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Korn inequalities for shells with zero Gaussian curvature

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Abstract

We consider shells with zero Gaussian curvature, namely shells with one principal curvature zero and the other one having a constant sign. Our particular interests are shells that are diffeomorphic to a circular cylindrical shell with zero principal longitudinal curvature and positive circumferential curvature, including, for example, cylindrical and conical shells with arbitrary convex cross sections. We prove that the best constant in the first Korn inequality scales like thickness to the power $3/2$ for a wide range of boundary conditions at the thin edges of the shell. Our methodology is to prove, for each of the three mutually orthogonal two-dimensional cross-sections of the shell, a "first-and-a-half Korn inequality"—a hybrid between the classical first and second Korn inequalities. These three two-dimensional inequalities assemble into a three-dimensional one, which, in turn, implies the asymptotically sharp first Korn inequality for the shell. This work is a part of mathematically rigorous analysis of extreme sensitivity of the buckling load of axially compressed cylindrical shells to shape imperfections.

Keywords: Korn's inequality; Shells; Nonlinear elasticity; Cones; Cylinders

1. Introduction

Classical first and second Korn inequalities have been known to play a central role in the theory of linear elasticity and recently they have found very important applications in the problems of buckling of slender structures [6,4,5]. Let us recall the classical first and second Korn inequalities, that actually date back to 1908, [8,9]. To that end we denote

$$\mathfrak{euc}(n) = \{\boldsymbol{u} : \mathbb{R}^n \to \mathbb{R}^n : \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b},\ \boldsymbol{A} \in \mathrm{Skew}(\mathbb{R}^n),\ \boldsymbol{b} \in \mathbb{R}^n\}$$

be the set of all infinitesimal motions, i.e., a Lie algebra of the group of all Euclidean transformations (rigid body motions). Let Ω be an open connected subset of $\mathbb{R}^n$ and $\boldsymbol{u} \in W^{1,2}(\Omega; \mathbb{R}^n)$. We denote[1] by $\operatorname{grad}\boldsymbol{u}$ and $(\operatorname{grad}\boldsymbol{u})_{\mathrm{sym}}$ the gradient and the symmetric part of the gradient, respectively, of a vector field $\boldsymbol{u}$. It is well-known that $(\operatorname{grad}\boldsymbol{u})_{\mathrm{sym}} = 0$

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[1] We reserve more streamlined notations $\nabla\boldsymbol{u}$ and $e(\boldsymbol{u})$ for "simplified" gradient and symmetrized gradient, respectively, that will be our main characters in the technical part of the paper.

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in Ω (in the sense of distributions) if and only if $\boldsymbol{u} \in \mathfrak{euc}(n)$. This is an immediate consequence of a simple observation (also very well-known) that all partial derivatives of the gradient $\boldsymbol{G} = \operatorname{grad} \boldsymbol{u}$ can be expressed as a linear combination of partial derivatives of the symmetric part of the gradient $\boldsymbol{E} = (\operatorname{grad} \boldsymbol{u})_{\mathrm{sym}}$:

$$\frac{\partial G_{jk}}{\partial x^i} = \frac{\partial E_{jk}}{\partial x^i} + \frac{\partial E_{ij}}{\partial x^k} - \frac{\partial E_{ik}}{\partial x^j}.$$

The classical first Korn inequality (e.g., as stated in [14]) quantifies this result by describing how large $(\operatorname{grad} \boldsymbol{u})_{\mathrm{sym}}$ must be if $\boldsymbol{u}$ lies in a closed subspace $V \subset W^{1,2}(\Omega; \mathbb{R}^n)$ that has trivial intersection with $\mathfrak{euc}(n)$. If Ω is a Lipschitz domain, then there exists a constant $K(\Omega, V)$, such that for every $\boldsymbol{u} \in V$

$$\|(\operatorname{grad} \boldsymbol{u})_{\mathrm{sym}}\|^2 \geq K(\Omega, V)\|\operatorname{grad} \boldsymbol{u}\|^2, \tag{1.1}$$

where $\|\cdot\|$ is the L^2-norm. The Korn constant $K(\Omega, V)$ measures the distance between the subspace V and $\mathfrak{euc}(n)$. The classical second Korn inequality asserts that the standard $W^{1,2}$ norm topology can be equivalently defined by replacing $\operatorname{grad} \boldsymbol{u}$ with $(\operatorname{grad} \boldsymbol{u})_{\mathrm{sym}}$:

$$\|\operatorname{grad} \boldsymbol{u}\|^2 \leq C(\Omega)(\|(\operatorname{grad} \boldsymbol{u})_{\mathrm{sym}}\|^2 + \|\boldsymbol{u}\|^2), \quad \boldsymbol{u} \in W^{1,2}(\Omega; \mathbb{R}^n). \tag{1.2}$$

Originally, Korn inequalities were used to prove existence, uniqueness and well-posedness of boundary value problems of linear elasticity (see e.g., [11,1]). Nowadays, often, as in our particular case, it is the best Korn constant $K(\Omega, V)$ in the first Korn inequality that is of central importance (e.g., [2,12,13,15,16,10]). Specifically, we are interested in the asymptotic behavior of the Korn constant $K(\Omega_h, V_h)$ for shells with zero Gaussian curvature as a function of their thickness h for subspaces V_h of $W^{1,2}$ functions satisfying various boundary conditions at the thin edges of the shell. In [6,5] we have shown that $K(\Omega_h, V_h)$ represents an absolute lower bound on safe loads for any slender structure. For a classical circular cylindrical shell we have proved in [3] that $K(\Omega_h, V_h) \sim h^{3/2}$ for a broad class of boundary conditions at the thin edges of the shell.

The motivation for this work comes from the fact that the experimentally measured buckling loads of axially compressed cylindrical shells behave in a paradoxical way, dramatically disagreeing with predictions of classical shell theory. The universal consensus is that such behavior is due to the extreme sensitivity of shells to imperfections of shape and load. This study is a part of rigorous analytical investigation of the influence of shape on the structural behavior of cylindrical shells. It looks like (and this will be addressed in future work) the determining factor of the effect of shape imperfections is the Gaussian curvature of the shell's mid-surface as the Ansatzen in [17] suggest. In this paper we show that if the shell has a vanishing principal curvature (yielding zero Gaussian curvature), as in circular cylindrical shells, then the scaling of the Korn constant $K(\Omega_h, V_h)$ will remain unaffected, provided the nonzero principal curvature has a constant sign. Our analysis also shows that if both principal curvatures are zero on any open subset of the shell's mid-surface, then $K(\Omega_h, V_h) \sim h^2$. We conjecture that $K(\Omega_h, V_h) \sim h$ for shells of uniformly positive Gaussian curvature, while $K(\Omega_h, V_h) \sim h^{4/3}$ if the Gaussian curvature is negative on any open subset of the shell's middle surface, as suggested by test functions constructed in [17]. These conjectures will be addressed elsewhere.

The goal of this paper is to show that the tools developed in [3] for circular cylindrical shells, and extended and developed further in [7], possess enough flexibility to be applicable to a wide family of shells, and especially to cylindrically-shaped shells (the ones that have no boundary in one of the principal directions). The main idea is to first prove an inequality that is a hybrid between the first and second Korn inequalities (we call it "first-and-a-half Korn inequality" for this reason) by "assembling" it from its two-dimensional versions corresponding to cross-sections of the shell by curvilinear coordinate surfaces. The first Korn inequality is then a consequence of the first-and-a-half Korn inequality and an estimate on the normal component of $\boldsymbol{u} \in V_h$. We believe that this general methodology will work for broad classes of shells, even though the Gaussian curvature does affect both the final results and the validity of some of the technical steps in the proof, which must be suitably modified in each particular case. For example, the assumption of zero Gaussian curvature is essential for all main results in Section 3.

