The inverse of the above Grammian can be found by using the inversion formula for the Cauchy matrix (see [33] for details). It is:

$$G_o^{-1} = V \begin{bmatrix} \frac{b_{11}\lambda_1}{V_{1t}^2} & \frac{b_{12}\lambda_2}{V_{2t}V_{1t}} & \cdots & \frac{b_{1n}\lambda_n}{V_{nt}V_{1t}} \\ \frac{b_{21}\lambda_1}{V_{1t}V_{2t}} & \frac{b_{22}\lambda_2}{V_{2t}^2} & \cdots & \frac{b_{2n}\lambda_n}{V_{nt}V_{2t}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{b_{n1}\lambda_1}{V_{1t}V_{nt}} & \frac{b_{n2}\lambda_2}{V_{2t}V_{nt}} & \cdots & \frac{b_{nn}\lambda_n}{V_{nt}^2} \end{bmatrix} V^T,$$

where $b_{ij} = -\frac{\prod_{k=1}^{n}(\frac{1}{\lambda_j}-\lambda_k)(\frac{1}{\lambda_k}-\lambda_i)}{(\frac{1}{\lambda_j}-\lambda_i)(\prod_{1\le k\le n, k\neq j}(\frac{1}{\lambda_j}-\frac{1}{\lambda_k}))(\prod_{1\le k\le n, k\neq i}(\lambda_i+\frac{1}{\lambda_k}))}$.

The source estimation fidelity metric can directly be computed from the inverse of the observability Gramian, as $F = e_s^T G_0^{-1} e_s$. Thus, the above spectral expression for G_0^{-1} directly gives some insights into source estimation. First, the expression shows that source estimation is difficult if any two eigenvalues whose corresponding right eigenvectors have non-zero entries at the source location are near each other. Likewise, source estimation is difficult if any eigenvector has a large entry at the source location but not the target location. These insights match with the basic intuition that source estimation is impossible whenever a poorly-observed subspace has a projection at the source, ortwo different modal responses are indistinguishable from the measurement location.

The determinant of the observability Gramian is useful as a global measure of the ease of observability. The following theorem specifies the determinant in terms of the spectrum:

Theorem 3: The determinant of G_o is:

$$det(G_o) = \frac{2}{det(V)}(\prod_{j=1}^{n} \frac{V_{jc}}{\lambda_j})(det(B)),$$

where $det(B) = \frac{\prod_{i=2}^{n}\prod_{j=1}^{i-1}\left(\frac{\lambda_j^2+\lambda_i^2}{\lambda_i\lambda_j}-2\right)}{\prod_{i=1}^{n}\prod_{j=1}^{n}\left(\frac{1-\lambda_i\lambda_j}{\lambda_i}\right)}$.
Details are omitted since the analysis is similar to that given in [33].

Exponential Fall Off of Eigenvalues:

The eigenvalue-based term in the determinant expression ($det(B)$) has been studied in several previous works focused on the determinants of Cauchy matrices [19]. These studies show that the determinant necessarily falls off super-exponentially with respect to n whenever the eigenvalues are real and bounded away from the unit circle. It can be readily shown, in consequence, that $\mathbf{p}^T G_o \mathbf{p}$ must be small (falling of exponentially with n) for some vector $\mathbf{p}$. Any source location that has a projection in this direction $\mathbf{p}$ thus is difficult to estimate. Of course, at least one source location must have a projection in this direction, and typically many source locations will have a non-negligible projection.

The result showing that source estimation is intrinsically difficult for symmetric networks can be refined to show that, in fact, a subspace of significant dimension necessarily becomes hard to estimate. This can be demonstrated by showing that the eigenvalues of the observability Gramian fall off exponentially, and hence that a significant number of Gramian eigenvalues are below a threshold. This in turn implies that the hard-to-observe subspace has significant dimension, and hence that any source location with a projection in these numerous directions is hard to estimate. The exponential fall-off in the eigenvalues of the Gramians has been demonstrated in the important work [22]. This characterization can be exploited to show that the fraction of eigenvalues below any threshold necessarily increases toward 1 with the dimension of the system, which means that the dimension of the effectively-unobservable subspace gets proportionally larger as the system dimension increases. This implies that source observability becomes impossible for any large system with a symmetric state matrix.

