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**Yury Grabovsky**

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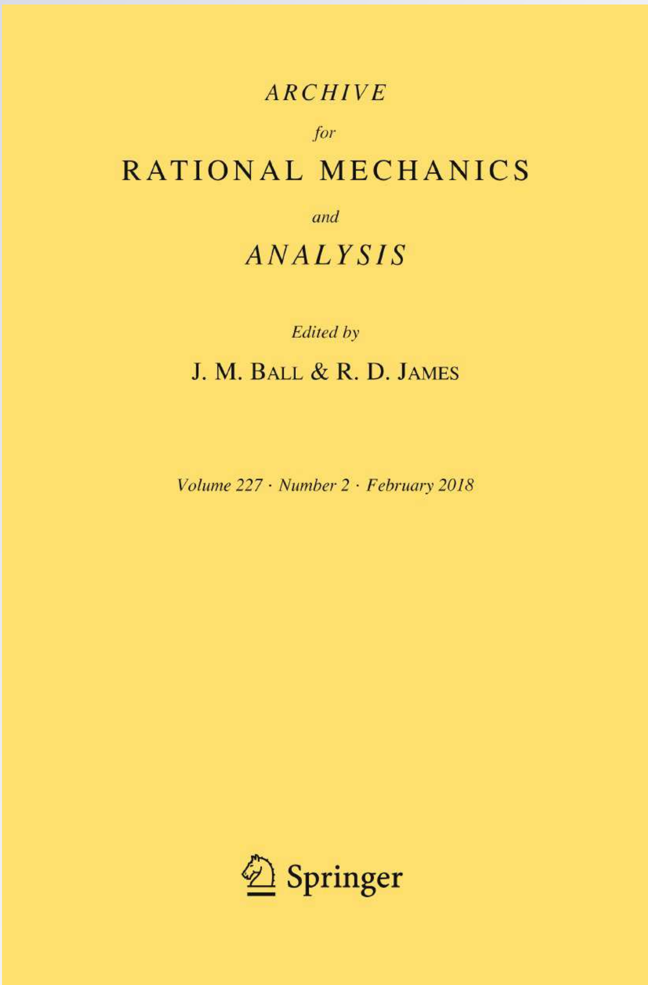

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# *From Microstructure-Independent Formulas for Composite Materials to Rank-One Convex, Non-quasiconvex Functions*

Yury Grabovsky

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## Abstract

Examples of non-quasiconvex functions that are rank-one convex are rare. In this paper we construct a family of such functions by means of the algebraic methods of the theory of exact relations for polycrystalline composite materials, developed to identify G-closed sets of positive codimensions. The algebraic methods are used to construct a set of materials of positive codimension that is closed under lamination but is not G-closed. The well-known link between G-closed sets and quasiconvex functions and sets closed under lamination and rank-one convex functions is then used to construct a family of rotationally invariant, nonnegative, and 2-homogeneous rank-one convex functions, that are not quasiconvex.

## Contents



## 1. Introduction

Problems of existence and necessary and sufficient conditions for minimizers in variational problems with multiple integrals lead to the concept of quasiconvexity [2,43].

**Definition 1.1.** We say that a function $W : \mathbb{R}^{m\times d} \to \mathbb{R}$ is quasiconvex at $\boldsymbol{F} \in \mathbb{R}^{m\times d}$ if

$$\int_{\mathbb{R}^d} \{W(\boldsymbol{F} + \nabla\boldsymbol{\phi}) - W(\boldsymbol{F})\} d\boldsymbol{x} \geqq 0$$

for every $\boldsymbol{\phi} \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^m)$. A function $W(\boldsymbol{F})$ is called quasiconvex if it is quasiconvex at every $\boldsymbol{F} \in \mathbb{R}^{m\times d}$.

It is well-known that quasiconvex functions have to be rank-one convex [43] (that is convex along any line joining $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$, provided $\boldsymbol{F}_1 - \boldsymbol{F}_2$ has rank 1). Whether or not the converse is true was an open question for a long time, until Šverák's counterexample [49] settled it. Even so, examples like Šverák's are rare, and the cases $m = 2$, $d \geqq 2$ are still open. There are also no examples of rotationally invariant nonnegative functions $W(\boldsymbol{F})$, that is the ones that satisfy $W(\boldsymbol{F}\boldsymbol{R}) = W(\boldsymbol{F})$ for all $\boldsymbol{F} \in \mathbb{R}^{m\times d}$ and all $\boldsymbol{R} \in SO(d)$. In this paper we give an example of a rotationally invariant function $W(\boldsymbol{F})$ for the case $d = 2$, $m = 8$. Our example has an intriguing $2 \times 2$ "flavor". Specifically, we regard $8 \times 2$ matrices $\boldsymbol{F}$ as $2 \times 2$ *quaternionic* matrices via a natural identification between $\mathbb{R}^4$ and $\mathbb{H}$—the set of all quaternions:

$$\mathbb{R}^4 \ni \boldsymbol{q} = (q_0, q_1, q_2, q_3) \mapsto q = q_0 + \mathfrak{i}q_1 + \mathfrak{j}q_2 + \mathfrak{k}q_3 \in \mathbb{H}. \tag{1.1}$$

The image of a vector, denoted by a bold letter, under the map (1.1), will consistently be denoted by the same letter in normal font.

To give a simple and explicit formula for one of our examples $W(\boldsymbol{F})$, it will be helpful to think of $\boldsymbol{F}$ as the gradient of $\boldsymbol{f}(x, y) = (u(x, y), v(x, y))$, where functions $u(x, y)$ and $v(x, y)$ are quaternion-valued. Then the $2 \times 2$ quaternionic matrix

$$\nabla\boldsymbol{f} = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix}$$

is identified with a real $8 \times 2$ matrix $\boldsymbol{F}$, via (1.1). In order to streamline the notation, we define a quaternion-valued "inner product" on $\mathbb{H}^2$:

$$\left(\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ v_2 \end{bmatrix}\right)_{\mathbb{H}^2} = u_1\overline{u_2} + v_1\overline{v_2}, \tag{1.2}$$

and the corresponding norm $\|\boldsymbol{f}\|_{\mathbb{H}^2}^2 = (\boldsymbol{f}, \boldsymbol{f})_{\mathbb{H}^2}$, where quaternionic conjugation is defined by

$$\bar{q} = \overline{q_0 + \mathfrak{i}q_1 + \mathfrak{j}q_2 + \mathfrak{k}q_3} = q_0 - \mathfrak{i}q_1 - \mathfrak{j}q_2 - \mathfrak{k}q_3. \tag{1.3}$$

We will prove that the function $W(\boldsymbol{F})$, given by

$$W(\nabla\boldsymbol{f}) = \sqrt{\det_{\mathbb{H}}((\nabla\boldsymbol{f})^T\overline{\nabla\boldsymbol{f}})} = \sqrt{\left\|\frac{\partial\boldsymbol{f}}{\partial x}\right\|_{\mathbb{H}^2}^2 \left\|\frac{\partial\boldsymbol{f}}{\partial y}\right\|_{\mathbb{H}^2}^2 - \left|\left(\frac{\partial\boldsymbol{f}}{\partial x}, \frac{\partial\boldsymbol{f}}{\partial y}\right)_{\mathbb{H}^2}\right|^2} \tag{1.4}$$

is rank-one convex, but not quasiconvex.[1] Specifically, we will show that $W(\boldsymbol{F})$ is not quasiconvex at $\boldsymbol{F} = \boldsymbol{I}_2$—the quaternionic $2 \times 2$ identity matrix. We note that the non-commutativity of quaternion multiplication plays a key role here. Indeed, if $\boldsymbol{f}$ were $\mathbb{C}^2$-valued, then the determinant of the product of two complex matrices can be written as the product of their determinants, and $W(\nabla \boldsymbol{f})$ would be equal to $|\det(\nabla \boldsymbol{f})|$, which is, obviously, polyconvex. Another observation is that $W(\boldsymbol{F}) \geqq 0$ with equality if and only if either

$$F_{11} = F_{12} = 0, \text{ or } F_{11} = F_{21} = 0, \text{ or } F_{12} = F_{22} = 0, \text{ or } F_{11}^{-1} F_{21} = F_{12}^{-1} F_{22}. \tag{1.5}$$

This statement is easy to obtain from the well-known conditions of equality for the triangle and the Cauchy-Schwarz inequalities in Euclidean spaces:

$$|(\boldsymbol{u}, \boldsymbol{v})_{\mathbb{H}^2}| = |u_1 \overline{v_1} + u_2 \overline{v_2}| \leqq |u_1||\overline{v_1}| + |u_2||\overline{v_2}| \leqq \|\boldsymbol{u}\|_{\mathbb{H}^2} \|\boldsymbol{v}\|_{\mathbb{H}^2}.$$

Conditions (1.5) include the case when the corresponding $8 \times 2$ matrix $\boldsymbol{F}$ has rank 1 (this is characterized by (1.5) with the additional requirement that $F_{11} F_{12}^{-1}$ be real), but describe a larger, 12-dimensional, cone in $\mathbb{R}^{8\times 2}$.

The relatively non-technical proof of failure of quasiconvexity of (1.4) presented here comes as a consequence of the theory of exact relations for composite materials [13,14,18]. A direct proof of rank-one convexity of (1.4) is also given in Appendix B. The connection between homogenization and quasiconvexity is well-known [25–27] (see also [41, Section 31.4]), where the corresponding problem is to produce a microstructure whose effective behavior cannot be attained by laminates made with the same constituents. The homogenization problem is harder, since it can be regarded as a particular case of quasiconvexification. In fact, Milton's example of a composite that cannot be mimicked by a laminate [41, Sections 31.8–9] uses Šverák's counterexample to guide the explicit construction. In this paper we solve a purportedly harder problem: finding SO(2)-invariant (polycrystalline) exact relations that are valid for all laminates but not for arbitrary microstructures.

As the name implies, the term "exact relation" refers to a microstructure-independent (that is exact) relation linking effective tensors of composite materials with tensors of material properties of their constituents. It is well-known that properties of composite materials depend strongly on the microstructure. In fact, in a generic case the knowledge of properties of constituent materials and their volume fractions alone cannot be used to determine a single equation that must be satisfied by effective tensors of composites. Nevertheless, the literature on composite materials abounds with beautiful microstructure-independent formulas that hold in special, non-generic circumstances. Examples are known in virtually every physical context, such as conductivity [11,12,23], elasticity [10,20,31,32], piezoelectricity [4,6], thermoelasticity [19,28,45,47], thermoelectricity [1,48] or even for thermoelectroelastic composites [5,9]. (See a review by Milton [40].) The general theory of exact relations, developed in [13,14,18], created a machinery for systematic

[1] Of course one can also write this function in conventional notation. However, once the quaternionic products are expanded out, the expression under the square root becomes unwieldy, losing both its simplicity and structure.

computation of *all* such formulas. The idea was to identify equations satisfied by effective tensors of simple *laminates*, whereby two constituent materials are combined in layers perpendicular to a given unit vector (lamination direction). Once such an equation is discovered, one needs to decide whether or not it is valid for all, not just laminar, microstructures. The general theory provides a simple algebraic sufficient condition. While there is a strong algebraic evidence that this sufficient condition should not be a consequence of stability under lamination, each and every laminate exact relation (or L-relation), within classical physical contexts mentioned above, is known to satisfy them. This "mystery" is explained by the fact that our physical examples have a relatively low dimensionality, from the algebraic point of view, with "no room" for counterexamples. Hence, in order to produce a desired example of an L-relation, which is not exact, we consider multifield composite materials [34–36], coupling 4 curl-free and 4 divergence-free fields in two space dimensions.[2] The main tool is a different, purely algebraic condition, derived in [14], that is necessary for an equation to hold for all microstructures, and that does not come from the study of laminates. Once the example of an L-relation that is not exact has been found, we utilize the well-known connection between homogenization and quasiconvexification to produce explicit examples of rotationally invariant rank-one convex, non-quasiconvex functions, one of which is given in (1.4). We remark that our method produces rotationally invariant 2-homogeneous functions that are certifiably rank-one convex or quasiconvex in a systematic manner. The method can in principle be reversed to produce a direct, albeit long and arduous proof of failure of quasiconvexity of particular functions. A direct proof of rank-one convexity of (1.4), given in Appendix B, is ad hoc and unrelated to the construction process.

The paper is organized as follows. In Section 2 we state and prove all necessary facts from the theory of exact relations. The interested reader can consult the books [17,41] for discussions of the origin and other applications of some of the ideas and constructions from the theory. In Section 3 we introduce multifield materials coupling 4 curl-free fields to 4 divergence-free fields and discuss exact and L-relations for composites in that context. In particular, we exhibit an L-relation that is not exact. In Section 4 we use the well-known links between homogenization and quasiconvexification to construct a family of nonnegative, rotationally invariant, rank-one convex, but non-quasiconvex functions of which function (1.4) is a member.

## 2. General Theory of Exact Relations

### *2.1. Periodic Composites*

The standard references [3,21] for the mathematical theory of composite materials emphasize homogenization theorems and deal primarily with conducting

[2] Curiously, the same construction in three space dimensions does not produce any counterexamples, because, as was proved in [18], all L-relations for three dimensional multifield polycrystalline composites are exact.

and elastic composites. Homogenization in other physical contexts, such as piezoelectricity or thermoelasticity, is very similar. This similarity has been noted and incorporated into an abstract Hilbert space framework [8,24,37–39,42] encompassing all coupled field composites. In this framework materials are assumed to respond linearly to an applied field $\boldsymbol{E}(\boldsymbol{x})$, producing in response a "flux field" $\boldsymbol{J}(\boldsymbol{x}) = \mathsf{L}\boldsymbol{E}(\boldsymbol{x})$, whereby their material properties are described by a linear operator $\mathsf{L}$ on a finite dimensional inner product space $\mathcal{T}$, where the physical fields take their values. For example, $\mathcal{T} = \mathbb{R}^d$ for $d$-dimensional conducting composites, because the electric field $\boldsymbol{e}(\boldsymbol{x})$ and the resulting current field $\boldsymbol{j}(\boldsymbol{x})$ are $\mathbb{R}^d$-valued. In $d$-dimensional elasticity, $\mathcal{T} = \mathrm{Sym}(\mathbb{R}^d)$—the space of symmetric $d \times d$ matrices, because the strain field $\boldsymbol{\varepsilon}(\boldsymbol{x})$ and the resulting stress field $\boldsymbol{\sigma}(\boldsymbol{x})$ are $\mathrm{Sym}(\mathbb{R}^d)$-valued. In almost every physical context the tensor of material properties $\mathsf{L}$ is a symmetric operator on $\mathcal{T}$. It is also required to be positive definite, that is $\mathsf{L} \in \mathrm{Sym}^+(\mathcal{T})$. The linear homogeneous differential constraints satisfied by the physical fields $\boldsymbol{E}$ and $\boldsymbol{J}$ can be conveniently written as linear algebraic constraints satisfied by the formal Fourier transforms of these fields: $\hat{\boldsymbol{E}}(\boldsymbol{\xi}) \in \mathcal{E}_{\boldsymbol{n}} \otimes \mathbb{C}$, $\hat{\boldsymbol{J}}(\boldsymbol{\xi}) \in \mathcal{J}_{\boldsymbol{n}} \otimes \mathbb{C}$, where $\boldsymbol{n} = \boldsymbol{\xi}/|\boldsymbol{\xi}|$. For example, in the context of conductivity an electric field $\boldsymbol{E}$ must be curl-free and the current field $\boldsymbol{J}$, divergence-free. In this case,

$$\mathcal{E}_{\boldsymbol{n}} = \mathbb{R}\boldsymbol{n}, \qquad \mathcal{J}_{\boldsymbol{n}} = \{\boldsymbol{j} \in \mathbb{R}^d : \boldsymbol{j} \cdot \boldsymbol{n} = 0\}. \tag{2.1}$$

Similarly, for linear elasticity,

$$\mathcal{E}_{\boldsymbol{n}} = \{\boldsymbol{u} \otimes \boldsymbol{n} + \boldsymbol{n} \otimes \boldsymbol{u} : \boldsymbol{u} \in \mathbb{R}^d\}, \qquad \mathcal{J}_{\boldsymbol{n}} = \{\boldsymbol{\sigma} \in \mathrm{Sym}(\mathbb{R}^d) : \boldsymbol{\sigma}\boldsymbol{n} = 0\}, \tag{2.2}$$

corresponding to the differential constraints $\boldsymbol{\varepsilon} = (\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)/2$, $\nabla \cdot \boldsymbol{\sigma} = 0$ for the strain and the stress, respectively.

