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The stable Galois correspondence for real closed fields

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Abstract. In previous work [**7**], the authors constructed and studied a lift of the Galois correspondence to stable homotopy categories. In particular, if L/k is a finite Galois extension of fields with Galois group G, there is a functor $c^*_{L/k} : \mathrm{SH}^G \to \mathrm{SH}_k$ from the G-equivariant stable homotopy category to the stable motivic homotopy category over k such that $c^*_{L/k}(G/H_+) = \mathrm{Spec}(L^H)_+$. The main theorem of [**7**] %citeHO:E2M says that when k is a real closed field and $L = k[i]$, the restriction of $c^*_{L/k}$ to the η-complete subcategory is full and faithful. Here we "uncomplete" this theorem so that it applies to $c^*_{L/k}$ itself. Our main tools are Bachmann's theorem on the $(2,\eta)$-periodic stable motivic homotopy category and an isomorphism range for the map $\pi^{\mathbb{R}}_\star \mathbb{S}_{\mathbb{R}} \to \pi^{C_2}_\star \mathbb{S}_{C_2}$ induced by C_2-equivariant Betti realization.

1. Introduction

In [**9**], Levine showed that the "constant" functor $c^* : \mathrm{SH} \to \mathrm{SH}_k$ from the classical stable homotopy category to the motivic stable homotopy category over an algebraically closed field of characteristic zero is a full and faithful embedding. Inspired by his result, in [**7**] we introduced and studied functors $c^*_{L/k} : \mathrm{SH}^G \to \mathrm{SH}_k$, where L/k is a Galois extension with Galois group G. We showed that if k is real closed and $L = k[i]$, then *after completing* at a prime p and at η, if $p \neq 2$, the functor $c^*_{L/k}$ is full and faithful. The need for the completion arose from our lack of knowledge about certain homotopy groups of the motivic sphere over $\mathbb{R}$. In the meantime advances have been made. Ananyevskiy-Levine-Panin [**1**] established a motivic version of Serre's finiteness theorem, which in particular implies that $c^*_{L/k}$ is full and faithful after η-completion. The purpose of this paper is to use Bachmann's recent results [**3**], about a localization of $\pi^{\mathbb{R}}_\star(\mathbb{S}_{\mathbb{R}})$, to remove the η-completions in the main theorem of [**7**].

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THEOREM 1.1. *Let k be a real closed field and $L = k[i]$ be its algebraic closure. Then the functor*

$$c^*_{L/k} : \mathrm{SH}^{C_2} \to \mathrm{SH}_k$$

is a full and faithful embedding.

One of the primary tools in the proof of Theorem 1.1 is the C_2-equivariant Betti realization functor $\mathrm{Re}^{C_2}_B : \mathrm{SH}_{\mathbb{R}} \to \mathrm{SH}^{C_2}$ which extends the functor taking a smooth $\mathbb{R}$-scheme to its $\mathbb{C}$-points with complex conjugation action. In [**6**], Dugger-Isaksen study the effect of this functor on the bigraded homotopy groups of the real motivic sphere spectrum, proving that $\mathrm{Re}^{C_2}_B : \pi^{\mathbb{R}}_{m+n\alpha}(\mathbb{S}_{\mathbb{R}})^\wedge_2 \to \pi^{C_2}_{m+n\sigma}(\mathbb{S}_{C_2})^\wedge_2$ is an isomorphism when $m \geq 2n-5$. We use odd-primary Adams spectral sequences and Bachmann's results to produce a range of bigradings in which the integral version of this map is an isomorphism. In particular, we prove the following.

THEOREM 1.2 (Theorem 3.12). *The map $\pi^{\mathbb{R}}_{m+n\alpha}(\mathbb{S}_{\mathbb{R}}) \to \pi^{C_2}_{m+n\sigma}(\mathbb{S}_{C_2})$ is*

(i) *an injection if $m = 2n-6$ and $n \geq 0$, and*
(ii) *an isomorphism if $m \geq 2n-5$ and $n \geq 0$.*

Outline. In Section 2 we recall Bachmann's theorem and deduce some consequences for the C_2-equivariant Betti realization functor and Morel's $\pm$-splitting of the 2-periodic stable motivic homotopy category. In Section 3 we adapt the methods of [**6**] to odd-primary Adams spectral sequences. Via an arithmetic fracture square and the results of Section 2, we deduce Theorem 3.12. Finally, in Section 4 we recall how to bootstrap Theorem 3.12 into a proof of Theorem 1.1.

Notation. We use the following notation throughout the paper.