2. Preliminaries

Consider a shell whose mid-surface is of class C^2. Suppose z and θ are coordinates on the mid-surface of the shell, such that $z =$ constant and $\theta =$ constant are the lines of principal curvatures. Here θ will denote the circumferential

coordinate and z—the longitudinal for cylindrically shaped shells. In the case of a straight circular cylinder, θ and z are the standard cylindrical coordinates. Let $\boldsymbol{r}(\theta, z)$ be the position vector of the shell's mid-surface. Introducing the normal coordinate t, we obtain the set of orthogonal curvilinear coordinates (t, θ, z), related to Cartesian coordinates via

$$\boldsymbol{x} = \boldsymbol{R}(t, \theta, z) = \boldsymbol{r}(z, \theta) + t\boldsymbol{n}(z, \theta),$$

where $\boldsymbol{n}$ is the *outward* unit normal, and $\boldsymbol{R}(t, \theta, z)$ is the position vector of a point in space with coordinates (t, θ, z). In this paper we will study shells of uniform thickness h, given in (t, θ, z) coordinates by

$$\mathcal{C}_h = \{\boldsymbol{R}(t, \theta, z) : t \in I_h,\ \theta \in [0, p],\ z \in [l, L],\ \boldsymbol{R}(t, 0, z) = \boldsymbol{R}(t, p, z)\},\ I_h = \left[-\frac{h}{2}, \frac{h}{2}\right]. \tag{2.1}$$

We denote

$$A_z^2 = |\boldsymbol{r}_{,z}|^2, \qquad A_\theta^2 = |\boldsymbol{r}_{,\theta}|^2$$

the two nonzero components[2] of the metric tensor of the middle surface. The two principal curvatures will be denoted by κ_z and κ_θ. Their signs are chosen in such a way that k_z and k_θ are positive for a barrel-shaped shells, like a sphere. The four functions A_θ, A_z, κ_θ, and κ_z satisfy the Codazzi–Gauss relations (see e.g. [17, Section 1.1])

$$\frac{\partial \kappa_z}{\partial \theta} = (\kappa_\theta - \kappa_z)\frac{A_{z,\theta}}{A_z}, \qquad \frac{\partial \kappa_\theta}{\partial z} = (\kappa_z - \kappa_\theta)\frac{A_{\theta,z}}{A_\theta}, \tag{2.2}$$

$$\frac{\partial}{\partial z}\left(\frac{A_{\theta,z}}{A_z}\right) + \frac{\partial}{\partial \theta}\left(\frac{A_{z,\theta}}{A_\theta}\right) = -A_z A_\theta \kappa_z \kappa_\theta, \tag{2.3}$$

and define the Levi-Civita connection on the middle surface of the shell via the following derivation formulas

$$\nabla_{\boldsymbol{e}_z}\boldsymbol{e}_z = -\frac{1}{A_z A_\theta}\frac{\partial A_z}{\partial \theta}\boldsymbol{e}_\theta - \kappa_z \boldsymbol{n}, \qquad \nabla_{\boldsymbol{e}_z}\boldsymbol{e}_\theta = \frac{1}{A_z A_\theta}\frac{\partial A_z}{\partial \theta}\boldsymbol{e}_z, \qquad \nabla_{\boldsymbol{e}_z}\boldsymbol{n} = \kappa_z \boldsymbol{e}_z,$$

$$\nabla_{\boldsymbol{e}_\theta}\boldsymbol{e}_\theta = -\frac{1}{A_z A_\theta}\frac{\partial A_\theta}{\partial z}\boldsymbol{e}_z - \kappa_\theta \boldsymbol{n}, \qquad \nabla_{\boldsymbol{e}_\theta}\boldsymbol{e}_z = \frac{1}{A_z A_\theta}\frac{\partial A_\theta}{\partial z}\boldsymbol{e}_\theta, \qquad \nabla_{\boldsymbol{e}_\theta}\boldsymbol{n} = \kappa_\theta \boldsymbol{e}_\theta.$$

Using these formulas we can compute the components of $\operatorname{grad}\boldsymbol{u}$ in the orthonormal basis $\boldsymbol{e}_t$, $\boldsymbol{e}_\theta$, $\boldsymbol{e}_z$:

$$\operatorname{grad}\boldsymbol{u} = \begin{bmatrix} u_{t,t} & \dfrac{u_{t,\theta} - A_\theta \kappa_\theta u_\theta}{A_\theta(1 + t\kappa_\theta)} & \dfrac{u_{t,z} - A_z \kappa_z u_z}{A_z(1 + t\kappa_z)} \\ u_{\theta,t} & \dfrac{A_z u_{\theta,\theta} + A_z A_\theta \kappa_\theta u_t + A_{\theta,z} u_z}{A_z A_\theta(1 + t\kappa_\theta)} & \dfrac{A_\theta u_{\theta,z} - A_{z,\theta} u_z}{A_z A_\theta(1 + t\kappa_z)} \\ u_{z,t} & \dfrac{A_z u_{z,\theta} - A_{\theta,z} u_\theta}{A_z A_\theta(1 + t\kappa_\theta)} & \dfrac{A_\theta u_{z,z} + A_z A_\theta \kappa_z u_t + A_{z,\theta} u_\theta}{A_z A_\theta(1 + t\kappa_z)} \end{bmatrix}.$$

We will now specialize to the particular case of zero Gaussian curvature $\kappa_z = 0$. In this case, equations (2.2)–(2.3) can be solved explicitly in terms of four arbitrary smooth functions $B(z)$, $a(\theta)$, $b(\theta)$, $c(\theta)$:

$$A_z = B'(z), \qquad A_\theta = a(\theta)B(z) + b(\theta), \qquad \kappa_\theta = \frac{c(\theta)}{A_\theta}. \tag{2.4}$$

We require that A_z, A_θ and $c(\theta)$ be strictly positive functions of their variables on the mid-surface of the shell. Hence, for shells of zero Gaussian curvature the formula for $\operatorname{grad}\boldsymbol{u}$ simplifies:

$$\operatorname{grad}\boldsymbol{u} = \begin{bmatrix} u_{t,t} & \dfrac{u_{t,\theta} - c(\theta)u_\theta}{A_\theta + tc(\theta)} & \dfrac{u_{t,z}}{A_z} \\ u_{\theta,t} & \dfrac{u_{\theta,\theta} + c(\theta)u_t + a(\theta)u_z}{A_\theta + tc(\theta)} & \dfrac{u_{\theta,z}}{A_z} \\ u_{z,t} & \dfrac{u_{z,\theta} - a(\theta)u_\theta}{A_\theta + tc(\theta)} & \dfrac{u_{z,z}}{A_z} \end{bmatrix}. \tag{2.5}$$

[2] The principal directions are mutually orthogonal.

Table 1
Functions $B(z)$, $a(\theta)$, $b(\theta)$ and $c(\theta)$ for cylinders and cones.

	$B(z)$	$a(\theta)$	$b(\theta)$	$c(\theta)$
cylinders	z	0	1	$\kappa(\theta)$
cones	z	1	0	$(\boldsymbol{\sigma}(\theta), \boldsymbol{\sigma}'(\theta), \boldsymbol{\sigma}''(\theta))$

In the case of shells, the thickness variable t is uniformly small. We therefore introduce the simplified gradient

$$\nabla \boldsymbol{u} = \begin{bmatrix} u_{t,t} & \dfrac{u_{t,\theta} - c(\theta)u_\theta}{A_\theta} & \dfrac{u_{t,z}}{A_z} \\ u_{\theta,t} & \dfrac{u_{\theta,\theta} + c(\theta)u_t + a(\theta)u_z}{A_\theta} & \dfrac{u_{\theta,z}}{A_z} \\ u_{z,t} & \dfrac{u_{z,\theta} - a(\theta)u_\theta}{A_\theta} & \dfrac{u_{z,z}}{A_z} \end{bmatrix}. \tag{2.6}$$

We note that in (2.6) the components u_t, u_θ and u_z are still functions of (t, θ, z).

To be more specific we give two examples of zero Gaussian curvature shells: cylinders and cones. A cylinder is described by a simple, smooth closed curve of length p in the xy-plane. Let $\boldsymbol{\rho}(\theta)$, $\theta \in [0, p)$ be the position vector of this curve, parametrized by its arc-length. The position vector of the middle surface of the shell is then given by $\boldsymbol{r}(\theta, z) = \boldsymbol{\rho}(\theta) + z\boldsymbol{e}_z$, where $\boldsymbol{e}_z$ is the unit vector perpendicular to the xy-plane, i.e. the unit vector in the z-direction. It is easy to verify that $\theta =$ constant and $z =$ constant are lines of curvature and $\kappa_\theta = \kappa(\theta)$ is the curvature of curve $\boldsymbol{\rho}(\theta)$ in the plane, whose sign is chosen to be positive for a circle.