It is easy to verify for both conductivity and elasticity that $\mathcal{E}_{\boldsymbol{n}}$ and $\mathcal{J}_{\boldsymbol{n}}$ are orthogonal complements of one another:

$$\mathcal{T} = \mathcal{E}_{\boldsymbol{n}} \oplus \mathcal{J}_{\boldsymbol{n}}, \qquad \boldsymbol{n} \in \mathbb{R}^d, \ |\boldsymbol{n}| = 1. \tag{2.3}$$

It turns out that this property is universal, as it holds in every other physical context, such as piezoelectricity or thermoelasticity [39].

The microstructure of a periodic composite with period cell $Q = [0, 1]^d$ is completely described by the *local tensor* $\mathsf{L} : Q \to \mathrm{Sym}^+(\mathcal{T})$.[3] The effective tensor $\mathsf{L}^* \in \mathrm{Sym}^+(\mathcal{T})$ of such a periodic composite can be defined as an H-limit [44] of the sequence $\mathsf{L}_p(n\boldsymbol{x})$, as $n \to \infty$, where $\mathsf{L}_p$ stands for the $Q$-periodic extension of $\mathsf{L}(\boldsymbol{x})$ to $\mathbb{R}^d$. The effective tensor can be determined either by solving a periodic "cell problem" [3,41] (see equation (A.5) in Appendix A), or, and this is what we will use instead, by an explicit formula for the Milton W-transformation of $\mathsf{L}^*$ [39]. The W-transformation is an invertible fractional-linear transformation defined on $\mathrm{Sym}^+(\mathcal{T})$, and it involves an arbitrary reference tensor $\mathsf{L}_0 \in \mathrm{Sym}^+(\mathcal{T})$, that can be regarded as a "preconditioner", since the effective tensor $\mathsf{L}^*$ does not

[3] The period cell $Q$ can be an arbitrary parallelepiped. We choose $Q = [0, 1]^d$, corresponding to the lattice $\mathbb{Z}^d$, because the dual lattice is also $\mathbb{Z}^d$ in this case, resulting in simpler notation. However, all results can be reformulated for an arbitrary parallelepiped of periods.

depend on it, and which can thus be chosen to make the calculation of $\mathsf{L}^*$ more robust, for example, or to serve any other purpose.

In order to define the W-transformation we introduce a family of symmetric linear maps (see Appendix A)

$$\boldsymbol{\Gamma}_0(\boldsymbol{n}) = \mathsf{L}_0^{-1}\boldsymbol{\Gamma}'(\boldsymbol{n}) \in \mathrm{Sym}(\mathcal{T}), \qquad \boldsymbol{n} \in \mathbb{R}^d, \ |\boldsymbol{n}| = 1, \tag{2.4}$$

where $\boldsymbol{\Gamma}'(\boldsymbol{n})$ is a possibly non-orthogonal projection operator onto $\mathsf{L}_0\mathcal{E}_{\boldsymbol{n}}$ along $\mathcal{J}_{\boldsymbol{n}}$. Following [39], we define transformations $W_{\boldsymbol{n}} : \mathrm{Sym}^+(\mathcal{T}) \to \mathrm{Sym}(\mathcal{T})$, $|\boldsymbol{n}| = 1$, by

$$W_{\boldsymbol{n}}(\mathsf{L}) = \left[\mathsf{I} + (\mathsf{L} - \mathsf{L}_0)\boldsymbol{\Gamma}_0(\boldsymbol{n})\right]^{-1}(\mathsf{L} - \mathsf{L}_0) = (\mathsf{L} - \mathsf{L}_0)\left[\mathsf{I} + \boldsymbol{\Gamma}_0(\boldsymbol{n})(\mathsf{L} - \mathsf{L}_0)\right]^{-1}, \tag{2.5}$$

where $\mathsf{I}$ denotes the identity operator on $\mathcal{T}$. Lemma A.1 shows that the linear maps $\mathsf{I} + (\mathsf{L} - \mathsf{L}_0)\boldsymbol{\Gamma}_0(\boldsymbol{n})$ are invertible for *any* $\mathsf{L} \in \mathrm{Sym}^+(\mathcal{T})$. Moreover, the map $W_{\boldsymbol{n}}(\mathsf{L})$ is a diffeomorphism from $\mathrm{Sym}^+(\mathcal{T})$ onto its image. This is proved in Lemma A.2. Finally, Lemma A.3 establishes the formula for $\mathsf{L}^*$ that will be used in the subsequent analysis:

$$W_{\boldsymbol{n}}(\mathsf{L}^*) = \langle W_{\boldsymbol{n}}(\mathsf{L}(\boldsymbol{x}))(\mathsf{I} - \Lambda_{\boldsymbol{n}} W_{\boldsymbol{n}}(\mathsf{L}(\boldsymbol{x})))^{-1}\rangle, \tag{2.6}$$

where $\langle\cdot\rangle$ denotes average over the period cell $[0, 1]^d$. Operators $\Lambda_{\boldsymbol{n}}$, $|\boldsymbol{n}| = 1$ are Fourier multiplier operators on $L^2_{\mathrm{per}}([0, 1]^d; \mathcal{T})$—the set of $[0, 1]^d$-periodic locally $L^2$ vector fields, defined by

$$\widehat{\Lambda_{\boldsymbol{n}}\boldsymbol{h}}(\boldsymbol{k}) = \begin{cases} \mathsf{A}_{\boldsymbol{n}}(\boldsymbol{k})\widehat{\boldsymbol{h}}(\boldsymbol{k}), & \boldsymbol{k} \in \mathbb{Z}^d \setminus \{0\} \\ 0, & \boldsymbol{k} = 0, \end{cases} \tag{2.7}$$

where

$$\mathsf{A}_{\boldsymbol{n}}(\boldsymbol{k}) = \boldsymbol{\Gamma}_0(\boldsymbol{n}) - \boldsymbol{\Gamma}_0\left(\frac{\boldsymbol{k}}{|\boldsymbol{k}|}\right), \qquad \boldsymbol{k} \in \mathbb{Z}^d \setminus \{0\}. \tag{2.8}$$

Formula (2.6) is understood in the sense that for any constant vector $\boldsymbol{t} \in \mathcal{T}$,

$$W_{\boldsymbol{n}}(\mathsf{L}^*)\boldsymbol{t} = \langle W_{\boldsymbol{n}}(\mathsf{L}(\boldsymbol{x}))\boldsymbol{u}(\boldsymbol{x})\rangle,$$

where $\boldsymbol{u}(\boldsymbol{x})$ is the unique $L^2_{\mathrm{per}}([0, 1]^d; \mathcal{T})$ solution of the operator equation

$$(\mathsf{I} - \Lambda_{\boldsymbol{n}} W_{\boldsymbol{n}}(\mathsf{L}))\boldsymbol{u} = \boldsymbol{t}, \tag{2.9}$$

where $\boldsymbol{t}$ is now understood as a (constant) vector field in $L^2_{\mathrm{per}}([0, 1]^d; \mathcal{T})$. The unique solvability of (2.9) is established in Lemma A.3.

We remark, that even though the mapping $W_{\boldsymbol{n}}(\mathsf{L})$ and the operator $\Lambda_{\boldsymbol{n}}$ involve a unit vector $\boldsymbol{n}$ and a reference medium $\mathsf{L}_0$, the effective tensor $\mathsf{L}^*$ defined by (2.6) is independent of both. We now recall the definition of G-closure of a set of materials, [29].

**Definition 2.1.** The G-closure $G(U)$ of a compact subset $U \subset \mathrm{Sym}^+(\mathcal{T})$ is the relative closure in $\mathrm{Sym}^+(\mathcal{T})$ of the set of all effective tensors $\mathsf{L}^*$ of all possible periodic composites, made with materials from the set $U$.[4] A subset of $\mathrm{Sym}^+(\mathcal{T})$ is G-closed if it is relatively closed in $\mathrm{Sym}^+(\mathcal{T})$ and contains G-closure of any of its compact subsets.

In this paper we deal exclusively with polycrystalline composites for which the set $U$ of admissible materials must be $SO(d)$ invariant.

Generically, the set $G(U)$ has a nonempty interior in $\mathrm{Sym}^+(\mathcal{T})$, even if $U$ consists of only 2 points [30]. We are interested in special, non-generic situations, where $G(U)$ is a submanifold of $\mathrm{Sym}^+(\mathcal{T})$ of nonzero co-dimension.

**Definition 2.2.** The submanifold $\mathbb{M}$ of $\mathrm{Sym}^+(\mathcal{T})$ of positive codimension is called *an exact relation* if the effective tensor $\mathsf{L}^*$ of a periodic composite, whose constituents are taken from any compact subset of $\mathbb{M}$, must lie in $\mathbb{M}$, *regardless of the microstructure*. Equivalently, submanifold $\mathbb{M}$ is an exact relation if and only if it is G-closed.

### *2.2. L-Relations and Jordan Multialgebras*

In order to identify all exact relations we test a prospective submanifold $\mathbb{M}$ by taking two arbitrary points $\{\mathsf{L}_1, \mathsf{L}_2\} \subset \mathbb{M}$ and forming a simple laminate—a composite consisting of layers of material $\mathsf{L}_1$ alternating with layers of material $\mathsf{L}_2$. The geometry of a simple laminate is described by the direction of lamination $\boldsymbol{n} \in \mathbb{S}^{d-1}$ ($d = 2$ or 3) and the volume fractions $\theta_1$, $\theta_2 = 1 - \theta_1$ of $\mathsf{L}_1$ and $\mathsf{L}_2$, respectively. Every simple laminate can be regarded as a periodic composite, if we choose a period cell to be a cube with $\boldsymbol{n}$ being normal to one of its faces. By analogy with G-closed sets we define L-closed sets.

**Definition 2.3.** A set of materials $U \subset \mathrm{Sym}^+(\mathcal{T})$ is called *L-closed* if $U$ is relatively closed in $\mathrm{Sym}^+(\mathcal{T})$ and contains effective tensors of all simple laminates made with any two materials $\{\mathsf{L}_1, \mathsf{L}_2\} \subset U$, taken in any volume fraction and arbitrary orientation of layers.

Restricting our attention only to laminate microstructures we formulate the notion of lamination exact relation or *L-relation*.

**Definition 2.4.** A submanifold $\mathbb{M}$ of positive co-dimension in $\mathrm{Sym}^+(\mathcal{T})$ is called an *L-relation* if the effective tensor $\mathsf{L}^*$ of a simple laminate made with any $\{\mathsf{L}_1, \mathsf{L}_2\} \subset \mathbb{M}$ is in $\mathbb{M}$ for any choice of lamination direction and volume faction. Equivalently, submanifold $\mathbb{M}$ is an L-relation if and only if it is L-closed.

If $\mathsf{L}(\boldsymbol{x})$ is the local tensor of a simple laminate with lamination direction $\boldsymbol{n}$, then $\mathsf{L}(\boldsymbol{x})$ depends only on $\boldsymbol{x} \cdot \boldsymbol{n}$. Therefore, since $\mathsf{A}_{\boldsymbol{n}}(\boldsymbol{n}) = 0$, we have $\Lambda_{\boldsymbol{n}} W_{\boldsymbol{n}}(\mathsf{L}(\boldsymbol{x})) = 0$, due to (2.7). In this case formula (2.6) simplifies [39,41]:

$$W_{\boldsymbol{n}}(\mathsf{L}^*) = \theta_1 W_{\boldsymbol{n}}(\mathsf{L}_1) + \theta_2 W_{\boldsymbol{n}}(\mathsf{L}_2). \tag{2.10}$$

[4] G-closure of any set is independent of the choice of a period cell parallelepiped.

Geometrically, this means that $W_{\boldsymbol{n}}$-images of G-closed sets must be convex (for any choice of $\boldsymbol{n}$ and $\mathsf{L}_0$). It also means (together with other results from Appendix A) that L-closed sets (and a fortiori G-closed sets) are diffeomorphic images of convex sets. In particular, the image of $\mathrm{Sym}^+(\mathcal{T})$ under the map $W_{\boldsymbol{n}}$ is an open convex subset of $\mathrm{Sym}(\mathcal{T})$, containing $0 = W_{\boldsymbol{n}}(\mathsf{L}_0)$.

If a submanifold $\mathbb{M}$ is an L-relation, then (2.10) implies that $W_{\boldsymbol{n}}(\mathbb{M})$ must be a convex subset of $\mathrm{Sym}(\mathcal{T})$ and, at the same time, a submanifold of $\mathrm{Sym}(\mathcal{T})$ of the same dimension as $\mathbb{M}$. Therefore, $W_{\boldsymbol{n}}(\mathbb{M})$ must be a convex subset of an affine subspace of $\mathrm{Sym}(\mathcal{T})$. If we choose the reference tensor $\mathsf{L}_0$ so that $\mathsf{L}_0 \in \mathbb{M}$, then $W_{\boldsymbol{n}}(\mathbb{M})$ will be a convex subset, with nonempty relative interior, of a subspace

$$\Pi_{\boldsymbol{n}} = \mathrm{Span}\{W_{\boldsymbol{n}}(\mathsf{L}) : \mathsf{L} \in \mathbb{M}\} \subset \mathrm{Sym}(\mathcal{T}),$$

since $W_{\boldsymbol{n}}(\mathsf{L}_0) = 0$, according to (2.5). In that case the differential of $W_{\boldsymbol{n}}$ at $\mathsf{L}_0$ will be an isomorphism between the tangent space to $\mathbb{M}$ at $\mathsf{L}_0$ and $\Pi_{\boldsymbol{n}}$. We easily compute that the differential of $W_{\boldsymbol{n}}$ at $\mathsf{L}_0$ is the identity transformation. Thus, the subspaces $\Pi_{\boldsymbol{n}}$ do not depend on $\boldsymbol{n}$, since they all coincide with the tangent space to $\mathbb{M}$ at $\mathsf{L}_0$. Accordingly, we will denote by $\Pi$ the tangent space to $\mathbb{M}$ at $\mathsf{L}_0$. Then, the transformation $\Phi_{\boldsymbol{m},\boldsymbol{n}} = W_{\boldsymbol{m}} \circ W_{\boldsymbol{n}}^{-1}$ would map a small neighborhood $\mathcal{O}$ of $0 \in \Pi$ to a neighborhood of $0 \in \Pi$. We compute

$$\Phi_{\boldsymbol{m},\boldsymbol{n}}(\mathsf{K}) = [\mathsf{I} - \mathsf{K}\mathsf{A}_{\boldsymbol{n}}(\boldsymbol{m})]^{-1}\mathsf{K}, \qquad \mathsf{K} \in \mathcal{O} \subset \Pi. \tag{2.11}$$

For sufficiently small $\mathsf{K}$ we can expand (2.11) into the Neumann series and conclude that $\mathsf{K}\mathsf{A}_{\boldsymbol{n}}(\boldsymbol{m})\mathsf{K} \in \Pi$ for all $\mathsf{K} \in \Pi$ and all unit vectors $\boldsymbol{m}$. From this it is not difficult to obtain the characterization of all L-relations. (See [18] or [41, Chapter 17] for details.)

**Theorem 2.5.** *Let $\Pi$ be a subspace in $\mathrm{Sym}(\mathcal{T})$ and $\boldsymbol{n}_0$ be a fixed unit vector. We also define the subspace*

$$\mathcal{A} = \mathrm{Span}\{\boldsymbol{\Gamma}_0(\boldsymbol{n}) - \boldsymbol{\Gamma}_0(\boldsymbol{n}_0) : |\boldsymbol{n}| = 1\}, \tag{2.12}$$

*where $\boldsymbol{\Gamma}_0(\boldsymbol{n})$ is defined in (2.4). The submanifold $\mathbb{M}$, given by the formula*

$$\mathbb{M} = \{\mathsf{L} \in \mathrm{Sym}^+(\mathcal{T}) : W_{\boldsymbol{n}_0}(\mathsf{L}) \in \Pi\}, \tag{2.13}$$

*is an L-relation if and only if the subspace $\Pi$ is a Jordan $\mathcal{A}$-multialgebra, meaning that*

$$\mathsf{K}_1 *_{\mathsf{A}} \mathsf{K}_2 = \frac{1}{2}(\mathsf{K}_1\mathsf{A}\mathsf{K}_2 + \mathsf{K}_2\mathsf{A}\mathsf{K}_1) \in \Pi, \quad \forall\{\mathsf{K}_1, \mathsf{K}_2\} \subset \Pi, \ \mathsf{A} \in \mathcal{A}. \tag{2.14}$$

Jordan algebras first appeared in early versions of quantum mechanics [22]. In particular, subspaces of $\mathrm{Sym}(\mathcal{T})$ that are closed with respect to any of the multiplications (2.14), are examples of Jordan algebras. Theorem 2.5 requires subspaces $\Pi$, defining L-relations, to be closed with respect to an entire family of Jordan multiplications. For this reason we call such subspaces *Jordan $\mathcal{A}$-multialgebras*.

*2.3. Stability Under Homogenization*

From formula (2.6) we can obtain a criterion of stability under homogenization via the Neumann series, [18].

