

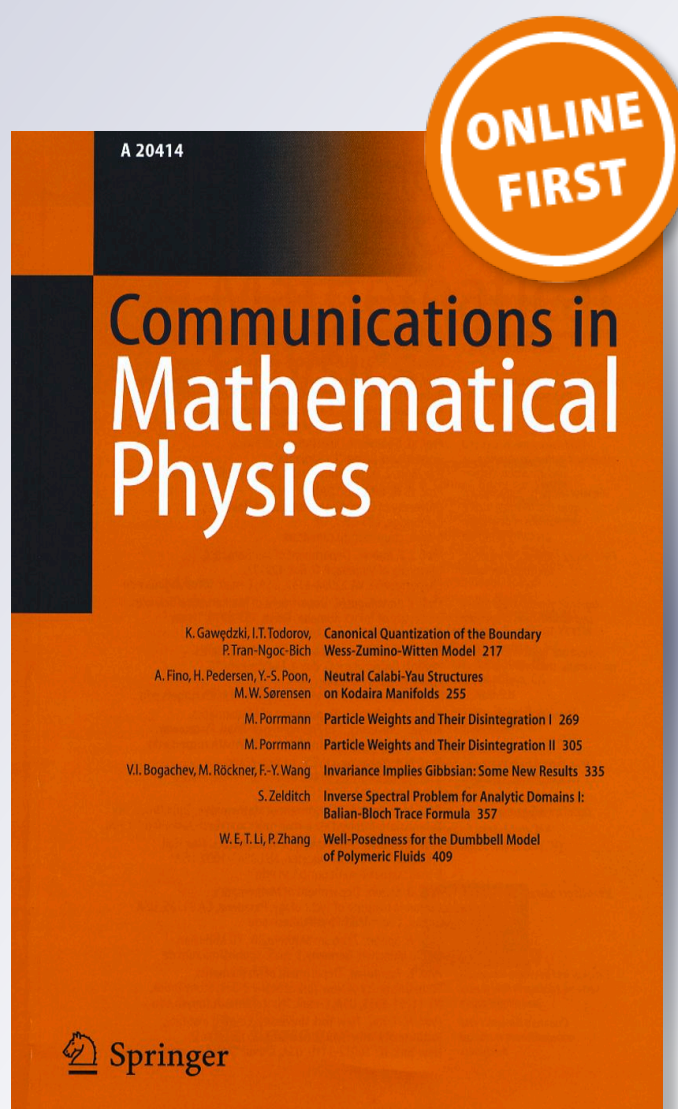
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The Existence of Stable BGK Waves

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Abstract: The 1D Vlasov-Poisson system is the simplest kinetic model for describing an electrostatic collisionless plasma, and the BGK waves are its famous exact steady solutions. They play an important role on the long time dynamics of a collisionless plasma as potential “final states” or “attractors”, thanks to many numerical simulations and observations. Despite their importance, the existence of stable BGK waves has been an open problem since the discovery of BGK waves in 1957. In this paper, linearly stable BGK waves are constructed near homogeneous states.

1. Introduction

The 1D Vlasov-Poisson (VP) system is the simplest kinetic model for describing a collisionless plasma:

$$\partial_t f_{\pm} + v \partial_x f_{\pm} \pm E \partial_v f_{\pm} = 0, \quad E_x = \int [f_+ - f_-] dv, \quad (1)$$

where $f_{\pm}(t, x, v)$ denote distribution functions for ions (+) and electrons (−) respectively, with their self-consistent electric field E . In plasma physics literature, 1D VP for the electrons with a fixed ion background is often studied. The equation becomes

$$\partial_t f + v \partial_x f - E \partial_v f = 0, \quad E_x = - \int_{-\infty}^{+\infty} f dv + 1, \quad (2a)$$

where 1 is the fixed ion density. There are important physical phenomena from the study of Vlasov models. They include Landau damping ([7, 20, 23]), and kinetic instabilities such as two-stream instabilities ([2]). The BGK waves were discovered by Bernstein-Greene-Kruskal in 1957 ([6]), as exact, spatially periodic steady state solutions to the

Vlasov-Poisson system. Consider general BGK waves with non-even distribution functions for ions (+) and electrons (−)

$$f_{\pm}^{\beta}(x, v) = \begin{cases} \mu_{\pm,+}(e_{\pm}) & \text{when } v > 0 \\ \mu_{\pm,-}(e_{\pm}) & \text{when } v < 0 \end{cases}, \quad (3)$$

where $e_{\pm} = \frac{v^2}{2} \pm \beta$. The self-consistent potential β satisfies

$$\begin{aligned} -\beta_{xx} &= \int_{v>0} \mu_{+,+}(e_+) dv + \int_{v<0} \mu_{+,-}(e_+) dv \\ &\quad - \int_{v>0} \mu_{-,+}(e_-) dv - \int_{v<0} \mu_{-,-}(e_-) dv \equiv h(\beta). \end{aligned} \quad (4)$$

Denote the homogeneous state

$$f_{0,\pm}(v) = \begin{cases} \mu_{\pm,+}(\frac{1}{2}v^2) & \text{when } v > 0 \\ \mu_{\pm,-}(\frac{1}{2}v^2) & \text{when } v < 0 \end{cases}. \quad (5)$$

For the case with fixed ion background, let the electron distribution be

$$f^{\beta}(x, v) = \begin{cases} \mu_{-,+}(e_-) & \text{when } v > 0 \\ \mu_{-,-}(e_-) & \text{when } v < 0 \end{cases},$$

and

$$-\beta_{xx} = 1 - \int_{v>0} \mu_{-,+}(e_-) dv - \int_{v<0} \mu_{-,-}(e_-) dv \equiv h(\beta), \quad (6)$$

where 1 is the ion density. Denote

$$f_0(v) = \begin{cases} \mu_{-,+}(\frac{1}{2}v^2) & \text{when } v > 0 \\ \mu_{-,-}(\frac{1}{2}v^2) & \text{when } v < 0 \end{cases}. \quad (7)$$

The existence of small BGK waves satisfying (4) (6) follows from the Liapunov center theorem (see e.g. [14]).

Lemma 1. *Consider the ODE $-\beta_{xx} = h(\beta)$ where $h \in C^1$. If $h(0) = 0$, $h'(0) = \left(\frac{2\pi}{P_0}\right)^2 > 0$, then there exist a family of periodic solutions with $\|\beta\|_{\infty} = \varepsilon \ll 1$ with minimum period $P_{\beta} \rightarrow P_0$ when $\varepsilon \rightarrow 0$. We can normalize $\beta(x)$ to be even in $\left[-\frac{P_{\beta}}{2}, \frac{P_{\beta}}{2}\right]$ with maximum at $x = 0$, minimum at $x = \pm \frac{P_{\beta}}{2}$, $\min \beta = -\max \beta$, so that*

$$\beta(x) = \varepsilon \cos \frac{2\pi}{P_{\beta}} x + O(\varepsilon^2). \quad (8)$$

Remarkably, these BGK waves play crucial roles in understanding long time dynamics of the Vlasov-Poisson system, which have been an important topic in plasma physics. Many numerical simulations [5, 11, 12, 21, 22, 24, 26, 28] indicate that for initial data near a stable homogeneous state including Maxwellian, the asymptotic behavior of approaching a BGK wave or superposition of BGK waves is usually observed. Moreover, BGK waves also appear as the ‘attractor’ or “final states” for the saturation of an unstable homogeneous state ([2, 4, 8–10, 12, 13]). For example, in [13], starting near a BGK

wave with double period which is unstable, the authors observed the gradual evolution to another BGK wave of minimal period. To understand such long time behaviors, an important first step is to construct stable BGK waves.

Ever since the discovery of BGK waves, their stability has been an active research area. There has been a lot of formal analysis in the physical literature. Instability of BGK waves to perturbations of multiple periods was proved in [14] for waves of small amplitude and in [18, 19] for waves of large amplitude. Unfortunately, despite intense efforts, no stable BGK waves to perturbations of minimum period have been found since 1957.

One difficulty is that stable BGK waves cannot be obtained by the traditional energy-Casimir method, which was first used by Newcomb in the 1950s ([27]) to prove nonlinear stability of Maxwellian. This method requires the profiles $\mu_{\pm, \pm}$ to be monotone decreasing to $e_{\pm}$, which implies that $h(\beta)$ defined in (4) or (6) is a decreasing function of β . So, by differentiating (4) or (6) and integrating it with β_x , we get

$$\int \left[(\beta_{xx})^2 - h'(\beta) (\beta_x)^2 \right] dx = 0,$$

and thus $\beta_x \equiv 0$ (i.e. homogeneous states). So for any nontrivial BGK waves, the profiles $\mu_{\pm, \pm}$ cannot be monotone and thus the energy-Casimir method does not work. For the homogeneous equilibria, due to the separation of Fourier modes, a simple dispersion relation function can be analyzed to get the Penrose stability criterion ([25]). However, even for small BGK waves, due to the coupling of infinitely many modes, the dispersion relation is difficult to study for linear stability.

In the rest of this paper, we assume $\mu'_{\pm, +}(\theta)$, $\mu'_{\pm, -}(\theta) \equiv 0$ for $|\theta| \leq \sigma_{\pm}$ and denote $\sigma = \min\{\sigma_+, \sigma_-\}$. That is, the distribution function is assumed to be flat near 0. First, this simplifies some technical steps in our construction. Second, this assumption is also physically relevant. It was known that in the long time evolution of VP near a homogeneous state, the distribution function can develop a plateau due to the resonant particles ([3]).

Our first result shows that small BGK waves with non-even distribution are generally spectrally stable, that is, the spectra of the linearized VP operator lies in the imaginary axis.

Theorem 2. (i) (Uneven and Two species) Assume $\mu_{\pm, \pm} \in C^3(\mathbf{R})$ are nonnegative and

$$\max_{1 \leq i \leq 3} |\mu_{\pm, \pm}^{(i)}(y)| \leq C(1 + |y|)^{-\gamma} \quad (\gamma > 1), \quad \mu'_{\pm, +}(\theta), \quad \mu'_{\pm, -}(\theta) \equiv 0 \text{ for } |\theta| \leq \sigma_{\pm}. \quad (9)$$

Define $f_{0, \pm}(v)$ by (5) and assume:

$$\int f_{0, +}(v) dv = \int f_{0, -}(v) dv, \quad \int \frac{f'_{0, +}(v) + f'_{0, -}(v)}{v} = \left(\frac{2\pi}{P_0} \right)^2, \quad (10)$$

$$\int \frac{f'_{0, +}(v) + f'_{0, -}(v)}{v^2} \neq 0, \quad (11)$$

$$\int \frac{f'_{0, +}(v) + f'_{0, -}(v)}{v - v_r} dv < \left(\frac{2\pi}{P_0} \right)^2, \quad (12)$$

for any critical point of v_r of $f_{0, +}(v) + f_{0, -}(v)$ with $|v_r| \geq \sigma$, and

$$\int \frac{f'_{0, +}(v) + f'_{0, -}(v)}{v - v_r} dv \neq \left(\frac{2\pi}{P_0} \right)^2, \quad \text{when } |v_r| \leq \sigma. \quad (13)$$

Then when $\varepsilon = \|\beta\|_\infty$ is small enough, the BGK wave $[f_\pm^\beta, -\beta_x]$ satisfying (4) is spectrally stable against P_β -periodic perturbations.

(ii) (Uneven and Fixed ion background) Assume $\mu_{-, \pm} \in C^3(\mathbf{R})$ are nonnegative and

$$\max_{1 \leq i \leq 3} |(\mu_{-, \pm})^{(i)}(y)| \leq (1 + |y|)^{-\gamma} (\gamma > 1), \quad (\mu_{-, \pm})'(\theta) \equiv 0 \text{ for } |\theta| \leq \sigma_\pm. \quad (14)$$

Define $f_0(v)$ by (7) and assume: $\int \frac{f_0'(v)}{v^2} dv \neq 0$,

$$\int f_0(v) dv = 1, \quad \int \frac{f_0'(v)}{v} dv = \left(\frac{2\pi}{P_0}\right)^2, \quad (15)$$

$$\int \frac{f_0'(v)}{v - v_r} dv < \left(\frac{2\pi}{P_0}\right)^2 \quad (16)$$

for any critical point of $f_0(v)$ with $|v_r| \geq \sigma$, and

$$\int \frac{f_0'(v)}{v - v_r} dv \neq \left(\frac{2\pi}{P_0}\right)^2, \text{ when } |v_r| \leq \sigma. \quad (17)$$

Then small BGK waves $[f^\beta, -\beta_x]$ satisfying (6) are spectrally stable against P_β -periodic perturbations.

The conditions in the above Theorem are quite natural and general: (10) is the bifurcation condition of small BGK waves; (11) is a non-degeneracy condition which is true for generic non-even profiles; (12) is the Penrose stability condition (at period P_0) for the flat homogeneous states $f_{0, \pm}$. The condition (13) is to ensure that 0 is the only discrete eigenvalue for the homogeneous profile with period P_0 . For the fixed ion case, the conditions are similar with $f_{0, \pm}$ being replaced by f_0 . We refer to the final section for more explicit construction of examples. Theorem 2 shows that for general non-even Penrose stable profiles flat near 0, the small BGK waves are linearly stable.

Next, we give a sharp stability criterion for small BGK waves with even profiles.

Theorem 3. (i) (Even and Two species) Assume $\mu_{\pm, -} = \mu_{\pm, +} = \mu_\pm \in C^3(\mathbf{R})$ are nonnegative and

$$\max_{1 \leq i \leq 3} |\mu_\pm^{(i)}(y)| \leq (1 + |y|)^{-\gamma} (\gamma > 1), \quad \mu_\pm'(\theta) \equiv 0 \text{ for } |\theta| \leq \sigma_\pm.$$

Denote $f_{0, \pm}(v) = \mu_\pm(\frac{1}{2}v^2)$ and assume: (10), (12), (13) and

$$\int \frac{[\mu_+' + \mu_-'](\frac{1}{2}v^2)}{v^2} dv > 0. \quad (18)$$

Then the small BGK waves $[\mu_\pm(e_\pm), -\beta_x]$ satisfying (4) are spectrally stable if

$$\int v^{-2} [\mu_+'(\frac{v^2}{2}) - \mu_-'](\frac{v^2}{2}) dv > 0 \text{ (equivalently } P'_\beta < 0),$$

and unstable if

$$\int v^{-2} [\mu_+'(\frac{v^2}{2}) - \mu_-'](\frac{v^2}{2}) dv < 0 \text{ (equivalently } P'_\beta > 0).$$

Here, the derivative P'_β is respect to $\varepsilon = \max |\beta|$. Moreover, for the stable case, there exists a pair of nonzero imaginary eigenvalues of the linearized VP operator around $[\mu_\pm(e_\pm), -\beta_x]$.

(ii) (Even and Fixed ion background) Assume $\mu_{-,+} = \mu_{-,-} = \mu \in C^3(\mathbf{R})$ is nonnegative and

$$\max_{1 \leq i \leq 3} |\mu^{(i)}(y)| \leq (1 + |y|)^{-\gamma} (\gamma > 1), \quad \mu'(\theta) \equiv 0 \text{ for } |\theta| \leq \sigma.$$

Denote $f_0(v) = \mu(\frac{1}{2}v^2)$ and assume: (15), (16), (17) and

$$\int \frac{\mu'(\frac{1}{2}v^2)}{v^2} dv > 0. \quad (19)$$

Then the small BGK waves $[\mu(e_-), -\beta_x]$ satisfying (6) are unstable.

Let $\sigma_\pm$ be the width of flatness near 0 for $\mu_\pm$. It is shown in Section 6 that when $\sigma_\pm$ is small, (18) is always satisfied. Moreover, when $\sigma_- \gg \sigma_+$ ($\sigma_- \ll \sigma_+$), the stability (instability) condition $P'_\beta < 0$ ($P'_\beta > 0$) is satisfied. So for the even and two-species case, we can construct both stable and unstable small BGK waves. For the stable case, there exist a pair of purely imaginary nonzero eigenvalues of the linearized VP operator. This suggests that there exists a time periodic solution of the linearized VP equation, for which the electric field does not decay in time. So there is no Landau damping even at the linear level. In contrast, for linearly stable BGK waves with uneven profiles as in Theorem 2 (i), such nonzero purely imaginary eigenvalues do not exist and the Landau damping might be true. The problems of constructing nonlinear time periodic solutions for the even case and proving Landau damping for the uneven case are currently under investigation.

