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Graph-Theoretic Analysis of Estimators for Stochastically-Driven Diffusive Network Processes

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***Abstract*—Monitoring of a linear diffusive network dynamics that is subject to a stationary stochastic input is considered, from a graph-theoretic perspective. Specifically, the performance of minimum mean square error (MMSE) estimators of the stochastic input and network state, based on remote noisy measurements, is studied. Using a graph-theoretic characterization of frequency responses in the diffusive network model, we show that the performance of an off-line (noncausal) estimator exhibits an exact topological pattern, which is related to vertex cuts and paths in the network's graph. For on-line (causal) estimation, graph-theoretic results are obtained for the case where the measurement noise is small.**

I. Introduction

Several applications require estimation of network dynamics and models from sparse ambient measurements [1], [3], [4]. A number of model-based and model-free algorithms for estimation from ambient data have been developed in recent years, which broadly draw on traditional linear-systems concepts along with specialized understanding of the application domains. Some studies have also aimed to develop bounds or confidence intervals on the estimated quantities [5]. While these efforts are a promising start, several challenges remain in estimating network processes from ambient data. First, the scale, complexity, and uncertainty of the processes often dictate that simple graphical rubrics rather than precise algorithms are needed for estimation and performance evaluation [6]. Additionally, methods for resource-constrained sensor placement need to be developed [7], [8].

The purpose of this study is to develop graph-theoretic insights into the estimation of network processes from ambient data, with the view of informing performance analysis and sensor selection. Specifically, a network synchronization or diffusion process that is subject to a persistent stochastic stimulation is considered, and both input and state estimation from noisy local measurements is pursued. The performance of minimum-mean-square-error filters and smoothers are compared, for different sensor locations. The estimator performance is shown to exhibit an exact topological pattern in the case of off-line estimation, which relates to cuts or paths in the graph. Meanwhile, in the case of on-line estimation, the network's topology determines whether or not perfect estimation is possible in the low-noise limit. Implications in budget-constrained sensor placement are briefly discussed.

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This study connects with a recent literature on optimal budget-constrained sensor and actuator placement in networks [8]–[11]. As a whole, these studies have demonstrated that optimal sensor placement in networks is a computationally difficult problem, but in some cases performance-bounded suboptimal solutions can be found using simple algorithms (e.g., greedy algorithms). Beyond the computational difficulty, sensor design is also complicated by the lack of an explicit solution to the Riccati equation which gives the performance (achieved error covariance) of a sensing scheme. Relative to this literature, our work puts forth an optimistic perspective, that specially-structured network processes may sometimes admit structural insights into estimator performance which can support good (albeit perhaps suboptimal) sensor placement.

Our work also contributes to a research effort on observability, estimation/detection, and input-output behaviors of network processes [7], [12]–[16]. It is particularly aligned with graph-theoretic studies which consider metrics for estimation [11], [14]. However, these earlier studies have largely considered performance only in the presence of observation noise, in which case the Riccati equation admits an explicit solution. In contrast, here the estimation of dynamics driven by ambient process noise is considered. For this case, comparisons of transfer functions for different input-output channels are used to identify topological patterns in estimation performance. This study also connects to notions of isotropy in networks [17], and to frequency-domain analysis of string-stability [18].

The article is organized as follows. The estimation problem and estimator performance analysis are formulated in Section II. Graph-theoretic analyses of network transfer functions, which enable characterization of estimator performance, are developed in Section III. The main results characterizing estimator performance are presented in Section IV.

II. Problem Formulation

A linear diffusive network dynamics that is subject to a stationary stochastic input signal (process noise) at a single network node is considered. Formally, a network with n nodes, labeled $1,\dots,n$, is considered. Each node i has a scalar state $x_i(t)$, which evolves in continuous time. The network's dynamics are defined by a weighted digraph Γ with n vertices labeled $1,\dots,n$, which correspond to the nodes. The weight of the directed edge from a vertex i to a vertex j is labeled a_{ij}, and is assumed positive. If the graph does not have an edge from i to j, then the weight is considered to be $a_{ij}=0$. The network graph Γ is assumed to be strongly connected, i.e. there is a directed path between any two vertices.