**Lemma 2.6.** *A submanifold* $\mathbb{M}$*, given by* (2.13)*, is G-closed if and only if*

$$\langle (\mathsf{K}(\boldsymbol{x})\Lambda_{\boldsymbol{n}_0})^k \mathsf{K}(\boldsymbol{x})\rangle \in \Pi_{\mathbb{C}} \tag{2.15}$$

*for every* $k \geqq 0$ *and* $\mathsf{K} \in L^\infty([0,1]^d; \Pi_{\mathbb{C}})$*, where* $\Pi_{\mathbb{C}} = \{\mathsf{K}_1 + i\mathsf{K}_2 \mid \{\mathsf{K}_1, \mathsf{K}_2\} \subset \Pi\}$ *is the complexification of* $\Pi$*, and* $\langle\cdot\rangle$ *denotes average over the period cell.*

We remark, that when $k = 1$ condition (2.15) is equivalent to (2.14). In order to apply Neumann series expansion to (2.6) we need to relate periodic composites with arbitrary $\mathsf{K}(\boldsymbol{x})$ to the ones where $\mathsf{K}(\boldsymbol{x})$ is uniformly small.

**Lemma 2.7.** *Suppose* $\mathbb{M}$ *is given by* (2.13)*, where* $\Pi$ *is a Jordan* $\mathcal{A}$*-multialgebra. Let us assume that there exists* $\epsilon > 0$*, such that* $\mathsf{L}^* \in \mathbb{M}$ *for every periodic composite* $\mathsf{L}(\boldsymbol{x}) \in \mathbb{M}$*, satisfying* $|W_{\boldsymbol{n}_0}(\mathsf{L}(\boldsymbol{x}))| < \epsilon$*. Then* $\mathbb{M}$ *is an exact relation in the sense of Definition* 2.2*.*

**Proof.** Let $\mathcal{K} \subset \mathbb{M}$ be a compact subset and $\mathsf{L}(\boldsymbol{x}) \in \mathcal{K}$ for all $\boldsymbol{x} \in Q$. Then $\mathsf{K}(\boldsymbol{x}) = W_{\boldsymbol{n}_0}(\mathsf{L}(\boldsymbol{x}))$ is a uniformly bounded function, satisfying $\mathsf{K}(\boldsymbol{x}) \in \Pi$. Thus, there exists $\delta > 0$ so small that for every $0 < \theta < \delta$ we have, $\theta|\mathsf{K}(\boldsymbol{x})| < \epsilon$. Let $\mathsf{L}_\theta(\boldsymbol{x}) = W_{\boldsymbol{n}_0}^{-1}(\theta\mathsf{K}(\boldsymbol{x}))$. Then, by the lamination formula (2.10), for each fixed $\boldsymbol{x} \in Q$, the effective tensor of the laminate of two materials $\mathsf{L}(\boldsymbol{x})$ and $\mathsf{L}_0$, taken in volume fractions $\theta$ and $1-\theta$, respectively, with lamination direction $\boldsymbol{n}_0$, will be $\mathsf{L}_\theta(\boldsymbol{x})$. Theorem 2.5 then implies that $\mathsf{L}_\theta(\boldsymbol{x}) \in \mathbb{M}$, for every $0 \leqq \theta \leqq 1$, and every $\boldsymbol{x} \in Q$. By assumption, $\mathsf{L}_\theta^* \in \mathbb{M}$ for every $0 \leqq \theta < \delta$. That means that $f(\theta) = (W_{\boldsymbol{n}_0}(\mathsf{L}_\theta^*), \mathsf{P})_{\mathrm{Sym}(\mathcal{T})} = 0$ for every $0 \leqq \theta < \delta$ and every $\mathsf{P} \in \Pi^\perp$— the orthogonal complement to $\Pi$ in $\mathrm{Sym}(\mathcal{T})$. By formula (2.6)

$$W_{\boldsymbol{n}_0}(\mathsf{L}_\theta^*) = \langle \mathsf{K}(\boldsymbol{x})(\theta^{-1}\mathsf{I} - \Lambda_{\boldsymbol{n}_0}\mathsf{K}(\boldsymbol{x}))^{-1}\rangle \quad \forall\theta \in (0, 1].$$

By Lemma A.3 all operators $\theta^{-1}\mathsf{I} - \Lambda_{\boldsymbol{n}_0}\mathsf{K}$ must be invertible on $L^2(Q; \mathcal{T})$ for all $\theta \in (0, 1]$. Thus, $\theta^{-1}$ is not in the spectrum of $\Lambda_{\boldsymbol{n}_0}\mathsf{K}$ for all $\theta \in (0, 1]$, but then the function $f(\theta)$ must be analytic on a neighborhood of $\theta \in [0, 1]$ in the complex plane [46]. It follows that $f(\theta) = 0$ for all $\theta \in [0, 1]$, since $f(\theta) = 0$ for all $0 \leqq \theta < \delta$. This proves that $W_{\boldsymbol{n}_0}(\mathsf{L}^*) \in \Pi$, which implies that $\mathsf{L}^* \in \mathbb{M}$. □

We are now ready to prove Lemma 2.6.

**Proof of Lemma 2.6.** If $\mathbb{M}$ is G-closed, and $\mathsf{K} \in L^\infty(Q; \Pi)$ then for sufficiently small (in absolute value) $\epsilon$ we have $\mathsf{L}_\epsilon(\boldsymbol{x}) = W_{\boldsymbol{n}_0}^{-1}(\epsilon\mathsf{K}(\boldsymbol{x})) \in \mathbb{M}$. Hence, $\mathsf{L}_\epsilon^* \in \mathbb{M}$, which, in turn, implies that $W_{\boldsymbol{n}_0}(\mathsf{L}_\epsilon^*) \in \Pi$. Expanding (2.6) into the Neumann series

$$W_{\boldsymbol{n}_0}(\mathsf{L}_\epsilon^*) = \sum_{k=0}^{\infty} \epsilon^{k+1}\langle (\mathsf{K}\Lambda_{\boldsymbol{n}_0})^k \mathsf{K}(\boldsymbol{x})\rangle, \tag{2.16}$$

we conclude that (2.15) must hold for every $\mathsf{K} \in L^\infty(Q; \Pi)$. To prove that (2.15) holds for every $\mathsf{K} \in L^\infty(Q; \Pi_{\mathbb{C}})$, we take any $\{\mathsf{K}_1, \mathsf{K}_2\} \subset L^\infty(Q; \Pi)$, $\lambda \in \mathbb{C}$, and define $\mathsf{K}_\lambda(\boldsymbol{x}) = \mathsf{K}_1(\boldsymbol{x}) + \lambda\mathsf{K}_2(\boldsymbol{x})$. Observe that $\mathsf{K}_\lambda \in L^\infty(Q; \Pi)$ for any $\lambda \in \mathbb{R}$, and therefore, for any $\mathsf{P} \in \Pi^\perp$, and any $k \geqq 1$ we have

$$p(\lambda) = (\langle(\mathsf{K}_\lambda \Lambda_{\boldsymbol{n}_0})^k \mathsf{K}_\lambda(\boldsymbol{x})\rangle, \mathsf{P})_{\mathrm{Sym}(\mathcal{T})} = 0, \qquad \lambda \in \mathbb{R}.$$

Observe that $p(\lambda)$ is a polynomial in $\lambda$. Therefore, if it vanishes on $\mathbb{R}$ it must also vanish on $\mathbb{C}$. Hence, $p(i) = 0$, and (2.15) is proved for any $\mathsf{K} \in L^\infty(Q; \Pi_{\mathbb{C}})$.

Conversely, if (2.15) holds for every $\mathsf{K} \in L^\infty(Q; \Pi_{\mathbb{C}})$, then formula (2.16) proves that $W_{\boldsymbol{n}_0}(\mathsf{L}^*) \in \Pi$, provided $\mathsf{L}(\boldsymbol{x}) \in \mathbb{M}$ and $\mathsf{K}(\boldsymbol{x}) = W_{\boldsymbol{n}_0}(\mathsf{L}(\boldsymbol{x}))$ is sufficiently small. Lemma 2.7 now guarantees that $\mathbb{M}$ is G-closed. □

We now turn to the formulation of a nice algebraic sufficient condition on the subspace $\Pi \subset \mathrm{Sym}(\mathcal{T})$ for (2.15) to hold for all $k \geqq 1$.

**Definition 2.8.** A subspace $\Pi' \subset \mathrm{End}(\mathcal{T})$ is called an associative $\mathcal{A}$-multialgebra, if $\mathsf{K}_1\mathsf{A}\mathsf{K}_2 \in \Pi'$ for all $\{\mathsf{K}_1, \mathsf{K}_2\} \subset \Pi'$ and all $\mathsf{A} \in \mathcal{A}$.

**Theorem 2.9.** *Suppose that $\Pi'$ is an associative $\mathcal{A}$-multialgebra. Let $\Pi = \Pi' \cap \mathrm{Sym}(\mathcal{T})$. Then the submanifold $\mathbb{M}$, given by (2.13) is an exact relation in the sense of Definition 2.2.*

**Proof.** The idea is to use the algebraic property of $\Pi$ to prove (2.15) for all $k \geqq 1$. Our first observation is that it is sufficient to prove that if $\mathsf{K} \in L^\infty([0, 1]^d; \Pi'_{\mathbb{C}})$ then $(\mathsf{K}\Lambda_{\boldsymbol{n}})^k\mathsf{K} \in L^2([0, 1]^d; \Pi'_{\mathbb{C}})$ for all $k \geqq 1$. This statement is proved by induction in $k$. It is amusing that it is the induction step that is almost trivial, while the case $k = 1$ is the only part that requires a proof. Indeed, suppose that $\mathsf{T}_k = (\mathsf{K}\Lambda_{\boldsymbol{n}})^k\mathsf{K} \in L^2([0, 1]^d; \Pi'_{\mathbb{C}})$. Then $\mathsf{T}_{k+1} = \mathsf{K}\Lambda_{\boldsymbol{n}}\mathsf{T}_k$. The conclusion for the induction step follows from from a slightly expanded statement for $k = 1$.

**Lemma 2.10.** *Suppose that $\mathsf{K}_1 \in L^\infty([0, 1]^d; \Pi'_{\mathbb{C}})$ and $\mathsf{K}_2 \in L^2([0, 1]^d; \Pi'_{\mathbb{C}})$. Then $\mathsf{K}_1\Lambda_{\boldsymbol{n}}\mathsf{K}_2 \in L^2([0, 1]^d; \Pi'_{\mathbb{C}})$.*

**Proof.** If $\mathsf{K}_2 \in L^2([0, 1]^d; \Pi'_{\mathbb{C}})$ then $\widehat{\mathsf{K}}_2(\boldsymbol{k}) \in \Pi'_{\mathbb{C}}$ for all $\boldsymbol{k} \in \mathbb{Z}^d$. We compute

$$(\mathsf{K}_1\Lambda_{\boldsymbol{n}}\mathsf{K}_2)(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d\setminus\{0\}} \mathsf{K}_1(\boldsymbol{x})\mathsf{A}_{\boldsymbol{n}}(\boldsymbol{k})\widehat{\mathsf{K}}_2(\boldsymbol{k})e^{2\pi i\boldsymbol{k}\cdot\boldsymbol{x}}.$$

It remains to observe that every term in the above expansion is in $\Pi'_{\mathbb{C}}$ for almost every $\boldsymbol{x} \in [0, 1]^d$, since $\mathsf{A}_{\boldsymbol{n}}(\boldsymbol{k}) \in \mathcal{A}$. □

Now, if $\mathsf{L} \in L^\infty([0, 1]^d; \mathbb{M})$, then $\mathsf{K} = W_{\boldsymbol{n}}(\mathsf{L}) \in L^\infty([0, 1]^d; \Pi)$. We have proved that formula (2.6) implies that $W_{\boldsymbol{n}}(\mathsf{L}^*) \in \Pi'_{\mathbb{C}}$. But we also know that $W_{\boldsymbol{n}}(\mathsf{L}^*) \in \mathrm{Sym}(\mathcal{T})$. Thus,

$$W_{\boldsymbol{n}}(\mathsf{L}^*) \in \Pi'_{\mathbb{C}} \cap \mathrm{Sym}(\mathcal{T}) = \Pi.$$

The theorem is proved now. □

**Remark 2.11.** From an algebraic point of view it may be regarded as surprising that in every two and three dimensional physical context from conductivity to piezoelectricity every subspace satisfying (2.14) also satisfies conditions in Theorem 2.9 [14, 16, 18]. From the analysis point of view, this might not be as surprising, since, as this work shows, each L-relation that is not closed with respect to homogenization would generate examples of rank-one convex non-quasiconvex functions, and such examples are rare.

Now we are going to obtain a new algebraic condition that is necessary for stability under homogenization, but is not a consequence of (2.14). While our construction produces a new necessary condition in any number of space dimensions, only in 2D can it be formulated in a practically useful form.[5]

**Theorem 2.12.** *If $d = 2$ and $\mathbb{M}$, given by (2.13), is an exact relation, then, in addition to (2.14), it must satisfy*

$$\mathsf{K}_1\mathsf{A}_1\mathsf{K}_2\mathsf{A}_2\mathsf{K}_3 + \mathsf{K}_3\mathsf{A}_2\mathsf{K}_2\mathsf{A}_1\mathsf{K}_1 \in \Pi. \tag{2.17}$$

*for any* $\{\mathsf{K}_1, \mathsf{K}_2, \mathsf{K}_3\} \subset \Pi$ *and* $\{\mathsf{A}_1, \mathsf{A}_2\} \subset \mathcal{A}$.

This theorem was proved in [14] as a consequence of a more sophisticated result, which we do not need here. For this reason we give a direct (and simpler) Proof of Theorem 2.12.

**Proof.** To prove the theorem we choose

$$\mathsf{K}(\boldsymbol{x}) = \mathsf{K}_1 e^{2\pi i \boldsymbol{l}_1\cdot\boldsymbol{x}} + \mathsf{K}_2 e^{2\pi i \boldsymbol{l}_2\cdot\boldsymbol{x}} + \mathsf{K}_3 e^{2\pi i \boldsymbol{l}_3\cdot\boldsymbol{x}} \in L^\infty([0, 1]^2; \Pi_{\mathbb{C}}), \tag{2.18}$$

where $\{\mathsf{K}_1, \mathsf{K}_2, \mathsf{K}_3\} \subset \Pi$ and $\{\boldsymbol{l}_1, \boldsymbol{l}_2, \boldsymbol{l}_3\} \subset \mathbb{Z}^2$ will be specified now.

First we observe that any unit vector $\boldsymbol{n} \in \mathbb{R}^2$ can be approximated with any degree of accuracy by a vector $\tilde{\boldsymbol{n}} = \boldsymbol{k}_0/|\boldsymbol{k}_0|$ for some $\boldsymbol{k}_0 \in \mathbb{Z}^2$. In that case $\mathsf{A}_{\tilde{\boldsymbol{n}}}(\boldsymbol{k}_0) = 0$, according to (2.8). Next we choose $\boldsymbol{k}_1 \in \mathbb{Z}^2$ that is linearly independent with $\boldsymbol{k}_0$. Finally we choose arbitrary $\{m_0, m_1\} \subset \mathbb{Z}\backslash\{0\}$ and define

$$\boldsymbol{l}_1 = m_1\boldsymbol{k}_1, \quad \boldsymbol{l}_2 = -m_0\boldsymbol{k}_0 - m_1\boldsymbol{k}_1, \quad \boldsymbol{l}_3 = m_0\boldsymbol{k}_0.$$

It is obvious that with the choices described above, all 3 vectors $\boldsymbol{l}_j$ are nonzero and distinct. Now we substitute (2.18) into (2.15) for $k = 3$ and $\boldsymbol{n}$ replaced with $\tilde{\boldsymbol{n}}$ and obtain (taking into account that $\mathsf{A}_{\tilde{\boldsymbol{n}}}(\boldsymbol{l}_3) = 0$)

$$\mathsf{K}_2\mathsf{A}_{\tilde{\boldsymbol{n}}}(\boldsymbol{l}_2)\mathsf{K}_3\mathsf{A}_{\tilde{\boldsymbol{n}}}(\boldsymbol{l}_1)\mathsf{K}_1 + \mathsf{K}_1\mathsf{A}_{\tilde{\boldsymbol{n}}}(\boldsymbol{l}_1)\mathsf{K}_3\mathsf{A}_{\tilde{\boldsymbol{n}}}(\boldsymbol{l}_2)\mathsf{K}_2 \in \Pi.$$

Next we note that

$$\mathsf{A}_{\tilde{\boldsymbol{n}}}(\boldsymbol{l}_1) = \mathsf{A}_{\tilde{\boldsymbol{n}}}\left(\frac{\boldsymbol{k}_1}{|\boldsymbol{k}_1|}\right), \qquad \mathsf{A}_{\tilde{\boldsymbol{n}}}(\boldsymbol{l}_2) = \mathsf{A}_{\tilde{\boldsymbol{n}}}\left(\frac{m_0\boldsymbol{k}_0 + m_1\boldsymbol{k}_1}{|m_0\boldsymbol{k}_0 + m_1\boldsymbol{k}_1|}\right).$$

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[5] Just for the record, the corresponding condition in three dimensions is (2.17), except the subspace $\mathcal{A}$ is replaced with a family of subspaces $\mathcal{A}_{\boldsymbol{p}}$, $|\boldsymbol{p}| = 1$, where $\mathcal{A}_{\boldsymbol{p}}$ is defined by (2.12) in which both vectors $\boldsymbol{n}$ and $\boldsymbol{n}_0$ must be orthogonal to $\boldsymbol{p}$.