- k is a field and L/k is a finite Galois extension of fields with Galois group G.
- SH_k is Voevodsky's stable motivic homotopy category [**13**]; hom sets in SH_k are denoted $[\ ,\ ]_k$.
- SH^G is the G-equivariant stable homotopy category in the sense of [**10**]; hom sets in SH^G are denoted $[\ ,\ ]_G$.
- $c^*_{L/k} : \mathrm{SH}^G \to \mathrm{SH}_k$ is the functor induced by the classical Galois correspondence $G/H \mapsto \mathrm{Spec}(L^H)$, constructed in [**7**, Section 4.3].[1]
- $\mathbb{S}_k$ is the sphere spectrum in SH_k and $\mathbb{S}_G$ is the sphere spectrum in SH^G.
- For integers m, n, $S^{m+n\alpha} = (S^1)^{\wedge m} \wedge (\mathbb{A}^1 \smallsetminus 0)^{\wedge n}$. If $G = C_2$, $S^{m+n\sigma} = (S^1)^{\wedge m} \wedge (S^\sigma)^{\wedge n}$ where S^σ is the one-point compactification of the real sign representation.
- For $X \in \mathrm{SH}_k$, $\pi^k_{m+n\alpha}X := [S^{m+n\alpha}, X]_k$ is the $(m+n\alpha)$-th homotopy group of X.
- For $X \in \mathrm{SH}^{C_2}$, $\pi^{C_2}_{m+n\sigma}X := [S^{m+n\sigma}, X]_{C_2}$ is the $(m+n\sigma)$-th homotopy group of X.
- For $X \in \mathrm{SH}_k$, $\pi^k_* X := \bigoplus_{m\in\mathbb{Z}} \pi_{m+0\alpha}X$ and similarly for $\pi^{C_2}_* Y$ when $Y \in \mathrm{SH}^G$.

[1] In *loc. cit.* we state that the category GsSet of G-simplicial sets is equivalent to the category $\mathrm{sPre}(\mathrm{Or}_G)$ of simplicial presheaves on the orbit category. This isn't quite true: $G\mathrm{sSet} \subseteq \mathrm{sPre}(\mathrm{Or}_G)$ is only a full subcategory. (Thanks to Tom Bachmann for drawing our attention to this inaccuracy.) This has no effect on any of the subsequent mathematics in *loc. cit.* because what is used is that the associated homotopy categories are equivalent and this is true by Elmendorf's Theorem.

- Given an embedding $\phi : k \hookrightarrow \mathbb{R}$, $\mathrm{Re}^{C_2}_{B,\phi} = \mathrm{Re}^{C_2}_B : \mathrm{SH}_k \to \mathrm{SH}^{C_2}$ is the C_2-equivariant Betti realization which extends the functor taking a smooth k-scheme X to $X(\mathbb{C})$ with the conjugation action [**7**, §4.4].
- Depending on context, η is either the motivic Hopf map arising from $\mathbb{A}^2 \smallsetminus 0 \to \mathbb{P}^1$ or the C_2-equivariant Hopf map.
- $(\ )^\wedge_2$ is the 2-completion functor and $(\ )^\wedge_\eta$ is the η-completion functor. If $a = 2$ or η we have $X^\wedge_a = \mathrm{holim}\, X/a^n$.
- We write $X[a^{-1}]$ or $X[1/a]$ for the homotopy colimit of $X \xrightarrow{a} X \xrightarrow{a} \cdots$. If $X \simeq X[a^{-1}]$ we say that X is *a-periodic*.

2. Preliminaries

The main new input we use in this paper is a recent theorem of Bachmann [**3**, Theorem 31]. His theorem compares a localization of $\mathrm{SH}_\mathbb{R}$ with the classical stable homotopy category. The most convenient form of his result for us is the following recasting.

THEOREM 2.1 (Bachmann). *Betti realization induces an equivalence of triangulated categories*

$$\mathrm{Re}^{C_2}_B : \mathrm{SH}_\mathbb{R}[1/2, \eta^{-1}] \xrightarrow{\simeq} \mathrm{SH}^{C_2}[1/2, \eta^{-1}].$$

PROOF. Consider the functor $\mathrm{Re}_\mathbb{R} : \mathrm{SH}_\mathbb{R} \to \mathrm{SH}$ which extends the functor $\mathrm{Sm}_\mathbb{R} \to \mathrm{Top}$, sending X to $X(\mathbb{R})$. Consider as well the composite $\Phi^{C_2} \circ \mathrm{Re}^{C_2}_B$, where $\Phi^{C_2} : \mathrm{SH}^{C_2} \to \mathrm{SH}$ is the geometric fixed points functor. Both $\mathrm{Re}_\mathbb{R}$ and $\Phi^{C_2} \circ \mathrm{Re}^{C_2}_B$ preserve homotopy colimits and if $X \in \mathrm{Sm}_\mathbb{R}$ and $n \in \mathbb{Z}$, they both send $\Sigma^n_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+$ to the spectrum $\Sigma^n\Sigma^\infty X(\mathbb{R})_+$. It follows that $\mathrm{Re}_\mathbb{R} = \Phi^{C_2} \circ \mathrm{Re}^{C_2}_B$ and so we have the commutative triangle of functors

$$\begin{array}{ccc} \mathrm{SH}_\mathbb{R}[1/2, \eta^{-1}] & \xrightarrow{\mathrm{Re}^{C_2}_B} & \mathrm{SH}^{C_2}[1/2, \eta^{-1}] \\ & \searrow^{\mathrm{Re}_\mathbb{R}} & \downarrow{\scriptstyle \Phi^{C_2}} \\ & & \mathrm{SH}[1/2]. \end{array}$$