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The network states evolve according to:

$$\dot{\mathbf{x}} = -L(\Gamma)\mathbf{x} + \mathbf{e}_s w(t). \quad (1)$$

In Equation (1), $L(\Gamma)$ is an (asymmetric) Laplacian or grounded Laplacian matrix associated with the directed graph Γ. That is, $L_{ij} = -a_{ij}$ for $i \neq j$, and the diagonal entries $L_{ii} \geq -\sum_{j\neq i} L_{ij}$. Also, $\mathbf{e}_s$ is 0–1 indicator vector with entry s equal to 1. The input $w(t)$ is assumed to be a stationary zero-mean stochastic input with autocorrelation $R_w(t)$, which has finite power (i.e., the integral of the power spectrum is finite). We refer to s as the source or input node, and the corresponding graph vertex as the source or input vertex.

Noisy measurements of the network dynamics at a single node q are considered. The measurement signal is:

$$y(t) = \mathbf{e}_q^T \mathbf{x} + v(t), \quad (2)$$

where $v(t)$ is a zero-mean stationary stochastic signal with autocorrelation $R_v(t)$.

The problem of interest is to recover the input $w(t)$ and the state process $\mathbf{x}(t)$ from the measurement signal $y(t)$. Minimum mean-square error (MMSE) estimation is considered, and both off-line and on-line estimation tasks are pursued. Our primary focus is on characterizing the performance of the estimators. The following four problems are studied:

1) Off-line input estimation. The goal is to find the MMSE estimate $\widehat{w}_{off}(t)$ of the input $w(t)$, using the measurement signal $y(t)$ over the infinite horizon. The estimator's performance is denoted as $P_{w-off} = E[(w(t) - \widehat{w}_{off}(t))^2]$.

2) On-line input estimation. The goal is to find the MMSE estimate $\widehat{w}_{on}(t)$ of the input $w(t)$ at each time t, using the measurement prior to time t ($y(\tau)$ for $\tau \leq t$). The estimator's performance is denoted as $P_{w-on} = E[(w(t) - \widehat{w}_{on}(t))^2]$.

3) Off-line state estimation. The goal is to find the MMSE estimate $\widehat{z}_{off}(t)$ of a state projection $z(t) = \mathbf{c}^T\mathbf{x}(t)$, using the measurement $y(t)$ over the infinite horizon. The estimator's performance is denoted as $P_{z-off} = E[(z(t) - \widehat{z}_{off}(t))^2]$.

4) On-line state estimation. The goal is to find the MMSE estimate $\widehat{z}_{on}(t)$ of a state projection $z(t) = \mathbf{c}^T\mathbf{x}(t)$ at each time t, using prior measurements ($y(\tau)$ for $\tau \leq t$). The estimator performance is denoted as $P_{z-on} = E[(z(t) - \widehat{z}_{on}(t))^2]$.

The four estimation problems are classical filtering and smoothing problems for linear systems driven by wide-sense-stationary inputs, which can be solved using the Wiener filtering theory [19]. Our aim here is to characterize the estimators' performance in terms of the graph Γ, and the positions of the input and measurement relative to the graph. Many of the analyses compare the performance for different measurement locations (nodes). For such comparisons, we often delineate the measurement location in the performance measure, e.g. as $P_{z-on}(q^*)$ when the measurement location is $q = q^*$. One result concerned with measurement at multiple nodes is also presented, see Section IV.

III. Graph-Theoretic Analyses of Network Input-Output Dynamics

We are concerned with MMSE estimators for inputs/states of a linear system with a stationary drive. The Wiener filtering theory relates the estimator performance to the frequency response of the driven system [19]. Thus, to develop graph-theoretic results on MMSE estimator performance, we first develop graph-theoretic characterizations of the transfer function and frequency response from the driving input to the measurement in the diffusive network model.