A second example is a cone with vertex at the origin. A cone is described by a simple, smooth closed curve of length p lying in the northern hemisphere of a unit sphere centered at the origin. Let $\boldsymbol{\sigma}(\theta)$, $\theta \in [0, p)$ be the arc-length parametrization of this curve. In this case the middle surface of the shell is given by $\boldsymbol{r}(\theta, z) = z\boldsymbol{\sigma}(\theta)$. Once again, it is easy to verify that $\theta =$ constant and $z =$ constant are lines of curvature and

$$\kappa_\theta = \frac{(\boldsymbol{\sigma}(\theta), \boldsymbol{\sigma}'(\theta), \boldsymbol{\sigma}''(\theta))}{z},$$

where $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) = \boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$ is the triple-product of 3 vectors in space. We summarize the data for cylinders and cones in Table 1.

In this paper all norms $\|\cdot\|$ are L^2 norms. However, because of the curvilinear coordinates we will use several different flavors of the L^2 inner product and the corresponding norm. For $f, g\colon \mathcal{C}_h \to \mathbb{R}$ we define the L^2 inner product

$$(f, g)_{\mathcal{C}_h} = \int_{\mathcal{C}_h} f(\boldsymbol{x})g(\boldsymbol{x})d\boldsymbol{x} = \int_{I_h}\int_l^L\int_0^p A_z A_\theta fg d\theta dz dt,$$

which gives rise to the norm $\|f\|$.

$$\|f\|^2 = (f, f)_{\mathcal{C}_h} = \int_{I_h}\int_l^L\int_0^p A_z A_\theta f^2 d\theta dz dt. \tag{2.7}$$

In cross-sections $\theta =$ constant we use

$$\|f\|_\theta^2 = \int_{I_h}\int_l^L A_z f^2(t, \theta, z) dz dt. \tag{2.8}$$

We will also use the Euclidean version of the norm on cross-sections

$$\|f\|_0^2 = \int_{\alpha_1}^{\alpha_2}\int_{\beta_1}^{\beta_2} f^2(\alpha,\beta,\gamma)d\alpha d\beta, \tag{2.9}$$

where $\{\alpha,\beta,\gamma\} = \{t,\theta,z\}$ and $\alpha_i, \beta_i, i = 1,2$ are the corresponding limits of integration. In each case it will be clear which variable t, θ or z plays the role of the fixed variable γ. Of course, due to uniform positivity of A_θ and A_z the norms $\|f\|_\gamma$ and $\|f\|_0$ are obviously equivalent. In particular, all inequalities involving one type of norm will also be valid for another. Finally, all constants that are independent of $\boldsymbol{u}$ and h will be denoted by C. Once this is understood, such abuse of notation does not lead to any ambiguity.

3. Main results

We formulate our Korn inequalities for vector fields $\boldsymbol{u}$ satisfying specific boundary conditions at the two edges of the shell. We define

$$V_h^1 = \{\boldsymbol{u} \in W^{1,2}(\mathcal{C}_h;\mathbb{R}^3) : u_z(t,\theta,l) = u_\theta(t,\theta,l) = u_\theta(t,\theta,L) = 0\}, \tag{3.1}$$

and

$$V_h^2 = \{\boldsymbol{u} \in V_h^1 : u_t(t,\theta,l) = 0\}. \tag{3.2}$$

We note that p-periodicity of functions in V_h^1 is built in by definition of $\mathcal{C}_h$. We state our main results as a sequence of related theorems.

Theorem 3.1. *Suppose $\kappa_z = 0$ on $\mathcal{C}_h$. Then there exist a constant C independent of h, such that for every $\boldsymbol{u} \in V_h^2$,*

$$\|\operatorname{grad}\boldsymbol{u}\|^2 \le C\left(\frac{\|u_t\|\|(\operatorname{grad}\boldsymbol{u})_{\mathrm{sym}}\|}{h} + \|(\operatorname{grad}\boldsymbol{u})_{\mathrm{sym}}\|^2\right), \tag{3.3}$$

$$\|\operatorname{grad}\boldsymbol{u}\|^2 \le \frac{C}{h^2}\|(\operatorname{grad}\boldsymbol{u})_{\mathrm{sym}}\|^2, \tag{3.4}$$

for all $h \in (0,1)$.

We note that in Theorem 3.1 we placed no constraints on the behavior of the other principal curvature κ_θ. This freedom is compensated by requiring that $\boldsymbol{u}$ belong to a smaller subspace V_h^2 of $W^{1,2}(\mathcal{C}_h;\mathbb{R}^3)$. In fact, if κ_θ vanishes on an open subset of the middle surface of the shell then, according to Theorem 3.3, the bound in (3.4) is asymptotically sharp as $h \to 0$.

If we demand that the curvature κ_θ does not change sign (i.e., be uniformly positive for cylindrically-shaped shells), then not only can the first Korn inequality (3.4) be improved, but its domain of validity can be extended from V_h^2 to V_h^1.

Theorem 3.2. *Suppose that $\kappa_z = 0$ and $\kappa_\theta \neq 0$ on $\mathcal{C}_h$. Then there exists a constant C independent of h, such that for every $\boldsymbol{u} \in V_h^1$, inequalities (3.3) and*

$$\|\operatorname{grad}\boldsymbol{u}\|^2 \le \frac{C}{h\sqrt{h}}\|(\operatorname{grad}\boldsymbol{u})_{\mathrm{sym}}\|^2, \tag{3.5}$$

hold for all $h \in (0,1)$.

In fact, inequalities (3.3) and (3.5) are asymptotically sharp.

Theorem 3.3 *(Existence of optimal ansatzen).*

(i) *Suppose that both curvatures κ_θ and κ_z vanish on an open subset Ω of the middle surface of the shell. Then there exist C^∞ nonzero vector fields $\boldsymbol{u}^h$, compactly supported in $\Omega \times I_h$, and a constant C independent of h, such that*

$$\|(\operatorname{grad}\boldsymbol{u}^h)_{\mathrm{sym}}\|^2 \le Ch^2\|\operatorname{grad}\boldsymbol{u}^h\|^2, \quad \textit{for all} \quad h \in (0,1). \tag{3.6}$$

(ii) *Suppose that* $\kappa_z = 0$ *and* $\kappa_\theta > 0$ *on* $\mathcal{C}_h$. *Then there exist nonzero vector fields* $\boldsymbol{u}^h$, *vanishing at* $z = l, L$, *and a constant* C *independent of* h, *for which*

$$\|(\operatorname{grad} \boldsymbol{u}^h)_{\mathrm{sym}}\|^2 \le Ch\sqrt{h}\|\operatorname{grad} \boldsymbol{u}^h\|^2, \quad \textit{for all} \quad h \in (0,1). \tag{3.7}$$

Remark 3.4. Our results are formulated for shells cut along the coordinate surfaces. However, they are also valid for any shell $\mathcal{C}_h$ bounded by the surfaces $z = z(t,\theta)$ and $z = Z(t,\theta)$, where the spaces V_h^1, V_h^2 are defined by (3.1) and (3.2) respectively, except the indicated components of $\boldsymbol{u}$ vanish on the surfaces $z = z(t,\theta)$ and $z = Z(t,\theta)$, instead of $z = l$ and $z = L$. This is because there exists shells $\mathcal{C}_h^- \subset \mathcal{C}_h \subset \mathcal{C}_h^+$, such that the shells $\mathcal{C}_h^\pm$ are bounded by surfaces $z =$ constant. But then the ansatz from Theorem 3.3 supported in $\mathcal{C}_h^-$ gives an upper bound on the Korn constant of $\mathcal{C}_h$ that scales as $h^{3/2}$ (or as h^2). At the same time every function in V_h^1 or V_h^2 of $\mathcal{C}_h$ can be extended (by extending the relevant components of $\boldsymbol{u}$ by zero) to a function in V_h^1 or V_h^2 of $\mathcal{C}_h^+$, giving the lower bound on the Korn constant that scales as $h^{3/2}$ (or as h^2).