For any unit vector $\boldsymbol{u} \in \mathbb{R}^2$ we can choose $\boldsymbol{k}_1 \in \mathbb{Z}^2$ and linearly independent with $\boldsymbol{k}_0$, such that $\tilde{\boldsymbol{u}} = \boldsymbol{k}_1/|\boldsymbol{k}_1|$ approximates $\boldsymbol{u}$ with any degree of accuracy. Then

$$\frac{m_0\boldsymbol{k}_0 + m_1\boldsymbol{k}_1}{|m_0\boldsymbol{k}_0 + m_1\boldsymbol{k}_1|} = \frac{s\boldsymbol{N} + r\tilde{\boldsymbol{u}}}{|s\boldsymbol{N} + r\tilde{\boldsymbol{u}}|}, \quad \boldsymbol{N} = \frac{\tilde{\boldsymbol{n}}|\boldsymbol{k}_0|}{|\boldsymbol{k}_1|}, \quad r = \frac{m_1}{|m_0|}, \quad s = \mathrm{sign}(m_0).$$

By our construction the vectors $\tilde{\boldsymbol{u}}$ and $\boldsymbol{N}$ are linearly independent in $\mathbb{R}^2$. It is now clear that for any unit vector $\boldsymbol{v} \in \mathbb{R}^2$ we can choose the sign $s$ and a nonzero rational number $r$, so that $\tilde{\boldsymbol{v}} = (s\boldsymbol{N} + r\tilde{\boldsymbol{u}})/|s\boldsymbol{N} + r\tilde{\boldsymbol{u}}|$ approximates $\boldsymbol{v}$ with any degree of accuracy. Hence, we conclude that

$$\mathsf{K}_2\mathsf{A}_{\tilde{\boldsymbol{n}}}(\tilde{\boldsymbol{v}})\mathsf{K}_3\mathsf{A}_{\tilde{\boldsymbol{n}}}(\tilde{\boldsymbol{u}})\mathsf{K}_1 + \mathsf{K}_1\mathsf{A}_{\tilde{\boldsymbol{n}}}(\tilde{\boldsymbol{u}})\mathsf{K}_3\mathsf{A}_{\tilde{\boldsymbol{n}}}(\tilde{\boldsymbol{v}})\mathsf{K}_2 \in \Pi$$

The function

$$\mathbb{S}^1 \times \mathbb{S}^1 \ni (\boldsymbol{n}, \boldsymbol{m}) \mapsto \mathsf{A}_{\boldsymbol{n}}(\boldsymbol{m}) = \boldsymbol{\Gamma}_0(\boldsymbol{n}) - \boldsymbol{\Gamma}_0(\boldsymbol{m})$$

is continuous and therefore

$$\mathsf{K}_2\mathsf{A}_{\boldsymbol{n}}(\boldsymbol{v})\mathsf{K}_3\mathsf{A}_{\boldsymbol{n}}(\boldsymbol{u})\mathsf{K}_1 + \mathsf{K}_1\mathsf{A}_{\boldsymbol{n}}(\boldsymbol{u})\mathsf{K}_3\mathsf{A}_{\boldsymbol{n}}(\boldsymbol{v})\mathsf{K}_2 \in \Pi,$$

for any unit vectors $\boldsymbol{n}$, $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{R}^2$. Fixing $\boldsymbol{n}$ and $\boldsymbol{u}$ and varying $\boldsymbol{v}$ we obtain that

$$\mathsf{K}_2\mathsf{A}_1\mathsf{K}_3\mathsf{A}_{\boldsymbol{n}}(\boldsymbol{u})\mathsf{K}_1 + \mathsf{K}_1\mathsf{A}_{\boldsymbol{n}}(\boldsymbol{u})\mathsf{K}_3\mathsf{A}_1\mathsf{K}_2 \in \Pi$$

for any $\{\mathsf{K}_1, \mathsf{K}_2, \mathsf{K}_3\} \subset \Pi$, $\mathsf{A}_1 \in \mathcal{A}$ and any unit vectors $\boldsymbol{n}$ and $\boldsymbol{u}$. Fixing $\mathsf{A}_1$ and $\boldsymbol{n}$, and varying $\boldsymbol{u}$ we obtain (2.17). □

If the set of admissible constituent materials contains anisotropic ones, then it is usually natural not to insist that such a material be used only in one fixed orientation. Mathematically speaking, if $\mathsf{L}$ is a tensor of a constituent material, then every rotation of $\mathsf{L}$, denoted symbolically by $\boldsymbol{R} \cdot \mathsf{L}$, $\boldsymbol{R} \in SO(d)$, must be admissible. Composites like these are called *polycrystalline*. Restricting attention only to the polycrystalline composites means that we are interested only in rotationally invariant exact relation submanifolds $\mathbb{M}$. Since a polycrystal with statistically isotropic texture must be isotropic (fixed point of $\mathrm{SO}(d)$ action), we conclude that $\mathbb{M}$ must contain an isotropic tensor $\mathsf{L}_0$, which we will use as a reference medium in the definition of the W-transformation (2.5). In that case, it is easy to see that both subspaces $\Pi$ and $\mathcal{A}$ must be rotationally invariant.

## 3. Case Study: Multifield Composite Materials

Multifield materials were considered in [35,36]. In this context $N$ coupled potential fields $\boldsymbol{E} = (\nabla\phi_1, \ldots, \nabla\phi_N)$ induce $N$ conjugate fluxes $\boldsymbol{J} = (\boldsymbol{j}_1, \ldots, \boldsymbol{j}_N)$ satisfying

$$\nabla \cdot \boldsymbol{j}_1 = \ldots = \nabla \cdot \boldsymbol{j}_N = 0.$$

For example, thermoelectric materials fit in this context with $N = 2$.

Mathematically speaking, we choose

$$\mathcal{T} = \underbrace{\mathbb{R}^d \oplus \ldots \oplus \mathbb{R}^d}_{N} \cong \mathbb{R}^N \otimes \mathbb{R}^d, \qquad \boldsymbol{u} \otimes \boldsymbol{x} \leftrightarrow (u_1\boldsymbol{x}, \ldots, u_N\boldsymbol{x}) \in \mathcal{T}, \quad \boldsymbol{x} \in \mathbb{R}^d,$$

with the inner product, defined most succinctly by the formula

$$(\boldsymbol{u} \otimes \boldsymbol{x}, \boldsymbol{v} \otimes \boldsymbol{y})_{\mathcal{T}} = (\boldsymbol{u} \cdot \boldsymbol{v})(\boldsymbol{x} \cdot \boldsymbol{y}), \qquad \forall\{\boldsymbol{u}, \boldsymbol{v}\} \subset \mathbb{R}^N,\ \forall\{\boldsymbol{x}, \boldsymbol{y}\} \subset \mathbb{R}^d$$

in terms of the dot products in $\mathbb{R}^N$ and $\mathbb{R}^d$, respectively.[6] The family of subspaces $\mathcal{E}_{\boldsymbol{n}}$ is given by

$$\mathcal{E}_{\boldsymbol{n}} = \{\boldsymbol{u} \otimes \boldsymbol{n} : \boldsymbol{u} \in \mathbb{R}^N\}, \qquad \boldsymbol{n} \in \mathbb{S}^{d-1}.$$

Each member of the family of subspaces $\mathcal{J}_{\boldsymbol{n}}$ is the orthogonal complements of $\mathcal{E}_{\boldsymbol{n}}$. Explicitly,

$$\mathcal{J}_{\boldsymbol{n}} = \{(\boldsymbol{j}_1, \ldots, \boldsymbol{j}_N) \in \mathcal{T} : \boldsymbol{j}_1 \cdot \boldsymbol{n} = \cdots = \boldsymbol{j}_N \cdot \boldsymbol{n} = 0\}.$$

Rotations $\boldsymbol{R} \in SO(d)$ act simultaneously on each copy of $\mathbb{R}^d$ in $\mathcal{T}$. Specifically,

$$\boldsymbol{R} \cdot (\boldsymbol{u} \otimes \boldsymbol{x}) = \boldsymbol{u} \otimes \boldsymbol{R}\boldsymbol{x}, \quad \boldsymbol{u} \in \mathbb{R}^N, \quad \boldsymbol{x} \in \mathbb{R}^d.$$

For simplicity we chose $\mathsf{L}_0 = \mathsf{I}$—the identity operator on $\mathcal{T}$. In this case we easily compute [18]:

$$\mathcal{A} = \{\boldsymbol{I}_N \otimes \boldsymbol{A} : \boldsymbol{A} \in \mathrm{Sym}(\mathbb{R}^d),\ \mathrm{Tr}\,\boldsymbol{A} = 0\},$$

where $\boldsymbol{I}_N$ denotes the $N \times N$ identity matrix. Here for $\boldsymbol{X} \in \mathrm{End}(\mathbb{R}^N)$ and $\boldsymbol{Y} \in \mathrm{End}(\mathbb{R}^d)$ the operator $\boldsymbol{X} \otimes \boldsymbol{Y}$ on $\mathbb{R}^N \otimes \mathbb{R}^d$ is uniquely defined by the property $(\boldsymbol{X} \otimes \boldsymbol{Y})(\boldsymbol{u} \otimes \boldsymbol{x}) = \boldsymbol{X}\boldsymbol{u} \otimes \boldsymbol{Y}\boldsymbol{x}$.

### *3.1. Polycrystalline L-Relations*

In order to identify all polycrystalline L-relations we need to find all rotationally invariant Jordan $\mathcal{A}$-multialgebras in $\mathrm{Sym}(\mathcal{T})$. For the case $d = 3$, all SO(3)-invariant Jordan $\mathcal{A}$-multialgebras have been computed in [18], where it was shown that *all* SO(3)-invariant L-relations satisfy sufficient conditions of Theorem 2.9, and hence are exact in the sense of Definition 2.2.

When $d = 2$ it will be convenient to identify the physical space $\mathbb{R}^2$ with complex numbers, so that $\boldsymbol{x} = (x_1, x_2) \mapsto x = x_1 + ix_2 \in \mathbb{C}$.[7] Then

$$\mathcal{T} \cong \mathbb{R}^N \otimes \mathbb{R}^2 \cong \mathbb{R}^N \otimes \mathbb{C} \cong \mathbb{C}^N, \quad \boldsymbol{u} \otimes \boldsymbol{x} \mapsto \boldsymbol{u} \otimes x \mapsto (u_1x, \ldots, u_Nx) \in \mathbb{C}^N. \tag{3.1}$$

[6] If we identify $\mathcal{T} = \mathbb{R}^N \otimes \mathbb{R}^d$ with the space of $N \times d$ matrices, then this inner product coincides with the Frobenius inner product $\mathrm{Tr}\,(\boldsymbol{A}\boldsymbol{B}^T)$.

[7] The image in $\mathbb{C}$ of a vector in $\mathbb{R}^2$, denoted by a bold letter, is represented by the same letter in normal font.

The utility of this isomorphism of $2N$-dimensional real vector spaces comes from the alternative interpretation of $\mathbb{C}^N$ as a complex vector space. In order to characterize all rotationally invariant subspaces in $\mathrm{Sym}(\mathcal{T})$ we observe that rotations $\boldsymbol{R}_\theta$ of $\mathbb{R}^2$ through the angle $\theta$ counterclockwise act on vectors $\boldsymbol{u} \in \mathcal{T} \cong \mathbb{C}^N$ by $\boldsymbol{R}_\theta \cdot \boldsymbol{u} = e^{i\theta}\boldsymbol{u}$. Every $\mathsf{K} \in \mathrm{Sym}(\mathcal{T})$ is uniquely determined by a complex Hermitian $N \times N$ matrix $X$ and a complex symmetric $N \times N$ matrix $Y$ by the rule

$$\mathsf{K}\boldsymbol{u} = X\boldsymbol{u} + Y\overline{\boldsymbol{u}}, \qquad \boldsymbol{u} \in \mathbb{C}^N.$$

Henceforth, we will write $\mathsf{K}(X, Y)$ to indicate this parametrization of $\mathrm{Sym}(\mathcal{T})$. In this notation

$$\mathsf{K}(X_1, Y_1)\mathsf{K}(X_2, Y_2) = \mathsf{K}(X_1X_2 + Y_1Y_2^H, X_1Y_2 + Y_1X_2^T), \tag{3.2}$$

where $Y^H = \overline{Y}^T$ denotes Hermitian conjugation.[8] We easily compute the action of rotations $\boldsymbol{R}_\theta$ on $\mathsf{K}(X, Y)$:

$$\boldsymbol{R}_\theta \cdot \mathsf{K}(X, Y) = \mathsf{K}(X, e^{2i\theta}Y). \tag{3.3}$$

Therefore, if $\Pi$ is an SO(2)-invariant subspace of $\mathrm{Sym}(\mathcal{T})$ then

$$\Pi = \Pi_{V,W} = \{(X, Y) : X \in V \subset \mathcal{H}(\mathbb{C}^N),\ Y \in W \subset \mathrm{Sym}(\mathbb{C}^N)\},$$

where $V$ can be any subspace of $\mathcal{H}(\mathbb{C}^N)$—the set of all complex Hermitian $N \times N$ matrices, regarded as a real vector space, and $W$ can be any subspace of $\mathrm{Sym}(\mathbb{C}^N)$—the set of all complex symmetric $N \times N$ matrices, regarded as a complex vector space.

In order to identify L-relations we need to compute all Jordan $\mathcal{A}$-multialgebras, where in our new notation

$$\mathcal{A} = \{\mathsf{K}(0, z\boldsymbol{I}_N) : z \in \mathbb{C}\}.$$

Using multiplication rule (3.2) we determine that a subspace $\Pi_{V,W}$ is a Jordan $\mathcal{A}$-multialgebra if and only if

$$Y^2 + XX^T \in W, \quad YX + XY^H \in V \text{ for all } X \in V,\ Y \in W. \tag{3.4}$$

In contrast with the three dimensional case, where all rotationally invariant Jordan $\mathcal{A}$-multialgebras have a simple characterization (see [18]), the set of solutions of (3.4) is unknown, in general. It in not hard to verify that the necessary condition (2.17) is equivalent to

$$iX_1X_2^TX_3 + (iX_1X_2^TX_3)^H \in V, \quad \forall\{X_1, X_2, X_3\} \subset V, \tag{3.5}$$

provided that (3.4) holds as well.

---

[8] We do not use the standard notation $Y^*$ to avoid confusion with our notation for the effective tensor.

*3.2. L-Relation that is not Exact*

In this section we present an example of the subspace $\Pi_{V,W}$ that satisfies (3.4), but fails (3.5), when $N = 4$. The case $N = 4$ is special because we can regard vectors in $\mathbb{R}^4$ as quaternions via (1.1).

$$\mathcal{T} = \mathbb{R}^4 \otimes \mathbb{R}^2 \cong \mathbb{R}^2 \otimes \mathbb{R}^4 \cong \mathbb{R}^4 \oplus \mathbb{R}^4 \cong \mathbb{H}^2. \tag{3.6}$$

Explicitly, for $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$, $\boldsymbol{u} \in \mathbb{R}^4$

$$\boldsymbol{u} \otimes \boldsymbol{x} \leftrightarrow \boldsymbol{x} \otimes \boldsymbol{u} \leftrightarrow (x_1\boldsymbol{u}, x_2\boldsymbol{u}) \leftrightarrow (x_1 u, x_2 u) \in \mathbb{H}^2,$$

where $u \in \mathbb{H}$ corresponds to $\boldsymbol{u} \in \mathbb{R}^4$, via (1.1). The new representation (3.6) of $\mathcal{T}$ does not replace the old one (3.1). Instead, both will be used.