To develop the graph-theoretic analyses, we consider the diffusive network dynamics with a general input $u(t)$ at the node s, and a measurement $y(t)$ at the node q:

$$\begin{aligned} \dot{\mathbf{x}} &= -L(\Gamma)\mathbf{x} + \mathbf{e}_s u(t) \\ y(t) &= \mathbf{e}_q^T \mathbf{x}, \end{aligned} \quad (3)$$

The notation $H_q(s)$ (respectively $H_q(j\omega)$) is used for the transfer function (respectively frequency response) of this system (i.e. from u to y), where we have explicitly indicated the dependence on q to permit comparisons.

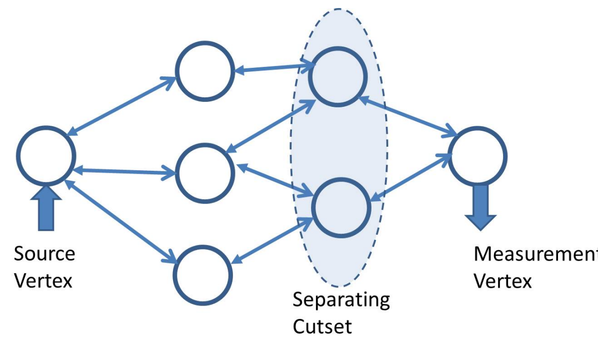

Fig. 1. A separating vertex cutset is illustrated.

A main result developed here is that the transfer gain of the diffusive network model at each frequency falls off in a certain sense, as the measurement location is moved away from the source (input) location. This concept can be formalized by considering a target measurement node $q = q^*$, and a vertex-cutset $\mathcal{Q}$ that separates the input vertex s from q^* in the sense that any directed path from s to q^* passes through $\mathcal{Q}$ (Fig. 1). Without loss of generality, we assume that $\mathcal{Q}$ has m vertices, say $q(1), \ldots, q(m)$. The spatial fall-off in the transfer gain can be formalized by comparing the frequency response at cutset nodes (network nodes corresponding to cutset vertices) with the frequency response at the target location, as follows:

Theorem 1: Consider the frequency response of the diffusive network model for the target node q^* and for the cutset nodes $\mathcal{Q} = q(1), \ldots, q(m)$. At each frequency $\omega > 0$, $|H_{q^*}(j\omega)| \leq |H_{q(i)}(j\omega)|$ for some $i = 1, \ldots, m$.

Proof: The proof is presented first for the case where L is a grounded Laplacian matrix. We consider driving the diffusive network model (3) with the persistent sinusoidal input $u(t) = cos(\omega t)$, $t \in R$. Since (3) is linear and asymptotically internally stable, the response at each node q is $x_q(t) = |H_q(j\omega)| cos(\omega t + \angle(H_q(j\omega)))$.

Next, as an alternative to analyzing the sinusoidal input response directly, let us imagine that the responses $x_q(t)$ for $q \in \mathcal{Q}$ (i.e. on the cutset) to the sinusoidal input are already known. Then consider finding the state response at only the nodes that are separated from the source s by the cutset $\mathcal{Q}$; we refer to this set of nodes as $\mathcal{P}$. The state responses $x_q(t)$ for $q \in \mathcal{P}$ can then be found without knowledge of the input $u(t)$ or the states of any nodes outside $\mathcal{P}$ and $\mathcal{Q}$. Using this approach, we aim to compare the extremal values of $x_{q^*}(t)$ with the extremal values $x_q(t)$ for $q \in \mathcal{Q}$.