Remark 3.5. Our main results hold not only for cylindrical shells (i.e. p-periodic displacements $\boldsymbol{u}$), but also for "open" shells, where periodicity of $\boldsymbol{u}$ is replaced either by

$$u_\theta(t,0,z) = u_\theta(t,p,z) = 0, \tag{3.8}$$

or by

$$u_t(t,0,z) = u_t(t,p,z) = u_z(t,0,z) = u_z(t,p,z) = 0. \tag{3.9}$$

In both cases we can apply our theory to $2p$-periodic vector fields, obtained by the odd $2p$-periodic extension of u_θ and even $2p$-periodic extension of all other functions, in the case of boundary conditions (3.8); and odd $2p$-periodic extension of both u_t and u_z, and even $2p$-periodic extension of all other functions, in the case of boundary conditions (3.9). In each case, it is easy to check that every component (in (t,θ,z) coordinate system) of $\nabla\boldsymbol{u}$ and $e(\boldsymbol{u}) = (\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T)/2$ will either be even or odd in θ, making squares of their L^2 norms on $I_h \times [-p,p] \times [l,L]$ be double those on $I_h \times [0,p] \times [l,L]$.

4. Proofs of Theorems 3.1 and 3.2

Our strategy is to prove a first-and-a-half Korn inequality (3.3) for the simplified version $\nabla\boldsymbol{u}$ of $\operatorname{grad}\boldsymbol{u}$, given by (2.6)

$$\|\nabla\boldsymbol{u}\|^2 \le C\left(\frac{\|u_t\|\|e(\boldsymbol{u})\|}{h} + \|e(\boldsymbol{u})\|^2\right), \tag{4.1}$$

where

$$e(\boldsymbol{u}) = \frac{1}{2}\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right).$$

We then show that (4.1) implies (3.3). In order to prove (4.1) we apply the method, introduced in [3], of assembling (4.1) from the analogous two-dimensional inequalities corresponding to the three coordinate surface cross-sections of the shell. Most of the proof is done for $\boldsymbol{u} \in V_h^1$ and without any additional assumptions on the sign of κ_θ. When additional assumptions become necessary, they will be explicitly stated. Also, in what follows we will use a classical density argument implicitly, by conducting the proofs for smooth vector fields and then observing that we can pass to the limit in the final inequalities when a $W^{1,2}$ vector field is approximated by a sequence of C^∞ ones with the same boundary conditions.

4.1. The $t =$ const *cross-section*

The Korn-type inequality corresponding to $t =$ const cross-section involves $\theta\theta$, θz, $z\theta$, and zz components of the gradient. The first-and-a-half Korn inequality in this case is stated in the following lemma.

Lemma 4.1. *The following inequality is valid whenever* $\boldsymbol{u} \in V_h^1$.

$$\|(\nabla \boldsymbol{u})_{\theta z}\|^2 + \|(\nabla \boldsymbol{u})_{z\theta}\|^2 \le C\|e(\boldsymbol{u})\|(\|e(\boldsymbol{u})\| + \|u_t\|). \tag{4.2}$$

Proof. Observing that

$$\|(\nabla \boldsymbol{u})_{\theta z}\|^2 + \|(\nabla \boldsymbol{u})_{z\theta}\|^2 = 4\|e(\boldsymbol{u})_{\theta z}\|^2 - 2((\nabla \boldsymbol{u})_{\theta z}, (\nabla \boldsymbol{u})_{z\theta})_{\mathcal{C}_h},$$

we conclude that it is sufficient to prove

$$|((\nabla \boldsymbol{u})_{\theta z}, (\nabla \boldsymbol{u})_{z\theta})_{\mathcal{C}_h}| \le C\|e(\boldsymbol{u})\|(\|e(\boldsymbol{u})\| + \|u_t\|). \tag{4.3}$$

We have, that

$$((\nabla \boldsymbol{u})_{\theta z}, (\nabla \boldsymbol{u})_{z\theta})_{\mathcal{C}_h} = \int_{I_h}\int_l^L\int_0^p A_\theta A_z \frac{u_{\theta,z}}{A_z}\frac{u_{z,\theta} - a(\theta)u_\theta}{A_\theta} d\theta dz dt = \int_{I_h} (I_1(t) - I_2(t))dt,$$

where

$$I_1(t) = \int_l^L\int_0^p u_{\theta,z}u_{z,\theta} d\theta dz,$$

and

$$I_2(t) = \int_l^L\int_0^p a(\theta)u_\theta u_{\theta,z} d\theta dz = \frac{1}{2}\int_0^p a(\theta)\int_l^L (u_\theta^2)_{,z} dz = 0,$$

since $u_\theta = 0$ at $z = l$ and $z = L$.

Let us estimate $I_1(t)$. The idea is first to observe that

$$\int_l^L\int_0^p u_{\theta,z}u_{z,\theta} d\theta dz = -\int_l^L\int_0^p u_\theta u_{z,\theta z} d\theta dz = \int_l^L\int_0^p u_{\theta,\theta}u_{z,z} d\theta dz,$$

where p-periodicity of u_z and u_θ has been used in the last equality, and then express $u_{\theta,\theta}$ and $u_{z,z}$ in terms of $(\nabla \boldsymbol{u})_{\theta\theta} = e(\boldsymbol{u})_{\theta\theta}$ and $(\nabla \boldsymbol{u})_{zz} = e(\boldsymbol{u})_{zz}$, respectively. Thus,

$$\int_{I_h} I_1(t)dt = \left(e(\boldsymbol{u})_{zz}, e(\boldsymbol{u})_{\theta\theta} - \frac{a(\theta)}{A_\theta}u_z - \kappa_\theta u_t\right)_{\mathcal{C}_h}. \tag{4.4}$$

Applying the Schwartz inequality we obtain

$$\left|\int_{I_h} I_1(t)dt\right| \le C\|e(\boldsymbol{u})\|(\|e(\boldsymbol{u})\| + \|u_t\| + \|u_z\|). \tag{4.5}$$

By the Poincaré inequality, applicable, since $u_z = 0$ at $z = l$,

$$\|u_z\| \le C\|u_{z,z}\| \le C\|e(\boldsymbol{u})\|. \tag{4.6}$$

Using this inequality in (4.5) we obtain the desired bound (4.2). □

4.2. *The $\theta = \text{const}$ cross-section*

Lemma 4.2. *The following inequality is valid whenever $u_z(t,\theta,l) = 0$.*

$$\|(\nabla \boldsymbol{u})_{tz}\|^2 + \|(\nabla \boldsymbol{u})_{zt}\|^2 \le C\left(\frac{\|u_t\| \cdot \|e(\boldsymbol{u})\|}{h} + \|e(\boldsymbol{u})\|^2\right). \tag{4.7}$$

Let us show that Lemma 4.2 is an immediate consequence of the same two-dimensional inequality in Cartesian coordinates, proved in [3, Theorem 3.1]. It states that if $h \in (0,1)$, $\boldsymbol{\phi} = (u,v) \in H^1(I_h \times (l,L);\mathbb{R}^2)$ satisfies $v(x,l) = 0$, $x \in I_h$ in the sense of traces, then

$$\|\nabla^c \boldsymbol{\phi}\|_0^2 \le C\|e^c(\boldsymbol{\phi})\|_0\left(\frac{\|u\|_0}{h} + \|e^c(\boldsymbol{\phi})\|_0\right), \tag{4.8}$$

where

$$\nabla^c \boldsymbol{\phi} = \begin{bmatrix} u_{,x} & u_{,y} \\ v_{,x} & v_{,y} \end{bmatrix}, \qquad e^c(\boldsymbol{\phi}) = \frac{1}{2}\left(\nabla^c \boldsymbol{\phi} + (\nabla^c \boldsymbol{\phi})^T\right).$$

Lemma 4.2 says that the same statement holds in our curvilinear coordinates, where $\nabla \boldsymbol{u}$ is given by (2.6).