Write $\rho : S^0 \to S^\sigma$ for the standard inclusion. Since $\eta^2\rho = -2\eta$, we have the equivalence of categories $\mathrm{SH}^{C_2}[1/2, \eta^{-1}] \simeq \mathrm{SH}^{C_2}[1/2, \rho^{-1}]$. Note that Φ^{C_2} induces an equivalence $\mathrm{SH}^{C_2}[\rho^{-1}] \to \mathrm{SH}$ by [**10**, Corollary 9.6] (for a modern proof see, *e.g.*, [**11**, Theorem 6.11]), and thus it follows that Φ^{C_2} is an equivalence $\mathrm{SH}^{C_2}[1/2, \eta^{-1}] \simeq \mathrm{SH}[1/2]$. A specialization of Bachmann's theorem [**3**, Theorem 31] says that $\mathrm{Re}_\mathbb{R}$ in the above diagram is an equivalence. We conclude that $\mathrm{Re}^{C_2}_B$ is an equivalence as well.

□

Recall Morel's $\pm$-operations in motivic homotopy theory (see [**4**, Section 16.2]). Let ε denote the stable map induced by the twist isomorphism $S^\alpha \wedge S^\alpha \simeq S^\alpha \wedge S^\alpha$. In the C_2-equivariant setting, let ε denote the twist $S^\sigma \wedge S^\sigma \simeq S^\sigma \wedge S^\sigma$, and note that $\mathrm{Re}^{C_2}_B(\varepsilon) = \varepsilon$. In either the motivic or equivariant setting, invert 2 and note that $e_+ := (\varepsilon - 1)/2$ and $e_- := (\varepsilon + 1)/2$ are orthogonal idempotents. Let the operation $(\ )^+$ denote inversion of e_+ and let $(\ )^-$ denote inversion of e_-, *i.e.*, $(\ )^\pm$ is the cofiber of the operation $e_\pm$. For any 2-periodic motivic or C_2-equivariant spectrum X there is a natural splitting $X \simeq X^+ \vee X^-$.

LEMMA 2.2. *If X is a 2-periodic motivic or C_2-equivariant spectrum, then $X^+ \simeq X^\wedge_\eta$ and $X^- \simeq X[\eta^{-1}]$.*

PROOF. We prove the motivic version of this statement, which easily adapts to the C_2-equivariant setting. Let X be a spectrum such that $X \simeq X[1/2]$. We have $\varepsilon = -1 - \rho\eta$ whence $e_- = \frac{-1}{2}\rho\eta$. Thus inverting e_- inverts both ρ and η. Since $(2+\rho\eta)\eta = 0$, this is the same as inverting 2 and η, whence $X^- \simeq X[\eta^{-1}]$. Now apply η-completion to the splitting $X \simeq X^+ \vee X^-$ to get $X^\wedge_\eta \simeq (X^+)^\wedge_\eta \vee (X^-)^\wedge_\eta$. Since $X^- \simeq X[\eta^{-1}]$, the second summand is trivial. Since $e_+\eta = 0$, X^+ is η-complete, *i.e.*, $(X^+)^\wedge_\eta \simeq X^+$. We conclude that $X^\wedge_\eta \simeq X^+$, as desired. □

As an interesting corollary (which we will not use in the remainder of this paper) we note the following.

PROPOSITION 2.3. *The natural map $\mathrm{Re}_B^{C_2}((\mathbb{S}_\mathbb{R})^\wedge_\eta) \to (\mathbb{S}_{C_2})^\wedge_\eta$ is an equivalence.*

PROOF. Let $\mathbb{S} = \mathbb{S}_\mathbb{R}$ and η-complete the 2-primary fracture square for $\mathbb{S}$ in order to produce the bicartesian square

$$\begin{array}{ccc} \mathbb{S}^\wedge_\eta & \longrightarrow & \mathbb{S}[1/2]^\wedge_\eta \\ \downarrow & & \downarrow \\ \mathbb{S}^\wedge_{2,\eta} & \longrightarrow & (\mathbb{S}^\wedge_2[1/2])^\wedge_\eta. \end{array}$$

Applying $\mathrm{Re}_B^{C_2}$ results in a homotopy pullback square which maps to the corresponding fracture square for $(\mathbb{S}_{C_2})^\wedge_\eta$. First consider the map $\mathrm{Re}_B^{C_2}(\mathbb{S}^\wedge_{2,\eta}) \to (\mathbb{S}_{C_2})^\wedge_{2,\eta}$ between lower left corners. We may check that this is an equivalence by computing its effect on Mackey functor homotopy groups, which in turn may be done by comparing the motivic and C_2-equivariant Adams spectral sequences as in [**7**, Proposition 2.11]. Note that we also conclude that $\mathrm{Re}_B^{C_2}(\mathbb{S}^\wedge_2) \to (\mathbb{S}_{C_2})^\wedge_2$ is an equivalence by [**8**, Theorem 1] and [**7**, Theorem 2.10]. The map between the upper right vertices are equivalences by Lemma 2.2. The equivalence between bottom right corners follows from Lemma 2.2 and that $\mathrm{Re}_B^{C_2}(\mathbb{S}^\wedge_2) \simeq (\mathbb{S}_{C_2})^\wedge_2$. Since the maps on these three corners are equivalences, we conclude that the map between homotopy pullbacks $\mathrm{Re}_B^{C_2}((\mathbb{S}_\mathbb{R})^\wedge_\eta \to (\mathbb{S}_{C_2})^\wedge_\eta$ is an equivalence. □