To develop a comparison, let us define $h = max_{q\in\mathcal{Q}}|H_q(j\omega)|$. We prove by contradiction that, for any $q \in \mathcal{P}$, it must be true that $|x_q(t)| \leq h$ for all t. Assuming the claim is false, there exists $q \in \mathcal{P}$ and a time t and $x_q(t) > h$ (since the signal is a pure sinusoid). Now let $\widehat{q}$ be the $q \in \mathcal{P}$ for which $x_q(t)$ is maximized, i.e. $\widehat{q} = arg\, max_q\, max_t x_q(t)$. Since additionally $x_{\widehat{q}}(t)$ is sinusoidal, there must exist a time $t = \bar{t}$ such that: 1) $x_{\widehat{q}}(\bar{t}) \geq 0$, 2) $x_{\widehat{q}}(\bar{t}) \geq x_q(\bar{t})$ for all $q \in \mathcal{Q} \cup \mathcal{P}$ and 3) $\frac{dx_{\widehat{q}}(\bar{t})}{dt} > 0$. However, notice that $\frac{dx_{\widehat{q}}(t)}{dt} = \sum_{j\neq\widehat{q}} L_{\widehat{q},j}(x_j - x_{\widehat{(q)}}) - \Delta x_{\widehat{q}}$, where $\Delta = L_{\widehat{q},\widehat{q}} - \sum_{j\neq\widehat{q}} L_{\widehat{q},j}$. It follows immediately from this expression that $\frac{dx_{\widehat{q}}(\bar{t})}{dt} \leq 0$. Hence, a contradiction is reached, and it follows that $|x_q(t)| \leq h$ for all t, for $q \in \mathcal{P}$. Since $q^* \in \mathcal{P}$, we immediately find that $|x_{q^*}(t)| \leq h$ for all t. Thus, it follows that $|H_{q^*}(j\omega)| = max_t x_{q^*}(t) \leq max_{q\in\mathcal{Q}}|H_q(j\omega)|$. The result has thus been proved for grounded-Laplacian L.

If L is a (true) Laplacian matrix, then it is easy to check that a sinusoidal input at any frequency other than $\omega = 0$ still produces only a sinusoidal output at the same frequency. The remainder of the proof follows as above. □

The theorem formalizes that the frequency response has smaller magnitude at the target node as compared to at least one cutset, at each non-zero frequency. This majorization also holds for the DC (zero-frequency) response, for a grounded-Laplacian state matrix. (For a true-Laplacian state matrix, the DC gain is infinite, hence there is no steady-state.)

Remark: Theorem 1 has a similar flavor to other topological majorizations of diffusive network processes [26], however the previous work was focused on transient rather than persistent responses. Transfer functions of linear systems with grounded-Laplacian state matrices have also been characterized in [27], although for networks with inputs at all nodes.

The cutset-based characterization of the frequency response can also be readily phrased in terms of paths from the source to the target vertex, as formalized in the following corollary:

Corollary 1: Consider the magnitude of the frequency response of the diffusive network model at a target node q^*, i.e. $|H_{q^*}(j\omega)|$. For each frequency $\omega > 0$, there exists a path from the source to target node, say $(s, r(p), \ldots, r(1), q^*)$, such that the frequency response is non-increasing along the path: $|H_s(j\omega)| \geq |H_{r(p)}(j\omega)| \geq \ldots \geq |H_{r(1)}(j\omega)| \geq |H_{q^*}(j\omega)|$.

Proof:

The proof is by induction. First consider the set of neighbors of the target vertex q^* in Γ (i.e., vertices from which there is a directed link to q^*). Since this set of vertices is a cutset separating the source and the target, it follows from Theorem 1 that $|H_q(j\omega)| \geq |H_{q^*}(j\omega)|$ for some q^* in the set.

Now say that there is a connected subgraph of Γ containing the $\widehat{p}$ vertices Ω, which has the following four properties: 1) $|H_q(j\omega)| \geq |H_{q^*}(j\omega)|$ for each vertex $q \in \Omega$, 2) the subset of vertices $\overline{\Omega}$ within Ω for which $|H_q(j\omega)|$ is maximized induce a connected subgraph of Γ, 3) there is a path from q^* to $\overline{\Omega}$ along which $|H_q(j\omega)|$ is non-decreasing, and 4) the source vertex is not within the set. Now consider the set of neighbors of $\overline{\Omega}$. This cutset either separates the source vertex from $\overline{\Omega}$, or contains the source vertex. If the set contains the source vertex, it follows immediately that there is a directed path from q^* to s along which $|H_q(j\omega)|$ is non-decreasing, since the frequency response is necessarily maximized at the source vertex per Theorem 1. Alternately, consider the case the cutset separates the source vertex from $\overline{\Omega}$. Then from Theorem 1, the cutset contains at least one vertex q for which $|H_q(j\omega)|$ is greater than or equal to the frequency-response magnitude for a vertex in $\overline{\Omega}$. Further, this vertex is not contained in Ω, since the vertices in Ω that are not in $\overline{\Omega}$ have smaller frequency-response magnitude. Including this new vertex, we have thus found a set of $\widehat{p} + 1$ vertices which also satisfy the four properties listed above. The iteration is repeated until the source vertex is included, and hence the corollary is proved. □