Proof of Lemma 4.2. We first prove inequality (4.7) for each fixed $\theta \in [0,p]$:

$$\|(\nabla \boldsymbol{u})_{tz}\|_\theta^2 + \|(\nabla \boldsymbol{u})_{zt}\|_\theta^2 \le C\left(\frac{\|u_t\|_\theta \cdot \|e_\theta(\boldsymbol{u})\|_\theta}{h} + \|e_\theta(\boldsymbol{u})\|_\theta^2\right), \tag{4.9}$$

where

$$\|f\|_\theta^2 = \int_{I_h}\int_l^L A_z f(t,\theta,z)^2 dz dt, \quad e_\theta(\boldsymbol{u}) = \begin{bmatrix} u_{t,t} & \frac{1}{2}\left(\frac{u_{t,z}}{A_z} + u_{z,t}\right) \\ \frac{1}{2}\left(\frac{u_{t,z}}{A_z} + u_{z,t}\right) & \frac{u_{z,z}}{A_z} \end{bmatrix}.$$

Then inequality (4.7) is obtained by integrating (4.9) over $\theta \in [0,p]$, and using uniform positivity and boundedness of A_θ and A_z together with the Schwartz inequality, so that

$$\int_0^p \|f\|_\theta \|g\|_\theta d\theta \le C\|f\|\|g\|.$$

Let

$$\boldsymbol{G} = \begin{bmatrix} u_{t,t} & u_{t,z} \\ A_z u_{z,t} & (A_z u_z)_{,z} \end{bmatrix}, \quad \boldsymbol{E} = \frac{1}{2}(\boldsymbol{G} + \boldsymbol{G}^T).$$

Then, by (4.8) applied to $\boldsymbol{\phi} = (u_t, A_z u_z)$, we obtain

$$\|\boldsymbol{G}\|_0^2 \le C\|\boldsymbol{E}\|_0\left(\frac{\|u_t\|_0}{h} + \|\boldsymbol{E}\|_0\right). \tag{4.10}$$

By uniform positivity and boundedness of A_θ, A_z norms $\|\cdot\|_0$ and $\|\cdot\|_\theta$ are equivalent. Hence,

$$\|(\nabla \boldsymbol{u})_{tz}\|_\theta^2 + \|(\nabla \boldsymbol{u})_{zt}\|_\theta^2 \le C(\|\boldsymbol{G}_{tz}\|_0^2 + (\|\boldsymbol{G}_{zt}\|_0^2) \le C\|\boldsymbol{G}\|_0^2.$$

Applying (4.10), we prove the lemma, if we show that

$$\|\boldsymbol{E}\|_0 \le C\|e_\theta(\boldsymbol{u})\|_\theta. \tag{4.11}$$

We estimate

$$\|\boldsymbol{E}\|_0^2 \le C(\|u_{t,t}\|_\theta^2 + \|u_{z,z}\|_\theta^2 + \|u_z\|_\theta^2) + \frac{1}{2}\|u_{t,z} + A_z u_{z,t}\|_0^2.$$

By the Poincaré inequality $\|u_z\|_\theta^2 \le C\|u_{z,z}\|_\theta^2$, so that

$$\|\boldsymbol{E}\|_0^2 \le C\|e_\theta(\boldsymbol{u})\|_\theta^2 + \frac{1}{2}\|u_{t,z} + A_z u_{z,t}\|_0^2.$$

It remains to observe that

$$\|u_{t,z} + A_z u_{z,t}\|_0 = \left\|\sqrt{A_z}\left(\frac{u_{t,z}}{A_z} + u_{z,t}\right)\right\|_\theta \le C\|e_\theta(\boldsymbol{u})\|_\theta. \qquad \square$$

4.3. The $z =$ const cross-section

Lemma 4.3. *Let $\boldsymbol{u} \in V_h^1$. Then there exists a constant $C > 0$, independent of h, so that*

$$\|(\nabla \boldsymbol{u})_{t\theta}\|^2 + \|(\nabla \boldsymbol{u})_{\theta t}\|^2 \le C\left(\frac{\|u_t\| \cdot \|e(\boldsymbol{u})\|}{h} + \|u_t\|^2 + \|e(\boldsymbol{u})\|^2\right). \tag{4.12}$$

Proof. As before, we will show that (4.23) is a consequence of a two-dimensional Korn-type inequality. However, before we can proceed with this strategy, we observe that the term with u_z in the $\theta\theta$-component of $\nabla \boldsymbol{u}$ can be easily discarded due to the Poincaré inequality (4.6). Indeed, suppose we have proved (4.23), where $e(\boldsymbol{u})$ is replaced with $e'(\boldsymbol{u})$, where all components are the same as in $e(\boldsymbol{u})$, except

$$e'(\boldsymbol{u})_{\theta\theta} = \frac{u_{\theta,\theta} + c(\theta)u_t}{A_\theta}.$$

Then

$$\|e'(\boldsymbol{u})\| \le \|e(\boldsymbol{u})\| + C\|u_z\| \le C\|e(\boldsymbol{u})\|,$$

so that

$$\begin{aligned}\|(\nabla \boldsymbol{u})_{t\theta}\|^2 + \|(\nabla \boldsymbol{u})_{\theta t}\|^2 &\le C\left(\frac{\|u_t\| \cdot \|e'(\boldsymbol{u})\|}{h} + \|u_t\|^2 + \|e'(\boldsymbol{u})\|^2\right)\\ &\le C\left(\frac{\|u_t\| \cdot \|e(\boldsymbol{u})\|}{h} + \|u_t\|^2 + \|e(\boldsymbol{u})\|^2\right).\end{aligned}$$

Next we prove the two-dimensional Korn-type inequality.

Theorem 4.4. *Let*

$$V = \left\{\boldsymbol{\phi} = (u, v) \in W^{1,2}(I_h \times (0, p); \mathbb{R}^2) : u(x, \cdot) \text{ is } p-\text{periodic}\right\}$$

and let

$$\boldsymbol{G}(\boldsymbol{\phi}) = \begin{bmatrix} u_{,x} & a_1(y)u_{,y} + b_1(y)v \\ v_{,x} & a_2(y)v_{,y} + b_2(y)u \end{bmatrix}, \qquad \boldsymbol{E}(\boldsymbol{\phi}) = \frac{1}{2}(\boldsymbol{G} + \boldsymbol{G}^T), \tag{4.13}$$

where a_1, a_2, b_1, b_2 are Lipschitz continuous functions in $[0, p]$, such that $a_1(y)$ and $a_2(y)$ do not vanish on $[0, p]$. Then,

$$\|\boldsymbol{G}\|^2 \le C\left(\frac{\|u\|\|\boldsymbol{E}\|}{h} + \|\boldsymbol{E}\|^2 + \|\boldsymbol{\phi}\|^2\right), \tag{4.14}$$

where

$$\|f\|^2 = \int_{-h/2}^{h/2}\int_0^p f^2(x, y)\,dy\,dx.$$

The proof is based on the sharp inequality for harmonic functions [3, Lemma 4.3], see also [7, Theorem 1.1], which we formulate here for the sake of completeness.

Lemma 4.5. *Let $R_h = (-\frac{h}{2}, \frac{h}{2}) \times (0, p)$. Suppose $w \in H^1(R_h)$ is harmonic in R_h and satisfies the boundary condition $w(x, 0) = w(x, p)$ in the sense of traces. Then*

$$\|w_{,y}\|^2 \le \frac{2\sqrt{3}}{h}\|w\|\|w_{,x}\| + \|w_{,x}\|^2. \tag{4.15}$$

Proof of Theorem 4.4. The first step is to replace $u(x, y)$ by its harmonic extension in R_h by defining $w \in H^1(R_h)$ to be the unique solution of the Dirichlet boundary value problem

$$\begin{cases} \triangle w = 0, & (x, y) \in R_h \\ w = u, & (x, y) \in \partial R_h. \end{cases} \tag{4.16}$$