The utility of the isomorphism (3.6) of 8-dimensional real vector spaces comes from the multiplicative properties of quaternions. Using the identification (1.1) between $\mathbb{H}$ and $\mathbb{R}^4$ we first define an $\mathbb{R}$-linear transformation $Q : \mathbb{H} \to \mathrm{End}(\mathbb{R}^4)$ by

$$Q(q)\boldsymbol{h} = \boldsymbol{g}, \quad g = qh, \qquad \{\boldsymbol{h}, \boldsymbol{g}\} \subset \mathbb{R}^4, \ \{g, h, q\} \subset \mathbb{H}. \tag{3.7}$$

It is easy to see that $Q(q)Q(h) = Q(qh)$ and $Q(q)^T = Q(\overline{q})$. Next, we regard $\mathrm{End}(\mathbb{R}^4)$ as a subset of $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^4)$ by regarding real entries in $4 \times 4$ matrices in $\mathrm{End}(\mathbb{R}^4)$ as complex numbers. Thus, every operator in $\mathrm{End}(\mathbb{R}^4)$ can be canonically viewed as an operator in $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^4)$. Applying this interpretation to operators $Q(h) \in \mathrm{End}(\mathbb{R}^4)$ and using the original representation (3.1) of $\mathcal{T}$ as $\mathbb{C}^4$ (understood as a real vector space), we obtain the mapping $\mathfrak{Q} : \mathbb{H} \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}^4) \subset \mathrm{End}_{\mathbb{R}}(\mathcal{T})$.

To describe our example, we take $\Pi = \Pi_{V,W}$, where

$$V = \{i\mathfrak{Q}(q) : q \in \mathbb{H}, \ \mathfrak{Re}(q) = 0\} \subset \mathcal{H}(\mathbb{C}^4), \qquad W = \{a\boldsymbol{I}_4 : a \in \mathbb{C}\} \subset \mathrm{Sym}(\mathbb{C}^4). \tag{3.8}$$

Let us verify that $\Pi$ is a Jordan $\mathcal{A}$-multialgebra by checking (3.4). For $X = i\mathfrak{Q}(q)$ and $Y = a\boldsymbol{I}_4$ we compute

$$\begin{aligned} Y^2 + XX^T &= a^2\boldsymbol{I}_4 + i\mathfrak{Q}(q)i\mathfrak{Q}(\overline{q}) = a^2\boldsymbol{I}_4 - \mathfrak{Q}(|q|^2) = (a^2 - |q|^2)\boldsymbol{I}_4 \in W, \\ YX + XY^H &= ai\mathfrak{Q}(q) + \overline{a}i\mathfrak{Q}(q) = 2i\mathfrak{Re}(a)\mathfrak{Q}(q) \in V. \end{aligned}$$

We also verify that (3.5) fails. For this purpose we take

$$X_1 = i\mathfrak{Q}(\mathfrak{i}), \qquad X_2 = i\mathfrak{Q}(\mathfrak{j}), \qquad X_3 = i\mathfrak{Q}(\mathfrak{k}),$$

where $\mathfrak{i}$, $\mathfrak{j}$, $\mathfrak{k}$ are the imaginary quaternionic units. We compute

$$iX_1X_2^TX_3 = i(i\mathfrak{Q}(\mathfrak{i})i\mathfrak{Q}(-\mathfrak{j})i\mathfrak{Q}(\mathfrak{k})) = i^4\mathfrak{Q}(-\mathfrak{i}\mathfrak{j}\mathfrak{k}) = \mathfrak{Q}(1).$$

Thus,

$$iX_1X_2^TX_3 + (iX_1X_2^TX_3)^H = \mathfrak{Q}(2) \notin V.$$

In order to describe the submanifold $\mathbb{M}$ determined by SO(2)-invariant Jordan $\mathcal{A}$-multialgebra $\Pi_{V,W}$ and $\mathsf{L}_0 = \mathsf{I}$, given by (3.8) we use representation (3.6) of $\mathcal{T}$. Then, the action of operators $\mathsf{K}(X, Y) = \mathsf{K}(i\mathfrak{Q}(q), (a_1 + ia_2)\boldsymbol{I}_4) \in \Pi_{V,W}$

on $\mathcal{T} \cong \mathbb{C}^4$ can also be described as the action of $2 \times 2$ *quaternionic* Hermitian matrices

$$\mathsf{K}(X, Y) = \begin{bmatrix} a_1 & a_2 - q \\ a_2 + q & -a_1 \end{bmatrix} \tag{3.9}$$

on $\mathcal{T} \cong \mathbb{H}^2$.[9] Observing that $\boldsymbol{\Gamma}_0(\boldsymbol{n}) = \boldsymbol{n} \otimes \boldsymbol{n} \in \mathbb{H}^{2\times 2}$, where $\boldsymbol{n} \in \mathbb{R}^2$ is now interpreted as a vector in $\mathbb{H}^2$ both of whose components happen to be real, we can compute $\mathbb{M}$, given by (2.13), working only with $2 \times 2$ quaternionic matrices. The calculations are fairly straightforward and entirely analogous to 2D conductivity [13–15], where $\Pi = \{\boldsymbol{K} \in \mathrm{Sym}(\mathbb{R}^2) : \mathrm{Tr}\,\boldsymbol{K} = 0\}$ was shown to correspond the well-known Keller-Dykhne exact relation [11,12,23]. In our case of 2D "quaternionic materials"

$$\mathbb{M} = \left\{ \mathsf{L} = \begin{bmatrix} \lambda & h \\ \overline{h} & \mu \end{bmatrix} : \lambda > 0,\ \mu > 0,\ h \in \mathbb{H},\ \det_{\mathbb{H}}(\mathsf{L}) = \lambda\mu - |h|^2 = 1 \right\}. \tag{3.10}$$

Of course, $\det_{\mathbb{H}}(\mathsf{L})$ is set to 1 in (3.10) only for simplicity. It can be any positive constant.

The consequence of our theory of exact relations developed in Section 2 is the following corollary.

**Corollary 3.1.** *Let $\mathbb{M}$ be given by (3.10). Then, for every $\delta > 0$ there exists a measurable function $\mathsf{L} : [0, 1]^2 \to \mathbb{M}$, such that $\|\mathsf{L} - \mathsf{I}\|_{L^\infty} < \delta$ and $\mathsf{L}^* \notin \mathbb{M}$. However, effective tensors of all simple laminates made with any materials $\{\mathsf{L}_1, \mathsf{L}_2\} \subset \mathbb{M}$, taken in any volume fraction and arbitrary orientation of layers, belong to $\mathbb{M}$.*

Indeed, by construction of the manifold $\mathbb{M}$ and Theorem 2.12, the manifold $\mathbb{M}$ is *not* G-closed. Then, the first statement in Corollary 3.1 is a consequence of Lemma 2.7 and the fact that $W_{\boldsymbol{n}}$ (for any $\boldsymbol{n}$ and $\mathsf{L}_0 = \mathsf{I}$) maps a small neighborhood of $\mathsf{I} \in \mathbb{M}$ into a small neighborhood of $0 \in \Pi$. The second statement in Corollary 3.1 follows from Theorem 2.5.

For our purposes Corollary 3.1 needs to be augmented by a positive result, placing the effective tensor $\mathsf{L}^*$ into a submanifold $\widetilde{\mathbb{M}}$ that contains $\mathbb{M}$ but has only one more dimension than $\mathbb{M}$. This submanifold corresponds to the Jordan $\mathcal{A}$-multialgebra

$$\widetilde{\Pi} = \left\{ \begin{bmatrix} \alpha & q \\ \overline{q} & \beta \end{bmatrix} : \{\alpha, \beta\} \subset \mathbb{R},\ q \in \mathbb{H} \right\},$$

of which $\Pi_{V,W}$, given by (3.8), is a subspace in view of (3.9). It is easy to see that computation of $\widetilde{\mathbb{M}}$ via (2.13) can proceed entirely in the framework of $2 \times 2$

[9] They are Hermitian because in (3.8) $q$ must be purely imaginary and hence, $a_2 - q = \overline{a_2 + q}$.

quaternionic matrices resulting in

$$\widetilde{\mathbb{M}} = \left\{ \mathsf{L} = \begin{bmatrix} \lambda & h \\ \overline{h} & \mu \end{bmatrix} : \lambda > 0,\ \mu > 0,\ h \in \mathbb{H},\ \det_{\mathbb{H}}(\mathsf{L}) > 0 \right\}. \tag{3.11}$$

The fact that $\widetilde{\mathbb{M}}$ is an exact relation is a consequence of Theorem 2.9, where $\Pi' = \mathbb{H}^{2\times 2}$ is the algebra of all $2 \times 2$ quaternionic matrices. It is also an associative $\mathcal{A}$-multialgebra, since $\mathcal{A} \subset \mathbb{H}^{2\times 2}$. It only remains to notice that $\widetilde{\Pi} = \Pi' \cap \mathrm{Sym}(\mathcal{T})$.

## 4. Homogenization, Rank-One Convexity and Quasiconvexity

We are finally ready to construct examples of rank-one convex, non-quasiconvex functions, of which (1.4) will be a member. We recall the well-known connection between rank-one convex functions and L-closed sets, and quasiconvex functions and G-closed sets [41, Sections 31.4–5]. In the context of multifield materials considered in Section 3 these results (or rather those results that we need) are summarized in the following theorem.

**Theorem 4.1.** *In the context of multifield materials of Section 3, let $U \subset \mathrm{Sym}^+(\mathcal{T})$ be a compact subset. For any $\boldsymbol{e} \in \mathcal{T} = \mathbb{R}^N \otimes \mathbb{R}^d$ we define*

$$W(\boldsymbol{e}) = \min_{\mathsf{L}\in U} \frac{1}{2}(\mathsf{L}\boldsymbol{e}, \boldsymbol{e})_{\mathcal{T}}.$$

i. *If $U$ is L-closed then $W(\boldsymbol{e})$ is rank-one convex.*
ii. *In general*

$$QW(\boldsymbol{e}) = \min_{\mathsf{L}\in G(U)} \frac{1}{2}(\mathsf{L}\boldsymbol{e}, \boldsymbol{e})_{\mathcal{T}}, \tag{4.1}$$

*where $QW(\boldsymbol{e})$ is the quasiconvex envelope of $W(\boldsymbol{e})$ [7], and $G(U)$ is the G-closure of $U$.*[10]

In Section 3.2 we have constructed the set $\mathbb{M}$, given by (3.10) that is L-closed, but not G-closed. We cannot apply Theorem 4.1 to $\mathbb{M}$ directly, because $\mathbb{M}$ is noncompact. However, recall that the sets

$$\mathcal{B}_\gamma = \left\{ \mathsf{L} \in \mathrm{Sym}^+(\mathcal{T}) : \gamma \mathsf{I} \leqq \mathsf{L} \leqq \gamma^{-1}\mathsf{I} \right\}, \qquad \gamma \in (0, 1)$$

are G-closed [44].[11] Thus, the sets

$$\mathbb{M}_\gamma = \mathbb{M} \cap \mathcal{B}_\gamma, \quad \gamma \in (0, 1)$$

---

[10] This statement follows from the variational principle for the energy of a periodic composite: $(\mathsf{L}^*\boldsymbol{e}, \boldsymbol{e})_{\mathcal{T}} = \min_{\boldsymbol{\phi}} \langle (\mathsf{L}(\boldsymbol{x})(\nabla\boldsymbol{\phi} + \boldsymbol{e}), \nabla\boldsymbol{\phi} + \boldsymbol{e})_{\mathcal{T}} \rangle$, where the minimum is taken over $[0, 1]^d$-periodic vector fields $\boldsymbol{\phi} \in W^{1,2}([0, 1]^d; \mathbb{R}^N)$.

[11] Due to symmetry of operators in $\mathcal{B}_\gamma$ we can also write that $\mathcal{B}_\gamma = \left\{ \mathsf{L} : \mathsf{L} \geqq \gamma\mathsf{I},\ \mathsf{L}^{-1} \geqq \gamma\mathsf{I} \right\} \cap \mathrm{Sym}^+(\mathcal{T})$, representing $\mathcal{B}_\gamma$ as an intersection of two G-closed sets.

are L-closed, but not G-closed (due to Corollary 3.1), while the sets

$$\widetilde{\mathbb{M}}_\gamma = \widetilde{\mathbb{M}} \cap \mathcal{B}_\gamma, \quad \gamma \in (0, 1)$$

are G-closed.

To produce our example we will use the "doubling trick" of Milton, where $\mathsf{L} \in \mathrm{Sym}^+(\mathcal{T})$ acts on $\mathcal{T} \oplus \mathcal{T}$, via

$$\mathcal{T} \oplus \mathcal{T} \ni (\boldsymbol{e}_1, \boldsymbol{e}_2) \mapsto \mathbb{D}(\mathsf{L})(\boldsymbol{e}_1, \boldsymbol{e}_2) = (\mathsf{L}\boldsymbol{e}_1, \mathsf{L}\boldsymbol{e}_2) \in \mathcal{T} \oplus \mathcal{T}. \tag{4.2}$$

The key observation is that "doubling" commutes with homogenization: $\mathbb{D}(\mathsf{L})^* = \mathbb{D}(\mathsf{L}^*)$ for any periodic composite with local tensor $\mathsf{L}(\boldsymbol{x})$. Hence, instead of $\mathcal{T} \cong \mathbb{H}^2$ we work with $\mathcal{T} \oplus \mathcal{T} \cong \mathbb{H}^{2\times 2}$, where the pair $(\boldsymbol{e}_1, \boldsymbol{e}_2) \in \mathcal{T} \oplus \mathcal{T} \cong \mathbb{H}^2 \oplus \mathbb{H}^2$ is identified with the $2 \times 2$ quaternionic matrix $[\boldsymbol{e}_1, \boldsymbol{e}_2] \in \mathbb{H}^{2\times 2}$, whose columns are vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$. In particular, for $\boldsymbol{E} = [\boldsymbol{e}_1, \boldsymbol{e}_2] \in \mathbb{H}^{2\times 2}$ we define

$$\begin{aligned}\Phi_\gamma(\boldsymbol{E}) &= \frac{1}{2} \min_{\mathsf{L}\in\mathbb{M}_\gamma} (\mathbb{D}(\mathsf{L})(\boldsymbol{e}_1, \boldsymbol{e}_2), (\boldsymbol{e}_1, \boldsymbol{e}_2))_{\mathcal{T}\oplus\mathcal{T}} \\ &= \frac{1}{2} \min_{\mathsf{L}\in\mathbb{M}_\gamma} \{(\mathsf{L}\boldsymbol{e}_1, \boldsymbol{e}_1)_{\mathcal{T}} + (\mathsf{L}\boldsymbol{e}_2, \boldsymbol{e}_2)_{\mathcal{T}}\}.^{12}\end{aligned}$$

We can rewrite $\Phi_\gamma(\boldsymbol{E})$ in terms of the quaternion-valued inner product $(\cdot,\cdot)_{\mathbb{H}^2}$, defined in (1.2):

$$\Phi_\gamma(\boldsymbol{E}) = \frac{1}{2} \min_{\mathsf{L}\in\mathbb{M}_\gamma} \mathfrak{Re}\{(\mathsf{L}\boldsymbol{e}_1, \boldsymbol{e}_1)_{\mathbb{H}^2} + (\mathsf{L}\boldsymbol{e}_2, \boldsymbol{e}_2)_{\mathbb{H}^2}\},$$

which, in turn, allows us to represent $\Phi_\gamma(\boldsymbol{E})$ in terms of products of $2 \times 2$ quaternionic matrices:

$$\Phi_\gamma(\boldsymbol{E}) = \frac{1}{2} \min_{\mathsf{L}\in\mathbb{M}_\gamma} \mathfrak{Re}\mathrm{Tr}_{\mathbb{H}}(\mathsf{L}\boldsymbol{E}\boldsymbol{E}^H), \qquad \gamma > 0, \tag{4.3}$$

where $\mathrm{Tr}_{\mathbb{H}}(\boldsymbol{X})$ denotes the sum of the two quaternions on the main diagonal of a $2 \times 2$ quaternionic matrix $\boldsymbol{X}$. The Hermitian conjugate $\boldsymbol{E}^H$ is defined in the usual way, except complex conjugation is replaced with quaternionic conjugation (1.3).

Each column of $\boldsymbol{E}$ is an element of $\mathcal{T}$—the space of field values of 4-tuples of curl-free fields $(\nabla\phi_1, \dots, \nabla\phi_4)$. If we regard vector $\boldsymbol{u} = (\phi_1, \phi_2, \phi_3, \phi_4)$ as a quaternion $u$, then in representation (3.6) $\nabla\boldsymbol{u} \in \mathcal{T}$ is identified with $(u_x, u_y) \in \mathbb{H}^2$, where subscripts indicate partial derivatives. Using another $\mathbb{H}$-valued function $v(x, y)$ to generate the second copy of $\mathcal{T}$ we obtain the underlying interpretation of the $2 \times 2$ quaternionic matrix $\boldsymbol{E}$:

$$\boldsymbol{E} = \begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix} \in \mathbb{H}^{2\times 2} = (\nabla \boldsymbol{f})^T, \tag{4.4}$$

where $\boldsymbol{f}(x, y) = (u(x, y), v(x, y)) \in \mathbb{H}^2$.

[12] The inner product on a direct sum $V \oplus W$ of inner product spaces is canonically defined by $((\boldsymbol{v}, \boldsymbol{w}), (\boldsymbol{v}', \boldsymbol{w}'))_{V\oplus W} = (\boldsymbol{v}, \boldsymbol{v}')_V + (\boldsymbol{w}, \boldsymbol{w}')_W$.