The above results show that the frequency response degrades along some paths away from the source. However, that the paths along which the frequency response is decreasing may be different at each frequency. Conceptually, different frequencies may be filtered differentially by the network.

The performance of the causal Wiener filter, which is used for on-line estimation of stationary time series, is also closely related to the stability and finite-zero structure of the input-to-measurement transfer function [19]–[21]. For the diffusive network model, the transfer function $H_q(s)$ is necessarily bounded-input bounded-output stable, provided if state matrix $L(\Gamma)$ is a grounded Laplacian. The transfer function, however, may either be minimum-phase or nonminimum-phase depending on the input and measurement locations. Graph-theoretic conditions which indicate minimum-phase or non-minimum-phase dynamics have been developed in recent work [7], [16], [22], [24], [25]. Broadly, minimum-phase dynamics are guaranteed if there is a single path between the input and output, or the shortest input-output path is dominant. In contrast, if the network graph has alternate input-output paths that are sufficiently long and strong, the dynamics are nonminimum phase, see [16]. In the following section, we will also draw on these results to characterize the MMSE estimation performance.

IV. Graph-Theoretic Analysis of Estimation Performance

The transfer-function analysis developed in Section III is a starting point for developing graph-theoretic understandings of estimation performance for the off-line and on-line estimation problems considered here. In this section, several graph-theoretic results for the estimation performance are developed, based on classical Wiener filtering theory [19].

We consider off-line estimation first. The first result shows that measurements on single-node cuts near the input permit good estimation compared to remote measurements:

Theorem 2: Consider a diffusive network model that has single-node cut, say at node q^*. Let $\widehat{q}$ be any vertex in Γ that is separated from the source vertex s by the cut, in the sense that any path between s and $\widehat{q}$ passes through q^*. Off-line input estimation using measurements from node q^* achieves better performance than estimation using measurements from node $\widehat{q}$, i.e. $P_{w-off}(q^*) \leq P_{w-off}(\widehat{q})$. Likewise, off-line state estimation using measurements from node q^* achieves a better performance than estimation using measurements from node $\widehat{q}$, i.e. $P_{z-off}(q^*) \leq P_{z-off}(\widehat{q})$ for any projection vector $\mathbf{c}$.

Proof: The off-line MMSE estimator for the input $w(t)$ from the measurement signal $y(t)$ is the non-causal Wiener filter. For the linear model (1,2), time- and frequency- domain expressions for the Wiener filter's performance (mean-square error) are well known. Specifically, the performance of the off-line estimator for a measurement at node q^* is given by:

$$P_{w-off}(q^*) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{S_{ww}(\omega)S_{vv}(\omega)}{S_{ww}(\omega)|H_{q^*}(j\omega)|^2 + S_{vv}(\omega)}\, d\omega, \tag{4}$$

where $S_{ww}(\omega)$ and $S_{vv}(\omega)$ are the power spectra of $w(t)$ and $v(t)$, . The performance for a measurement at node $\widehat{q}$ is:

$$P_{w-off}(\widehat{q}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{S_{ww}(\omega)S_{vv}(\omega)}{S_{ww}(\omega)|H_{\widehat{q}}(j\omega)|^2 + S_{vv}(\omega)} . \, d\omega \tag{5}$$

From Theorem 1, we have that $|H_{\widehat{q}}(j\omega)|^2 \geq |H_{q^*}(j\omega)|^2$ for $\omega \in R$. Since the power spectra $S_{ww}(\omega)$ and $S_{vv}(\omega)$ are real and positive, it follows that $P_{w-off}(\widehat{q}) \leq P_{w-off}(q^*)$.