Last, we discuss some key ideas in the proof. Our analysis relies on a delicate perturbation argument from a stable homogeneous equilibria. It is well-known that the original spectra analysis around small BGK waves is difficult due to unbounded perturbation $\beta_x \partial_v g_\pm$ in the following eigenfunction equation at a BGK wave $(f_\pm^\beta(x, v), \beta)$ satisfying (3) and (4):

$$\lambda g_\pm + v \partial_x g_\pm \mp \beta_x \partial_v g_\pm \mp \phi_x v \mu'_{\pm,+} = 0, \quad v > 0 \quad (20)$$

$$\lambda g_\pm + v \partial_x g_\pm \mp \beta_x \partial_v g_\pm \mp \phi_x v \mu'_{\pm,-} = 0, \quad v < 0$$

$$-\phi_{xx} = \rho = \int (g_+ - g_-) dv, \quad (21)$$

where λ is the eigenvalue and $(g_\pm, \phi)$ is the eigenfunction. For an unstable eigenvalue λ with $\text{Re } \lambda > 0$, it is possible to 'integrate' the Vlasov equation (20) and study a nice operator on the electric potential ϕ to exclude unstable eigenvalues away from 0. The situation is much more subtle when λ is near zero, which is exactly the focus of the current stability analysis.

There are three new ingredients in our resolution to such an open question. The first is to use action-angle variables to integrate the Vlasov equation (20) and define a charge operator $\rho(\lambda; \varepsilon)$ acting on electric potential ϕ , where $\varepsilon = |\beta|_\infty$. Remarkably, we observe that if $\mu_{\pm,\pm}$ are flat near the origin, then the operator $\rho(\lambda; \varepsilon)$ is analytic in λ near 0 and continuous in ε . The second ingredient is to prove the number of eigenvalue λ near zero is no more than two for the non-even BGK waves and no more than four for the even

BGK waves, thanks to a new abstract lemma on stability of eigenvalues and the counting of the multiplicity of zero eigenvalue for the homogeneous state. The last ingredient is to use the Hamiltonian structure of the linearized Vlasov-Poisson system and the zero eigenmode due to translation. Combining such a structure with the eigenvalue counting near 0, we can rule out unstable eigenvalues for the uneven steady states. For the even states, the even and odd perturbations can be studied separately. For odd perturbations, the unstable eigenvalues can be ruled out by the counting as in the uneven case. For even perturbations, the possible unstable eigenvalues must be real, from which a sharp stability criterion can be derived.

Since neutrally stable spectra can easily become unstable under perturbations, it is generally difficult to construct stable steady states in Hamiltonian systems via a perturbation method. Our successful construction provides a general approach to find stability criteria to ensure that the zero eigenvalue can only bifurcate to stable ones for Hamiltonian systems with certain natural symmetry. The linearized Vlasov-Poisson system near BGK waves is a linear Hamiltonian system $\mathcal{J}\mathcal{L}$ (28) with an indefinite energy functional $\langle \bar{\mathcal{L}}\bar{g}, \bar{g} \rangle$, for which there are very few methods to study the stability issues. Our approach could be useful for other problems with an indefinite energy functional.

2. Stability of Spectra

First, we give an abstract lemma about the stability of eigenvalues near 0. Let $K(\lambda, \varepsilon)$ be a family of bounded linear operators from Hilbert space X to X , where $\lambda \in \mathbf{C}$, $\varepsilon \in \mathbf{R}$. We assume that:

$$\text{for } \varepsilon \ll 1, K(\lambda, \varepsilon) \text{ is analytic near } \lambda = 0. \quad (22)$$

That is, for $\varepsilon \ll 1$, the map $\lambda \rightarrow K(\lambda, \varepsilon)$ is analytic as an operator-valued function on a small disk $B_{R(\varepsilon)}(0) \subset \mathbf{C}$. This is equivalent to that $\lambda \rightarrow (K(\lambda, \varepsilon)u, v)$ is analytic for any $u, v \in X$ (see [16]). We investigate the set of generalized eigenvalues $\Lambda^\varepsilon = \{\lambda \in \mathbf{C} \text{ such that there is } 0 \neq r \in X \text{ such that } (\mathbf{I} + K(\lambda, \varepsilon))r = 0\}$.

Lemma 4. (*Stability of Spectra*) Assume (22) and:

$$1) \ker\{\mathbf{I} + K(0, 0)\} = \text{span}\{r_1, r_2\}.$$

2) $\mathbf{I} + K(0, 0) : (I - \mathbf{P})X \rightarrow (I - \mathbf{P})X$ is invertible, where $\mathbf{P}$ is the projection to the span of $\{r_1, r_2\}$ and

$$\det((\mathbf{I} + K(\lambda, 0))r_j, r_i) \sim \lambda^m, \quad (i, j = 1, 2)$$

near $\lambda = 0$.

$$3) K(\lambda, 0) \text{ is continuous in } \lambda \text{ and } K(\lambda, 0) : (I - \mathbf{P})X \rightarrow (I - \mathbf{P})X.$$

$$4) \text{ For any } \lambda \in \mathbf{C} \text{ with } |\lambda| < 1,$$

$$\lim_{\varepsilon \rightarrow 0} \|K(\lambda, \varepsilon) - K(\lambda, 0)\|_{L(X, X)} = 0.$$

Then there exists $\alpha > 0$ such that for all $\varepsilon \ll 1$, $\#(\Lambda^\varepsilon \cap \{|\lambda| < \alpha\}) \leq m$.

Proof. The proof uses the Liapunov-Schmidt reduction on $\ker\{\mathbf{I} + K(0, 0)\}$ and its complement space. By assumptions 2) and 3), there exists $\alpha > 0$ such that for any $\lambda \neq 0$ with $|\lambda| < \alpha$, $(\mathbf{I} + K(\lambda, 0))|_{(I - \mathbf{P})X}$ is invertible. We assume $|\lambda| < \alpha$ below.

Let $\lambda \in \Lambda^\varepsilon$, so that there is $r_\varepsilon = r \neq 0$ such that

$$(\mathbf{I} + K(\lambda, \varepsilon))r = \{\mathbf{I} + K(\lambda, 0) + [K(\lambda, \varepsilon) - K(\lambda, 0)]\}r = 0. \quad (23)$$

Let $r = r^\perp + r^\parallel$, where $r^\parallel = a_1 r_1 + a_2 r_2$ is the projection of r to $\{r_1, r_2\}$. Then projecting (23) to $(I - \mathbf{P})X$, we get

$$(\mathbf{I} + K(\lambda, 0)) r^\perp + (\mathbf{I} - \mathbf{P}) [K(\lambda, \varepsilon) - K(\lambda, 0)] (r^\perp + r^\parallel) = 0.$$

Solving $r^\perp$ in terms of $r^\parallel$, we get

$$\begin{aligned} r^\perp &= -[\mathbf{I} + (\mathbf{I} + K(\lambda, 0))^{-1} (\mathbf{I} - \mathbf{P}) (K(\lambda, \varepsilon) - K(\lambda, 0))]^{-1} \\ &\quad (\mathbf{I} + K(\lambda, 0))^{-1} (\mathbf{I} - \mathbf{P}) [K(\lambda, \varepsilon) - K(\lambda, 0)] r^\parallel \\ &\equiv Z^\perp(\lambda, \varepsilon) \{a_1 r_1 + a_2 r_2\}, \end{aligned}$$

where $\mathbf{P}$ is the projection to $\ker(\mathbf{I} + K(0, 0)) = \{r_1, r_2\}$ and

$$\begin{aligned} Z^\perp(\lambda, \varepsilon) &\equiv -[\mathbf{I} + (\mathbf{I} + K(\lambda, 0))^{-1} (\mathbf{I} - \mathbf{P}) (K(\lambda, \varepsilon) - K(\lambda, 0))]^{-1} \\ &\quad (\mathbf{I} + K(\lambda, 0))^{-1} (\mathbf{I} - \mathbf{P}) [K(\lambda, \varepsilon) - K(\lambda, 0)]. \end{aligned} \quad (24)$$

Plugging above formula to the equation (23), we have

$$\begin{aligned} 0 &= [K(\lambda, \varepsilon) - K(\lambda, 0)] (Z^\perp(\lambda, \varepsilon) + \mathbf{I}) (a_1 r_1 + a_2 r_2) \\ &\quad + (\mathbf{I} + K(\lambda, 0)) (a_1 r_1 + a_2 r_2) + (\mathbf{I} + K(\lambda, 0)) ((\mathbf{I} - \mathbf{P}) r). \end{aligned} \quad (25)$$

Taking inner product of above equation with r_1 and r_2 respectively, we get

$$\sum_{j=1}^2 ((\mathbf{I} + K(\lambda, 0)) r_j, r_i) a_j + \sum_j B_{ij}(\lambda, \varepsilon) a_j = 0, \quad i = 1, 2, \quad (26)$$

where

$$B_{ij}(\lambda, \varepsilon) = ([K(\lambda, \varepsilon) - K(\lambda, 0)] [Z^\perp(\lambda, \varepsilon) + \mathbf{I}] r_j, r_i).$$

Here, in the above we use the fact that

$$((\mathbf{I} + K(\lambda, 0))(\mathbf{I} - \mathbf{P}) r, r_i) = 0, \quad i = 1, 2.$$

Define the 2 by 2 matrix $A(\lambda, \varepsilon) = (A_{ij}(\lambda, \varepsilon))$ by

$$A_{ij} = ((\mathbf{I} + K(\lambda, 0)) r_j, r_i) + B_{ij}(\lambda, \varepsilon), \quad i, j = 1, 2,$$

then the eigenvalue problem (23) is equivalent to $\det A(\lambda, \varepsilon) = 0$.

By assumption (22), $K(\lambda, \varepsilon)$ is analytic near $\lambda = 0$, it follows that $\det A(\lambda, \varepsilon)$ is analytic in λ near 0 for $\varepsilon < 1$. By 2),

$$\det A(\lambda, 0) = \det((\mathbf{I} + K(\lambda, 0)) r_j, r_i) \sim \lambda^m.$$

Moreover, by 4) we have $\lim_{\varepsilon \rightarrow 0} |\det A(\lambda, \varepsilon) - \det A(\lambda, 0)| = 0$. It follows from the analytical function theory, there exists $\alpha > 0$ such that there are at most m distinct λ with $|\lambda| < \alpha$ satisfying $\det A(\lambda, \varepsilon) = 0$. Thus $\#(\Lambda^\varepsilon \cap \{|\lambda| < \alpha\}) \leq m$. $\square$

By the same proof, we have a similar result when $\ker(\mathbf{I} + K(0, 0))$ is one-dimensional.

Corollary 5. Assume (22) and:

- 1) $\ker\{\mathbf{I} + K(0, 0)\} = \text{span}\{r\}$.
- 2) $\{\mathbf{I} + K(0, 0)\}$ is invertible from $\{\mathbf{I} - \mathbf{P}\}X \rightarrow \{\mathbf{I} - \mathbf{P}\}X$, where $\mathbf{P}$ is the projection to $\text{span}\{r\}$, and $(\{I + K(\lambda, 0)\}r, r) \sim \lambda^m$.
- 3) $\{\mathbf{I} + K(\lambda, 0)\}$ is continuous in λ and $\{\mathbf{I} + K(\lambda, 0)\}$ maps from $\{\mathbf{I} - \mathbf{P}\}X$ to $\{\mathbf{I} - \mathbf{P}\}X$.
- 4) For any $\lambda \in \mathbb{C}$, $\lim_{\varepsilon \rightarrow 0} \|K(\lambda, \varepsilon) - K(\lambda, 0)\| \rightarrow 0$.

Then there exists a $\alpha > 0$ such that if $|\varepsilon| < 1$, $\#\{\Lambda^\varepsilon \cap \{|\lambda| < \alpha\} \leq m$.

3. Vlasov Spectra

In this section, we use the Hamiltonian structure of the linearized Vlasov-Poisson operator to show that the Vlasov spectra is symmetric to both real and imaginary axes. We only consider the two-species case and the same is true for the fixed ion case. Define the linearized Vlasov-Poisson operator

$$\mathcal{A} \begin{pmatrix} g_+ \\ g_- \end{pmatrix} = \begin{pmatrix} -(v\partial_x - \beta_x \partial_v) g_+ + \partial_x \phi \partial_v \mu_+ \\ -(v\partial_x + \beta_x \partial_v) g_- - \partial_x \phi \partial_v \mu_- \end{pmatrix}, \quad (27)$$

with

$$\phi = (\partial_x^2)^{-1} \left(\int \{g_+ - g_-\} dv \right),$$

where $\mu_\pm \equiv \mu_\pm(\frac{1}{2}|v|^2 \pm \beta)$. Define $X_\pm$ to be the $|\mu'_\pm|$ weighted L^2 space and $X = X_+ \times X_-$. We consider the spectra of the operator $\mathcal{A}$ in X .

Lemma 6. (Structure of Vlasov Spectra) The essential spectrum of the linearized Vlasov-Poisson operator $\mathcal{A}$ is in the imaginary axis. Let λ be an eigenvalue of $\mathcal{A}$, then $\bar{\lambda}$, $-\lambda$, $-\bar{\lambda}$ must also be eigenvalues.

Proof. We first note that the transport operator $-(v\partial_x \mp \beta_x \partial_v)$ is an anti-symmetric closed operator with imaginary axis being its essential spectrum, while $\pm \partial_x \phi \partial_v \mu_\pm$ with

$$\phi = (\partial_x^2)^{-1} \left(\int \{g_+ - g_-\} dv \right)$$

is a relative compact perturbation. Thus by Weyl's Theorem (Th. 5.35 in [17]), the essential spectrum of $\mathcal{A}$ remains the same, with possible additional discrete eigenvalues outside the imaginary axis. The same is true for the adjoint operator $\mathcal{A}^*$.

Define the operators

$$\mathcal{L}_\pm g_\pm = \frac{g_\pm}{\mu'_\pm}, \quad \mathcal{B}f = (\partial_x^2)^{-1} \left(\int f dv \right), \quad \mathcal{J}_\pm = -\mu'_\pm (v\partial_x \mp \beta_x \partial_v).$$

Then formally the operator $\mathcal{A} = \mathcal{J}\mathcal{L}$ is of Hamiltonian form, where the operators

$$\mathcal{J} = \begin{pmatrix} \mathcal{J}_+ & 0 \\ 0 & \mathcal{J}_- \end{pmatrix}, \quad \mathcal{L} = \begin{pmatrix} \mathcal{L}_+ - \mathcal{B} & \mathcal{B} \\ \mathcal{B} & \mathcal{L}_- - \mathcal{B} \end{pmatrix} \quad (28)$$

are anti-symmetric and symmetric respectively on X . We choose λ to be an eigenvalue of $\mathcal{A}$. Since $\mathcal{A}$ maps real functions to real functions, if λ is an eigenvalue of $\mathcal{A}$, then so

is $\bar{\lambda}$. To show that $-\lambda$ is also an eigenvalue of $\mathcal{A}$, it suffices to assume that $\text{Re } \lambda \neq 0$ since otherwise $-\lambda = \bar{\lambda}$ is already an eigenvalue. Assume $\text{Re } \lambda > 0$ and $\vec{g} = (g_+, g_-)$ is an eigenfunction. Then from $\mathcal{A}\vec{g} = \mathcal{J}\mathcal{L}\vec{g} = \lambda\vec{g}$, clearly $\mathcal{L}\vec{g} \neq 0$ since otherwise $\lambda\vec{g} = \mathcal{J}\mathcal{L}\vec{g} = 0$ so that $\lambda = 0$, a contradiction. Since $\mathcal{J}^* = -\mathcal{J}$ and $\mathcal{L}^* = \mathcal{L}$, $\mathcal{A}^* = -\mathcal{L}\mathcal{J}$. Define $\vec{h} = \mathcal{L}\vec{g} \neq 0$, then

$$\mathcal{A}^*\vec{h} = -(\mathcal{L}\mathcal{J})\mathcal{L}\vec{g} = -\mathcal{L}(\mathcal{J}\mathcal{L})\vec{g} = -\lambda\mathcal{L}\vec{g} = -\lambda\vec{h},$$

so $-\lambda$ is an eigenvalue of $\mathcal{A}^*$. Since the essential spectrum of $\mathcal{A}^*$ is in the imaginary axis and $\text{Re } (-\lambda) \neq 0$, so $-\lambda$ is an isolated eigenvalue of $\mathcal{A}^*$ with finite multiplicity. This implies that $-\bar{\lambda} \in \sigma(\mathcal{A}) = \overline{\sigma(\mathcal{A}^*)}$ is also an isolated eigenvalue of $\mathcal{A}$ with the same multiplicity (see [17] p. 184). Then by complex conjugate, $-\lambda$ is an eigenvalue of $\mathcal{A}$. $\square$

4. Action-Angle Reformulation

To use Lemma 4 on the stability of spectra, we reformulate the eigenvalue problem (20) to a Fredholm operator for the potential function ϕ . To achieve this, we solve f in terms of ϕ by using the action-angle variables of the steady trajectory. Below, we treat the two-species case. The fixed ion case is similar.