By the Poincaré inequality,

$$\|u - w\| \le h\|\nabla(u - w)\|. \tag{4.17}$$

Next, we express $\triangle(u - w) = \triangle u$ in terms of $\boldsymbol{E}(x, y)$, defined in (4.13), by eliminating all derivatives, except $u_{,y}$:

$$\triangle(u - w) = E_{11,x} + \frac{2E_{12,y}}{a_1} - \frac{1}{a_1 a_2}(E_{22,x} - b_2 E_{11} + b_1 E_{22}) + R(x, y), \tag{4.18}$$

where

$$R(x, y) = \frac{b_1 b_2 u - a_2 a_1' u_{,y} - a_2 b_1' v}{a_1 a_2}.$$

Now we multiply (4.18) by $u - w$ and integrate by parts over R_h using the fact that $u - w$ vanishes on ∂R_h:

$$\|\nabla(u - w)\|^2 = \int_{R_h} \left[E_{11}(u_{,x} - w_{,x}) + \frac{2E_{12}(u_{,y} - w_{,y})}{a_1} - \frac{E_{22}(u_{,x} - w_{,x})}{a_1 a_2} + (u - w)Q(x, y) \right] dxdy,$$

where $Q(x, y)$ is a linear combination of E_{11}, E_{12}, E_{22}, u, v and $u_{,y}$ with uniformly bounded coefficients. Estimating $\|u - w\|$ by (4.17) we obtain, after division by $\|\nabla(u - w)\|$,

$$\|\nabla(u - w)\| \le Ch\left(\frac{\|\boldsymbol{E}\|}{h} + \|u\| + \|v\| + \|u_{,y}\|\right). \tag{4.19}$$

Our last task is to estimate $\|u_{,y}\|$. This is done by replacing u with w, estimating $\|w_{,y}\|$ using (4.15), and returning back to u, while controlling the incurred errors by (4.17) and (4.19).

$$\begin{aligned}
\|u_{,y}\|^2 &\le 2\|u_{,y} - w_{,y}\|^2 + 2\|w_{,y}\|^2 \le C\left(\|\nabla(u - w)\|^2 + \frac{\|w\|\|w_{,x}\|}{h} + \|w_{,x}\|^2\right) \\
&\le C\left(\|\nabla(u - w)\|^2 + \|u_{,x}\|^2 + \frac{1}{h}(\|u\| + h\|\nabla(u - w)\|)(\|u_{,x}\| + \|\nabla(u - w)\|)\right) \\
&\le C\left(\|\nabla(u - w)\|^2 + \|u_{,x}\|^2 + \frac{\|u\|\|u_{,x}\|}{h} + \frac{\|u\|\|\nabla(u - w)\|}{h}\right) \\
&\le C\left(h^2\|u_{,y}\|^2 + \|\boldsymbol{E}\|^2 + \frac{\|u\|\|\boldsymbol{E}\|}{h} + \|u\|^2 + \|v\|^2 + \|u\|\|u_{,y}\|\right),
\end{aligned}$$

where we took into account that $u_{,x} = E_{11}$. Estimating

$$C\|u\|\|u_{,y}\| \le \frac{1}{2}\|u_{,y}\|^2 + \frac{C^2\|u\|^2}{2}$$

and choosing h so small that $Ch^2 < 1/4$ we obtain the inequality

$$\|u_{,y}\|^2 \le C\left(\|\boldsymbol{E}\|^2 + \frac{\|u\|\|\boldsymbol{E}\|}{h} + \|u\|^2 + \|v\|^2\right), \tag{4.20}$$

which holds for all sufficiently small $h > 0$. Observing that

$$\|\boldsymbol{G}\|^2 = \|E_{11}\|^2 + \|E_{22}\|^2 + \|G_{12}\|^2 + \|2E_{12} - G_{12}\|^2,$$

we get the bound

$$\|\boldsymbol{G}\|^2 \le 7\|\boldsymbol{E}\|^2 + 3\|G_{12}\|^2,$$

while

$$\|G_{12}\|^2 = \|a_1(y)u_{,y} + b_1(y)v\|^2 \le C(\|u_{,y}\|^2 + \|v\|^2).$$

This shows that (4.20) implies (4.14). □

If we apply this theorem to $\boldsymbol{\phi}(t,\theta) = (u_t(t,\theta,z), u_\theta(t,\theta,z))$ (for each fixed value of z) and then integrate over $z \in [l, L]$, we obtain the inequality (taking into account the equivalence between the curvilinear norm (2.7) and the Euclidean norm)

$$\|(\nabla \boldsymbol{u})_{t\theta}\|^2 + \|(\nabla \boldsymbol{u})_{\theta t}\|^2 \le C\left(\frac{\|u_t\| \cdot \|e(\boldsymbol{u})\|}{h} + \|e(\boldsymbol{u})\|^2 + \|u_t\|^2 + \|u_\theta\|^2\right). \tag{4.21}$$

By the Poincaré inequality and (4.2) we obtain

$$\|u_\theta\|^2 \le C\|(\nabla \boldsymbol{u})_{\theta z}\|^2 \le C\|e(\boldsymbol{u})\|(\|e(\boldsymbol{u})\| + \|u_t\|). \tag{4.22}$$

Combining this with (4.21) we obtain (4.12), which completes the proof of Lemma 4.3. □

Lemma 4.6. *Suppose that either $\boldsymbol{u} \in V_h^2$ or $\boldsymbol{u} \in V_h^1$ and $\kappa_\theta > 0$ on $[0, p] \times [l, L]$. Then there exists a constant $C > 0$, independent of h, such that*

$$\|(\nabla \boldsymbol{u})_{t\theta}\|^2 + \|(\nabla \boldsymbol{u})_{\theta t}\|^2 \le C\left(\frac{\|u_t\| \cdot \|e(\boldsymbol{u})\|}{h} + \|e(\boldsymbol{u})\|^2\right). \tag{4.23}$$

Proof. If $\boldsymbol{u} \in V_h^2$ then we can just use the Poincaré inequality and (4.7) to estimate

$$\|u_t\|^2 \le C\|(\nabla \boldsymbol{u})_{tz}\|^2 \le C\left(\frac{\|u_t\| \cdot \|e(\boldsymbol{u})\|}{h} + \|e(\boldsymbol{u})\|^2\right). \tag{4.24}$$

If $\boldsymbol{u} \in V_h^1$, then the Poincaré inequality cannot be used. Instead we use uniform positivity of κ_θ and estimate u_t by expressing it in terms of $e'(\boldsymbol{u})_{\theta\theta}$ as follows:

$$u_t = \frac{A_\theta e'(\boldsymbol{u})_{\theta\theta} - u_{\theta,\theta}}{A_\theta \kappa_\theta}. \tag{4.25}$$

Multiplying both sides of (4.25) by $A_\theta \kappa_\theta u_t$ we obtain

$$A_\theta \kappa_\theta u_t^2 = A_\theta u_t e'(\boldsymbol{u})_{\theta\theta} - (u_\theta u_t)_{,\theta} + u_\theta u_{t,\theta}.$$

Replacing the obtained $u_{t,\theta}$ by its expression $u_{t,\theta} = A_\theta(\nabla \boldsymbol{u})_{t\theta} + c(\theta)u_\theta$, and integrating over $I_h \times [0, p] \times [l, L]$ we arrive at the estimate (using equivalence of various norms)

$$\|u_t\|^2 \le C(\|u_t\|\|e(\boldsymbol{u})\| + \|u_\theta\|^2 + \|(\nabla \boldsymbol{u})_{t\theta}\|\|u_\theta\|).$$

Hence, replacing $\|u_\theta\|^2$ by its estimate from (4.22) and then using the inequality $Cab \le a^2/2 + C^2b^2/2$, we obtain the bound

$$\|u_t\|^2 \le C(\|e(\boldsymbol{u})\|^2 + \|(\nabla \boldsymbol{u})_{t\theta}\|\|u_\theta\|). \tag{4.26}$$

Using this inequality to estimate the term $\|u_t\|^2$ in (4.12), we obtain

$$\|(\nabla \boldsymbol{u})_{t\theta}\|^2 + \|(\nabla \boldsymbol{u})_{\theta t}\|^2 \le C\left(\frac{\|u_t\| \cdot \|e(\boldsymbol{u})\|}{h} + \|(\nabla \boldsymbol{u})_{t\theta}\|\|u_\theta\| + \|e(\boldsymbol{u})\|^2\right).$$

Once again we apply the inequality $Cab \le a^2/2 + C^2b^2/2$ to obtain

$$\|(\nabla \boldsymbol{u})_{t\theta}\|^2 + \|(\nabla \boldsymbol{u})_{\theta t}\|^2 \le C\left(\frac{\|u_t\| \cdot \|e(\boldsymbol{u})\|}{h} + \|u_\theta\|^2 + \|e(\boldsymbol{u})\|^2\right).$$

Estimating $\|u_\theta\|^2$ via (4.22), we obtain the desired inequality (4.23). □

4.4. Conclusion of the proof

The conclusion of the proof is conducted under the assumptions of Lemma 4.6, covering both Theorems 3.1 and 3.2. Combining the estimates (4.2), (4.7) and (4.23) we arrive at (4.1). However, $\nabla \boldsymbol{u}$ and $e(\boldsymbol{u})$ are the simplified versions of $\operatorname{grad} \boldsymbol{u}$ and $(\operatorname{grad} \boldsymbol{u})_{\mathrm{sym}}$. Thus, we need to show that (3.3) follows from (4.1). We will show that under the assumptions of Theorems 3.1 it is a consequence of (4.24), while under the assumptions of Theorems 3.2 it is a consequence of (4.26).