To characterize the MMSE estimator performance for the state projection $z(t) = \mathbf{c}^T\mathbf{x}(t)$, notice that $z(t)$ can be computed from $w(t)$ via processing by a linear causal filter; we use $F(j\omega)$ for the frequency response of this filter. Then

$$P_{z-off}(q^*) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|F(j\omega)|^2 S_{ww}(\omega)S_{vv}(\omega)}{S_{ww}(\omega)|H_{q^*}(j\omega)|^2 + S_{vv}(\omega)}\, d\omega \tag{6}$$

and

$$P_{z-off}(\widehat{q}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|F(j\omega)|^2 S_{ww}(\omega)S_{vv}(\omega)}{S_{ww}(\omega)|H_{\widehat{q}}(j\omega)|^2 + S_{vv}(\omega)}\, d\omega. \tag{7}$$

Thus, $P_{z-off}(\widehat{q}) \leq P_{z-off}(q^*)$. □

The majorization developed in the previous theorem holds regardless of the autocorrelation functions of the input and measurement noise, provided that: 1) the input has finite power and 2) the measurement noise model does not change with the measurement location; these properties hold in many applications, and are assumed in our formulation. An immediate consequence of the theorem is that measurements at the source location permit better input and state estimation as compared to any other node.

Let us next consider the special case that the network graph has a unique path between any two nodes, i.e. the network graph without edge directions is a tree. The single-node cutset result can be applied repeatedly to demonstrate degradation in estimation performance along paths in the network graph. Specifically, the following corollary is immediately obtained:

Corollary 2: Consider a diffusive network model that has a unique path between any two nodes. Consider a path in the network graph starting from and moving away from the source node, say vertices $s, q(1), \ldots, q(m)$. Off-line input estimation using measurements from the corresponding nodes degrades along the path. That is, $P_{w-off}(s) \leq P_{w-off}(q_1) \leq \ldots \leq P_{w-off}(q_m)$. Likewise, off-line state estimation using measurements from the corresponding nodes degrades along the path. That is, $P_{z-off}(s) \leq P_{z-off}(q_1) \leq \ldots \leq P_{z-off}(q_m)$.

The graph-theoretic performance analysis can also be generalized to the multi-node-cut case, by comparing estimation using measurements from all nodes on the cut with estimation using a single downstream measurement. The analysis requires considering off-line MMSE estimation of the input and state projections from noisy measurements of multiple nodes. Specifically, let us consider that state measurements are available for a set of nodes $\mathcal{Q}$, subject to statistically-identical additive noise. Formally, the measurements $y_q = \mathbf{e}_q^T\mathbf{x} + v_q$ are assumed to be available for each $q \in \mathcal{Q}$, where the v_q are stationary white-noise signals which have the same autocorrelation as $v(t)$. We again refer to MMSE estimation of the input and state projections from these measurement signals over the infinite horizon as the off-line estimation problem, and use the notation $P_{w-off}(\mathcal{Q})$ and $P_{z-off}(\mathcal{Q})$ for the performance (MMSE) of the optimal estimator.

This performance comparison for the multi-node-cut case is formalized in the following theorem:

Theorem 3: Consider a diffusive network model that has a multi-node cut $\mathcal{Q} = (q(1), \ldots, q(p))$. Let $\widehat{q}$ be any vertex in the graph that is separated from the source vertex s by the cut, i.e. any path between s and $\widehat{q}$ passes through the cut. Off-line input estimation using measurements from all nodes along the cut achieves better performance than estimation using measurements from node $\widehat{q}$, i.e. $P_{w-off}(\mathcal{Q}) \leq P_{w-off}(\widehat{q})$. Likewise, off-line state estimation using measurements from all nodes along the cut achieves better performance than estimation using measurements from node $\widehat{q}$. That is, $P_{z-off}(\mathcal{Q}) \leq P_{z-off}(\widehat{q})$, for any projection vector $\mathbf{c}$.