3. Comparing stable stems

In this section we establish a range of bidegrees in which the map on stable stems $\pi^\mathbb{R}_{m+n\alpha}(\mathbb{S}_\mathbb{R}) \to \pi^{C_2}_{m+n\sigma}(\mathbb{S}_{C_2})$ induced by equivariant Betti realization is an isomorphism.

Recall that Dugger-Isaksen establish a range in which 2-complete stems are isomorphic.

THEOREM 3.1 ([**6**, Theorem 4.1]). *The map $\pi^\mathbb{R}_{m+n\alpha}((\mathbb{S}_\mathbb{R})^\wedge_2) \to \pi^{C_2}_{m+n\alpha}((\mathbb{S}_{C_2})^\wedge_2)$ is an isomorphism if $m \geq 2n-5$ and an injection if $m = 2n-6$.*

REMARK 3.2. There are isomorphisms $\pi_{m+n\alpha}((\mathbb{S}_\mathbb{R})^\wedge_2) \cong \pi_{m+n\alpha}((\mathbb{S}_\mathbb{R})^\wedge_{2,\eta})$ for all m, n by [**8**, Theorem 1]. Similarly there are isomorphisms $\pi^{C_2}_{m+n\sigma}((\mathbb{S}_{C_2})^\wedge_2) \cong \pi^{C_2}_{m+n\sigma}((\mathbb{S}_{C_2})^\wedge_{2,\eta})$ for all m, n by [**7**, Theorem 2.10]. Dugger-Isaksen's result can thus equivalently be stated as a comparison between $(2,\eta)$-complete stable stems.

Recall the discussion of motivic and C_2-equivariant cobar complexes from [**6**, §3], noting that all of these constructions may be made with $H\mathbb{F}_p$, p odd, in place of $H\mathbb{F}_2$. The significance of the p-primary cobar complex is that it forms the E_1-page of the p-primary Adams spectral sequence (in the motivic or equivariant context). In order to concisely express the properties of these spectral sequences, let $\mathbb{S}$ be $\mathbb{S}_k$ (k a field) or $\mathbb{S}_{C_2}$, let $\mathbb{M}_p$ denote the homology of a point with coefficients in $\mathbb{F}_p$ or $\underline{\mathbb{F}}_p$, let $\mathcal{A}_p$ denote the p-primary motivic or C_2-equivariant dual Steenrod algebra, let C_p^* denote the p-primary motivic or C_2-equivariant cobar complex, let γ be α or σ, depending on context, and let $\mathrm{Ext}_{\mathcal{A}_p}^{*,*+*\gamma}(\mathbb{M}_p, \mathbb{M}_p)$ denote the homology of C_p^* (which is also Ext in the category of $\mathcal{A}_p$-comodules).

THEOREM 3.3 ([**8**, Theorem 1] and [**7**, Theorem 2.10]). *The (motivic or C_2-equivariant) p-primary Adams spectral sequence has E_1-page C_p^* and E_2-page $\mathrm{Ext}_{\mathcal{A}_p}^{*,*+*\gamma}(\mathbb{M}_p, \mathbb{M}_p)$; it is strongly convergent with*

$$E_2^{s,m+n\gamma} \implies \pi_{m-s+n\gamma}\mathbb{S}^\wedge_{p,\eta}.$$

When $p = 2$, we have $\mathbb{S}^\wedge_{2,\eta} \simeq \mathbb{S}^\wedge_2$ as long as $\mathrm{cd}_2(k[i]) < \infty$ (in the motivic case).

The Dugger-Isaksen result is obtained by comparing cobar complexes. We first extend this method to odd p to obtain an isomorphism range on (p, η)-complete stems. We begin by recalling some facts about the motivic and equivariant Steenrod algebras at odd primes. Write $\mathcal{A}_{\mathbb{F}_p}$ for the classical mod-p dual Steenrod algebra. Recall that

$$\mathcal{A}_{\mathbb{F}_p} = \mathrm{Sym}_{\mathbb{F}_p}(\tau_0, \tau_1, \ldots, \xi_1, \xi_2, \ldots)$$

is a free graded commutative algebra, where $|\tau_i| = p^i$ and $|\xi_i| = p^i - 1$.