The main observation in either case is that components of $\operatorname{grad} \boldsymbol{u}$ and $\nabla \boldsymbol{u}$ are multiples of one another with coefficients that are independent of $\boldsymbol{u}$. Thus, by direct calculation, we estimate

$$\|\operatorname{grad} \boldsymbol{u} - \nabla \boldsymbol{u}\| \le Ch \|\operatorname{grad} \boldsymbol{u}\|, \quad \forall \boldsymbol{u} \in W^{1,2}(\mathcal{C}_h; \mathbb{R}^3), \tag{4.27}$$

from which we get additionally,

$$\|(\operatorname{grad} \boldsymbol{u})_{\mathrm{sym}} - e(\boldsymbol{u})\| \le \|\operatorname{grad} \boldsymbol{u} - \nabla \boldsymbol{u}\| \le Ch \|\operatorname{grad} \boldsymbol{u}\|, \quad \forall \boldsymbol{u} \in W^{1,2}(\mathcal{C}_h; \mathbb{R}^3). \tag{4.28}$$

Proof of Theorem 3.1. Observe, that tt, tz, zt and zz components of $\operatorname{grad} \boldsymbol{u}$ and $\nabla \boldsymbol{u}$ coincide. The analysis for the cross section $\theta = \text{const}$ involved only these components of the gradient. Thus, estimate (4.7) holds for $\operatorname{grad} \boldsymbol{u}$ in place of $\nabla \boldsymbol{u}$. This implies (4.24) for $(\operatorname{grad} \boldsymbol{u})_{\mathrm{sym}}$ in place of $e(\boldsymbol{u})$, i.e., we have

$$\|u_t\| \le \frac{C}{h} \|(\operatorname{grad} \boldsymbol{u})_{\mathrm{sym}}\|. \tag{4.29}$$

This allows us to show that (4.1) implies (3.3). Combining (4.1), (4.27) and (4.28), we obtain

$$\|\operatorname{grad} \boldsymbol{u}\|^2 \le C \left(\frac{\|u_t\| \|(\operatorname{grad} \boldsymbol{u})_{\mathrm{sym}}\|}{h} + \|u_t\| \|\operatorname{grad} \boldsymbol{u}\| + \|(\operatorname{grad} \boldsymbol{u})_{\mathrm{sym}}\|^2 \right). \tag{4.30}$$

Estimating

$$C \|u_t\| \|\operatorname{grad} \boldsymbol{u}\| \le \frac{1}{2} \|\operatorname{grad} \boldsymbol{u}\|^2 + \frac{C^2}{2} \|u_t\|^2$$

we get

$$\|\operatorname{grad} \boldsymbol{u}\|^2 \le C \left(\frac{\|u_t\| \|(\operatorname{grad} \boldsymbol{u})_{\mathrm{sym}}\|}{h} + \|u_t\|^2 + \|(\operatorname{grad} \boldsymbol{u})_{\mathrm{sym}}\|^2 \right).$$

Finally, by (4.29)

$$\|u_t\|^2 \le \frac{\|u_t\| \|(\operatorname{grad} \boldsymbol{u})_{\mathrm{sym}}\|}{h},$$

and (3.3) follows. Combining estimates (4.29) and (3.3) we obtain (3.4). □

Proof of Theorem 3.2. In this case we proceed in the same way as in [3], proving the following lemma.

Lemma 4.7. *Inequalities (4.1), (4.22), and (4.26) taken together, imply*

$$\|\nabla \boldsymbol{u}\|^2 \le \frac{C}{h\sqrt{h}} \|e(\boldsymbol{u})\|^2. \tag{4.31}$$

We postpone the proof of this virtually algebraic lemma and show that it implies (3.3) and (3.5).

Combining (4.31) with (4.28) we obtain

$$\|\operatorname{grad} \boldsymbol{u}\|^2 \le C \|\nabla \boldsymbol{u}\|^2 \le \frac{C}{h\sqrt{h}} (\|(\operatorname{grad} \boldsymbol{u})_{\mathrm{sym}}\|^2 + h^2 \|\operatorname{grad} \boldsymbol{u}\|^2),$$

proving the first Korn inequality (3.5). Now, inequality (4.1) and (3.5) imply (3.3). Indeed, using the estimate

$$\|u_t\| \|\operatorname{grad} \boldsymbol{u}\| \le \frac{C \|u_t\| \|(\operatorname{grad} \boldsymbol{u})_{\mathrm{sym}}\|}{h^{3/4}} \le \frac{C \|u_t\| \|(\operatorname{grad} \boldsymbol{u})_{\mathrm{sym}}\|}{h}$$

in (4.30) we obtain (3.3). □

Proof of Lemma 4.7. We begin with the inequality (4.26),

$$\|u_t\|^2 \le C(\|e(\boldsymbol{u})\|^2 + 2\|\nabla \boldsymbol{u}\|\|u_\theta\|) \le C\left(\|e(\boldsymbol{u})\|^2 + \epsilon^2\|\nabla \boldsymbol{u}\|^2 + \frac{1}{\epsilon^2}\|u_\theta\|^2\right)$$

for any $\epsilon > 0$. The small parameter $\epsilon \in (0,1)$ will be chosen later to optimize the resulting inequality. Estimating $\|u_\theta\|^2$ by (4.22) we obtain for sufficiently small ϵ,

$$\|u_t\|^2 \le C\left(\|e(\boldsymbol{u})\|^2 + \epsilon^2\|\nabla \boldsymbol{u}\|^2 + \frac{1}{\epsilon^2}(\|e(\boldsymbol{u})\|^2 + \|e(\boldsymbol{u})\|\|u_t\|)\right).$$

Estimating

$$\frac{C}{\epsilon^2}\|e(\boldsymbol{u})\|\|u_t\| \le \frac{1}{2}\|u_t\|^2 + \frac{C^2}{2\epsilon^4}\|e(\boldsymbol{u})\|^2,$$

we obtain

$$\|u_t\|^2 \le C\left(\frac{\|e(\boldsymbol{u})\|^2}{\epsilon^4} + \epsilon^2\|\nabla \boldsymbol{u}\|^2\right).$$

Thus,

$$\|u_t\| \le C\left(\frac{\|e(\boldsymbol{u})\|}{\epsilon^2} + \epsilon\|\nabla \boldsymbol{u}\|\right). \tag{4.32}$$

Substituting this inequality into (4.1), we obtain

$$\|\nabla \boldsymbol{u}\|^2 \le C\left(\frac{\|e(\boldsymbol{u})\|^2}{h\epsilon^2} + \frac{\epsilon\|\nabla \boldsymbol{u}\|\|e(\boldsymbol{u})\|}{h}\right).$$

Estimating

$$\frac{C\epsilon\|\nabla \boldsymbol{u}\|\|e(\boldsymbol{u})\|}{h} \le \frac{1}{2}\|\nabla \boldsymbol{u}\|^2 + \frac{C^2\epsilon^2\|e(\boldsymbol{u})\|^2}{2h^2},$$

we obtain the inequality

$$\|\nabla \boldsymbol{u}\|^2 \le C\left(\frac{1}{h\epsilon^2} + \frac{\epsilon^2}{h^2}\right)\|e(\boldsymbol{u})\|^2.$$