Action-Angle Formulation: The construction of the action-angle variables follows from Section 50.B in [1].

Inside separatrix: When the initial state is in the trapped region, that is,

$$(x, v) \in \Omega_0 = \{e_{\pm} < \max \beta = -\min \beta\}, \quad (29)$$

the particle is trapped in the interval $[-\alpha_{\pm}(e_{\pm}), \alpha_{\pm}(e_{\pm})]$, with the period

$$T_{\pm}(e_{\pm}) = 2 \int_{-\alpha_{\pm}(e_{\pm})}^{\alpha_{\pm}(e_{\pm})} \frac{dx'}{\sqrt{2(e_{\pm} \mp \beta(x'))}},$$

where $e_{\pm} = \pm\beta(\alpha_{\pm}(e_{\pm}))$. Define the action variable

$$I_{\pm}(e_{\pm}) = \frac{1}{2\pi} \int_{-\max \beta}^{e_{\pm}} T_{\pm}(e'_{\pm}) de'_{\pm},$$

and the angle variable

$$\theta_{\pm} = \frac{2\pi}{T_{\pm}(e_{\pm})} \int_{-\alpha_{\pm}}^x \frac{dx'}{\sqrt{2(e_{\pm} \mp \beta(x'))}}, \quad v > 0,$$

and

$$\theta_{\pm} = 2\pi - \frac{2\pi}{T_{\pm}(e_{\pm})} \int_{-\alpha_{\pm}}^x \frac{dx'}{\sqrt{2(e_{\pm} \mp \beta(x'))}}, \quad v < 0.$$

On the separatrix:

When $(x, v) \in \{e_{\pm} = -\min \beta\}$, the particle takes infinite time to approach the saddle point $\left(\frac{P_{\beta}}{2}, 0\right)$ for electrons and $(0, 0)$ for ions.

Outside separatrix:

When the initial state is in the upper untrapped region, that is,

$$(x, v) \in \Omega_+ = \{e_{\pm} > -\min \beta, v > 0\}, \quad (30)$$

or in the lower untrapped region, that is,

$$(x, v) \in \Omega_- = \{e_{\pm} > -\min \beta, v < 0\}, \quad (31)$$

the particle goes through the whole interval $[0, P_{\beta}]$ without changing its direction. Then the period of the particle motion is

$$T_{\pm}(e_{\pm}) = \int_0^{P_{\beta}} \frac{dx'}{\sqrt{2(e_{\pm} \mp \beta(x'))}}.$$

We define the action and angle variables by

$$\begin{aligned} I_{\pm}(e_{\pm}) &= \frac{1}{2\pi} \int_{-\min \beta}^{e_{\pm}} T_{\pm}(e'_{\pm}) de'_{\pm}, \\ \theta_{\pm} &= \frac{2\pi}{T_{\pm}(e_{\pm})} \int_0^x \frac{dx'}{\sqrt{2(e_{\pm} \mp \beta(x'))}}, \end{aligned} \quad (32)$$

and denote

$$\omega_{\pm}(I_{\pm}) = \frac{2\pi}{T_{\pm}(e_{\pm}(I_{\pm}))}$$

to be the frequency. We list some basic properties of action-angle variables (see [1]). First, for both trapped region Ω_0 and untrapped regions $\Omega_{\pm}$, the action-angle transform $(x, v) \rightarrow (I_{\pm}, \theta_{\pm})$ is a smooth diffeomorphism with Jacobian 1. Second, in the coordinates $(I_{\pm}, \theta_{\pm})$, the particle motion equation $\dot{X}_{\pm} = V_{\pm}$, $\dot{V}_{\pm} = \mp \beta_x(X_{\pm})$ becomes $\dot{I}_{\pm} = 0$, $\dot{\theta}_{\pm} = \omega_{\pm}(I_{\pm})$ for trapped particles; for free particle, it becomes $\dot{I}_{\pm} = 0$, $\dot{\theta}_{\pm} = \omega_{\pm}(I_{\pm})$, when $V_{\pm}(0) > 0$ and $\dot{I}_{\pm} = 0$, $\dot{\theta}_{\pm} = -\omega_{\pm}(I_{\pm})$, when $V_{\pm}(0) < 0$. So the particle trajectory $(X_{\pm}(t; x, v), V_{\pm}(t; x, v))$ becomes: $(I_{\pm}, \theta_{\pm} + t\omega_{\pm}(I_{\pm}))$ inside the separatrix; $(I_{\pm}, \theta_{\pm} + t\omega_{\pm}(I_{\pm}))$ for $v > 0$ and $(I_{\pm}, \theta_{\pm} - t\omega_{\pm}(I_{\pm}))$ for $v < 0$, outside the separatrix. Here, $(I_{\pm}, \theta_{\pm})$ are the action-angle variables for the initial position $(X_{\pm}(0), V_{\pm}(0)) = (x, v)$. Correspondingly, we have the following relations of the transport operators in (x, v) and $(I_{\pm}, \theta_{\pm})$:

$$v\partial_x \mp \beta_x\partial_v = \omega_{\pm}(I_{\pm})\partial_{\theta_{\pm}} \quad (33)$$

inside the separatrix, and

$$v\partial_x \mp \beta_x\partial_v = \begin{cases} \omega_{\pm}(I_{\pm})\partial_{\theta_{\pm}} & \text{for } v > 0 \\ -\omega_{\pm}(I_{\pm})\partial_{\theta_{\pm}} & \text{for } v < 0 \end{cases} \quad (34)$$

outside the separatrix. We summarize main properties of the action-angle transform in the following lemma.

Lemma 7. *In the angle-action variables $(I_{\pm}, \theta_{\pm})$, we have*

- (i) $0 \leq \omega_{\pm} < \infty$,
- (ii) $\lim_{e_{\pm} \rightarrow -\min \beta} T_{\pm}(e_{\pm}) = \infty$, $\lim_{e_{\pm} \rightarrow -\min \beta} \omega_{\pm}(e_{\pm}) = 0$,
- (iii) *Inside the trapped region Ω_0 :*

$$v \partial_x \mp \beta_x \partial_v = \omega_{\pm}(I_{\pm}) \partial_{\theta_{\pm}}.$$

(iv) *Outside the trapped region:*

$$\begin{aligned} v \partial_x \mp \beta_x \partial_v &= \omega_{\pm}(I_{\pm}) \partial_{\theta_{\pm}}, \quad \text{for } v > 0, \\ v \partial_x \mp \beta_x \partial_v &= -\omega_{\pm}(I_{\pm}) \partial_{\theta_{\pm}}, \quad \text{for } v < 0. \end{aligned}$$

Recall that in this paper, the profiles $\mu_{\pm}$ of BGK waves are assumed to be flat near zero in an interval $[-\sigma_{\pm}, \sigma_{\pm}]$. Let $\sigma = \min\{\sigma_+, \sigma_-\}$. Below, the notation $f \lesssim g$ ($f \gtrsim g$) stands for $f \leq Cg$ ($f \geq Cg$), for a generic constant $C > 0$ independent of $\varepsilon = |\beta|_{\infty}$.

Lemma 8. *Assume $\varepsilon = |\beta|_{\infty} < \sigma$. For $|e_{\pm}| \geq \frac{\sigma}{2}$ (outside of separatrix), we have $|\frac{\partial x}{\partial \theta}| \lesssim 1$,*

$$\begin{aligned} &\left| \omega'_{\pm}(I_{\pm}) - \left(\frac{2\pi}{P_{\beta}} \right)^2 \right|, \quad |\partial_{I_{\pm}} x|, \quad \left| \omega_{\pm}(I_{\pm}) - \frac{2\pi}{P_0} |v| \right|, \\ &\left| \frac{d}{dI_{\pm}} \left(\frac{1}{\omega'_{\pm}(I_{\pm})} \right) \right|, \quad \left| x - \frac{P_{\beta}}{2\pi} \theta_{\pm} \right| \lesssim \varepsilon, \end{aligned}$$

and

$$\omega_{\pm}(I_{\pm}) = \frac{2\pi}{P_{\beta}} \sqrt{2e_{\pm}} + O\left(\frac{\varepsilon}{\sqrt{2e_{\pm}}}\right). \quad (35)$$

Proof. Fix $e_{\pm} \geq \frac{\sigma}{2} \gg \varepsilon$, from $\beta = \varepsilon \cos \frac{2\pi}{P_{\beta}} x + O(\varepsilon^2)$, we get

$$T_{\pm}(e_{\pm}) = \int_0^{P_{\beta}} \frac{dx'}{\sqrt{2(e_{\pm} \mp \beta(x'))}}} = \frac{P_{\beta}}{\sqrt{2e_{\pm}}} + O\left(\frac{\varepsilon}{(e_{\pm})^{\frac{3}{2}}}\right), \quad (36)$$

since by Taylor expansion

$$\left\{ 2(e_{\pm} \mp \beta(x')) \right\}^{-1/2} = \{2e_{\pm}\}^{-1/2} \left\{ 1 \mp \frac{\varepsilon}{2e_{\pm}} \cos \frac{2\pi x'}{P_{\beta}} + O\left(\left(\frac{\varepsilon}{e_{\pm}}\right)^2\right) \right\}. \quad (37)$$

Similarly, since $\frac{P_{\beta}}{T_{\pm}(e_{\pm})\sqrt{2e_{\pm}}} = 1 + O\left(\frac{\varepsilon}{e_{\pm}}\right)$,

$$\begin{aligned} \theta_{\pm} &= \frac{2\pi}{T_{\pm}(e_{\pm})} \int_0^x \frac{dx'}{\sqrt{2(e_{\pm} \mp \beta(x'))}}} \\ &= \frac{2\pi}{T_{\pm}(e)\sqrt{2e_{\pm}}} \left\{ x \mp \frac{\varepsilon P_{\beta}}{4\pi e_{\pm}} \sin \frac{2\pi}{P_{\beta}} x + O\left(\left(\frac{\varepsilon}{e_{\pm}}\right)^2\right) \right\} \\ &= \frac{2\pi}{P_{\beta}} x \mp \frac{\varepsilon}{2e_{\pm}} \sin \frac{2\pi}{P_{\beta}} x + O(\varepsilon^2), \end{aligned} \quad (38)$$

and thus $\left| \frac{\partial x}{\partial \theta} \right| \lesssim 1$, $\left| x - \frac{P_\beta}{2\pi} \theta_\pm \right|, \partial_{I_\pm} x \lesssim \varepsilon$. By (36) and (37), we have

$$\omega_\pm(I_\pm) = \frac{2\pi}{P_\beta} \sqrt{2e_\pm} + O\left(\frac{\varepsilon}{\sqrt{2e_\pm}}\right).$$

Combined above with $|v| = \sqrt{2e_\pm} + O(\varepsilon)$, we get $\left| \omega_\pm(I_\pm) - \frac{2\pi}{P_0} |v| \right| \lesssim \varepsilon$. Note that

$$\frac{dI_\pm}{de_\pm} = \frac{T_\pm(e_\pm)}{2\pi} = \frac{1}{\omega_\pm(I_\pm)},$$

and from (36)

$$T'_\pm(e_\pm) = -\frac{P_\beta}{(\sqrt{2e_\pm})^3} + O(\varepsilon) = -\frac{1}{P_\beta^2} T_\pm^3(e_\pm) + O(\varepsilon),$$

so

$$\omega'_\pm(I_\pm) = -\frac{2\pi}{T_\pm^2(e_\pm)} T'_\pm(e_\pm) \frac{de_\pm}{dI_\pm} = \left(\frac{2\pi}{P_\beta} \right)^2 + O(\varepsilon)$$

which implies that

$$\left| \omega'_\pm(I_\pm) - \left(\frac{2\pi}{P_\beta} \right)^2 \right|, \left| \frac{d}{dI_\pm} \left(\frac{1}{\omega'_\pm(I_\pm)} \right) \right| \lesssim \varepsilon.$$

This finishes the proof of the lemma. $\square$

The density operator $\rho(\lambda, \varepsilon)$: Consider a BGK wave solution

$$[\mu_{+,\pm}(e_+), \mu_{-,\pm}(e_-), -\beta_x]$$

as in Theorems 2 and 3. For an eigenvalue $\lambda = a + bi$, let $(g_\pm, \phi)$ be the eigenfunction satisfying (20) and (21). We will do the Fourier expansion of $g_\pm$ and ϕ in both action-angle variables $(I_\pm, \theta_\pm)$.

Define the spaces

$$H_\varepsilon^1 = \{P_\beta - \text{periodic } H^1 \text{ functions with zero mean}\},$$

and

$$H_0^1 = \{P_0 - \text{periodic } H^1 \text{ functions with zero mean}\}.$$

For any potential $\phi \in H_\varepsilon^1$, we expand

$$\phi = \sum_{k \in \mathbf{Z}} \phi_k^\pm(I_\pm) e^{ik\theta_\pm}, \quad \text{where } \phi_k^\pm(I_\pm) = \frac{1}{2\pi} \int_0^{2\pi} \phi(x) e^{-ik\theta_\pm} d\theta_\pm. \quad (39)$$

Then, we expand

$$g_\pm = \sum_k g_k^\pm(I_\pm) e^{ik\theta_\pm}, \quad k \in \mathbf{Z}. \quad (40)$$

Inside the separatrix ($e_{\pm}(x, v) < -\min \beta$): Since $|e_{\pm}(x, v)| = |\frac{1}{2}v^2 \pm \beta| \leq \varepsilon < \sigma$ so that $\phi_x v \mu'_{\pm, \pm} \equiv 0$ in (20), thanks to the flatness assumption for $\mu_{\pm, \pm}$. By (33),

$$v \partial_x g_{\pm} \mp \beta_x \partial_v g_{\pm} = \omega_{\pm}(I_{\pm}) \partial_{\theta_{\pm}} g_{\pm}, \quad (41)$$

in the trapped region, so $\lambda g_{\pm} + \omega_{\pm}(I_{\pm}) \partial_{\theta_{\pm}} g_{\pm} = 0$ from (20). We therefore deduce that $g_k^{\pm}(I_{\pm}) \equiv 0$ inside separatrix, for all $k \in \mathbb{Z}$.

Outside the separatrix ($e_{\pm}(x, v) > -\min \beta$): By (34), the equation (20) becomes

$$\lambda g_{+} \pm \omega_{+}(I_{+}) \partial_{\theta_{+}} g_{+} \mp \mu'_{+, \pm}(e_{+}) \omega_{+}(I_{+}) \partial_{\theta_{+}} \phi = 0, \text{ for } \pm v > 0, \quad (42)$$

$$\lambda g_{-} \pm \omega_{-}(I_{-}) \partial_{\theta_{-}} g_{-} \pm \mu'_{-, \pm}(e_{-}) \omega_{-}(I_{-}) \partial_{\theta_{-}} \phi = 0, \text{ for } \pm v > 0, \quad (43)$$

in the untrapped region. We may solve (42–43) by using the expansions (39–40) to obtain: If $\lambda \neq 0$, then $g_k^{\pm}(I) \equiv 0$ for $k = 0$ and for $k \neq 0$,

$$g_k^{\pm}(I_{+}) = \frac{\mu'_{+, \pm}(e_{+}) \omega_{+}(I_{+}) \phi_k^{\pm}(I_{+})}{\omega_{+} \pm \lambda / ik} \text{ for } \pm v > 0, \quad (44)$$

and

$$g_k^{-}(I_{-}) = -\frac{\mu'_{-, \pm}(e_{-}) \omega_{-}(I_{-}) \phi_k^{-}(I_{-})}{\omega_{+} \pm \lambda / ik} \text{ for } \pm v > 0. \quad (45)$$

We note that (44–45) are also valid inside the separatrix thanks to the flatness of $\mu_{\pm, \pm}$.