Proof:

To prove the result, we consider MMSE estimation of the input signal from a signal $\overline{y}(t)$, which is computed by filtering and combining the measurements on the cut. Precisely, we compute $\overline{y}(t)$ in the frequency domain as $\overline{Y}(j\omega) = \sum_{i=1}^{p} B_i(j\omega)Y_{q(i)}(j\omega)$. The filters $B_i(j\omega)$, $i = 1, \ldots, p$, are defined as follows. For each frequency ω, we choose one index $i = \widehat{i}$ such that $|H_{q(\widehat{i})}(j\omega)| \geq |H_{q^*}(j\omega)|$, and set $B_{\widehat{i}}(j\omega) = 1$ for this index i, while setting $B_i(j\omega) = 0$ for $i \neq \widehat{i}$. We notice that there is necessarily such a $\widehat{i}$, from Theorem 1.

By expressing the measurements on the cutset in terms of the input and noise signals in the frequency domain, the signal $\overline{Y}(j\omega)$ can be rewritten as $\overline{Y}(j\omega) = \overline{H}(j\omega)W(j\omega) + \overline{V}(j\omega)$, where $\overline{H}(j\omega) = \sum_{i=1}^{p} B_i(j\omega)H_{q(i)}(j\omega)$ and $\overline{V}(j\omega) =$

$\sum_{i=1}^{p} B_i(j\omega)V_{q(i)}(j\omega)$. From the definition of the filters $B_i(j\omega)$, it is immediate that $\overline{V}(j\omega)$ is a wide-sense stationary stochastic process with the power spectrum $S_{vv}(\omega)$, i.e. the same power spectrum as $v(t)$. Additionally, the filter definition yields that $|\overline{H}(j\omega)| \geq |H_{q^*}(j\omega)|$ at each frequency ω.

Next, from the Wiener filtering analysis, the MMSE in off-line estimation of $w(t)$ using $\overline{y}(t)$, which we denote by $\overline{P}_{w-off}(\mathcal{Q})$, is given by the following:

$$\overline{P}_{w-off}(\mathcal{Q}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{S_{ww}(\omega)S_{vv}(\omega)}{S_{ww}(\omega)|\overline{H}(j\omega)|^2 + S_{vv}(\omega)} d\omega. \quad (8)$$

Comparing this expression with the off-line estimator performance using measurements from node q^* (Equation 6), it follows that $\overline{P}_{w-off}(\mathcal{Q}) \leq P_{w-off}(\mathcal{Q})$. Since the MMSE estimator which uses measurements from all nodes in $\mathcal{Q}$ performs at least as well as the MMSE estimator using $\overline{y}(t)$, it further follows that $P_{w-off}(\mathcal{Q}) \leq \overline{P}_{w-off}(\mathcal{Q})$. Combining the two inequalities, we have that $P_{w-off}(\mathcal{Q}) \leq P_{w-off}(\mathcal{Q})$, and the result is proved for off-line input estimation. The result for off-line state-projection estimation follows according to the same logic as Theorem 2. □

The graph-theoretic analysis of estimator performance developed in the above theorems informs sensor placement. Broadly, the results show that sensors placed close to the stochastic input source are effective for off-line input and state estimation. More precisely, measurements on separating cutsets of the network necessarily outperform downstream sensing schemes. Thus, for resource-constrained sensor design needs, an effective strategy is to find small node cutsets that isolate the stochastic input (source location). We leave it to future work to develop high-performance sensor-selection algorithms based on this principle.