This result can be derived from the work presented in [22]. Specifically, for a matrix that satisfies a discrete-time Lyapunov function, [22] provides an eigenvalue decay bound which only depends on the condition number of the coefficient matrix. Note that both observability and controllability Grammians over the infinite horizon (i.e. G_o and G_c) of system (2) also satisfy discrete-time Lyapunov equations with coefficient matrix A. A similar eigenvalue decay bound for these Grammians (denoted G in general) can also be obtained. For simplicity, we limit ourselves to the case where A is symmetric and stable since the condition number of a symmetric matrix has a nicer form. We begin with a lemma that follows from the work in [22] without proof.

Lemma 7: Consider system (2) where A is symmetric and stable. Let G be an infinite-horizon Grammian matrix associated with this system. Then,

$$\lambda_{i+1}(G) \leq \left(\prod_{j=0}^{i-1} \frac{a^{(2j+1)/(2i)}-1}{a^{(2j+1)/(2i)}+1}\right)^2 \lambda_1(G),$$

where $\lambda_i(G)$, $i = 1, \cdots, n$, are the nonincreasingly ordered eigenvalues of G, a is the condition number of A specified as $a = \frac{(\lambda_{min}(A)-1)(\lambda_{max}(A)+1)}{(\lambda_{min}(A)+1)(\lambda_{max}(A)-1)}$.
Note that since A is symmetric and stable, its condition number $a > 1$. Now let S be a set of Grammians matrices of stable and symmetric systems. We note that the number of the states in each system may differ (A, as well as G, may have different dimensions). Further, all such A matrices have a fixed maximum eigenvalue as $\lambda_{max}(A) = \lambda_{max}$ and a fixed condition number a.

Theorem 4: For any $\varepsilon > 0$, there exists a positive integer n such that every matrix $G \in S$ of dimension larger than n has at least one eigenvalue below ε.

This result compares a class of systems who differ in the number of states and we find that the drop off of their eigenvalues can be guaranteed by looking at larger systems. Theorem 4 also indicates that the number of eigenvalues below a threshold is increasing if the system's dimension is sufficiently large. In fact, the fraction of eigenvalues below the threshold can be arbitrarily close to 1. Here is the result:

Corollary 1: In S, given any $\varepsilon > 0$, there exists a positive integer n such that every matrix $G \in S$ of dimension n has a fraction of eigenvalues below ε arbitrarily close to 1.

This analysis confirms that source estimation is necessarily hard for some source locations, for large symmetric networks. In the case where the eigenvalues are complex-valued, the determinant does not always fall of rapidly with the network size, however several works have shown that the determinant falls off rapidly unless the eigenvalues are roughly equally spaced around the unit circle. Thus, source estimation may be expected to be difficult for most asymmetric networks also.

VI. Conclusions

This work has considered notions of *target reachability* and *source observability*, which explore the ability to drive or estimate a single network node. We show that target reachability is often easy while source observability is usually hard. The reason for this asymmetry between these concepts is the fact that one subspace is for states we *can* reach, while the other is for states we *can not* observe. The asymmetry potentially may have several implications in the design and management of complex dynamical network. For instance, in considering security of network dynamics, most research focuses on the idea of *privacy*, a notion very related to source observability. Paradoxically however, this work suggest that vulnerabilities in most complex interconnected systems arise from the potential for driving particular components of the state vector to desired values, not in the potential of revealing particular components of the state vector to a remote estimator. This opposite interpretation in otherwise dual mathematical constructs yields a significant difference in security implications.

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