By using (44–45), we define the charge density operator as

$$\begin{aligned} \rho(\lambda, \varepsilon) \phi &= \sum_{k \neq 0, \pm} \left(\int_{v > 0} e^{ik\theta_{\pm}} \frac{\omega_{\pm} \mu'_{\pm, \pm}(e_{\pm})}{\omega_{\pm} \pm \frac{\lambda}{ik}} \phi_k^{\pm}(I_{\pm}) dv + \int_{v < 0} e^{ik\theta_{\pm}} \frac{\omega_{\pm} \mu'_{\pm, -}(e_{\pm})}{\omega_{\pm} - \frac{\lambda}{ik}} \phi_k^{\pm}(I_{\pm}) dv \right). \end{aligned} \quad (46)$$

In the formula (46), we note that $(I_{\pm}, \theta_{\pm})$ in the right hand side can be restricted to the untrapped region since $\mu'_{\pm, -}(e_{\pm}), \mu'_{\pm, +}(e_{\pm}) = 0$ in the trapped region. We also note that for any $\lambda \neq 0$,

$$\begin{aligned} \int_0^{P_{\beta}} \rho(\lambda, \varepsilon) \phi \, dx &= \int \int (g_{+} - g_{-}) \, dx dv \\ &= \int \int g_{+} dI_{+} d\theta_{+} - \int \int g_{-} dI_{-} d\theta_{-} = 0, \end{aligned}$$

by using that $g_k^{\pm} = 0$ when $k = 0$. Hence $-\partial_x^{-2} \rho(\lambda, \varepsilon) \phi$ is well-defined and the self-consistent Poisson equation (21) is reduced to $(\mathbf{I} + \partial_x^{-2} \rho(\lambda, \varepsilon)) \phi = 0$. So we conclude that for nonzero eigenvalues, the eigenspaces of (20–21) are equivalent to null spaces of the operator

$$\mathbf{I} + \partial_x^{-2} \rho(\lambda, \varepsilon) : H_{\varepsilon}^1 \rightarrow H_{\varepsilon}^1,$$

where ∂_x^{-2} denotes twice anti-derivatives with zero mean. To apply Lemma 4, we rescale above operators to be defined in the same function space H_0^1 . Let $G_{\beta}^{-1} \phi = \phi(\frac{P_0}{P_{\beta}} x)$ be the mapping from $H_0^1 \rightarrow H_{\varepsilon}^1$, and define the operator

$$K(\lambda, \varepsilon) = G_\beta \partial_x^{-2} \rho(\lambda, \varepsilon) G_\beta^{-1}$$

in H_0^1 .

To study the properties of the operators $K(\lambda, \varepsilon)$, we introduce two lemmas.

Lemma 9. *If $u(v) \in W^{s,p}(\mathbf{R})$ ($p > 1, s > \frac{1}{p}$), then for any $z \in \mathbf{C}$ with $\operatorname{Re} z \neq 0$, we have*

$$\left| \int_{\mathbf{R}} \frac{u(v)}{v-z} dv \right| \leq C \|u\|_{W^{s,p}(\mathbf{R})},$$

for some constant C independent of z .

Proof. Let $z = a + ib$, with $a, b \in \mathbf{R}$, $b \neq 0$. Then

$$\int_{\mathbf{R}} \frac{u(v)}{v-z} dv = \int_{\mathbf{R}} \frac{u(v)(v-a) + ibu(v)}{(v-a)^2 + b^2} dv.$$

Since $s > \frac{1}{p}$, the space $W^{s,p}(\mathbf{R})$ is embedded to the Hölder space $C^{0,\gamma}$ with $\gamma \in (0, s - \frac{1}{p})$. So

$$|u(v) - u(a)| \leq |v-a|^\gamma \|u\|_{C^{0,\gamma}} \leq C \|u\|_{W^{s,p}} |v-a|^\gamma.$$

The real part is estimated by

$$\begin{aligned} & \left| \int_{\mathbf{R}} \frac{u(v)(v-a)}{(v-a)^2 + b^2} dv \right| \\ & \leq \left| \int_{-1+a}^{1+a} \frac{(u(v) - u(a))(v-a)}{(v-a)^2 + b^2} dv \right| + \int_{|v-a| \geq 1} \left| \frac{u(v)}{v-a} \right| dv \\ & \leq \int_{-1+a}^{1+a} \left| \frac{u(v) - u(a)}{v-a} \right| dv + \left(\int_{|v-a| \geq 1} \frac{1}{|v-a|^{p'}} dv \right)^{\frac{1}{p'}} \|u\|_{L^p} \\ & \lesssim \|u\|_{W^{s,p}} \int_{-1+a}^{1+a} |v-a|^{-1+\gamma} dv + \|u\|_{L^p} \lesssim \|u\|_{W^{s,p}}. \end{aligned}$$

Similarly, for the imaginary part, we have

$$\begin{aligned} & \left| \int_{\mathbf{R}} \frac{bu(v)}{(v-a)^2 + b^2} dv \right| \\ & \leq \left| \int_{-1+a}^{1+a} \frac{(u(v) - u(a))b}{(v-a)^2 + b^2} dv \right| + \pi |u(a)| + \int_{|v-a| \geq 1} \left| \frac{u(v)}{v-a} \right| dv \\ & \lesssim \|u\|_{W^{s,p}}. \end{aligned}$$

Here, in the above we use

$$\left| \int_{-1+a}^{1+a} \frac{b}{(v-a)^2 + b^2} dv \right| \leq \int_{\mathbf{R}} \frac{1}{1+y^2} dy = \pi.$$

□

Lemma 10. Given $\phi \in H^1(0, P_\beta)$ and $\phi = \sum_k \phi_k^\pm(I_\pm) e^{ik\theta_\pm}$,

i) If $w : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ and $\int w \left(\frac{1}{2}v^2\right) dv < \infty$, then

$$\sum_k \int w(e_\pm) |\phi_k^\pm(I_\pm)|^2 dI_\pm \lesssim \|\phi\|_{L^2}^2, \quad (47)$$

and

$$\sum_k \int w(e_\pm) |\phi_k^{\pm'}(I_\pm)|^2 dI_\pm \lesssim \|\beta\|_{L^\infty}^2 \|\phi_x\|_{L^2}^2. \quad (48)$$

2) If $w : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ and $\int v^2 w \left(\frac{1}{2}v^2\right) dv < \infty$, then

$$\sum_{k \neq 0} k^2 \int \omega_\pm^2(I_\pm) w(e_\pm) |\phi_k^\pm(I_\pm)|^2 dI_\pm \lesssim \|\phi_x\|_{L^2}^2.$$

Proof. Proof of i): Since $(x, v) \rightarrow (I_\pm, \theta_\pm)$ has Jacobian 1, so

$$\begin{aligned} \sum_k \int w(e_\pm) |\phi_k^\pm(I_\pm)|^2 dI_\pm &= \int \int w(e_\pm) |\phi(x)|^2 dx dv \\ &\leq \sup_x \int w(e_\pm) dv \|\phi\|_{L^2}^2 \lesssim \|\phi\|_{L^2}^2. \end{aligned}$$

To prove (48), we notice that

$$\phi_k^{\pm'}(I_\pm) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\theta_\pm} \phi'(x) \frac{\partial x}{\partial I_\pm} d\theta_\pm,$$

and by Lemma 8, $\left| \frac{\partial x}{\partial I_\pm} \right| \lesssim \varepsilon$. So

$$\begin{aligned} \sum_k \int w(e_\pm) |\phi_k^{\pm'}(I_\pm)|^2 dI_\pm &= \int \int w(e_\pm) |\phi'(x)|^2 \left| \frac{\partial x}{\partial I_\pm} \right|^2 dx dv \\ &\lesssim \varepsilon^2 \sup_x \int w(e_\pm) dv \|\phi'\|_{L^2}^2 \lesssim \|\beta\|_{L^\infty}^2 \|\phi'\|_{L^2}^2. \end{aligned}$$

Proof of ii): We note that by Lemma 7,

$$v\phi_x = v\partial_x \mp \beta_x \partial_v = \begin{cases} \omega_\pm(I_\pm) \partial_{\theta_\pm} \phi & \text{when } (x, v) \in \Omega_0 \cup \Omega_+ \\ -\omega_\pm(I_\pm) \partial_{\theta_\pm} \phi & \text{when } (x, v) \in \Omega_- \end{cases},$$

where $\Omega_0, \Omega_\pm$ are defined in (29), (30), (31). Thus,

$$\begin{aligned} &\sum_{k \neq 0} k^2 \int \omega_\pm^2(I_\pm) w(e_\pm) |\phi_k^\pm(I_\pm)|^2 dI_\pm \\ &= \int \int w(e_\pm) |v\phi_x|^2 dx dv \\ &\leq \sup_x \int \int w(e_\pm) |v|^2 dv \|\phi'\|_{L^2}^2 \lesssim \|\phi_x\|_{L^2}^2. \end{aligned}$$

□

Lemma 11. (i) $0 \neq \lambda$ is an eigenvalue of the linearized VP operator iff there exists $0 \neq \phi \in H_0^1$ such that $\{I + K(\lambda, \varepsilon)\}\phi = 0$.

(ii) For $\varepsilon \ll \sigma$ and any $\lambda \in \mathbf{C}$ with $\text{Re } \lambda > 0$, the operator $K(\lambda, \varepsilon) : H_0^1 \rightarrow H_0^1$ is uniformly bounded to λ .

(iii) For $\varepsilon \ll \sigma$, $K(\lambda, \varepsilon) : H_0^1 \rightarrow H_0^1$ is analytic in λ when $|\lambda| \ll 1$.

Proof. (i) follows from the definition of the operator $K(\lambda, \varepsilon)$. Indeed, λ is a non-zero eigenvalue for the Vlasov-Poisson operator satisfying (20) (21), if and only if (42), (43), (44) and (45) are valid, which by the Poisson equation (21) is equivalent to $\phi_{xx} = \rho(\lambda, \varepsilon)\phi$ or $(I + \partial_x^{-2}\rho(\lambda, \varepsilon))\phi = 0$.

Proof of (ii): By the change of variable from $[0, P_0]$ to $[0, P_\beta]$, it is equivalent to show that the operator

$$\partial_x^{-2}\rho(\lambda, \varepsilon) : H_\varepsilon^1 \rightarrow H_\varepsilon^1$$

is uniformly bounded. For any $\phi \in H_\varepsilon^1$, let $\Phi = \partial_x^{-2}\rho(\lambda, \varepsilon)\phi$. Since $\Phi(x)$ has zero mean, so

$$\|\Phi\|_{H^1} \approx \|\Phi_x\|_{L^2} = \sup_{\substack{\psi \in L^2 \\ \|\psi\|_{L^2}=1}} (\Phi_x, \psi).$$

Let $\Psi = \partial_x^{-1}\psi$ be the anti-derivative of ψ with zero mean, then $\|\Psi\|_{H^1} \approx \|\psi\|_{L^2}$. Let

$$\Psi(x) = \sum_k e^{ik\theta_\pm} \Psi_k^\pm(I_\pm),$$

then

$$\begin{aligned} (\Phi_x, \psi) &= -(\Phi_{xx}, \Psi) = -(\rho(\lambda, \varepsilon)\phi, \Psi) \\ &= -\sum_{k \neq 0, \pm} \left(\int \frac{\omega_\pm \mu'_{\pm,+}}{\omega_\pm + \frac{\lambda}{ik}} \phi_k^\pm(I_\pm) \bar{\Psi}_k^\pm(I_\pm) dI_\pm + \int \frac{\omega_\pm \mu'_{\pm,-}}{\omega_\pm - \frac{\lambda}{ik}} \phi_k^\pm \bar{\Psi}_k^\pm dI_\pm \right). \end{aligned} \quad (49)$$

To estimate the above integrals, we change the integration variable to $\omega_\pm$ with $dI_\pm = \frac{1}{\omega'_\pm(I_\pm)} d\omega_\pm$. Define

$$H_k^{\pm,+}(\omega_\pm) = \omega_\pm \mu'_{\pm,+} \phi_k^\pm(I_\pm) \bar{\Psi}_k^\pm(I_\pm) \frac{1}{\omega'_\pm(I_\pm)}. \quad (50)$$

Noting that $\mu'_{\pm,\pm} \equiv 0$ for $|e_\pm| \leq \sigma$, so $H_k^{\pm,+}(\omega_\pm) = 0$ for $\omega_\pm$ near 0 (i.e. near separatrix). By zero extension, we can think of $H_k^{\pm,+}(\omega_\pm)$ as a function defined in $\mathbf{R}$. Then the first integral in (49) can be estimated by

$$\left| \int_{\mathbf{R}} \frac{H_k^{\pm,+}(\omega_\pm)}{\omega_\pm + \frac{\lambda}{ik}} d\omega_\pm \right| \lesssim \|H_k^{\pm,+}\|_{H^1(\mathbf{R})},$$

by Lemma 9. Since $\frac{\partial x}{\partial \theta}$ is bounded outside of separatrix, so

$$|\phi_k^\pm(I_\pm)| = \left| \frac{1}{2\pi k i} \int_0^{2\pi} e^{-ik\theta_\pm} \phi'(x) \frac{\partial x}{\partial \theta_\pm} d\theta_\pm \right| \leq \frac{1}{2\pi k} \|\phi\|_{W^{1,1}} \lesssim \frac{1}{k} \|\phi\|_{H^1}. \quad (51)$$

Similarly, $|\bar{\Psi}_k^\pm(I_\pm)| \lesssim \frac{1}{k} \|\Psi\|_{H^1}$. By assumption (9),

$$\int \sqrt{e_\pm} (\mu'_{\pm,+})^2 d e_\pm < \infty,$$

thus from

$$\frac{d\omega_\pm}{de_\pm} = \frac{d\omega_\pm}{dI_\pm} \frac{dI_\pm}{de_\pm} = \frac{\omega'_\pm}{\omega_\pm},$$

and (35)

$$\int (\omega_\pm \mu'_{\pm,+})^2 d\omega_\pm \lesssim \int \sqrt{e_\pm} (\mu'_{\pm,+})^2 d e_\pm < \infty$$

Thanks to Lemma 8, we know that $\omega'_\pm(I_\pm) \sim 1$ when $\mu'_{\pm,\pm} \neq 0$, since the supports of $\mu'_{\pm,\pm}$ are outside the separatrix. So

$$\begin{aligned} d\omega_\pm &= \omega'_\pm(I_\pm) dI_\pm \sim dI_\pm, \\ \frac{d}{d\omega_\pm} &= \frac{dI_\pm}{d\omega_\pm} \frac{d}{dI_\pm} = \frac{1}{\omega'_\pm(I_\pm)} \frac{d}{dI_\pm} \sim \frac{d}{dI_\pm}, \end{aligned} \quad (52)$$

when $\mu'_{\pm,\pm} \neq 0$. Combining above, we get

$$\|H_k^{\pm,+}\|_{L^2(\mathbf{R})} \lesssim \frac{1}{k^2} \|\phi\|_{H^1} \|\Psi\|_{H^1},$$

where $H_k^{\pm,+}$ is defined in (50). By using Lemma 8 and (51), we have

$$\begin{aligned} \left\| \frac{d}{d\omega_\pm} H_k^{\pm,+} \right\|_{L^2(\mathbf{R})} &\lesssim \frac{1}{k} \left(\|\phi\|_{H^1} \left(\int (\omega_\pm \mu'_{\pm,+})^2 |\bar{\Psi}_k^{\pm'}|^2 dI_\pm \right)^{\frac{1}{2}} \right. \\ &\quad \left. + \|\Psi\|_{H^1} \left(\int (\omega_\pm \mu'_{\pm,+})^2 |\phi_k^{\pm'}|^2 dI_\pm \right)^{\frac{1}{2}} \right) + \frac{1}{k^2} \|\phi\|_{H^1} \|\Psi\|_{H^1}. \end{aligned}$$