Recall that $\mathbb{M}_p^{\mathbb{R}} := \bigoplus_{m,n\in\mathbb{Z}} \pi^{\mathbb{R}}_{m+n\alpha} H\mathbb{F}_p = \mathbb{F}_p[\theta]$ where θ has degree $2-2\alpha$. This follows from the affirmative resolution of the Bloch-Kato conjecture [**15**, Theorem 6.1] together with [**12**, Theorem 7.4]. In the equivariant case we have $\mathbb{M}_p^{C_2} := \bigoplus_{m,n\in\mathbb{Z}} \pi^{C_2}_{m+n\sigma} H\underline{\mathbb{F}}_p = \mathbb{F}_p[\theta, \theta^{-1}]$, see *e.g.* [**5**, Theorem 2.8]. The dual motivic Steenrod algebra over $\mathbb{R}$ is equal to

$$\mathcal{A}_{\mathbb{R}} \cong \mathbb{M}_p^{\mathbb{R}} \otimes_{\mathbb{F}_p} \mathcal{A}_{\mathbb{F}_p} = \mathbb{M}_p^{\mathbb{R}}[\tau_0, \tau_1, \ldots, \xi_1, \xi_2, \cdots]/(\tau_0^2, \tau_1^2, \ldots),$$

where the elements of $\mathcal{A}_{\mathbb{F}_p}$ are considered to be bigraded by assigning weights so that $|\tau_i| = p^i + (p^i - 1)\alpha$ and $|\xi_i| = (p^i - 1) + (p^i - 1)\alpha$, see [**14**, Remark 12.12].

Similarly, the dual C_2-equivariant Steenrod algebra is equal to

$$\mathcal{A}_{C_2} \cong \mathbb{M}_p^{C_2} \otimes_{\mathbb{F}_p} \mathcal{A}_{\mathbb{F}_p} = \mathbb{M}_p^{C_2}[\tau_0, \tau_1, \ldots, \xi_1, \xi_2, \cdots]/(\tau_0^2, \tau_1^2, \ldots),$$

where in this case elements of $\mathcal{A}_{\mathbb{F}_p}$ are considered to be bigraded by assigning weights so that $|\tau_i| = p^i + (p^i - 1)\sigma$ and $|\xi_i| = (p^i - 1) + (p^i - 1)\sigma$.

Equivariant Betti realization $\mathrm{Re}_B^{C_2}$ induces maps $\mathbb{M}_p^{\mathbb{R}} \to \mathbb{M}_p^{C_2}$ and $\mathcal{A}_{\mathbb{R}} \to \mathcal{A}_{C_2}$ which have the obvious effects on the above named elements, *i.e.*, $\theta \mapsto \theta$, $\tau_i \mapsto \tau_i$ and $\xi_i \mapsto \xi_i$.

Write $\mathbb{M}_p$ for either $\mathbb{M}_p^{\mathbb{R}}$ or $\mathbb{M}_p^{C_2}$. In both cases, the dual Steenrod algebra is free over $\mathbb{M}_p$, and a basis is given by monomials $\tau_0^{\epsilon_0}\tau_1^{\epsilon_1}\cdots\xi_1^{n_1}\xi_2^{n_2}\cdots$, where $\epsilon_i \in \{0, 1\}$ and n_i is a nonnegative integer. We write $\tau^\epsilon\xi^n$ for such a monomial.

LEMMA 3.4. *Suppose that $|\tau^\epsilon\xi^n| = k + \ell\alpha$. Then $k \leq \dfrac{p}{p-1}\ell + 1$.*

PROOF. The bidegree of ξ_i satisfies $k < \frac{p}{p-1}\ell$ and the bidegree of τ_i satisfies $k \leq \frac{p}{p-1}\ell$ provided $i \geq 1$. It follows that if $\epsilon_0 = 0$ then the bidegree of $\tau^\epsilon \xi^n$ satisfies $k \leq \frac{p}{p-1}\ell$. If $\epsilon_0 = 1$ then write $\tau^\epsilon \xi^n = \tau_0 \tau^{\epsilon'} \xi^n$, where $\epsilon'_0 = 0$. This element thus satisfies the inequality $k \leq \frac{p}{p-1}\ell + 1$. □

LEMMA 3.5. *The map $C^f_{\mathbb{R}} \to C^f_{C_2}$ is*

(i) *an injection in all degrees, and*

(ii) *an isomorphism if $k - f \geq \left\lfloor \frac{p}{p-1}\ell - \frac{4p-2}{p-1} \right\rfloor + 1$.*

PROOF. We have that $C^f_{C_2} \cong \mathbb{M}_p^{C_2} \otimes_{M_p^{\mathbb{R}}} C^f_{\mathbb{R}}$, that $C^f_{\mathbb{R}}$ is free over $\mathbb{M}_p^{\mathbb{R}}$, and that the map $\mathbb{M}_{\mathbb{R}}^p \to \mathbb{M}_{C_2}^p$ is injective. It follows that $C^f_{\mathbb{R}} \to C^f_{C_2}$ is injective.