**Theorem 4.2.** *For every* $\gamma \in (0, 1)$ *functions* $\Phi_\gamma(\boldsymbol{E})$ *are rank-one convex, but not quasiconvex, where* $\boldsymbol{E}$ *is interpreted via* (4.4). *Specifically,* $Q\Phi_\gamma(\boldsymbol{I}_2) < \Phi_\gamma(\boldsymbol{I}_2)$.

Rank-one convexity of $\Phi_\gamma$ follows from Theorem 4.1(i). In order to prove that these functions are not quasiconvex we will use part (ii) of Theorem 4.1, which requires more information about geometry of $G(\mathbb{M}_\gamma)$ beyond the facts that $G(\mathbb{M}_\gamma) \not\subset \mathbb{M}$ and $G(\mathbb{M}_\gamma) \subset \widetilde{\mathbb{M}}_\gamma$. We will organize the proof of non-quasiconvexity of functions $\Phi_\gamma$ into a sequence of lemmas.

The key result here is the following analog of Mendelson's duality link [33].

**Lemma 4.3.** *Let* $\mathsf{L} \in L^\infty([0, 1]^2, \mathbb{M})$. *Let* $\widetilde{\mathsf{L}}(\boldsymbol{x}) = \overline{\mathsf{L}(\boldsymbol{x})}$, *be the tensor, whose components are quaternionic conjugates of the corresponding components of* $\mathsf{L}(\boldsymbol{x})$. *Then*

$$(\widetilde{\mathsf{L}})^* = \frac{\overline{\mathsf{L}^*}}{\det_{\mathbb{H}}(\mathsf{L}^*)}, \tag{4.5}$$

*where* $\det_{\mathbb{H}}$ *is defined in* (3.10). *In particular,*

$$\det_{\mathbb{H}}(\mathsf{L}^*)\det_{\mathbb{H}}((\widetilde{\mathsf{L}})^*) = 1. \tag{4.6}$$

**Proof.** Suppose that the $[0, 1]^2$-periodic fields $\boldsymbol{e}, \boldsymbol{j} \in L^2([0, 1]^2; \mathcal{T})$ solve the periodic cell problem

$$\boldsymbol{j}(\boldsymbol{x}) = \mathsf{L}(\boldsymbol{x})\boldsymbol{e}(\boldsymbol{x}), \quad \nabla \cdot \boldsymbol{j} = 0, \quad \nabla \times \boldsymbol{e} = 0, \quad \langle \boldsymbol{e} \rangle = \boldsymbol{e}_0, \tag{4.7}$$

where $\langle \cdot \rangle$ denotes the average over the period cell. Then $\langle \boldsymbol{j} \rangle = \mathsf{L}^*\boldsymbol{e}_0$. Let

$$\mathsf{R}_\perp = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \mathbb{H}^{2\times 2}.$$

Then $\boldsymbol{e}' = \mathsf{R}_\perp \boldsymbol{j}$ is a curl-free field and $\boldsymbol{j}' = \mathsf{R}_\perp \boldsymbol{e}$ is divergence-free. Thus, $\boldsymbol{e}'(\boldsymbol{x})$ and $\boldsymbol{j}'(\boldsymbol{x})$ also solve the cell problem (4.7), where $\mathsf{L}(\boldsymbol{x})$ is replaced with

$$\widetilde{\mathsf{L}}(\boldsymbol{x}) = \mathsf{R}_\perp \mathsf{L}(\boldsymbol{x})^{-1}\mathsf{R}_\perp^{-1} = \frac{\overline{\mathsf{L}(\boldsymbol{x})}}{\det_{\mathbb{H}} \mathsf{L}(\boldsymbol{x})} \tag{4.8}$$

and $\boldsymbol{e}_0$ with

$$\boldsymbol{e}_0' = \mathsf{R}_\perp \langle \boldsymbol{j} \rangle = \mathsf{R}_\perp \mathsf{L}^* \boldsymbol{e}_0.$$

Computing $\langle \boldsymbol{j}' \rangle = \mathsf{R}_\perp \boldsymbol{e}_0$ we conclude that

$$(\widetilde{\mathsf{L}})^* \mathsf{R}_\perp \mathsf{L}^* \boldsymbol{e}_0 = \mathsf{R}_\perp \boldsymbol{e}_0.$$

Since, $\boldsymbol{e}_0 \in \mathbb{H}^2$ is arbitrary we conclude that

$$(\widetilde{\mathsf{L}})^* = \mathsf{R}_\perp (\mathsf{L}^*)^{-1}\mathsf{R}_\perp^{-1} = \frac{\overline{\mathsf{L}^*}}{\det_{\mathbb{H}} \mathsf{L}^*}.$$

It remains to observe that if $\mathsf{L}(\boldsymbol{x}) \in \mathbb{M}$ then $\widetilde{\mathsf{L}}(\boldsymbol{x})$, given by (4.8) is equal to $\overline{\mathsf{L}(\boldsymbol{x})}$, and the lemma is proved. □

**Lemma 4.4.** *For every* $\gamma \in (0, 1)$ *there are tensors* $\{\mathsf{L}_+, \mathsf{L}_-\} \subset G(\mathbb{M}_\gamma) \subset \widetilde{\mathbb{M}}_\gamma$ *such that* $\det_{\mathbb{H}} \mathsf{L}_+ > 1$ *and* $\det_{\mathbb{H}} \mathsf{L}_- < 1$.

**Proof.** Observing that for every $\gamma \in (0, 1)$ the sets $\mathcal{B}_\gamma$ contain $\mathsf{I}$ in their interior, we apply Corollary 3.1 and conclude that there exists a measurable function $\mathsf{L} \in L^\infty([0, 1]^2; \mathbb{M}_\gamma)$, such that $\mathsf{L}^* \notin \mathbb{M}$. By construction, $\mathsf{L}^* \in G(\mathbb{M}_\gamma) \subset \widetilde{\mathbb{M}}$. Hence, $\det_{\mathbb{H}} \mathsf{L}^* \neq 1$. Observe that $\widetilde{\mathsf{L}} = \overline{\mathsf{L}} \in L^\infty([0, 1]^2; \mathbb{M}_\gamma)$. But then $(\widetilde{\mathsf{L}})^* \in G(\mathbb{M}_\gamma) \subset \widetilde{\mathbb{M}}$. By Lemma 4.3

$$\det_{\mathbb{H}}(\widetilde{\mathsf{L}})^* = \frac{1}{\det_{\mathbb{H}} \mathsf{L}^*}.$$

Thus, if $\det_{\mathbb{H}} \mathsf{L}^* < 1$, then $\det_{\mathbb{H}}(\widetilde{\mathsf{L}})^* > 1$, and if $\det_{\mathbb{H}} \mathsf{L}^* > 1$, then $\det_{\mathbb{H}}(\widetilde{\mathsf{L}})^* < 1$. The lemma is proved. □

**Lemma 4.5.** *For every* $\gamma \in (0, 1)$ *there exists* $\delta > 0$ *such that* $\{(1+\delta)\mathsf{I}, (1-\delta)\mathsf{I}\} \subset G(\mathbb{M}_\gamma)$.

**Proof.** Let $\{\mathsf{L}_+, \mathsf{L}_-\} \subset G(\mathbb{M}_\gamma)$ be as in Lemma 4.4. Let $\boldsymbol{n}_0$ be a fixed unit vector and $\mathsf{K}_\pm = W_{\boldsymbol{n}_0}(\mathsf{L}_\pm)$. Then $\{\mathsf{K}_+, \mathsf{K}_-\} \subset \widetilde{\Pi} \setminus \Pi$. In particular, it is easy to verify that $\mathrm{Tr}_{\mathbb{H}}\mathsf{K}_+ > 0$ and $\mathrm{Tr}_{\mathbb{H}}\mathsf{K}_- < 0$. Next, we note that the set $W_{\boldsymbol{n}_0}(G(\mathbb{M}_\gamma))$ is convex, and contains $W_{\boldsymbol{n}_0}(\mathbb{M}_\gamma)$, which is a neighborhood of 0 in $\Pi = \{\mathsf{K} \in \widetilde{\Pi} : \mathrm{Tr}_{\mathbb{H}}\mathsf{K} = 0\}$. It also contains $\mathsf{K}_\pm$. It is clear that the convex hull of $\mathsf{K}_+$, $\mathsf{K}_-$ and $W_{\boldsymbol{n}_0}(\mathbb{M}_\gamma)$ contains a neighborhood of 0 in $\widetilde{\Pi}$. But then $G(\mathbb{M}_\gamma)$ must contain a neighborhood of $\mathsf{I}$ in $\widetilde{\mathbb{M}}$. The statement of the lemma follows. □

**Proof of Theorem 4.2.** To prove Theorem 4.2 we are going to show that $Q\Phi_\gamma(\boldsymbol{I}_2) < \Phi_\gamma(\boldsymbol{I}_2)$ for every $\gamma \in (0, 1)$. Indeed, observing that $\mathsf{I} \in \mathbb{M}_\gamma$ for every $\gamma \in (0, 1)$, we have

$$\frac{1}{2}\inf_{\mathsf{L}\in\mathbb{M}} \mathrm{Tr}_{\mathbb{H}}\mathsf{L} \leqq \frac{1}{2}\inf_{\mathsf{L}\in\mathbb{M}_\gamma} \mathrm{Tr}_{\mathbb{H}}\mathsf{L} = \Phi_\gamma(\boldsymbol{I}_2) \leqq 1.$$

It only remains to observe that

$$\frac{1}{2}\inf_{\mathsf{L}\in\mathbb{M}} \mathrm{Tr}_{\mathbb{H}}\mathsf{L} = 1,$$

and is achieved at $\mathsf{L} = \mathsf{I}$. We can now use Theorem 4.1(ii) and obtain

$$Q\Phi_\gamma(\boldsymbol{I}_2) = \frac{1}{2}\min_{\mathsf{L}\in G(\mathbb{M}_\gamma)} \mathrm{Tr}_{\mathbb{H}}\mathsf{L} \leqq 1 - \delta < 1 = \Phi_\gamma(\boldsymbol{I}_2),$$

where we have used the fact that $\mathsf{L} = (1 - \delta)\mathsf{I} \in G(\mathbb{M}_\gamma)$, according to Lemma 4.5. □

We observe that the sets $\mathbb{M}_\gamma$ become larger when $\gamma$ is decreasing. Therefore $\Phi_{\gamma_1}(\boldsymbol{E}) \leqq \Phi_{\gamma_2}(\boldsymbol{E})$, when $\gamma_1 < \gamma_2$. Then

$$\Phi_0(\boldsymbol{E}) = \lim_{\gamma\to 0} \Phi_\gamma(\boldsymbol{E}) = \inf_{\gamma\in(0,1)} \Phi_\gamma(\boldsymbol{E}) = \frac{1}{2}\inf_{\mathsf{L}\in\mathbb{M}} \Re\mathfrak{e}\mathrm{Tr}_{\mathbb{H}}(\mathsf{L}\boldsymbol{E}\boldsymbol{E}^H). \qquad (4.9)$$

Thus,

$$Q\Phi_0(\boldsymbol{I}_2) \leqq Q\Phi_\gamma(\boldsymbol{I}_2) < \Phi_\gamma(\boldsymbol{I}_2) = 1 = \Phi_0(\boldsymbol{I}_2).$$

This shows that the function $\Phi_0(\boldsymbol{E})$ defined by (4.9) is not quasiconvex at $\boldsymbol{E} = \boldsymbol{I}_2$. However, it must be rank-one convex as a limit of rank-one convex functions. To conclude, it only remains to compute the explicit form of functions $\Phi_\gamma(\boldsymbol{E})$, $\gamma \in [0, 1)$. It will be convenient to give the answer in terms of the components of

$$\boldsymbol{E}\boldsymbol{E}^H = \begin{bmatrix} \alpha_1 & q \\ \overline{q} & \alpha_2 \end{bmatrix}.$$

**Lemma 4.6.**

$$\Phi_0(\boldsymbol{E}) = \sqrt{\det_{\mathbb{H}}(\boldsymbol{E}\boldsymbol{E}^H)} = \sqrt{\alpha_1\alpha_2 - |q|^2}, \tag{4.10}$$

$$\Phi_\gamma(\boldsymbol{E}) = \begin{cases} \sqrt{\alpha_1\alpha_2 - |q|^2}, & \text{if } \alpha_1\alpha_2 - |q|^2 \geqq \left(\frac{\alpha_1+\alpha_2}{2J_+(\gamma)}\right)^2, \\ J_+(\gamma)\dfrac{\alpha_1+\alpha_2}{2} - J_-(\gamma)\sqrt{|q|^2 + \left(\frac{\alpha_1-\alpha_2}{2}\right)^2}, & \text{otherwise}, \end{cases} \tag{4.11}$$

*where*

$$J_\pm(\gamma) = \frac{1}{2}\left(\frac{1}{\gamma} \pm \gamma\right), \qquad \gamma \in (0, 1). \tag{4.12}$$

**Proof.** We first observe that

$$\mathbb{M}_\gamma = \left\{\begin{bmatrix} \lambda & h \\ \overline{h} & \mu \end{bmatrix} : \lambda\mu - |h|^2 = 1,\ \lambda > 0,\ \mu > 0, \lambda + \mu \leqq 2J_+(\gamma)\right\},$$

where $J_+(\gamma)$ is the Joukowski function defined in (4.12). If

$$\mathsf{L} = \begin{bmatrix} \lambda & h \\ \overline{h} & \mu \end{bmatrix}, \qquad \boldsymbol{E}\boldsymbol{E}^H = \begin{bmatrix} \alpha_1 & q \\ \overline{q} & \alpha_2 \end{bmatrix},$$

then

$$\mathfrak{Re}\mathrm{Tr}\,(\mathsf{L}\boldsymbol{E}\boldsymbol{E}^H) = \lambda\alpha_1 + \mu\alpha_2 + 2\mathfrak{Re}(h\overline{q}). \tag{4.13}$$

If we fix $\lambda > 0$ and $\mu > 0$, so that $\lambda + \mu \leqq 2J_+(\gamma)$, then $\mathsf{L} \in \mathbb{M}_\gamma$ for any $h \in \mathbb{H}$ for which $|h| = \sqrt{\lambda\mu - 1}$. Observing that $\mathfrak{Re}(h\overline{q})$ is just a dot product of corresponding vectors $\{\boldsymbol{h}, \boldsymbol{q}\} \subset \mathbb{R}^4$ and $|h|$ is just the length of the vector $\boldsymbol{h}$, we can minimize $\mathfrak{Re}(h\overline{q})$ over all directions of $\boldsymbol{h}$, obtaining the formula

$$2\Phi_\gamma(\boldsymbol{E}) = \min_{\substack{0<\lambda+\mu\leqq 2J_+(\gamma) \\ \lambda\mu\geqq 1}} \{\lambda\alpha_1 + \mu\alpha_2 - 2\sqrt{\lambda\mu - 1}|q|\},$$

The minimum is achieved at a critical point

$$\lambda_* = \frac{\alpha_2}{\sqrt{\alpha_1\alpha_2 - |q|^2}}, \qquad \mu_* = \frac{\alpha_1}{\sqrt{\alpha_1\alpha_2 - |q|^2}}, \tag{4.14}$$

when $\lambda_* + \mu_* < 2J_+(\gamma)$, or equivalently, when

$$|q|^2 \leqq \alpha_1\alpha_2 - \left(\frac{\alpha_1+\alpha_2}{2J_+(\gamma)}\right)^2. \tag{4.15}$$

Otherwise, the minimum is achieved on the line $\lambda + \mu = 2J_+(\gamma)$ at

$$\lambda = J_+(\gamma) - \frac{J_-(\gamma)(\alpha_1-\alpha_2)}{\sqrt{4|q|^2+(\alpha_1-\alpha_2)^2}}, \qquad \mu = J_+(\gamma) + \frac{J_-(\gamma)(\alpha_1-\alpha_2)}{\sqrt{4|q|^2+(\alpha_1-\alpha_2)^2}}$$

Substituting these values into (4.13) we obtain (4.11). □

In order to connect (4.10) with (1.4) we use formula (4.4) and compute

$$\boldsymbol{E}\boldsymbol{E}^H = \begin{bmatrix} \|\boldsymbol{f}_x\|^2_{\mathbb{H}^2} & (\boldsymbol{f}_x, \boldsymbol{f}_y)_{\mathbb{H}^2} \\ (\boldsymbol{f}_x, \boldsymbol{f}_y)_{\mathbb{H}^2} & \|\boldsymbol{f}_y\|^2_{\mathbb{H}^2} \end{bmatrix}, \qquad \Phi_0(\boldsymbol{E}) = \sqrt{\det_{\mathbb{H}}((\nabla\boldsymbol{f})^T\overline{\nabla\boldsymbol{f}})}.$$