Thus

$$\begin{aligned} \left| \sum_{k \neq 0} \int_{\mathbf{R}} \frac{H_k^{\pm,+}(\omega_\pm)}{\omega_\pm + \frac{\lambda}{ik}} d\omega_\pm \right| &\lesssim \sum_{k \neq 0} \|H_k^{\pm,+}\|_{H^1(\mathbf{R})} \\ &\lesssim \sum_{k \neq 0} \left(\frac{1}{k} \|\phi\|_{H^1} \left(\int (\omega_\pm \mu'_{\pm,+})^2 |\bar{\Psi}_k^{\pm'}|^2 dI_\pm \right)^{\frac{1}{2}} \right. \\ &\quad \left. + \frac{1}{k} \|\Psi\|_{H^1} \left(\int (\omega_\pm \mu'_{\pm,+})^2 |\phi_k^{\pm'}|^2 dI_\pm \right)^{\frac{1}{2}} + \frac{1}{k^2} \|\phi\|_{H^1} \|\Psi\|_{H^1} \right) \\ &\lesssim \|\phi\|_{H^1} \left(\sum_k \int (\omega_\pm \mu'_{\pm,+})^2 |\bar{\Psi}_k^{\pm'}|^2 dI_\pm \right)^{\frac{1}{2}} \\ &\quad + \|\Psi\|_{H^1} \left(\sum_k \int (\omega_\pm \mu'_{\pm,+})^2 |\phi_k^{\pm'}|^2 dI_\pm \right)^{\frac{1}{2}} + \|\phi\|_{H^1} \|\Psi\|_{H^1} \\ &\lesssim \|\phi\|_{H^1} \|\Psi\|_{H^1}, \end{aligned}$$

by Lemma 10. The second term in (49) can be estimated in the same way. So we have

$$\left\| \partial_x^{-2} \rho(\lambda, \varepsilon) \phi \right\|_{H^1} = \|\Phi\|_{H^1} \lesssim \|\phi\|_{H^1}.$$

Proof of (iii): By (36), when $|e_{\pm}| \geq \sigma$, we have

$$\omega_{\pm} = 2\pi/T_{\pm}(e_{\pm}) \gtrsim \sqrt{\sigma}.$$

So for $|\lambda| \ll \sqrt{\sigma}$,

$$\left| \frac{\lambda}{ik} + \omega_{\pm} \right| \gtrsim \sqrt{\sigma}, \text{ uniformly for any } |k| \geq 1. \quad (53)$$

Hence, the integrals in (49) are clearly bounded by $\|\phi\|_{H^1} \|\Psi\|_{H^1}$. We further take complex λ derivatives of (49) to get

$$\begin{aligned} & (\partial_{\lambda} (\rho(\lambda, \varepsilon) \phi), \Psi) \\ &= - \sum_{k \neq 0, \pm} \left(- \int \frac{\omega_{\pm} \mu'_{\pm, +}}{ik(\omega_{\pm} + \frac{\lambda}{ik})^2} \phi_k^{\pm}(I_{\pm}) \bar{\Psi}_k^{\pm}(I_{\pm}) dI_{\pm} + \int \frac{\omega_{\pm} \mu'_{\pm, -}}{(\omega_{\pm} - \frac{\lambda}{ik})^2} \phi_k^{\pm} \bar{\Psi}_k^{\pm} dI_{\pm} \right), \end{aligned}$$

which by (53) again is bounded by $\|\phi\|_{H^1} \|\Psi\|_{H^1}$. This shows that $\partial_{\lambda} K(\lambda, \varepsilon)$ is bounded operator in H_0^1 and thus $K(\lambda, \varepsilon)$ is an analytic operator in H_0^1 when $|\lambda| \ll 1$. $\square$

Proposition 12. For $\varepsilon \ll \sigma$ and any $\lambda \in \mathbf{C}$ with $\text{Re } \lambda > 0$,

$$\|K(\lambda, \varepsilon) - K(\lambda, 0)\|_{L(H_0^1, H_0^1)} \leq C\sqrt{\varepsilon}, \quad (54)$$

where C is a positive constant independent of λ .

Proof. It is equivalent to show that

$$\left\| \partial_x^{-2} \rho(\lambda, \varepsilon) - \partial_x^{-2} \rho(\lambda, 0) \right\|_{L(H_{\varepsilon}^1, H_{\varepsilon}^1)} \leq C\sqrt{\varepsilon},$$

where $C > 0$ is independent of λ . Let $\phi, \Psi \in H_{\varepsilon}^1$, then it suffices to show that

$$|((\rho(\lambda, \varepsilon) - \rho(\lambda, 0)) \phi, \Psi)| \leq C\sqrt{\varepsilon} \|\phi\|_{H^1} \|\Psi\|_{H^1},$$

for some constant C independent of λ .

The proof is split into three steps.

Step 1. Representation. We note that for the homogeneous case ($\varepsilon = 0$) with period P_{β} , the action-angle variables become

$$\omega_{\pm} = \frac{2\pi}{P_{\beta}} |v|, \quad I_{\pm} = \frac{P_{\beta}}{2\pi} |v|, \quad \theta_{\pm} = \frac{2\pi}{P_{\beta}} x.$$

Then

$$\begin{aligned}
 & (\rho(\lambda, 0)\phi, \Psi) \\
 &= \sum_{k \neq 0, \pm} \left(\int \frac{\omega_{\pm} \mu'_{\pm,+}}{\omega_{\pm} + \frac{\lambda}{ik}} \phi_k^{\pm,0} \bar{\Psi}_k^{\pm,0} dI_{\pm} + \int \frac{\omega_{\pm} \mu'_{\pm,-}}{\omega_{\pm} - \frac{\lambda}{ik}} \phi_k^{\pm,0} \bar{\Psi}_k^{\pm,0} dI_{\pm} \right) \\
 &= \sum_{k \neq 0, \pm} \left(\frac{P_{\beta}}{2\pi} \right)^2 \left(\int \frac{\omega_{\pm} \mu'_{\pm,+}}{\omega_{\pm} + \frac{\lambda}{ik}} \phi_k^{\pm,0} \bar{\Psi}_k^{\pm,0} d\omega_{\pm} + \int \frac{\omega_{\pm} \mu'_{\pm,-}}{\omega_{\pm} - \frac{\lambda}{ik}} \phi_k^{\pm,0} \bar{\Psi}_k^{\pm,0} d\omega_{\pm} \right) \\
 &= \sum_{k \neq 0, \pm} \int \frac{H_k^{\pm,+;0}}{\omega_{\pm} + \frac{\lambda}{ik}} d\omega_{\pm} + \int \frac{H_k^{\pm,-;0}}{\omega_{\pm} - \frac{\lambda}{ik}} d\omega_{\pm},
 \end{aligned}$$

where

$$\begin{aligned}
 \mu'_{\pm,\pm} &= \mu'_{\pm,\pm} \left(\frac{1}{2} \left(\frac{P_{\beta}}{2\pi} \right)^2 \omega_{\pm}^2 \right) = \mu'_{\pm,\pm} \left(\frac{1}{2} v^2 \right), \\
 \phi_k^{\pm,0} &\equiv \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\theta_{\pm}} \phi \left(\frac{P_{\beta}}{2\pi} \theta_{\pm} \right) d\theta_{\pm}, \\
 \Psi_k^{\pm,0} &\equiv \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\theta_{\pm}} \Psi \left(\frac{P_{\beta}}{2\pi} \theta_{\pm} \right) d\theta_{\pm},
 \end{aligned}$$

and

$$\begin{aligned}
 H_k^{\pm,+;0}(\omega_{\pm}) &= \left(\frac{P_{\beta}}{2\pi} \right)^2 \omega_{\pm} \mu'_{\pm,+} \left(\frac{1}{2} \left(\frac{P_{\beta}}{2\pi} \right)^2 \omega_{\pm}^2 \right) \phi_k^{\pm,0} \bar{\Psi}_k^{\pm,0}, \\
 H_k^{\pm,-;0}(\omega_{\pm}) &= \left(\frac{P_{\beta}}{2\pi} \right)^2 \omega_{\pm} \mu'_{\pm,-} \left(\frac{1}{2} \left(\frac{P_{\beta}}{2\pi} \right)^2 \omega_{\pm}^2 \right) \phi_k^{\pm,0} \bar{\Psi}_k^{\pm,0}.
 \end{aligned}$$

In the case $\varepsilon > 0$, we use the same notations $(I_{\pm}, \theta_{\pm})$ and $\omega_{\pm}$ for action-angle variables and frequency, to make it convenient to estimate the difference $((\rho(\lambda, \varepsilon) - \rho(\lambda, 0))\phi, \Psi)$. Then as in the proof of Lemma 11, we can write

$$(\rho(\lambda, \varepsilon)\phi, \Psi) = \sum_{k \neq 0, \pm} \int \frac{H_k^{\pm,+;\varepsilon}(\omega_{\pm})}{\omega_{\pm} + \frac{\lambda}{ik}} d\omega_{\pm} + \int \frac{H_k^{\pm,-;\varepsilon}(\omega_{\pm})}{\omega_{\pm} - \frac{\lambda}{ik}} d\omega_{\pm},$$

where

$$\begin{aligned}
 H_k^{\pm,+;\varepsilon}(\omega_{\pm}) &= \omega_{\pm} \mu'_{\pm,+}(e_{\pm}) \phi_k^{\pm,\varepsilon}(I_{\pm}) \bar{\Psi}_k^{\pm,\varepsilon}(I_{\pm}) \frac{1}{\omega'_{\pm}(I_{\pm})}, \\
 H_k^{\pm,-;\varepsilon}(\omega_{\pm}) &= \omega_{\pm} \mu'_{\pm,-}(e_{\pm}) \phi_k^{\pm,\varepsilon}(I_{\pm}) \bar{\Psi}_k^{\pm,\varepsilon}(I_{\pm}) \frac{1}{\omega'_{\pm}(I_{\pm})},
 \end{aligned}$$

with

$$\begin{aligned}
 \phi_k^{\pm,\varepsilon}(I_{\pm}) &= \frac{1}{2\pi} \int_0^{2\pi} \phi(x(I_{\pm}, \theta_{\pm})) e^{-ik\theta_{\pm}} d\theta_{\pm}, \\
 \Psi_k^{\pm,\varepsilon}(I_{\pm}) &= \frac{1}{2\pi} \int_0^{2\pi} \Psi(x(I_{\pm}, \theta_{\pm})) e^{-ik\theta_{\pm}} d\theta_{\pm}.
 \end{aligned}$$

Here, in the above formula for $\phi_k^{\pm,\varepsilon}$, $\Psi_k^{\pm,\varepsilon}$, we use $(x(I_\pm, \theta_\pm), v(I_\pm, \theta_\pm))$ to denote the action-angle transform $(I_\pm, \theta_\pm) \rightarrow (x, v)$ in the case $\varepsilon > 0$. Defining

$$\begin{aligned} G_k^{\pm,+}(\omega_\pm) &= H_k^{\pm,+;\varepsilon}(\omega_\pm) - H_k^{\pm,+;0}(\omega_\pm), \\ G_k^{\pm,-}(\omega_\pm) &= H_k^{\pm,-;\varepsilon}(\omega_\pm) - H_k^{\pm,-;0}(\omega_\pm), \end{aligned} \quad (55)$$

then we get

$$((\rho(\lambda, \varepsilon) - \rho(\lambda, 0))\phi, \Psi) = \sum_{k \neq 0, \pm} \int \frac{G_k^{\pm,+}(\omega_\pm)}{\omega_\pm + \frac{\lambda}{ik}} d\omega_\pm + \int \frac{G_k^{\pm,-}(\omega_\pm)}{\omega_\pm - \frac{\lambda}{ik}} d\omega_\pm.$$

By Lemma 9, the proof is reduced to estimate $\|G_k^{\pm,\pm}\|_{H^1}$. We write

$$\begin{aligned} G_k^{\pm,+}(\omega_\pm) &= \omega_\pm \left(\mu'_{\pm,+}(e_\pm) - \mu'_{\pm,+} \left(\frac{1}{2} \left(\frac{P_\beta}{2\pi} \right)^2 \omega_\pm^2 \right) \right) \phi_k^{\pm,\varepsilon}(I_\pm) \bar{\Psi}_k^{\pm,\varepsilon}(I_\pm) \frac{1}{\omega'_\pm(I_\pm)} \\ &\quad + \omega_\pm \mu'_{\pm,+} \left(\frac{1}{2} \left(\frac{P_\beta}{2\pi} \right)^2 \omega_\pm^2 \right) \left(\frac{1}{\omega'_\pm(I_\pm)} - \left(\frac{P_\beta}{2\pi} \right)^2 \right) \phi_k^{\pm,\varepsilon}(I_\pm) \bar{\Psi}_k^{\pm,\varepsilon}(I_\pm) \\ &\quad + \omega_\pm \mu'_{\pm,+} \left(\frac{1}{2} \left(\frac{P_\beta}{2\pi} \right)^2 \omega_\pm^2 \right) \left(\frac{P_\beta}{2\pi} \right)^2 (\phi_k^{\pm,\varepsilon}(I_\pm) - \phi_k^{\pm,0}) \bar{\Psi}_k^{\pm,\varepsilon}(I_\pm) \\ &\quad + \omega_\pm \mu'_{\pm,+} \left(\frac{1}{2} \left(\frac{P_\beta}{2\pi} \right)^2 \omega_\pm^2 \right) \left(\frac{P_\beta}{2\pi} \right)^2 \phi_k^{\pm,0} (\bar{\Psi}_k^{\pm,\varepsilon}(I_\pm) - \bar{\Psi}_k^{\pm,0}) \\ &= \sum_{j=1}^4 G_k^{\pm,+;j}(\omega_\pm). \end{aligned} \quad (56)$$

Step 2. Estimates for $\phi_k^{\pm,\varepsilon} - \phi_k^{\pm,0}$ and $\psi_k^{\pm,\varepsilon} - \psi_k^{\pm,0}$ when $|e_\pm| \geq \frac{\sigma}{2}$.

We note that

$$\begin{aligned} \phi_k^{\pm,\varepsilon}(I_\pm) - \phi_k^{\pm,0} &= \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\theta_\pm} [\phi(x(I_\pm, \theta_\pm)) - \phi(\frac{P_\beta}{2\pi}\theta_\pm)] d\theta_\pm \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\theta_\pm} \left[\int_{\frac{P_\beta}{2\pi}\theta_\pm}^{x(I_\pm, \theta_\pm)} \phi_x(\zeta) d\zeta \right] d\theta_\pm, \end{aligned} \quad (57)$$

and $\left| x - \frac{P_\beta}{2\pi}\theta_\pm \right| \lesssim \varepsilon$ by Lemma 8, so

$$\left| \phi_k^{\pm,\varepsilon}(I_\pm) - \phi_k^{\pm,0} \right| \leq \max_{\theta_\pm} \left| \int_{\frac{P_\beta}{2\pi}\theta_\pm}^{x(I_\pm, \theta_\pm)} \phi_x(\zeta) d\zeta \right| \leq \min \{ \sqrt{\varepsilon} \|\phi\|_{H^1}, \varepsilon \|\phi_x\|_\infty \}, \quad (58)$$

and

$$\begin{aligned}
 & \sum_k \int (\omega_{\pm} \mu'_{\pm,+})^2 \left| \phi_k^{\pm,\varepsilon}(I_{\pm}) - \phi_k^{\pm,0} \right|^2 dI_{\pm} \\
 &= \int \int (\omega_{\pm} \mu'_{\pm,+})^2 \left| \int_{\frac{P_{\beta}}{2\pi} \theta_{\pm}(x,v)}^x \phi_x(\zeta) d\zeta \right|^2 dx dv \\
 &\leq \min \left\{ \varepsilon \|\phi\|_{H^1}^2, \varepsilon^2 \|\phi_x\|_{\infty}^2 \right\} \int \int (\omega_{\pm} \mu'_{\pm,+})^2 dx dv \\
 &\lesssim \min \left\{ \varepsilon \|\phi\|_{H^1}^2, \varepsilon^2 \|\phi_x\|_{\infty}^2 \right\}.
 \end{aligned} \tag{59}$$

Similarly, we have

$$\begin{aligned}
 \Psi_k^{\pm,\varepsilon}(I_{\pm}) - \Psi_k^{\pm,0} &= \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\theta_{\pm}} \left[\int_{\frac{P_{\beta}}{2\pi} \theta_{\pm}}^{x(I_{\pm}, \theta_{\pm})} \Psi_x(\zeta) d\zeta \right] d\theta_{\pm}, \\
 \left| \Psi_k^{\pm,\varepsilon}(I_{\pm}) - \Psi_k^{\pm,0} \right| &\leq \min \left\{ \sqrt{\varepsilon} \|\Psi\|_{H^1}, \varepsilon \|\Psi_x\|_{\infty} \right\},
 \end{aligned} \tag{60}$$

and

$$\sum_k \int (\omega_{\pm} \mu'_{\pm,+})^2 \left| \Psi_k^{\pm,\varepsilon}(I_{\pm}) - \Psi_k^{\pm,0} \right|^2 dI_{\pm} \lesssim \min \left\{ \varepsilon \|\Psi\|_{H^1}^2, \varepsilon^2 \|\Psi_x\|_{\infty}^2 \right\}. \tag{61}$$

This completes Step 2.