Let $Z = [z_1|z_2|\cdots|z_f]$ be a cobar element of Adams filtration f, where each z_i is of the form $\tau^\epsilon \xi^n$, of bidegree $k_i + \ell_i\alpha$. By Lemma 3.4 we have that $k_i \leq \frac{p}{p-1}\ell_i + 1$. Summing over i, we find that if $|Z| = k + \ell\alpha$, then $k \leq \frac{p}{p-1}\ell + f$. The cokernel of $C^f_{\mathbb{R}} \to C^f_{C_2}$ is generated by elements of the form $\theta^{-n}Z$ for $n \geq 1$. The elements θ^{-n} lie above the line of slope[2] $(p-1)/p$ passing through θ^{-1} and so we find that these satisfy the inequality $k \leq \frac{p}{p-1}\ell - \frac{4p-2}{p-1}$. It follows that the bidegree of $\theta^{-n}Z$ satisfies the inequality $k \leq \frac{p}{p-1}\ell + f - \frac{4p-2}{p-1}$. Thus the cokernel is zero in bidegrees satisfying $k > \frac{p}{p-1}\ell + f - \frac{4p-2}{p-1}$.

□

Recall the form of the motivic and C_2-equivariant Adams spectral sequences from Theorem 3.3. Equivariant Betti realization induces a map between these spectral sequences which we can now analyze.

PROPOSITION 3.6. *The map $\mathrm{Ext}_{\mathcal{A}_{\mathbb{R}}}^{(f,k+\ell\alpha)}(\mathbb{M}_p^{\mathbb{R}}, \mathbb{M}_p^{\mathbb{R}}) \to \mathrm{Ext}_{\mathcal{A}_{C_2}}^{(f,k+\ell\alpha)}(\mathbb{M}_p^{C_2}, \mathbb{M}_p^{C_2})$ is an injection if $k-f = \left\lfloor \frac{p}{p-1}\ell - \frac{4p-2}{p-1} \right\rfloor$ and an isomorphism if $k-f \geq \left\lfloor \frac{p}{p-1}\ell - \frac{4p-2}{p-1} \right\rfloor + 1$.*

PROOF. This follows from [**6**, Lemma 3.4] and Lemma 3.5. □

LEMMA 3.7. *$\mathrm{Ext}_{\mathcal{A}_{\mathbb{R}}}^{(f,k+\ell\alpha)}(\mathbb{M}_p^{\mathbb{R}}, \mathbb{M}_p^{\mathbb{R}})$ is a finite-dimensional $\mathbb{F}_p$-vector space for all f, k, ℓ.*

PROOF. Writing $C_{\mathbb{F}_p}$ for the classical cobar complex, we have $C_{\mathbb{R}} \cong \mathbb{M}_p^{\mathbb{R}} \otimes_{\mathbb{F}_p} C_{\mathbb{F}_p}$. This makes it clear that $C_{\mathbb{R}}^{(f,k+\ell\alpha)}$ is finite dimensional, and thus $\mathrm{Ext}_{\mathcal{A}_{\mathbb{R}}}^{(f,k+\ell\alpha)} \cong H^{(f,k+\ell\alpha)}(C_{\mathbb{R}})$ is finite dimensional as well. □

REMARK 3.8. The universal coefficient theorem further implies that

$$\mathrm{Ext}_{\mathcal{A}_{\mathbb{R}}}^{*,*+*\alpha}(\mathbb{M}_p^{\mathbb{R}}, \mathbb{M}_p^{\mathbb{R}}) \cong \mathrm{Ext}_{\mathcal{A}_{\mathbb{F}_p}}^{*,*}(\mathbb{F}_p, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathbb{M}_p^{\mathbb{R}}$$

up to a grading shift.

THEOREM 3.9. *The map $\pi^{\mathbb{R}}_{m+n\alpha}((\mathbb{S}_{\mathbb{R}})^\wedge_{(p,\eta)}) \to \pi^{C_2}_{m+n\sigma}((\mathbb{S}_{C_2})^\wedge_{(p,\eta)})$ is*

(i) *an injection if $m = \left\lfloor \frac{p}{p-1}n - \frac{4p-2}{p-1} \right\rfloor$, and*

(ii) *an isomorphism if $m \geq \left\lfloor \frac{p}{p-1}n - \frac{4p-2}{p-1} \right\rfloor + 1$.*

[2]We use the standard convention in which the k-axis is horizontal and the $\ell\alpha$-axis is vertical.

PROOF. This follows from Proposition 3.6 using the same argument as in [**6**, Theorem 4.1]. □

REMARK 3.10. In [**6**], Dugger-Isaksen establish a second isomorphism range on 2-complete stems. Namely, $\pi_{m+n\alpha}(\mathbb{S}_{\mathbb{R}})^\wedge_2 \to \pi_{m+n\sigma}(\mathbb{S}_{C_2})^\wedge_2$ is an isomorphism if $m+n \leq -1$. A version of this range appears to hold on (p,η)-complete stems, with the difference that the map might be only surjective when $m+n=-1$. However, this second isomorphism range doesn't extend to integral stems and we do not pursue it further here.

We now turn our attention to p-complete spheres.

PROPOSITION 3.11. *The map* $\pi^{\mathbb{R}}_{m+n\alpha}((\mathbb{S}_{\mathbb{R}})^\wedge_p) \to \pi^{C_2}_{m+n\sigma}((\mathbb{S}_{C_2})^\wedge_p)$ *is*

(1) *an injection if* $m = \left\lfloor \frac{p}{p-1}n - \frac{4p-2}{p-1} \right\rfloor$, *and*
(2) *an isomorphism if* $m \geq \left\lfloor \frac{p}{p-1}n - \frac{4p-2}{p-1} \right\rfloor + 1$.