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## Appendix A. The Equation for the Effective Tensor of a Periodic Composite

In this section we are going to give a rigorous derivation of (2.6) starting with the periodic cell problem. In order to formulate a general version of the cell problem we recall [39] that the linear differential constraints satisfied by the $[0,1]^d$-periodic fields $\boldsymbol{E}$ and $\boldsymbol{J}$ are described in terms of their Fourier coefficients

$$\widehat{\boldsymbol{E}}(\boldsymbol{k}) = \int_{[0,1]^d} \boldsymbol{E}(\boldsymbol{x})e^{-2\pi i\boldsymbol{x}\cdot\boldsymbol{k}}d\boldsymbol{x}, \qquad \widehat{\boldsymbol{J}}(\boldsymbol{k}) = \int_{[0,1]^d} \boldsymbol{J}(\boldsymbol{x})e^{-2\pi i\boldsymbol{x}\cdot\boldsymbol{k}}d\boldsymbol{x}, \quad \boldsymbol{k}\in\mathbb{Z}^d. \tag{A.1}$$

Namely,

$$\boldsymbol{E}\in\mathcal{E}_{\rm per}\oplus\mathcal{U}, \qquad \boldsymbol{J}\in\mathcal{E}_{\rm per}\oplus\mathcal{U},$$

where $\mathcal{U}$ is the space of uniform (constant) vector fields in $L^2_{\rm per}([0,1]^d;\mathcal{T})$ and

$$\mathcal{E}_{\rm per} = \left\{\boldsymbol{E}\in L^2_{\rm per}([0,1]^d;\mathcal{T}) : \widehat{\boldsymbol{E}}(0)=0,\ \widehat{\boldsymbol{E}}(\boldsymbol{k})\in\mathcal{E}_{\frac{\boldsymbol{k}}{|\boldsymbol{k}|}}\otimes\mathbb{C},\ \boldsymbol{k}\neq 0\right\}, \tag{A.2}$$

$$\mathcal{J}_{\rm per} = \left\{\boldsymbol{J}\in L^2_{\rm per}([0,1]^d;\mathcal{T}) : \widehat{\boldsymbol{J}}(0)=0,\ \widehat{\boldsymbol{J}}(\boldsymbol{k})\in\mathcal{J}_{\frac{\boldsymbol{k}}{|\boldsymbol{k}|}}\otimes\mathbb{C},\ \boldsymbol{k}\neq 0\right\}. \tag{A.3}$$

The subspaces $\mathcal{E}_{\boldsymbol{n}}$ and $\mathcal{J}_{\boldsymbol{n}}$ of $\mathcal{T}$ are required to satisfy (2.3). For example, $\mathcal{E}_{\boldsymbol{n}}$ and $\mathcal{J}_{\boldsymbol{n}}$ for conductivity and elasticity are given in (2.1) and (2.2), respectively.

We also have a corresponding orthogonal decomposition of $L^2_{\rm per}([0,1]^d;\mathcal{T})$:

$$L^2_{\rm per}([0,1]^d;\mathcal{T}) = \mathcal{E}_{\rm per} \oplus \mathcal{J}_{\rm per} \oplus \mathcal{U}.$$

Let the tensor of material properties in the period cell $\mathsf{L} \in L^\infty_{\rm per}([0,1]^d; \mathrm{Sym}(\mathcal{T}))$ be strictly, uniformly positive definite, i.e satisfy

$$(\mathsf{L}(\boldsymbol{x})\boldsymbol{e}, \boldsymbol{e})_{\mathcal{T}} \geqq \alpha_0 \|\boldsymbol{e}\|^2_{\mathcal{T}}, \quad \forall\, \boldsymbol{x} \in [0,1]^d. \tag{A.4}$$

Lax–Milgram lemma guarantees that the periodic cell problem

$$\boldsymbol{E} \in \mathcal{E}_{\rm per} \oplus \mathcal{U}, \quad \boldsymbol{J}(\boldsymbol{x}) = \mathsf{L}(\boldsymbol{x})\boldsymbol{E}(\boldsymbol{x}), \quad \boldsymbol{J} \in \mathcal{J}_{\rm per} \oplus \mathcal{U}, \quad \langle \boldsymbol{E} \rangle = \boldsymbol{e}_0 \tag{A.5}$$

has a unique solution for every given $\boldsymbol{e}_0 \in \mathcal{T}$. Then the effective tensor $\mathsf{L}^*$ is defined by its action on $\boldsymbol{e}_0$ via

$$\mathsf{L}^*\boldsymbol{e}_0 = \langle \mathsf{L}\boldsymbol{E} \rangle, \tag{A.6}$$

where $\langle\cdot\rangle$ denotes the average over the period cell. It is well-known that the effective tensor $\mathsf{L}^*$, defined by (A.6) is symmetric and positive definite. However, for our purposes we will need formula (2.6) for $\mathsf{L}^*$. Before we prove (2.6) we show that the W-transformation is always well-defined on $\mathrm{Sym}^+(\mathcal{T})$.

**Lemma A.1.** *Let* $\mathsf{L}_0 \in \mathrm{Sym}^+(\mathcal{T})$. *Then, for every unit vector* $\boldsymbol{n}$ *the operators*

$$\mathsf{M} = \mathsf{I} - (\mathsf{L}_0 - \mathsf{L})\boldsymbol{\Gamma}_0(\boldsymbol{n}) = \mathsf{I} - \boldsymbol{\Gamma}'(\boldsymbol{n}) + \mathsf{L}\mathsf{L}_0^{-1}\boldsymbol{\Gamma}'(\boldsymbol{n}) \tag{A.7}$$

*are invertible for every* $\mathsf{L} \in \mathrm{Sym}^+(\mathcal{T})$. *In addition,* $\{\boldsymbol{\Gamma}_0(\boldsymbol{n}), W_{\boldsymbol{n}}(\mathsf{L})\} \subset \mathrm{Sym}(\mathcal{T})$.

**Proof.** We note that $\mathcal{T} = \mathsf{L}_0\mathcal{E}_{\boldsymbol{n}} \oplus \mathcal{J}_{\boldsymbol{n}}$, since the $\dim(\mathsf{L}_0\mathcal{E}_{\boldsymbol{n}}) = \dim \mathcal{E}_{\boldsymbol{n}}$ and $\mathsf{L}_0\mathcal{E}_{\boldsymbol{n}} \cap \mathcal{J}_{\boldsymbol{n}} = \{0\}$. Indeed, if $\boldsymbol{j} = \mathsf{L}_0\boldsymbol{e} \in \mathcal{J}_{\boldsymbol{n}}$ for some $\boldsymbol{e} \in \mathcal{E}_{\boldsymbol{n}}$, then $(\mathsf{L}_0\boldsymbol{e}, \boldsymbol{e})_{\mathcal{T}} = (\boldsymbol{j}, \boldsymbol{e})_{\mathcal{T}} = 0$. Thus, $\boldsymbol{e} = 0$, since $\mathsf{L}_0 \in \mathrm{Sym}^+(\mathcal{T})$.
Now, let $\boldsymbol{t} \in \mathcal{T}$ be such that $\mathsf{M}\boldsymbol{t} = 0$. We write $\boldsymbol{t} = \mathsf{L}_0\boldsymbol{e} + \boldsymbol{j}$, where $\boldsymbol{e} \in \mathcal{E}_{\boldsymbol{n}}$ and $\boldsymbol{j} \in \mathcal{J}_{\boldsymbol{n}}$. Then $0 = \mathsf{M}\boldsymbol{t} = \mathsf{L}\boldsymbol{e} + \boldsymbol{j}$. Taking the inner product with $\boldsymbol{e}$ we obtain $(\mathsf{L}\boldsymbol{e}, \boldsymbol{e})_{\mathcal{T}} = 0$, so that $\boldsymbol{e} = 0$, since $\mathsf{L} \in \mathrm{Sym}^+(\mathcal{T})$. Thus, $\boldsymbol{t} = 0$ and, hence, the matrix $\mathsf{M}$ must be invertible.
To prove that $\boldsymbol{\Gamma}_0(\boldsymbol{n})$ is symmetric we write for any $\boldsymbol{u}_1, \boldsymbol{u}_2 \subset \mathcal{T}$

$$\boldsymbol{u}_1 = \boldsymbol{j}_1 + \mathsf{L}_0\boldsymbol{e}_1, \qquad \boldsymbol{u}_2 = \boldsymbol{j}_2 + \mathsf{L}_0\boldsymbol{e}_2, \qquad \{\boldsymbol{e}_1, \boldsymbol{e}_2\} \subset \mathcal{E}_{\boldsymbol{n}}, \quad \{\boldsymbol{j}_1, \boldsymbol{j}_2\} \subset \mathcal{J}_{\boldsymbol{n}}.$$

Then $\boldsymbol{\Gamma}_0(\boldsymbol{n})\boldsymbol{u}_1 = \boldsymbol{e}_1$, $\boldsymbol{\Gamma}_0(\boldsymbol{n})\boldsymbol{u}_2 = \boldsymbol{e}_2$, so that

$$(\boldsymbol{\Gamma}_0(\boldsymbol{n})\boldsymbol{u}_1, \boldsymbol{u}_2)_{\mathcal{T}} = (\boldsymbol{e}_1, \boldsymbol{u}_2)_{\mathcal{T}} = (\boldsymbol{e}_1, \mathsf{L}_0\boldsymbol{e}_2)_{\mathcal{T}} = (\mathsf{L}_0\boldsymbol{e}_1, \boldsymbol{e}_2)_{\mathcal{T}} = (\boldsymbol{u}_1, \boldsymbol{\Gamma}_0(\boldsymbol{n})\boldsymbol{u}_2)_{\mathcal{T}}.$$

Finally, the symmetry of $W_{\boldsymbol{n}}(\mathsf{L})$ follows from the symmetry of $\boldsymbol{\Gamma}_0(\boldsymbol{n})$ and the formula

$$W_{\boldsymbol{n}}(\mathsf{L}) = \left[\mathsf{I} + (\mathsf{L} - \mathsf{L}_0)\boldsymbol{\Gamma}_0(\boldsymbol{n})\right]^{-1}(\mathsf{L} - \mathsf{L}_0) = \left[(\mathsf{L} - \mathsf{L}_0)^{-1} + \boldsymbol{\Gamma}_0(\boldsymbol{n})\right]^{-1}$$

that holds on a dense subset of $\mathrm{Sym}^+(\mathcal{T})$. □

In fact, we can say more about the transformation $W_{\boldsymbol{n}}$.

**Lemma A.2.** *For any $\boldsymbol{n} \in \mathbb{S}^{d-1}$ and $\mathsf{L}_0 \in \mathrm{Sym}^+(\mathcal{T})$, the map $W_{\boldsymbol{n}}$ is a diffeomorphism from $\mathrm{Sym}^+(\mathcal{T})$ onto its image.*

**Proof.** Let us first show that $W_{\boldsymbol{n}}$ is a local diffeomorphism in the vicinity of any $\mathsf{L} \in \mathrm{Sym}^+(\mathcal{T})$. This follows from the inverse function theorem and the explicit calculation of the differential of $W_{\boldsymbol{n}}(\mathsf{L})$. Using the first representation of $W_{\boldsymbol{n}}(\mathsf{L})$ from (2.5) we compute, for any $\xi \in \mathrm{Sym}(\mathcal{T})$,

$$dW_{\boldsymbol{n}}(\mathsf{L})\xi = \left[\mathsf{I} + (\mathsf{L} - \mathsf{L}_0)\boldsymbol{\Gamma}_0(\boldsymbol{n})\right]^{-1} \xi(\mathsf{I} - \boldsymbol{\Gamma}_0(\boldsymbol{n})W_{\boldsymbol{n}}(\mathsf{L})).$$

Then, using the second representation of $W_{\boldsymbol{n}}(\mathsf{L})$ from (2.5) we obtain

$$dW_{\boldsymbol{n}}(\mathsf{L})\xi = \left[\mathsf{I} + (\mathsf{L} - \mathsf{L}_0)\boldsymbol{\Gamma}_0(\boldsymbol{n})\right]^{-1} \xi \left[\mathsf{I} + \boldsymbol{\Gamma}_0(\boldsymbol{n})(\mathsf{L} - \mathsf{L}_0)\right]^{-1}.$$

Lemma A.1 then implies that $dW_{\boldsymbol{n}}(\mathsf{L})$ is an isomorphism on $\mathrm{Sym}(\mathcal{T})$ for any $\mathsf{L} \in \mathrm{Sym}^+(\mathcal{T})$. To prove the lemma it only remains to show that $W_{\boldsymbol{n}}$ is a bijection onto its image. Indeed, if $\{\mathsf{L}, \mathsf{L}'\} \subset \mathrm{Sym}^+(\mathcal{T})$ and $W_{\boldsymbol{n}}(\mathsf{L}) = W_{\boldsymbol{n}}(\mathsf{L}')$, then, using both representations of $W_{\boldsymbol{n}}$ in (2.5), we obtain

$$\left[\mathsf{I} + (\mathsf{L} - \mathsf{L}_0)\boldsymbol{\Gamma}_0(\boldsymbol{n})\right]^{-1} (\mathsf{L} - \mathsf{L}_0) = (\mathsf{L}' - \mathsf{L}_0) \left[\mathsf{I} + \boldsymbol{\Gamma}_0(\boldsymbol{n})(\mathsf{L}' - \mathsf{L}_0)\right]^{-1}.$$

Multiplying this equality on the right and on the left by $\mathsf{I} + \boldsymbol{\Gamma}_0(\boldsymbol{n})(\mathsf{L}' - \mathsf{L}_0)$ and $\mathsf{I} + (\mathsf{L} - \mathsf{L}_0)\boldsymbol{\Gamma}_0(\boldsymbol{n})$, respectively, we obtain $\mathsf{L} = \mathsf{L}'$. □

We next prove that the operator inverse in (2.6) always exists.

**Lemma A.3.** *Let $\mathcal{K}$ be a compact subset of $\mathrm{Sym}^+(\mathcal{T})$. Suppose $\mathsf{L}(\boldsymbol{x}) \in \mathcal{K}$ for every $\boldsymbol{x} \in [0, 1]^d$. Then the operator $\mathfrak{T} = \mathsf{I} - \Lambda_{\boldsymbol{n}} W_{\boldsymbol{n}}(\mathsf{L})$ is invertible on $L^2_{\mathrm{per}}([0, 1]^d; \mathcal{T})$. Here $W_{\boldsymbol{n}}(\mathsf{L})$ is understood as the multiplication operator $L^2_{\mathrm{per}}([0, 1]^d; \mathcal{T}) \ni \boldsymbol{u} \mapsto W_{\boldsymbol{n}}(\mathsf{L}(\boldsymbol{x}))\boldsymbol{u}(\boldsymbol{x})$.*

**Proof.** By our assumptions and Lemma A.1 the operator $\mathfrak{T}$ is bounded. It remains to prove that for every $\boldsymbol{f} \in L^2_{\mathrm{per}}([0, 1]^d; \mathcal{T})$ there exists unique $\boldsymbol{u} \in L^2_{\mathrm{per}}([0, 1]^d; \mathcal{T})$, such that $\mathfrak{T}\boldsymbol{u} = \boldsymbol{f}$. Then, by the Banach invertibility theorem this would imply that $\mathfrak{T}^{-1}$ is a bounded operator on $L^2_{\mathrm{per}}([0, 1]^d; \mathcal{T})$. Let $\Gamma_0 = \mathsf{L}_0^{-1}\Gamma'$, where $\Gamma'$ is the projection onto $\mathsf{L}_0\mathcal{E}_{\mathrm{per}}$ along $\mathcal{J}_{\mathrm{per}} \oplus \mathcal{U}$. Explicitly,

$$\widehat{\Gamma_0 \boldsymbol{h}}(\boldsymbol{k}) = \begin{cases} \boldsymbol{\Gamma}_0\left(\frac{\boldsymbol{k}}{|\boldsymbol{k}|}\right)\widehat{\boldsymbol{h}}(\boldsymbol{k}), & \boldsymbol{k} \in \mathbb{Z}^d \setminus \{0\} \\ 0, & \boldsymbol{k} = 0, \end{cases} \tag{A.8}$$

Observe that, according to (2.7), operators $\Lambda_{\boldsymbol{n}}$ can be expressed in terms of $\Gamma_0$:

$$\Lambda_{\boldsymbol{n}}\boldsymbol{h} = \boldsymbol{\Gamma}_0(\boldsymbol{n})\boldsymbol{h} - \boldsymbol{\Gamma}_0(\boldsymbol{n})\langle \boldsymbol{h} \rangle - \Gamma_0\boldsymbol{h}, \quad \forall \boldsymbol{h} \in L^2_{\mathrm{per}}([0, 1]^d; \mathcal{T}). \tag{A.9}$$