Step 3. Estimate for $\|G^{\pm,\pm}\|_{H^1}$.

Here, the function $G^{\pm,\pm}$ is defined in (55). We will estimate $\|G^{\pm,+}\|_{H^1}$ and it is similar for $\|G^{\pm,-}\|_{H^1}$. we have

$$e_{\pm} - \frac{1}{2} \left(\frac{P_{\beta}}{2\pi} \right)^2 \omega_{\pm}^2 = O(\varepsilon),$$

and thus by (9)

$$\left| \mu'_{\pm,+}(e_{\pm}) - \mu'_{\pm,+} \left(\frac{1}{2} \left(\frac{P_{\beta}}{2\pi} \right)^2 \omega_{\pm}^2 \right) \right| \leq C \varepsilon (1 + \omega_{\pm}^2)^{-\gamma}, \quad \gamma > 1.$$

We recall that $G^{\pm,+} = \sum_{j=1}^4 G^{\pm,+,j}$ (see (56)) in the following estimates, where we also use (52) to transform the integrals $\int \cdots d\omega_{\pm}$ to $\int \cdots dI_{\pm}$. By (51),

$$\left\| G_k^{\pm,+,1} \right\|_{L^2} \lesssim \frac{\varepsilon}{k^2} \|\phi\|_{H^1} \|\Psi\|_{H^1}.$$

By Lemma 8 $\left| \frac{1}{\omega_{\pm}(I_{\pm})} - \left(\frac{P_{\beta}}{2\pi} \right)^2 \right| \lesssim \varepsilon$, so

$$\left\| G_k^{\pm,+,2} \right\|_{L^2} \lesssim \frac{\varepsilon}{k^2} \|\phi\|_{H^1} \|\Psi\|_{H^1}.$$

It is straightforward to get

$$\left\| G_k^{\pm,+,3} \right\|_{L^2} \lesssim \frac{1}{k} \|\Psi\|_{H^1} \left(\int (\omega_{\pm} \mu'_{\pm,+})^2 \left| \phi_k^{\pm,\varepsilon}(I_{\pm}) - \phi_k^{\pm,0} \right|^2 dI_{\pm} \right)^{\frac{1}{2}},$$

and

$$\left\| G_k^{\pm,+,4} \right\|_{L^2} \lesssim \frac{1}{k} \|\phi\|_{H^1} \left(\int (\omega_{\pm} \mu'_{\pm,+})^2 \left| \Psi_k^{\pm,\varepsilon}(I_{\pm}) - \Psi_k^{\pm,0} \right|^2 dI_{\pm} \right)^{\frac{1}{2}}.$$

Now we estimate $\left\| \frac{d}{d\omega_{\pm}} G_k^{\pm,+}(\omega_{\pm}) \right\|_{L^2}$. We note that

$$\begin{aligned} & \left| \frac{d}{d\omega_{\pm}} \left[\mu'_{\pm,+}(e_{\pm}) - \mu'_{\pm,+} \left(\frac{1}{2} \left(\frac{P_{\beta}}{2\pi} \right)^2 \omega_{\pm}^2 \right) \right] \right| \\ &= \left| \mu''_{\pm,+}(e_{\pm}) \frac{\omega_{\pm}}{\omega'_{\pm}} - \mu''_{\pm,+} \left(\frac{1}{2} \left(\frac{P_{\beta}}{2\pi} \right)^2 \omega_{\pm}^2 \right) \left(\frac{P_{\beta}}{2\pi} \right)^2 \omega_{\pm} \right| \\ &\lesssim \varepsilon \omega_{\pm} \left(1 + \omega_{\pm}^2 \right)^{-\gamma}, \end{aligned}$$

by (9) and Lemma 8. So we can get

$$\begin{aligned} & \left\| \frac{d}{d\omega_{\pm}} G_k^{\pm,+,1}(\omega_{\pm}) \right\|_{L^2} \\ &\lesssim \frac{\varepsilon}{k^2} \|\phi\|_{H^1} \|\Psi\|_{H^1} + \frac{1}{k} \|\phi\|_{H^1} \left(\int (\omega_{\pm} \mu'_{\pm,+})^2 \left| (\Psi_k^{\pm,\varepsilon})'(I_{\pm}) \right|^2 dI_{\pm} \right)^{\frac{1}{2}} \\ &\quad + \frac{1}{k} \|\Psi\|_{H^1} \left(\int (\omega_{\pm} \mu'_{\pm,+})^2 \left| (\phi_k^{\pm,\varepsilon})'(I_{\pm}) \right|^2 dI_{\pm} \right)^{\frac{1}{2}}. \end{aligned}$$

For $\left\| \frac{d}{d\omega_{\pm}} G_k^{\pm,+,2}(\omega_{\pm}) \right\|_{L^2}$, the above estimate also holds true by using

$$\left| \frac{d}{dI_{\pm}} \left(\frac{1}{\omega'_{\pm}(I_{\pm})} \right) \right| \lesssim \varepsilon.$$

By using (58) and (60), we can get that

$$\begin{aligned} & \left\| \frac{d}{d\omega_{\pm}} G_k^{\pm,+,3}(\omega_{\pm}) \right\|_{L^2} + \left\| \frac{d}{d\omega_{\pm}} G_k^{\pm,+,4}(\omega_{\pm}) \right\|_{L^2} \\ &\lesssim \frac{1}{k} \|\phi\|_{H^1} \left(\int (\omega_{\pm} \mu'_{\pm,+})^2 \left| (\Psi_k^{\pm,\varepsilon})'(I_{\pm}) \right|^2 dI_{\pm} \right)^{\frac{1}{2}} \\ &\quad + \frac{1}{k} \|\Psi\|_{H^1} \left(\int (\omega_{\pm} \mu'_{\pm,+})^2 \left| (\phi_k^{\pm,\varepsilon})'(I_{\pm}) \right|^2 dI_{\pm} \right)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
 & + \min \{ \sqrt{\varepsilon} \|\Psi\|_{H^1}, \varepsilon \|\Psi_x\|_{\infty} \} \left(\int \left(\omega_{\pm}^2 \mu''_{\pm,+} \right)^2 \left| \phi_k^{\pm,\varepsilon} \right|^2 dI_{\pm} \right)^{\frac{1}{2}} \\
 & + \min \{ \sqrt{\varepsilon} \|\phi\|_{H^1}, \varepsilon \|\phi_x\|_{\infty} \} \left(\int \left(\omega_{\pm}^2 \mu''_{\pm,+} \right)^2 \left| \Psi_k^{\pm,\varepsilon} \right|^2 dI_{\pm} \right)^{\frac{1}{2}}.
 \end{aligned}$$

Combining above, we have

$$\begin{aligned}
 \sum_k \left\| G_k^{\pm,+} \right\|_{H^1} & \lesssim \sum_k \left\{ \frac{\varepsilon}{k^2} \|\phi\|_{H^1} \|\Psi\|_{H^1} \right. \\
 & + \frac{1}{k} \|\Psi\|_{H^1} \left(\int \left(\omega_{\pm} \mu'_{\pm,+} \right)^2 \left| \phi_k^{\pm,\varepsilon}(I_{\pm}) - \phi_k^{\pm,0} \right|^2 dI_{\pm} \right)^{\frac{1}{2}} \\
 & + \frac{1}{k} \|\phi\|_{H^1} \left(\int \left(\omega_{\pm} \mu'_{\pm,+} \right)^2 \left| \Psi_k^{\pm,\varepsilon}(I_{\pm}) - \Psi_k^{\pm,0} \right|^2 dI_{\pm} \right)^{\frac{1}{2}} \\
 & + \frac{1}{k} \|\phi\|_{H^1} \left(\int \left(\omega_{\pm} \mu'_{\pm,+} \right)^2 \left| \left(\Psi_k^{\pm,\varepsilon} \right)'(I_{\pm}) \right|^2 dI_{\pm} \right)^{\frac{1}{2}} \\
 & + \frac{1}{k} \|\Psi\|_{H^1} \left(\int \left(\omega_{\pm} \mu'_{\pm,+} \right)^2 \left| \left(\phi_k^{\pm,\varepsilon} \right)'(I_{\pm}) \right|^2 dI_{\pm} \right)^{\frac{1}{2}} \\
 & + \min \{ \sqrt{\varepsilon} \|\Psi\|_{H^1}, \varepsilon \|\Psi_x\|_{\infty} \} \left(\int \left(\omega_{\pm}^2 \mu''_{\pm,+} \right)^2 \left| \phi_k^{\pm,\varepsilon} \right|^2 dI_{\pm} \right)^{\frac{1}{2}} \\
 & + \min \{ \sqrt{\varepsilon} \|\phi\|_{H^1}, \varepsilon \|\phi_x\|_{\infty} \} \left(\int \left(\omega_{\pm}^2 \mu''_{\pm,+} \right)^2 \left| \Psi_k^{\pm,\varepsilon} \right|^2 dI_{\pm} \right)^{\frac{1}{2}} \Big\} \\
 & \leq \sqrt{\varepsilon} \|\Psi\|_{H^1} \|\phi\|_{H^1},
 \end{aligned}$$

by using Lemma 10 and (59), (61). Similarly, we get

$$\sum_k \left\| G_k^{\pm,-} \right\|_{H^1} \lesssim \sqrt{\varepsilon} \|\Psi\|_{H^1} \|\phi\|_{H^1},$$

So by Lemma 9, we have

$$\begin{aligned}
 |((\rho(\lambda, \varepsilon) - \rho(\lambda, 0)) \phi, \Psi)| & \lesssim \sum_k \left\| G_k^{\pm,+} \right\|_{H^1} + \left\| G_k^{\pm,-} \right\|_{H^1} \\
 & \lesssim \sqrt{\varepsilon} \|\Psi\|_{H^1} \|\phi\|_{H^1}.
 \end{aligned}$$

This finishes the proof. $\square$

Lemma 13. Assume the conditions in Theorems 3 and 2. For any $\delta_0 > 0$, when ε is small enough, there exists no unstable eigenvalue λ of the linearized VP (20) (21) with $\text{Re } \lambda > 0$ and $|\lambda| \geq \delta_0$.

Proof. We note that $I + \partial_x^{-2} \rho(\lambda, 0)$ is uniformly invertible when $\text{Re } \lambda > 0$ and $|\lambda| \geq \delta_0$ (see Lemma 15). So the conclusion follows from Proposition 12. $\square$

The following lemma is used in the proof of Theorem 3.

Lemma 14. Assume $\mu_{\pm,+} = \mu_{\pm,-} = \mu_{\pm}$ (even case). Let $\lambda \in \mathbf{C}$ and $|\lambda| \ll \sqrt{\sigma}$, $\phi, \psi \in H_0^1$ and $\phi_x, \psi_x \in L^\infty$. Then

$$|(\partial_\lambda (K(\lambda, \varepsilon) - K(\lambda, 0)) \phi, \psi)| \leq C\varepsilon |\lambda| \|\phi_x\|_{L^\infty} \|\psi\|_{L^\infty}.$$

Proof. It suffices to estimate

$$\left| (\partial_\lambda \left(\partial_x^{-2} \rho(\lambda, \varepsilon) - \partial_x^{-2} \rho(\lambda, 0) \right) \phi, \psi) \right|$$

for $\phi, \psi \in H_\varepsilon^1$. Note that in the untrapped region, the points (x, v) and $(x, -v)$ have the same action-angle coordinates $(I_\pm, \theta_\pm)$. So when $\mu_{\pm,+} = \mu_{\pm,-} = \mu_{\pm}$, we can combine the integrals for $\rho(\lambda, \varepsilon)\phi$ in (46) as

$$\begin{aligned} & \sum_{k \neq 0, \pm} \left[\int_{v>0} \left(\frac{\omega_\pm}{\omega_\pm + \frac{\lambda}{ik}} + \frac{\omega_\pm}{\omega_\pm - \frac{\lambda}{ik}} \right) \mu'_\pm(e_\pm) \phi_k^\pm(I_\pm) e^{ik\theta_\pm} dv \right] \\ &= \sum_{k \neq 0, \pm} \int_{v>0} \frac{2\omega_\pm^2 \mu'_\pm(e_\pm)}{\omega_\pm^2 + \frac{\lambda^2}{k^2}} \phi_k^\pm(I_\pm) e^{ik\theta_\pm} dv, \end{aligned} \quad (62)$$

and then

$$\partial_\lambda \rho(\lambda, \varepsilon) \phi = -4 \sum_{k, \pm} \int_{v>0} \frac{\omega_\pm^2 \mu'_\pm(e_\pm) \frac{\lambda}{k^2}}{\left(\omega_\pm^2 + \frac{\lambda^2}{k^2} \right)^2} \phi_k^\pm(I_\pm) e^{ik\theta_\pm} dv.$$

Note that,

$$(\partial_x^{-2} \partial_\lambda \rho(\lambda, \varepsilon) \phi, \psi) = (\partial_\lambda \rho(\lambda, \varepsilon) \phi, \Psi),$$

where $\Psi = \partial_x^{-2} \psi$. Then

$$\begin{aligned} & (\partial_\lambda \left(\partial_x^{-2} \rho(\lambda, \varepsilon) - \partial_x^{-2} \rho(\lambda, 0) \right) \phi, \psi) \\ &= (\partial_\lambda (\rho(\lambda, \varepsilon) - \rho(\lambda, 0)) \phi, \Psi) \\ &= -4\lambda \sum_{k, \pm} \frac{1}{k^2} \int \frac{\omega_\pm^2}{[\omega_\pm^2 + \frac{\lambda^2}{k^2}]^2} G_k^\pm(\omega_\pm) d\omega_\pm, \end{aligned}$$

where

$$G_k^\pm(\omega_\pm) = G_k^{\pm,+}(\omega_\pm) = G_k^{\pm,-}(\omega_\pm)$$

is defined in (55). Note that $\omega_\pm \gtrsim \sqrt{\sigma}$ in the support of $G_k^\pm$ and $|\lambda| \ll \sqrt{\sigma}$, so

$$\left\| \frac{\omega_\pm^2}{[\omega_\pm^2 + \frac{\lambda^2}{k^2}]^2} \right\|_{L^2} \leq C_0 \text{ (independent of } \lambda \text{)}.$$

Thus

$$\begin{aligned} |(\partial_\lambda (\rho(\lambda, \varepsilon) - \rho(\lambda, 0)) \phi, \Psi)| &\leq \sum_{k, \pm} \frac{4|\lambda|}{k^2} \int \frac{\omega_\pm^2}{[\omega_\pm^2 + \frac{\lambda^2}{k^2}]^2} G_k^\pm(\omega_\pm) d\omega_\pm \\ &\lesssim \sum_{k, \pm} \frac{4|\lambda|}{k^2} \|G_k^\pm\|_{L^2} \lesssim \varepsilon |\lambda| \|\phi_x\|_{L^\infty} \|\Psi_x\|_{L^\infty}, \end{aligned}$$

by the same estimates for $\sum_k \|G_k^{\pm, \pm}\|_{L^2}$ as in the proof of Proposition 12 and by choosing L^∞ bounds in (59) and (61). This finishes the proof of the lemma. $\square$

5. Proof of Main Theorems

First, we study the spectrum of the linearized operator $\mathbf{I} + K(\lambda, 0)$ in H_0^1 for flat homogeneous states.