PROOF. Write $\mathbb{S}$ for either of $\mathbb{S}_{\mathbb{R}}$ or $\mathbb{S}_{C_2}$. If $p=2$, the statement of the proposition is Dugger-Isaksen's Theorem 3.1, so we can assume p is odd. In this case 2 is invertible in $\mathbb{S}/p^r$ and in $\mathbb{S}^\wedge_p$. By Lemma 2.2 we have that $(\mathbb{S}^\wedge_p)^+ \simeq \mathbb{S}^\wedge_{(p,\eta)}$ and $(\mathbb{S}^\wedge_p)^- \simeq \mathbb{S}^\wedge_p[\eta^{-1}]$ and so we have that

$$\mathbb{S}^\wedge_p \simeq \mathbb{S}^\wedge_{(p,\eta)} \vee \mathbb{S}^\wedge_p[\eta^{-1}].$$

Note that we have an isomorphism $(\mathbb{S}^\wedge_p)^- \cong \lim_r((\mathbb{S}/p^r)^-)$. It follows from Theorem 2.1 that the map $\pi^{\mathbb{R}}_{*+*\alpha}((\mathbb{S}_{\mathbb{R}})^\wedge_p[\eta^{-1}]) \to \pi^{C_2}_{*+*\sigma}((\mathbb{S}_{C_2})^\wedge_p[\eta^{-1}])$ is an isomorphism. The result thus follows from Theorem 3.9 and the direct sum decomposition of $\mathbb{S}^\wedge_p$ above. □

THEOREM 3.12. *The map* $\pi^{\mathbb{R}}_{m+n\alpha}(\mathbb{S}_{\mathbb{R}}) \to \pi^{C_2}_{m+n\sigma}(\mathbb{S}_{C_2})$ *is*

(i) *an injection if* $m = 2n-6$ *and* $n \geq 0$, *and*
(ii) *an isomorphism if* $m \geq 2n-5$ *and* $n \geq 0$.

PROOF. Consider the comparison of long exact sequences of homotopy groups induced by cofiber sequences

$$\mathbb{S} \to \Big(\prod_p \mathbb{S}^\wedge_p\Big) \vee \mathbb{S}_{\mathbb{Q}} \to \Big(\prod_p \mathbb{S}^\wedge_p\Big)_{\mathbb{Q}}$$

obtained from the arithmetic fracture squares for $\mathbb{S}_{\mathbb{R}}$ and $\mathbb{S}_{C_2}$. It suffices to show that the comparison map $\pi^{\mathbb{R}}_{m+n\alpha}(-) \to \pi^{C_2}_{m+n\sigma}(-)$ at the middle and the righthand terms are injections if $m = 2n-6$ and $n \geq 0$ and an isomorphism if $m \geq 2n-5$ and $n \geq 0$.

The reader may check that when $n > 0$ the inequalities for the injectivity and isomorphism ranges of p-complete stable stems from Proposition 3.11 for odd p are dominated by Dugger-Isaksen's inequalities in Theorem 3.1, for the 2-complete stable stems. It follows that $\pi^{\mathbb{R}}_{m+n\alpha}(\prod_p(\mathbb{S}_{\mathbb{R}})^\wedge_p) \to \pi^{C_2}_{m+n\alpha}(\prod_p(\mathbb{S}_{C_2})^\wedge_p)$ is an injection if $m = 2n-6$, $n > 0$, and an isomorphism if $m \geq 2n-5$, $n > 0$. In fact, these ranges extend to $n = 0$ as well. Indeed, injectivity on $\pi^{\mathbb{R}}_{-6}(\prod_p(\mathbb{S}_{\mathbb{R}})^\wedge_p)$ is trivial since the group in question is 0 by connectivity of the motivic sphere spectrum. Similarly, $\pi^{\mathbb{R}}_{-5}(\prod_p(\mathbb{S}_{\mathbb{R}})^\wedge_p) = 0$, and $\pi^{C_2}_{-5}(\prod_p(\mathbb{S}_{C_2})^\wedge_p) = 0$ as well by [**2**, Proposition 7.0]. This is the only value of m, n for which the 2-complete isomorphism range does not dominate, so the isomorphism range extends.

Since we have that the map $\pi^{\mathbb{R}}_{m+n\alpha}(\prod_p(\mathbb{S}_{\mathbb{R}})^\wedge_p)_{\mathbb{Q}} \to \pi^{C_2}_{m+n\alpha}(\prod_p(\mathbb{S}_{C_2})^\wedge_p)_{\mathbb{Q}}$ is a filtered colimit of these maps, it too is an injection if $m = 2n-6$, $n \geq 0$ and an isomorphism if $m \geq 2n-5$, $n \geq 0$.