We now choose $\epsilon = h^{1/4}$ to minimize the upper bound and obtain (4.31). □

5. Proof of Theorem 3.3

The ansatz in part (i) of the theorem is a classical Kirchhoff ansatz. The assumptions of part (i) say that the shell contains a plate, which means that we can introduce a local Cartesian coordinate system (x_1, x_2, x_3) in which the (sub)plate be described as

$$\mathcal{P}_h = \{(x_1, x_2) \in \Omega \subset \mathbb{R}^2,\ x_3 \in I_h\}.$$

In these Cartesian coordinates we construct the ansatz in terms of the function $\phi(x_1, x_2)$, compactly supported in Ω:

$$\begin{cases} u_1^h = -x_3\phi_{,x_1}, \\ u_2^h = -x_3\phi_{,x_2}, \\ u_3^h = \phi(x_1, x_2). \end{cases} \tag{5.1}$$

Then

$$\nabla \boldsymbol{u}^h = \begin{bmatrix} -x_3\phi_{,x_1x_1} & -x_3\phi_{,x_1x_2} & -\phi_{,x_1} \\ -x_3\phi_{,x_1x_2} & -x_3\phi_{,x_2x_2} & -\phi_{,x_2} \\ \phi_{,x_1} & \phi_{,x_2} & 0 \end{bmatrix}, \qquad e(\boldsymbol{u}^h) = \begin{bmatrix} -x_3\phi_{,x_1x_1} & -x_3\phi_{,x_1x_2} & 0 \\ -x_3\phi_{,x_1x_2} & -x_3\phi_{,x_2x_2} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This shows that

$$\|\nabla \boldsymbol{u}^h\|^2 = \|\nabla\phi\|^2 + \frac{h^2}{12}\|\nabla\nabla\phi\|^2, \qquad \|e(\boldsymbol{u}^h)\|^2 = \frac{h^2}{12}\|\nabla\nabla\phi\|^2.$$

Choosing a fixed nonzero $\phi \in C_0^2(\Omega)$ we establish (3.6). The ansatz (5.1) was found by looking for the ansatz in the form $\boldsymbol{u}^h = \boldsymbol{v}(x_1, x_2) + x_3\boldsymbol{w}(x_1, x_2)$. We then compute $e(\boldsymbol{u}^h) = \boldsymbol{E}_0(x_1, x_2) + x_3\boldsymbol{E}_1(x_1, x_2)$. The ansatz (5.1) is the general solution of the equations $\boldsymbol{E}_0(x_1, x_2) = 0$. The same idea could be applied to $\operatorname{grad}\boldsymbol{u}$, given by (2.5). However, the different structure of the gradient results only in trivial solutions of $\boldsymbol{E}_0(\theta, z) = 0$. Nevertheless, the same idea works if we relax our requirements. Specifically, we can find an ansatz in which all components of $\boldsymbol{E}_0(\theta, z)$ vanish, *except the zz-component*. Accordingly, we look for the ansatz in the form $\boldsymbol{u}^h = \boldsymbol{v}^h(\theta, z) + t\boldsymbol{w}^h(\theta, z)$, so that

$$(\operatorname{grad}\boldsymbol{u}^h)_{\text{sym}} = \boldsymbol{E}_0^h(\theta, z) + t\boldsymbol{E}_1^h(\theta, z) + O(t^2).$$

In accordance with our strategy we have the following system of equations

$$\begin{cases} w_t^h = 0, \\ w_\theta^h = -\dfrac{1}{A_\theta}\left(\dfrac{\partial v_t^h}{\partial\theta} + c(\theta)v_\theta^h\right), \\ w_z^h = -\dfrac{1}{A_z}\dfrac{\partial v_t^h}{\partial z}, \\ v_t^h = -\dfrac{v_{\theta,\theta}^h + a(\theta)v_z^h}{c(\theta)}, \\ -A_\theta v_{\theta,z}^h = A_z(v_{z,\theta}^h - a(\theta)v_\theta^h). \end{cases} \tag{5.2}$$

The first four equations in (5.2) express $\boldsymbol{w}^h$ and v_t^h in terms of only two functions v_θ^h and v_z^h. The last equation relates v_θ^h and v_z^h, and needs to be solved. There are two mutually exclusive cases

- Case 1:
$$\frac{A_z}{A_\theta} = \frac{H(\theta)}{G(z)}, \tag{5.3}$$
for some Lipschitz functions $H(\theta)$ and $G(z)$. It is easy to see from formulas (2.4) that (5.3) is equivalent to $a(\theta)$ and $b(\theta)$ being linearly dependent, i.e. there exists a constant scalar λ_0, such that either $a(\theta) = \lambda_0 b(\theta)$ or $b(\theta) = \lambda_0 a(\theta)$.
- Case 2: There exists an interval $I = (\theta_1, \theta_2) \subset (0, p)$, such that $a(\theta) \neq 0$ and $\rho'(\theta) \neq 0$ for all $\theta \in I$, where
$$\rho(\theta) = \frac{b(\theta)}{a(\theta)}.$$

Case 1. It is easy to see from Table 1 that all cylinders and cones fall into this case. Under the assumption (5.3) the last equation in (5.2) has a general solution

$$v_z^h = A_\theta G(z)H(\theta)\phi_{,z}^h, \qquad v_\theta^h = -A_\theta H(\theta)^2\phi_{,\theta}^h, \tag{5.4}$$

expressing the ansatz in terms of a single potential $\phi^h(z, \theta)$, which can be an arbitrary function with compact support.

Case 2. In this case we will assume that functions $a(\theta)$ and $b(\theta)$ are of class C^3. Solving the last equation in (5.2) with respect to $v_{z,\theta}^h$

$$v_{z,\theta}^h = \frac{1}{B'(z)}(a(\theta)B'(z)v_\theta^h - (a(\theta)B(z) + b(\theta))v_{\theta,z}^h)$$

we see that we need both $a(\theta)v_\theta^h$ and $b(\theta)v_\theta^h$ to be θ-derivatives of some θ-periodic smooth functions of (θ, z). Hence, we define

$$v_\theta^h = \frac{\psi_{,\theta}^h}{a(\theta)},$$

where $\psi^h(\theta, z)$ is supported on $I \times (l, L)$. But then, we also need that $\rho(\theta)\psi_{,\theta}^h$ be a θ-derivative of some θ-periodic smooth function of (θ, z). We then define

$$\psi^h = \frac{\phi_{,\theta}^h}{\rho'(\theta)},$$

where $\phi^h(\theta, z)$ is supported on $I \times (l, L)$. These formulas yield the explicit ansatz that can be expressed in terms of a single potential $\phi^h(z, \theta)$, which can be an arbitrary function with compact support. Specifically,

$$\begin{cases} v_\theta^h = \dfrac{1}{a(\theta)} \dfrac{\partial}{\partial\theta} \left(\dfrac{\phi_{,\theta}^h}{\rho'(\theta)} \right), \\ v_z^h = \dfrac{B'(z)\phi_{,\theta}^h + \rho'(\theta)\phi_{,z}^h - (B(z) + \rho(\theta))\phi_{,\theta z}^h}{B'(z)\rho'(\theta)} \end{cases}. \tag{5.5}$$

Finally, in order to obtain optimal upper bound on the Korn constant we use the same scaling analysis as in [3] and define $\phi^h(\theta, z)$ in terms of the smooth, non-constant p-periodic in θ function $\Phi(\theta, z)$. In Case 1 we just set

$$\phi^h(\theta, z) = \Phi(n(h)\theta, z),$$

where $n(h)$ is the integer part of $h^{-1/4}$. In Case 2 we define

$$\phi^h(\theta, z) = \eta(\theta, z)\Phi(n(h)\theta, z),$$

where $\eta(\theta, z)$ is a smooth p-periodic in θ function, supported on $I \times (l, L)$. In both cases the constructed ansatz yields the upper bound $K(V_h) \le Ch^{3/2}$ for any V_h containing all $W^{1,2}$ vector fields that vanish at $z = l, L$.

Conflict of interest statement

There is no conflict of interests.

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