Lemma 15. (i) (Even Case) Consider the flat homogeneous states satisfying the assumptions in Theorem 3. Then:

- (1) $\mathbf{I} + K(\lambda, 0)$ is invertible when $\text{Re } \lambda > 0$; for any $\delta > 0$, $(\mathbf{I} + K(\lambda, 0))^{-1}$ is uniformly bounded in $\{\text{Re } \lambda > 0, |\lambda| \geq \delta\}$;
- (2) $\mathbf{I} + K(\lambda, 0)$ is analytic near $\lambda = 0$; $\ker(\mathbf{I} + K(0, 0)) = \{e^{i\frac{2\pi}{P_0}x}, e^{-i\frac{2\pi}{P_0}x}\} = \{r_1, r_2\}$; $\mathbf{I} + K(0, 0) : (I - \mathbf{P})X \rightarrow (I - \mathbf{P})X$ is invertible, where $\mathbf{P}$ is the project to $\text{span}\{r_1, r_2\}$.
- (3) $\det((\mathbf{I} + K(\lambda, 0)) r_j, r_i) \sim \lambda^4$ near $\lambda = 0$.

(ii) (Uneven Case) Consider the flat homogeneous states satisfying the assumptions in Theorem 2. Then (1) and (2) are valid, and $\det((\mathbf{I} + K(\lambda, 0)) r_j, r_i) \sim \lambda^2$ near $\lambda = 0$.

Proof. (i) (Even Case) We only consider the two-species case and the proof for the fixed ion case is similar. Denote $\mu_{\pm, +} = \mu_{\pm, -} = \mu_\pm$.

Proof of (1): We note that when $\varepsilon = 0$, $\mathbf{I} + K(\lambda, 0)$ is the Penrose diagonal operator

$$\begin{aligned} (\mathbf{I} + K(\lambda, 0)) e^{ik\frac{2\pi}{P_0}x} &= \{\mathbf{I} + \partial_x^{-2} \rho(\lambda, 0)\} e^{ik\frac{2\pi}{P_0}x} \\ &= \left\{ 1 - \frac{4\pi^2}{P_0^2 k^2} \int \frac{v[\mu'_+ + \mu'_-]}{v + \frac{\lambda}{ik}} dv \right\} e^{ik\frac{2\pi}{P_0}x}, k \neq 0. \end{aligned} \quad (63)$$

When $|k| > 1$, by the Penrose stability condition (12), there is no unstable eigenvalue with $\text{Re } \lambda > 0$. So we restrict to the critical case $k = 1$. Define

$$F(w) \equiv \frac{4\pi^2}{P_0^2} \int \frac{v[\mu'_+ + \mu'_-]}{v - w} dv. \quad (64)$$

Then linear instability is equivalent to that $F(w) = 1$ for some w with $\text{Im } w > 0$. We define the Penrose curve: $\{(a, F(a + 0i))\}$, the image of the real axis in the complex plane. For the flat profiles, the Penrose curve is an interval in the real axis near $(1, 0)$, and under the condition (18) it passes beyond $(1, 0)$ to the right. Now we prove the linear stability of homogeneous equilibria. We assume the contrary, there is $w_0 = a + bi$ with $a \in \mathbf{R}$, $b > 0$ such that $F(w_0) = 1$. We now approximate $\mu_\pm$ by non-constant $\mu_\pm^n$ near

0, such that $\mu_{\pm}'' \rightarrow \mu_{\pm}$ in some strong topology, while $\mu_{\pm}''(0) = 0$, but $\mu_{\pm}''(\xi) < 0$ for $\xi > 0$. We define

$$F_n(w) \equiv \frac{4\pi^2}{P_0^2} \int \frac{v[\mu_+'' + \mu_-'']}{v - w} dv.$$

Now the Penrose curves for $F_n(w)$ break the constant interval so that $(1, 0)$ is a proper point of the boundary, and from (12), $(1, 0)$ is the single largest intersection of Penrose curves with real axis. Since $F(\lambda_0) = 1$, and $F_n(\lambda)$ are analytic near λ_0 , $\lim F_n(\lambda) = F(\lambda)$ for λ near λ_0 we therefore conclude that there is λ_n near λ_0 such that $F_n(\lambda_n) = 1$. By the open mapping theorem, there is a full neighborhood B_n of $(1, 0)$ such that $B_n \in F_n(\text{Im } \lambda > 0)$. This contradicts to the fact that $(1, 0)$ is a proper boundary point. The uniform boundedness of $(\mathbf{I} + K(\lambda, 0))^{-1}$ on $\{\text{Re } \lambda > 0, |\lambda| \geq \delta\}$ follows from the Penrose condition (12) and (13).

The proof of (2) is by straightforward calculations. To show (3), we note that $F(0) = F'(0) = 0$ and by (18)

$$F''(0) = \frac{8\pi^2}{P_0^2} \int \frac{\mu_+' + \mu_-'}{v^2} dv > 0. \quad (65)$$

Define the 2 by 2 matrix

$$A(\lambda, 0) = (\mathbf{I} + K(\lambda, 0)r_j, r_l), \quad j, l = 1, 2.$$

Then near $\lambda = 0$,

$$\det A(\lambda, 0) = F\left(\frac{\lambda}{i}\right) F\left(-\frac{\lambda}{i}\right) \sim \lambda^4.$$

(ii) (*Uneven Case*) The proof of (1) and (2) is similar. Finally, we note that $F(0) = 0$, $F'(0) \neq 0$ (by (11)), so $\det A(\lambda, 0) \sim \lambda^2$ near $\lambda = 0$. $\square$

Proof of Theorem 2. We only consider the two species case since the proof for the fixed ion case is similar. First, we study the eigenvalues near 0. Combining Lemma 15 (ii) and Lemma 4, we know that the number of eigenvalues for $\mathbf{I} + K(\lambda, \varepsilon)$ near 0 is at most two when ε is small. By Lemma 6, the nonzero eigenvalues must appear in pairs. But clearly the translation invariance leads to a zero eigenvector $[\mu_{+, \pm}'(e_+) \beta_x, -\mu_{-, \pm}'(e_-) \beta_x, \beta_x]$ for the linearized Vlasov-Poisson equation (20) (21), which gives rise to a zero eigenvector for $\mathbf{I} + K(0, \varepsilon)$ for all $\varepsilon > 0$. So we conclude that there exists no nonzero eigenvalues near 0, besides the translation mode. That is, there exists $\delta_0 > 0$, such that there is no nonzero eigenvalue λ with $|\lambda| \leq \delta_0$ for $\mathbf{I} + K(\lambda, \varepsilon)$, when $\varepsilon < 1$. By Lemma 13, when ε is small enough, there exists no unstable eigenvalue λ of linearized VP (20) (21) with $\text{Re } \lambda > 0$ and $|\lambda| \geq \delta_0$. This proves the spectral stability of small BGK waves. $\square$

Lastly, we prove Theorem 3 for the even case.

Proof of Theorem 3. We give the detailed proof for the two species case and make some comments on the fixed ion case at the end. Denote $\mu_{\pm, +} = \mu_{\pm, -} = \mu_{\pm}$. Again by Lemma 13, it suffices to exclude unstable eigenvalues near 0. First, it follows from Lemma 15 (i) and Lemma 4 that when $\varepsilon < 1$, there exist at most four eigenvalues near $\lambda = 0$ for $\mathbf{I} + K(\lambda, \varepsilon)$. By Lemma 6, besides the translation mode, any nonzero

eigenvalue near 0 must be either purely imaginary or real, since any eigenvalue with both nonzero real and imaginary parts must appear in quadruple.

Now we exclude real eigenvalues to show spectral stability. Assume $\lambda \in \mathbf{R}$ and $|\lambda| < 1$. First, we show that when $\mu'_{\pm}(e_{\pm})$ are even, the operator $\rho(\lambda, \varepsilon)\phi$ preserves parity. For an even function $\phi(x) \in H_{\varepsilon}^1$, let $\phi(x) = \sum_{k \in \mathbf{Z}} \phi_k^{\pm}(I_{\pm}) e^{ik\theta_{\pm}}$. Then for $k \neq 0$, $\phi_k^{\pm}(I_{\pm}) = \phi_{-k}^{\pm}(I_{\pm})$ for untrapped particles, since (x, v) and $(-x, v)$ have action-angle coordinates (I, θ) and $(I, 2\pi - \theta)$ respectively. Thus by (62),

$$\begin{aligned} \rho(\lambda, \varepsilon)\phi &= \sum_{k \neq 0, \pm} \int_{v>0} \frac{2\omega_{\pm}^2 \mu'_{\pm}(e_{\pm})}{\omega_{\pm}^2 + \frac{\lambda^2}{k^2}} \phi_k^{\pm}(I_{\pm}) e^{ik\theta_{\pm}} dv \\ &= 4 \sum_{k>0, \pm} \int_{v>0} \frac{\omega_{\pm}^2 \mu'_{\pm}(e_{\pm}) \phi_k^{\pm}(I_{\pm})}{\omega_{\pm}^2 + \frac{\lambda^2}{k^2}} \cos k\theta_{\pm} dv, \end{aligned} \quad (66)$$

is again an even function in x . Similarly, it can be shown that when ϕ is odd, so is $\rho(\lambda, \varepsilon)\phi$. Thus, we can consider $\mathbf{I} + K(\lambda, \varepsilon)$ in the even and odd spaces separately. In the odd subspace H_{odd}^1 of H_0^1 , we notice that $\mathbf{I} + K(0, 0)$ has a 1D kernel spanned by $r_1 = \sin \frac{2\pi}{P_0} x$. By (63) and noticing that for even profiles the function $F(w)$ defined by (64) is even, so

$$\begin{aligned} ((\mathbf{I} + K(\lambda, 0)) r_1, r_1) &= ((\mathbf{I} + \partial_x^{-2} \rho(\lambda, 0)) r_1, r_1) \\ &= 1 - F(i\lambda) \sim a_0 \lambda^2. \end{aligned} \quad (67)$$

where $a_0 = F''(0) > 0$ (see (65)) by the condition (18). We also note that for any $\varepsilon > 0$, $\mathbf{I} + K(0, \varepsilon)$ has a translation kernel which is odd. So by Corollary 5 and Lemma 6, there is no nonzero eigenvalue λ near zero for $\mathbf{I} + K(\lambda, \varepsilon)$ when $\varepsilon < 1$.

We can now restrict ourselves to the even subspace H_{even}^1 of H_0^1 , so that $\mathbf{I} + K(0, 0)$ has 1D kernel spanned by $r_0 = \cos \frac{2\pi}{P_0} x$. We note that (66) is valid for all $\lambda \in \mathbf{C}$ with $|\lambda| < 1$. So letting $v = \lambda^2$, we may rewrite $K(\lambda, \varepsilon)$ in terms of v . That is, we define

$$\bar{\rho}(v, \varepsilon)\phi = 4 \sum_{k>0, \pm} \int_{v>0} \frac{\omega_{\pm}^2 \mu'_{\pm}(e_{\pm}) \phi_k^{\pm}(I_{\pm})}{\omega_{\pm}^2 + \frac{v}{k^2}} \cos k\theta_{\pm} dv,$$

and

$$\bar{K}(v, \varepsilon) = G_{\beta} \partial_x^{-2} \bar{\rho}(v, \varepsilon) G_{\beta}^{-1}.$$

Then $\bar{K}(v, \varepsilon)$ is analytic for v near zero. Suppose $(\mathbf{I} + K(\lambda, \varepsilon))r = 0$ or equivalently $(\mathbf{I} + \bar{K}(v, \varepsilon))r = 0$, for $r \in H_{\text{even}}^1$ and let $\mathbf{P}r = ar_0$, where $\mathbf{P}$ is the projection to $\{r_0\}$. As in the proof of Lemma 4, we use the Liapunov-Schmidt reduction to solve $(\mathbf{I} + \bar{K}(v, \varepsilon))r = 0$ to obtain

$$(I + \bar{K}(v, 0))r + (\bar{K}(v, \varepsilon) - \bar{K}(v, 0))\{Z^{\perp}(v, \varepsilon) + \mathbf{I}\}\mathbf{P}r = 0,$$

which is equivalent to

$$\begin{aligned} 0 &= ((\mathbf{I} + \bar{K}(v, 0)) r_0, r_0) \\ &\quad + ((\bar{K}(v, \varepsilon) - \bar{K}(v, 0)) [Z^{\perp}(v, \varepsilon) + \mathbf{I}] r_0, r_0), \end{aligned} \quad (68)$$

where

$$Z^\perp(v, \varepsilon) \equiv -[\mathbf{I} + (\mathbf{I} + \bar{K}(v, 0))^{-1} (\mathbf{I} - \mathbf{P}) (\bar{K}(v, \varepsilon) - \bar{K}(v, 0))]^{-1} \\ (\mathbf{I} + \bar{K}(\lambda, 0))^{-1} (\mathbf{I} - \mathbf{P}) [\bar{K}(v, \varepsilon) - \bar{K}(v, 0)]$$

as defined in (24) is analytic in v near 0. Since the eigenvalue λ must be real or pure imaginary, so it suffices to study the equation (68) for $v = \lambda^2 \in \mathbf{R}$. As in (67), we have

$$((\mathbf{I} + \bar{K}(v, 0)) r_0, r_0) = ((\mathbf{I} + \bar{K}(v, 0)) r_0, r_0) \\ = 1 - F(i\lambda) \sim a_0 \lambda^2 = a_0 v, \quad (69)$$

with $a_0 > 0$. So from the implicit function theorem, for $\varepsilon \ll 1$ there is a unique $v(\varepsilon) \in \mathbf{R}$ so that (68) is valid, with $v(0) = 0$. There is a growing mode $\lambda(\varepsilon) > 0$ if and only if $v(\varepsilon) = \lambda(\varepsilon)^2 > 0$, and there is a purely imaginary eigenvalue $0 \neq \lambda(\varepsilon) \in i\mathbf{R}$ if and only if $v(\varepsilon) = \lambda(\varepsilon)^2 < 0$. By (69) and (68) we need

$$S(v, \varepsilon) \equiv \left((\bar{K}(v, \varepsilon) - \bar{K}(v, 0)) (Z^\perp(v, \varepsilon) + \mathbf{I}) r_0, r_0 \right) < 0$$

for $\varepsilon \ll 1$ to ensure that $v(\varepsilon) > 0$. Note that

$$S(v, \varepsilon) = [S(v, \varepsilon) - S(0, \varepsilon)] + S(0, \varepsilon)$$

and

$$|[S(v, \varepsilon) - S(0, \varepsilon)]| \leq \max_{v' \in (0, v)} |\partial_v S(v', \varepsilon)| |v|, \quad (70)$$

where

$$\partial_v S(v, \varepsilon) = \left(\partial_v [\bar{K}(v, \varepsilon) - \bar{K}(v, 0)] [Z^\perp(v, \varepsilon) + \mathbf{I}] r_0, r_0 \right) \\ + \left([\bar{K}(v, \varepsilon) - \bar{K}(v, 0)] \partial_v Z^\perp(v, \varepsilon) r_0, r_0 \right).$$

For $v = \lambda^2$ near 0, by lemma 14,

$$\left| \left(\partial_v (\bar{K}(v, \varepsilon) - \bar{K}(v, 0)) (Z^\perp(v, \varepsilon) + \mathbf{I}) r_0, r_0 \right) \right| \\ = \frac{1}{2|\lambda|} \left| \left(\partial_\lambda (K(\lambda, \varepsilon) - K(\lambda, 0)) (Z^\perp(v, \varepsilon) + \mathbf{I}) r_0, r_0 \right) \right| \\ \lesssim \varepsilon \left\| (Z^\perp(v, \varepsilon) + \mathbf{I}) r_0 \right\|_{W^{1, \infty}} \|r_0\|_{L^\infty} \lesssim \varepsilon,$$

and by Proposition 12,

$$\left| \left((\bar{K}(v, \varepsilon) - \bar{K}(v, 0)) \partial_v Z^\perp(v, \varepsilon) r_0, r_0 \right) \right| \\ = \left| \left((K(\lambda, \varepsilon) - K(\lambda, 0)) \partial_v Z^\perp(v, \varepsilon) r_0, r_0 \right) \right| \lesssim \sqrt{\varepsilon}.$$