By Theorem 2.1 and Lemma 2.2, $\pi^{\mathbb{R}}_{*+*\alpha}(\mathbb{S}_{\mathbb{R}})^-_{\mathbb{Q}} \to \pi^{C_2}_{*+*\alpha}(\mathbb{S}_{C_2})^-_{\mathbb{Q}}$ is an isomorphism. By [**4**, Theorems 11 and 16.2.13], $(\mathbb{S}_{\mathbb{R}})^+_{\mathbb{Q}} \simeq \mathrm{H}\mathbb{Q}$, where $\mathrm{H}\mathbb{Q}$ is the rationalized motivic cohomology spectrum. Thus, we have that $\pi^{\mathbb{R}}_{i+j\alpha}(\mathbb{S}_{\mathbb{R}})^+_{\mathbb{Q}} = 0$ whenever $j > 0$. When $j = 0$, we have that $\pi^{\mathbb{R}}_i(\mathbb{S}_{\mathbb{R}})^+_{\mathbb{Q}} = 0$ if $i \neq 0$ and $\pi^{\mathbb{R}}_0(\mathbb{S}_{\mathbb{R}})^+_{\mathbb{Q}} = \mathbb{Q}$. We have

$$\pi^{C_2}_{i+j\alpha}(\mathbb{S}_{C_2})^+_{\mathbb{Q}} = \begin{cases} \mathbb{Q} & j \text{ is even and } i+j=0 \\ 0 & \text{else.} \end{cases}$$

The region $j \geq 0$, $i \geq 2j-5$ is entirely contained in this vanishing region. The map $\pi^{\mathbb{R}}_0(\mathbb{S}_{\mathbb{R}})^+_{\mathbb{Q}} \to \pi^{C_2}_0(\mathbb{S}_{C_2})^+_{\mathbb{Q}}$ is an ismorphism. We conclude that $\pi^{\mathbb{R}}_{m+n\alpha}(\mathbb{S}_{\mathbb{R}})^+_{\mathbb{Q}} \to \pi^{C_2}_{m+n\sigma}(\mathbb{S}_{C_2})^+_{\mathbb{Q}}$ is an injection if $m = 2n-6$ and $n \geq 0$, and an isomorphism if $m \geq 2n-5$ and $n \geq 0$. □

4. Proof of Theorem 1.1

We finish by explaining how the comparison of stable stems in the previous section implies the embedding theorem.

Proposition 4.1. *If*

(i) $\mathrm{Re}^{C_2}_B : [S^n, \mathbb{S}_{\mathbb{R}}]_{\mathbb{R}} \xrightarrow{\cong} [S^n, \mathbb{S}_{C_2}]_{C_2}$, *and*
(ii) $\mathrm{Re}^{C_2}_B : [\mathrm{Spec}(\mathbb{C})_+ \wedge S^n, \mathbb{S}_{\mathbb{R}}]_{\mathbb{R}} \xrightarrow{\cong} [C_{2+} \wedge S^n, \mathbb{S}_{C_2}]_{C_2}$

are isomorphisms for all $n \in \mathbb{Z}$, then Theorem 1.1 *is true for any real closed field k.*

Proof. Let k be a real closed field and $L = k[i]$. To prove Theorem 1.1, it suffices to prove that

(a) $c^*_{L/k} : [S^n, \mathbb{S}_{C_2}]_{C_2} \xrightarrow{\cong} [S^n, \mathbb{S}_k]_k$, and
(b) $c^*_{L/k} : [C_{2+} \wedge S^n, \mathbb{S}_{C_2}]_{C_2} \xrightarrow{\cong} [\mathrm{Spec}(L)_+ \wedge S^n, \mathbb{S}_k]_k$

are isomorphisms for all $n \in \mathbb{Z}$, by the same argument as in the beginning of the proof of [**7**, Theorem 2.21].

To prove that the maps in (a), (b) are isomorphisms, we can assume that $k = \mathbb{R}$ and $L = \mathbb{C}$, by the same argument as in [**7**, Proposition 2.20]. We now consider the C_2-equivariant Betti realization functor $\mathrm{Re}^{C_2}_B : \mathrm{SH}_{\mathbb{R}} \to \mathrm{SH}^{C_2}$. Since $\mathrm{Re}^{C_2}_B \circ c^*_{\mathbb{C}/\mathbb{R}} = \mathrm{id}$, it follows that (a) and (b) are isomorphisms. □

Corollary 4.2 (Theorem 1.1). *Let k be a real closed field and $L = k[i]$ be its algebraic closure. Then the functor*

$$c^*_{L/k} : \mathrm{SH}^{C_2} \to \mathrm{SH}_k$$

is a full and faithful embedding.

Proof. If $i < 0$ then $\pi^{\mathbb{R}}_i(\mathbb{S}_{\mathbb{R}}) = \pi^{C_2}_i(\mathbb{S}_{C_2}) = 0$ and so the map in 4.1(i) is an isomorphism for $i < 0$. It is an isomorphism for $i \geq 0$ by setting $n = 0$ in Theorem 3.12. The map in 4.1(ii) is identical to the map $[S^n, \mathbb{S}_{\mathbb{C}}]_{\mathbb{C}} \to [S^n, \mathbb{S}]$ induced by complex Betti realization. This is an isomorphism by Levine's theorem [**9**]. □

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