Using formulas (A.9) and (2.5) and for $\Lambda_{\boldsymbol{n}}$ and $W_{\boldsymbol{n}}$, respectively, we can rewrite equation $\mathfrak{T}\boldsymbol{u} = \boldsymbol{f}$ as

$$\Gamma_0(W_{\boldsymbol{n}}(\mathsf{L})\boldsymbol{u}) = \boldsymbol{f} - \boldsymbol{\Gamma}_0(\boldsymbol{n})\langle W_{\boldsymbol{n}}(\mathsf{L})\boldsymbol{u} \rangle - (\mathsf{I} + \boldsymbol{\Gamma}_0(\boldsymbol{n})(\mathsf{L} - \mathsf{L}_0))^{-1}\boldsymbol{u}. \tag{A.10}$$

If equation (A.10) is satisfied then the right-hand side of (A.10) must belong to $\mathcal{E}_{\text{per}}$. Denoting it by $\boldsymbol{e}$ and the uniform field $\boldsymbol{\Gamma}_0(\boldsymbol{n})\langle W_{\boldsymbol{n}}(\mathsf{L})\boldsymbol{u}\rangle$ by $\boldsymbol{e}_0$, we obtain the relation between $\boldsymbol{u}$ and $\boldsymbol{e}$

$$\boldsymbol{u} = (\mathsf{I} + \boldsymbol{\Gamma}_0(\boldsymbol{n})(\mathsf{L} - \mathsf{L}_0))(\boldsymbol{f} - \boldsymbol{e} - \boldsymbol{e}_0). \tag{A.11}$$

Next, multiplying equation (A.10) on the left by $\mathsf{L}_0$ we obtain

$$\Gamma'(W_{\boldsymbol{n}}(\mathsf{L})\boldsymbol{u}) = \mathsf{L}_0(\boldsymbol{f} - \boldsymbol{e}_0) - \mathsf{L}_0(\mathsf{I} + \boldsymbol{\Gamma}_0(\boldsymbol{n})(\mathsf{L} - \mathsf{L}_0))^{-1}\boldsymbol{u}$$

Thus,

$$W_{\boldsymbol{n}}(\mathsf{L})\boldsymbol{u} - \mathsf{L}_0(\boldsymbol{f} - \boldsymbol{e}_0) + \mathsf{L}_0(\mathsf{I} + \boldsymbol{\Gamma}_0(\boldsymbol{n})(\mathsf{L} - \mathsf{L}_0))^{-1}\boldsymbol{u} \in \mathcal{J}_{\text{per}} \oplus \mathcal{U}. \tag{A.12}$$

Equation (A.12), together with $\boldsymbol{e} \in \mathcal{E}_{\text{per}}$ is equivalent to (A.10).
Denoting the orthogonal projection onto $\mathcal{E}_{\text{per}}$ by $\Gamma_{\text{per}}$ and applying it to (A.12), we obtain an equivalent form of (A.12):

$$\Gamma_{\text{per}}(\mathsf{L}(\mathsf{I} + \boldsymbol{\Gamma}_0(\boldsymbol{n})(\mathsf{L} - \mathsf{L}_0))^{-1}\boldsymbol{u}) = \Gamma_{\text{per}}(\mathsf{L}_0\boldsymbol{f}). \tag{A.13}$$

Eliminating $\boldsymbol{u}$, using (A.11) we rewrite (A.13) as

$$\Gamma_{\text{per}}(\mathsf{L}(\boldsymbol{e} + \boldsymbol{e}_0)) = \Gamma_{\text{per}}((\mathsf{L} - \mathsf{L}_0)\boldsymbol{f}), \qquad \boldsymbol{e} \in \mathcal{E}_{\text{per}}. \tag{A.14}$$

Equation (A.14), together with definition (A.11) of $\boldsymbol{e}$, and

$$\boldsymbol{e}_0 = \boldsymbol{\Gamma}_0(\boldsymbol{n})\langle W_{\boldsymbol{n}}(\mathsf{L})\boldsymbol{u}\rangle, \tag{A.15}$$

is equivalent to (A.10).
We note that by analogy with the cell problem (A.5) the Lax–Milgram lemma guarantees that the operator $\mathcal{E}_{\text{per}} \ni \boldsymbol{e} \mapsto \Gamma_{\text{per}}(\mathsf{L}\boldsymbol{e}) \in \mathcal{E}_{\text{per}}$ is invertible on $\mathcal{E}_{\text{per}}$. Hence, the solution $\boldsymbol{e}$ of (A.14) can be represented as $\boldsymbol{e} + \boldsymbol{e}_0 = \tilde{\boldsymbol{e}} + \boldsymbol{E}$, where $\tilde{\boldsymbol{e}} \in \mathcal{E}_{\text{per}}$ is the unique solution of

$$\Gamma_{\text{per}}(\mathsf{L}\tilde{\boldsymbol{e}}) = \Gamma_{\text{per}}((\mathsf{L} - \mathsf{L}_0)\boldsymbol{f}), \tag{A.16}$$

while $\boldsymbol{E} \in \mathcal{E}_{\text{per}} \oplus \mathcal{U}$ is the unique solution of the periodic cell problem (A.5), which can be written as

$$\Gamma_{\text{per}}(\mathsf{L}\boldsymbol{E}) = 0, \qquad \langle \boldsymbol{E}\rangle = \boldsymbol{e}_0, \qquad \boldsymbol{E} \in \mathcal{E}_{\text{per}} \oplus \mathcal{U}. \tag{A.17}$$

It only remains to find an explicit formula for $\boldsymbol{e}_0 \in \mathcal{T}$ in terms of $\boldsymbol{f}$.
In order to determine $\boldsymbol{e}_0$ we use (A.11) to compute

$$W_{\boldsymbol{n}}(\mathsf{L})\boldsymbol{u} = (\mathsf{L} - \mathsf{L}_0)(\boldsymbol{f} - \boldsymbol{e} - \boldsymbol{e}_0). \tag{A.18}$$

Averaging (A.18) over the period cell, multiplying by $\boldsymbol{\Gamma}_0(\boldsymbol{n})$, and using (A.15) we obtain

$$\boldsymbol{e}_0 = \boldsymbol{\Gamma}_0(\boldsymbol{n})\langle(\mathsf{L} - \mathsf{L}_0)(\boldsymbol{f} - \boldsymbol{e} - \boldsymbol{e}_0)\rangle.$$

Replacing $\boldsymbol{e} + \boldsymbol{e}_0$ with $\tilde{\boldsymbol{e}} + \boldsymbol{E}$ we obtain

$$\boldsymbol{e}_0 = \boldsymbol{\Gamma}_0(\boldsymbol{n})\langle(\mathsf{L} - \mathsf{L}_0)(\boldsymbol{f} - \tilde{\boldsymbol{e}})\rangle - \boldsymbol{\Gamma}_0(\boldsymbol{n})(\mathsf{L}^* - \mathsf{L}_0)\boldsymbol{e}_0,$$

where we took into account that $\boldsymbol{E}$ solves (A.5). This gives

$$\boldsymbol{e}_0 = (\mathsf{I} + \boldsymbol{\Gamma}_0(\boldsymbol{n})(\mathsf{L}^* - \mathsf{L}_0))^{-1}\boldsymbol{\Gamma}_0(\boldsymbol{n})\langle(\mathsf{L} - \mathsf{L}_0)(\boldsymbol{f} - \tilde{\boldsymbol{e}})\rangle. \tag{A.19}$$

We have now established uniqueness, by showing that if $\boldsymbol{u}$ solves $\mathfrak{T}\boldsymbol{u} = \boldsymbol{f}$, then it has to be given by

$$\boldsymbol{u} = (\mathsf{I} + \boldsymbol{\Gamma}_0(\boldsymbol{n})(\mathsf{L} - \mathsf{L}_0))(\boldsymbol{f} - \tilde{\boldsymbol{e}} + \boldsymbol{E}), \tag{A.20}$$

where $\tilde{\boldsymbol{e}}$ is the unique solution of (A.16) and $\boldsymbol{E}$ is the unique solution of (A.17), where $\boldsymbol{e}_0$ is given by (A.19). Existence follows by tracing our calculations back and showing that if $\boldsymbol{u}(\boldsymbol{x})$ is defined by (A.20), then it must also satisfy (A.10). The lemma is proved now. □

Formula (2.6) is now easily obtained by choosing $\boldsymbol{f} = \boldsymbol{f}_0 \in \mathcal{U}$, in which case

$$\langle(\mathsf{L} - \mathsf{L}_0)(\boldsymbol{f}_0 - \tilde{\boldsymbol{e}})\rangle = (\mathsf{L}^* - \mathsf{L}_0)\boldsymbol{f}_0,$$

and hence (A.19) results in

$$\boldsymbol{e}_0 = \boldsymbol{f}_0 - (\mathsf{I} + \boldsymbol{\Gamma}_0(\boldsymbol{n})(\mathsf{L}^* - \mathsf{L}_0))^{-1}\boldsymbol{f}_0. \tag{A.21}$$

Now, taking the average of (A.18) and replacing $\boldsymbol{u}$ with $\mathfrak{T}^{-1}\boldsymbol{f}_0$, we obtain

$$\langle W_{\boldsymbol{n}}(\mathsf{L})\mathfrak{T}^{-1}\boldsymbol{f}_0\rangle = (\mathsf{L}^* - \mathsf{L}_0)(\boldsymbol{f}_0 - \boldsymbol{e}_0) = W_{\boldsymbol{n}}(\mathsf{L}^*)\boldsymbol{f}_0, \tag{A.22}$$

where we used (A.21) to eliminate $\boldsymbol{e}_0$. Thus, formula (2.6) is established. We comment that (2.6) represents a slight abuse of notation. What is meant by (2.6) is (A.22), where on the left-hand side $\boldsymbol{f}_0 \in \mathcal{U}$ is understood as a uniform field in $L^2_{\text{per}}([0, 1]^d; \mathcal{T})$, while on the right-hand side $\boldsymbol{f}_0$ is understood as a vector in $\mathcal{T}$.

## Appendix B. A Direct Proof of Rank-One Convexity of (1.4)

In this section we will present a direct proof of rank-one convexity of

$$W(\boldsymbol{F}) = \sqrt{\det_{\mathbb{H}}(\bar{\boldsymbol{F}}\boldsymbol{F}^T)} = V(\boldsymbol{F}^T), \qquad V(\boldsymbol{E}) = \sqrt{\det_{\mathbb{H}}(\boldsymbol{E}\boldsymbol{E}^H)}.$$

The idea of the proof is based on the large group of symmetries of $W(\boldsymbol{F})$. One symmetry subgroup was built-in by the construction:

$$W(\boldsymbol{F}\boldsymbol{R}) = W(\boldsymbol{F}), \qquad \forall \boldsymbol{R} \in SO(2).$$

The other is intimately linked with quaternionic algebra:

$$V(\boldsymbol{E}\boldsymbol{Q}) = V(\boldsymbol{E}), \qquad \forall \boldsymbol{Q} \in \mathbb{H}^{2\times 2},\ \boldsymbol{Q}\boldsymbol{Q}^H = \boldsymbol{I}_2. \tag{B.1}$$

Both symmetries leave invariant the set of matrices $\boldsymbol{E} = \boldsymbol{n} \otimes \boldsymbol{u}$, $\boldsymbol{n} \in \mathbb{R}^2$, $\boldsymbol{u} \in \mathbb{H}^2$, corresponding to rank-one $8 \times 2$ matrices $\boldsymbol{F}$. It follows that it is sufficient to prove that $\phi(\epsilon)$ is locally convex near $\epsilon = 0$ for every $\boldsymbol{E} \in \mathbb{H}^{2\times 2}$, where $\phi(\epsilon) = V(\boldsymbol{E} + \epsilon\boldsymbol{e}_1 \otimes \boldsymbol{e}_1)$. Here the left $\boldsymbol{e}_1$ in $\boldsymbol{e}_1 \otimes \boldsymbol{e}_1$ is $(1, 0) \in \mathbb{S}^2$, while the right $\boldsymbol{e}_1$ is $(1, 0) \in \mathbb{H}^2$. This observation follows from the homogeneity of $W(\boldsymbol{F})$ and the fact that any $\boldsymbol{u} \in \mathbb{H}^2$ can be mapped to $|\boldsymbol{u}|\boldsymbol{e}_1$ by a transformation $\boldsymbol{Q}$, satisfying $\boldsymbol{Q}\boldsymbol{Q}^H = \boldsymbol{I}_2$.[12] (As well as the fact that any unit vector $\boldsymbol{n} \in \mathbb{R}^2$ can be mapped into $\boldsymbol{e}_1 \in \mathbb{S}^2$ by $\boldsymbol{R} \in SO(2)$.) For points $\boldsymbol{E}$, for which $V(\boldsymbol{E}) \neq 0$ it is not difficult to calculate $\phi''(0)$:

$$V(\boldsymbol{E} + \epsilon \boldsymbol{e}_1 \otimes \boldsymbol{e}_1)^2 = (|E_{21}|^2 + |E_{22}|^2)(|E_{12}|^2 + |E_{11} + \epsilon|^2) - |E_{12}\overline{E_{22}} + E_{11}\overline{E_{21}} + \epsilon \overline{E_{21}}|^2$$

Hence,

$$\phi(\epsilon) = \sqrt{\det_{\mathbb{H}}(\boldsymbol{E}\boldsymbol{E}^H) + 2\epsilon \mathfrak{Re}(|E_{22}|^2 E_{11} - E_{12}\overline{E_{22}}E_{21}) + \epsilon^2 |E_{22}|^2}. \quad \text{(B.2)}$$

It is now easy to compute

$$\phi''(0) = V(\boldsymbol{E})^{-3} \left( \det_{\mathbb{H}}(\boldsymbol{E}\boldsymbol{E}^H)|E_{22}|^2 - \left[ |E_{22}|^2 \mathfrak{Re}(E_{11}) - \mathfrak{Re}(E_{12}\overline{E_{22}}E_{21}) \right]^2 \right),$$

provided $V(\boldsymbol{E}) \neq 0$. Expanding $\det_{\mathbb{H}}(\boldsymbol{E}\boldsymbol{E}^H)$ we conclude that the inequality $\phi''(0) \geqq 0$ is equivalent to the inequality $Q(\boldsymbol{E}) \geqq 0$, where

$$Q(\boldsymbol{E}) = |E_{22}|^2 (|E_{11}|^2 |E_{22}|^2 + |E_{12}|^2 |E_{21}|^2 - 2\mathfrak{Re}(E_{11}\overline{E_{21}}E_{22}\overline{E_{12}})) - \left[ \mathfrak{Re}((E_{11}E_{22} - E_{21}E_{12})\overline{E_{22}}) \right]^2 .$$

We observe that $Q(\boldsymbol{E})$ is quadratic with respect to $E_{11}$. Minimizing $Q(\boldsymbol{E})$ over $E_{11} \in \mathbb{H}$ we conclude that $Q(\boldsymbol{E})$ does not depend on the real part of $E_{11}$ and is minimized at

$$E_{11} = \frac{\mathfrak{Im}(E_{12}\overline{E_{22}}E_{21})}{|E_{22}|^2},$$

where in contrast to complex numbers we define $\mathfrak{Im}(q) = q - \mathfrak{Re}(q)$. It is a simple calculation to verify that $Q(\boldsymbol{E}) = 0$ at the minimizer. This conclusion holds, provided $E_{22} \neq 0$. If $E_{22} = 0$, then $Q(\boldsymbol{E}) = 0$. To finish the proof of rank-one convexity we need to examine the remaining case $V(\boldsymbol{E}) = 0$. In this case it is not hard to show that $\phi(\epsilon) = |\epsilon||E_{22}|$, which is convex. Indeed, in this case one of the equations in (1.5) must hold, which implies that $\mathfrak{Re}(|E_{22}|^2 E_{11} - E_{12}\overline{E_{22}}E_{21}) = 0$, but then formula (B.2) becomes $\phi(\epsilon) = |\epsilon||E_{22}|$. The rank-one convexity of $W(\boldsymbol{F})$ has been established.

[12] The set of transformations $\boldsymbol{Q}$, satisfying (B.1) is a compact 10-dimensional Lie group $\mathfrak{G}$. The $\mathfrak{G}$-orbit of $\boldsymbol{e}_1$ is a compact submanifold of the 7-dimensional unit sphere is $\mathbb{H}^2$. A simple explicit calculation of the stabilizer of $\boldsymbol{e}_1 \in \mathbb{H}^2$ in $\mathfrak{G}$ shows that it is a 3-dimensional Lie subgroup of $\mathfrak{G}$ (isomorphic to SU(2)). Thus, the orbit $\mathfrak{G}\boldsymbol{e}_1$ must be the entire unit sphere in $\mathbb{H}^2$.

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YURY GRABOVSKY
Temple University,
Philadelphia, PA, USA.
e-mail: yury@temple.edu

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