Noting that $v(\varepsilon) = v'(0)\varepsilon + \frac{v''(0)\varepsilon^2}{2!} + \dots = O(\varepsilon)$, thus a combination of above estimates and (70) gives

$$|S(v, \varepsilon) - S(0, \varepsilon)| = O\left(\varepsilon^{\frac{3}{2}}\right).$$

Since $S(0, 0) = 0$, so a sharp *stability criterion* is for the first nonzero derivative:

$$\frac{d}{d\varepsilon} S(0, \varepsilon)|_{\varepsilon=0} < 0 \quad \text{for instability,}$$

and

$$\frac{d}{d\varepsilon} S(0, \varepsilon)|_{\varepsilon=0} > 0 \quad \text{for stability.}$$

Moreover, under the stability condition $\frac{d}{d\varepsilon} S(0, \varepsilon)|_{\varepsilon=0} > 0$, from our proof we have a pair of imaginary eigenvalues bifurcating from $\varepsilon = 0$. Note that since $Z^\perp(0, 0) = 0$,

$$\frac{d}{d\varepsilon} S(0, \varepsilon)|_{\varepsilon=0} = \frac{d}{d\varepsilon} (G_\beta \partial_x^{-2} \rho(0, \varepsilon) G_\beta^{-1} r_0, r_0)|_{\varepsilon=0}.$$

For $r_0 = \cos(\frac{2\pi}{P_0}x)$,

$$\partial_x^{-2} G_\beta^{-1} r_0 = \partial_x^{-2} \cos\left(\frac{P_0}{P_\beta} \frac{2\pi}{P_0} x\right) = -\frac{P_\beta^2}{4\pi^2} \cos\left(\frac{2\pi}{P_\beta} x\right).$$

Notice that for $\phi \in H_\varepsilon^1$,

$$\rho(0, \varepsilon)\phi = \int_{\mathbf{R}} (\mu'_+(e_+) + \mu'_-(e_-)) dv \phi.$$

So

$$\begin{aligned} S(0, \varepsilon) &= \frac{P_0}{P_\beta} (\partial_x^{-2} \rho(0, \varepsilon) G_\beta^{-1} r_0, G_\beta^{-1} r_0) \\ &= -\frac{P_0 P_\beta}{4\pi^2} \int_0^{P_\beta} \int (\mu'_+(e_+) + \mu'_-(e_-)) dv \cos^2\left(\frac{2\pi}{P_\beta} x\right) dx \\ &= -\frac{P_\beta^2}{4\pi^2} \int_0^{P_0} \int [\mu'_+ \left(\frac{1}{2} v^2 + \beta \left(\frac{P_\beta}{P_0} x\right)\right) \\ &\quad + \mu'_- \left(\frac{1}{2} v^2 - \beta \left(\frac{P_\beta}{P_0} x\right)\right)] dv \cos^2\left(\frac{2\pi}{P_0} x\right) dx. \end{aligned}$$

Notice that by (8),

$$\frac{d}{d\varepsilon} \beta \left(\frac{P_\beta}{P_0} x\right) |_{\varepsilon=0} = \cos\left(\frac{2\pi}{P_0} x\right).$$

So

$$\begin{aligned} &\frac{d}{d\varepsilon} S(0, \varepsilon)|_{\varepsilon=0} \\ &= -\frac{P_0 P'_\beta(0)}{2\pi^2} \int_0^{P_0} \int \left(\mu'_+ \left(\frac{1}{2} v^2\right) + \mu'_- \left(\frac{1}{2} v^2\right) \right) dv \cos^2\left(\frac{2\pi}{P_0} x\right) dx \\ &\quad - \frac{P_0^2}{4\pi^2} \int_0^{P_0} \int \left(\mu''_+ \left(\frac{1}{2} v^2\right) - \mu'_- \left(\frac{1}{2} v^2\right) \right) dv \cos^3\left(\frac{2\pi}{P_0} x\right) dx \\ &= -\frac{P_0^2 P'_\beta(0)}{4\pi^2} \int \left(\mu'_+ \left(\frac{1}{2} v^2\right) + \mu'_- \left(\frac{1}{2} v^2\right) \right) dv = -P'_\beta(0). \end{aligned}$$

So we deduce the stability criterion that $P'_\beta(0) < 0$ for stability and $P'_\beta(0) > 0$ for instability. Moreover, in the stable case ($P'_\beta > 0$), there exists a pair of nonzero purely imaginary eigenvalues.

Lastly, we calculate $P'_\beta(0)$. Recall

$$-\beta_{xx} = h(\beta) = \int \mu_+ \left(\frac{v^2}{2} + \beta \right) - \int \mu_- \left(\frac{v^2}{2} - \beta \right) = H'(\beta).$$

Since $H'(0) = 0$, $H''(0) = (\frac{2\pi}{P_0})^2$,

$$H'''(0) = \int \mu_+'' \left(\frac{v^2}{2} \right) - \int \mu_-'' \left(\frac{v^2}{2} \right) = \int \frac{\mu_+' \left(\frac{v^2}{2} \right)}{v^2} - \int \frac{\mu_-' \left(\frac{v^2}{2} \right)}{v^2}.$$

Let $a_2 = (\frac{2\pi}{P_0})^2$, $a_3 = H'''(0)$, and define $g(u) = 2H(u) - uh(u)$, then

$$\begin{aligned} P'_\beta|_{\beta=0} &= \lim_{\beta \rightarrow 0} \frac{4}{\beta} \int_0^\beta \frac{g(\beta) - g(u)}{(\sqrt{2(H(\beta) - H(u))})^3} du \\ &= \lim_{\beta \rightarrow 0} \frac{4}{\beta} \frac{1}{6} a_3 \int_0^\beta \frac{\beta^3 - u^3}{(a_2(\beta^2 - u^2))^{\frac{3}{2}}} du \\ &= \frac{1}{6} a_3 \int_0^1 \frac{1 - u^3}{(a_2(1 - u^2))^{\frac{3}{2}}} du. \end{aligned}$$

So we deduce instability when

$$\int \frac{\mu_+' \left(\frac{v^2}{2} \right)}{v^2} - \int \frac{\mu_-' \left(\frac{v^2}{2} \right)}{v^2} > 0 \quad \left(\text{equivalently } P'_\beta > 0 \right),$$

and stability when

$$\int \frac{\mu_+' \left(\frac{v^2}{2} \right)}{v^2} - \int \frac{\mu_-' \left(\frac{v^2}{2} \right)}{v^2} < 0 \quad \left(\text{equivalently } P'_\beta < 0 \right).$$

The proof for the fixed ion case is similar. Here, the condition (19) plays a dual role. It is the stability condition for the homogeneous state $(\mu(\frac{1}{2}v^2), 0)$ at the critical period P_0 , and is also equivalent to the instability condition $P'_\beta > 0$ for small BGK waves. $\square$

6. Examples

1. The flat profiles satisfying conditions in Theorems 2 and 3 can be easily constructed. We give an example for the uneven and one species case. Let

$$f(v) = c_0 \left(e^{-(v-v_1)^2} + e^{-(v+v_1)^2} \right),$$

where $v_1 > 0$ and $c_0 > 0$ is such that $\int f(v) dv = 1$. Note that $f(v)$ has two maximum at v_1 and $-v_1$ and one minimum at 0. If v_1 is large enough, then

$$\int \frac{f'(v)}{v \pm v_1} dv < \int \frac{f'(v)}{v} dv \quad \text{and} \quad \int \frac{f'(v)}{v} dv > 0.$$

We now modify $f(v)$ slightly to get $f_0(v)$ which is flat in a small interval $[-\sigma, \sigma]$ and slightly uneven such that $\int \frac{f'_0(v)}{v^2} dv \neq 0$. Let v_a and v_b be the two maximum points of $f_0(v)$, then we still have

$$\max \left\{ \int \frac{f'_0(v)}{v - v_a} dv, \int \frac{f'_0(v)}{v - v_b} dv \right\} < \int \frac{f'_0(v)}{v} dv = \left(\frac{2\pi}{P_0} \right)^2.$$

Since $\int \frac{f'_0(v)}{v^2} dv \neq 0$, when σ is small, the function $F(w) = \int \frac{f'_0(v)}{v-w} dv$ is monotone in $[-\sigma, \sigma]$ and the condition (17) is also satisfied. Thus, the profiles $f_0(v)$ satisfies all the conditions in Theorem 2. The small BGK waves bifurcating near $(f_0(v), 0)$ are spectrally stable.

2. For the even case, the condition $\int \frac{[\mu'_+ + \mu'_-]}{v^2} dv > 0$ is always true when the flatness width $\sigma_{\pm}$ are small enough. Consider $\mu_{\pm}$ to be the modification (flattening near 0) of non-flat profiles with local minimum at 0. Let $\sigma_0 = \max \{\sigma_+, \sigma_-\}$ and take an interval $I = [2\sigma_0, a]$ such that $\mu'_{\pm} \geq c_0 > 0$ in I . Then

$$\int_I \frac{[\mu'_+ + \mu'_-]}{v^2} dv \geq 2c_0 \int_I \frac{1}{v^2} = 2c_0 \left(\frac{1}{2\sigma_0} - \frac{1}{a} \right).$$

Since $\mu'_{\pm} \geq 0$ in $[0, 2\sigma_0]$,

$$\begin{aligned} \int \frac{[\mu'_+ + \mu'_-]}{v^2} dv &= 2 \int_{v>0} \frac{[\mu'_+ + \mu'_-]}{v^2} dv \\ &\geq c_0 \left(\frac{1}{2\sigma_0} - \frac{1}{a} \right) + \int_{v>a} \frac{[\mu'_+ + \mu'_-]}{v^2} dv \\ &> 0, \end{aligned}$$

when σ_0 is small enough. By the same argument, both stability and instability conditions

$$\int v^{-2} [\mu'_+(\frac{v^2}{2}) - \mu'_-(\frac{v^2}{2})] > 0 \quad (< 0) \quad (71)$$

can be satisfied by choosing different $\sigma_{\pm}$ for $\mu_{\pm}$. Indeed, when $1 \gg \sigma_- \gg \sigma_+$ ($\sigma_- \ll \sigma_+ \ll 1$), we get $+$ ($-$) sign in (71).

3. For the one species and even case, by the same arguments as above, the instability condition $\int \frac{\mu'(\frac{1}{2}v^2)}{v^2} dv > 0$ is always satisfied when the flatness width is small. So BGK waves near $\mu(\frac{1}{2}v^2)$ are linearly unstable. But for a slightly uneven profile $f_0(v)$ close to $\mu(\frac{1}{2}v^2)$, by Theorem 2, the BGK waves near $f_0(v)$ are linearly stable. These suggest that there exist a transition from stability to instability when we increase the amplitude of BGK waves bifurcated from $f_0(v)$ up to the one close to an unstable BGK wave bifurcated from $\mu(\frac{1}{2}v^2)$.

4. In Theorems 3 and 2, we construct linearly stable BGK waves for both even and uneven cases. However, there is a significant difference in their spectra which must lie in the imaginary axis. For the uneven case, there is no nonzero imaginary eigenvalue of the linearized VP operator at a stable small BGK wave. In contrast, for the even case, there exists a pair of nonzero imaginary eigenvalues for the stable small BGK waves. In particular, these imply that there is no linear damping for the even BGK waves due to the existence of nonzero time periodic solutions of linearized Vlasov-Poisson equation. For the uneven case, the linear damping is under investigation ([15]).

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References

1. Arnold, V.I.: Mathematical methods of classical mechanics. In: Translated from the Russian by Vogtmann K., Weinstein, A. (eds) Graduate Texts in Mathematics, 2nd edn. Springer, New York, pp 60 (1989)
2. Armstrong, T., Montgomery, D.: Asymptotic state of the two-stream instability. *J. Plasma Phys.* **1**(4), 425–433 (1967)
3. Bellan, P.M.: Fundamentals of Plasma Physics. Cambridge University Press, Cambridge (2006)
4. Manfredi, G., Bertrand, P.: Stability of Bernstein-Greene-Kruskal modes. *Phys. Plasmas.* **7**(6), 2425–2431 (2000)
5. Brunetti, M., Califano, F., Pegoraro, F.: Asymptotic evolution of nonlinear Landau damping. *Phys. Rev. E.* **62**, 4109–4114 (2000)
6. Bernstein, I., Greene, J., Kruskal, M.: Exact nonlinear plasma oscillations. *Phys. Rev.* **108**(3), 546–550 (1957)
7. Bedrossian, J., Masmoudi, N., Mouhot, C.: Landau damping: paraproducts and Gevrey regularity. *Ann. PDE.* **2**(1), 2:4 (2016)
8. Cheng, Y., Gamba, I.M., Morrison, P.J.: Study of conservation and recurrence of Runge–Kutta discontinuous Galerkin schemes for Vlasov-Poisson systems. *J. Sci. Comput.* **56**, 319–349 (2013)
9. Cheng, C.Z., Knorr, G.: The integration of the Vlasov equation in configuration space. *J. Comput. Phys.* **22**(3), 330–351 (1976)
10. Demeio, L., Zweifel, P.F.: Numerical simulations of perturbed Vlasov equilibria. *Phys. Fluids B.* **2**, 1252–1255 (1990)
11. Demeio, L., Holloway, J.P.: Numerical simulations of BGK modes. *J. Plasma Phys.* **46**, 63–84 (1991)
12. Danielson, J.R., Anderegg, F., Driscoll, C.F.: Measurement of Landau damping and the evolution to a BGK equilibrium. *Phys. Rev. Lett.* **92**, 245003-1-4 (2004)
13. Ghizzo, A., Izrar, B., Bertrand, P., Fijalkow, E., Feix, M.R., Shoucri, M.: Stability of Bernstein-Greene-Kruskal plasma equilibria. Numerical experiments over a long time. *Phys. Fluids.* **31**(1), 72–82 (1988)
14. Guo, Y., Strauss, W.: Instability of periodic BGK equilibria. *Comm. Pure Appl. Math.* **XLVIII**, 861–894 (1995)
15. Guo, Y., Lin, Z.: Linear Landau damping near stable BGK waves. Work in preparation
16. Hislop, P.D., Sigal, I.M.: Introduction to Spectral Theory. Springer, Berlin (1996)
17. Kato, T.: Perturbation theory for linear operators. In: Grundlehren der Mathematischen Wissenschaften, Band 132, 2nd edn. Springer, Berlin (1976)
18. Lin, Z.: Instability of periodic BGK waves. *Math. Res. Lett.* **8**, 521–534 (2001)
19. Lin, Z.: Nonlinear instability of periodic waves for Vlasov-Poisson system. *Commun. Pure Appl. Math.* **58**, 505–528 (2005)
20. Lin, Z., Zeng, C.: Small BGK waves and nonlinear Landau damping. *Commun. Math. Phys.* **306**, 291–331 (2011)
21. Luque, A., Schamel, H.: Electrostatic trapping as a key to the dynamics of plasmas, fluids and other collective systems. *Phys. Rep.* **415**, 261–359 (2005)
22. Medvedev, M.V., Diamond, P.H., Rosenbluth, M.N., Shevchenko, V.I.: Asymptotic theory of nonlinear landau damping and particle trapping in waves of finite amplitude. *Phys. Rev. Lett.* **81**, 5824 (1998)
23. Mouhot, C., Villani, C.: On Landau damping. *Acta Math.* **207**(1), 29–201 (2011)
24. Manfredi, G.: Long-time behavior of nonlinear Landau damping. *Phys. Rev. Lett.* **79**, 2815–2818 (1997)
25. Penrose, O.: Electrostatic instability of a non-Maxwellian plasma. *Phys. Fluids.* **3**, 258–265 (1960)
26. Schamel, H.: Cnoidal electron hole propagation: trapping, the forgotten nonlinearity in plasma and fluid dynamics. *Phys. Plasmas.* **19**, 020501 (2012)
27. Trivelpiece, A.W., Krall, N.A.: Principles of Plasma Physics. McGraw-Hill, New York (1973)
28. Valentini, F., Carbone, V., Veltri, P., Mangeney, A.: Wave-particle interaction and nonlinear Landau damping in collisionless electron plasmas. *Transp. Theory Stat. Phys.* **34**, 89–101 (2005)