Our analysis thus far has focused on comparing the off-line estimation performance for different measurement locations. We are also interested to compare the off-line estimation performance for different state projections, for a specified measurement location. In particular, we seek to compare the performance of estimators for different node's states, for a particular measurement location. For this development, the notation $P_{x_i-off}(q)$ is used for the performance (MMSE) achieved by an off-line estimator of node i's state response $x_i(t)$ from noisy measurements taken at node q. The following theorem shows that the node state estimator performance $P_{x_i-off}(q)$ for different estimated nodes also exhibits a topological pattern:

Theorem 4: Consider off-line estimation of node states using measurements at a fixed node q. In particular, let us consider the performance of the off-line estimator for a particular node $\widehat{i}$'s state, where $\widehat{i}$ specifies a single-vertex cutset in the network graph Γ. Also, let us consider the estimation performance for any node i that is separated the source vertex s by $\widehat{i}$. Then $P_{x_i-off}(q) \leq P_{x_{\widehat{i}}-off}(q)$.

Proof: Estimation of a node i's state is equivalent to estimation of the state projection $z(t) = \mathbf{e}_i^T \mathbf{x}(t)$. Thus, from (6), the off-line estimation performance the node i's state is given by:

$$P_{x_i}(q) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|H_i(j\omega)|^2 S_{ww}(\omega)S_{vv}(\omega)}{S_{ww}(\omega)|H_q(j\omega)|^2 + S_{vv}(\omega)} d\omega, \quad (9)$$

where we have used the fact that the transfer function from the input sequence $w(t)$ to node i's state $x_i(t)$ is given by $H_i(j\omega)$. Per Theorem 1, $|H_{\widehat{i}}(j\omega)| \geq |H_i(j\omega)|$ at all frequencies ω, for any vertex i that is separated from the the source vertex s by $\widehat{i}$. The result then follows directly from the expression for the estimation performance (9), since the integrand scales with $|H_i(j\omega)|^2$. □ The result clarifies that estimation of network states away from the stochastic input is easier, because these signals are smoothed and reduced in magnitude compared to states near the stochastic input.

Many ambient-data-based network monitoring applications require on-line algorithms. In contrast with the off-line case, the estimator performance for the on-line problem does not display a simple topological dependence on the measurement location. However, interesting graph-theoretic insights into the estimator performance can be developed, in the case where the measurement noise is small. Specifically, it is well known that the on-line estimator performance displays a binary behavior in the limit of small measurement noise, with the estimator error approaching zero in in some circumstances and reaching a non-zero aymptote in others [20], [21]. The limiting behavior of the estimator is depends precisely on whether or not the system transfer function is minimum phase. This general insight is specialized to the diffusive-network model setting in the following lemma, as a starting point toward developing graph-theoretic results.

Lemma 1: Consider the diffusive network model, with input (source) location s and measurement location q. Also, assume that the measurement noise signal has the form $v(t) = \alpha\widehat{v}(t)$, where $\widehat{v}(t)$ has fixed autocorrelation $R_{\widehat{v}}(t)$ and $\alpha > 0$ is a scaling parameter. In the limit $\alpha \to 0$, the on-line input-estimation performance approaches zero error, i.e. $P_{w-on}(q) \to 0$, if and only if the transfer function $H_q(s)$ is minimum phase. Likewise, the on-line state estimation performance approaches zero for all projection vectors $\mathbf{c}$, i.e. $P_{z-on}(q) \to 0$, if and only if $H_q(s)$ is minimum phase.

The proof of the lemma is immediate from [20], [21], so details are omitted. The lemma allows the development of graph-theoretic results for the on-line estimator performance, when the measurement noise is made small. In particular, topological conditions under which the diffusive network dynamics are minimum-phase or non-minimum phase can be exploited to obtain topological sufficient conditions for perfect or imperfect estimation in the low-noise limit. One representative result of this form is given here. In particular, using this approach, one can show that perfect on-line estimation is possible in the low-noise limit, in the case that there is a unique path between each pair of vertices in the network graph:

Theorem 5: Consider a diffusive network model which has a unique directed path between each pair of vertices. Also, assume that the measurement noise signal has the form $v(t) = \alpha\widehat{v}(t)$, where $\widehat{v}(t)$ has fixed autocorrelation $R_{\widehat{v}}(t)$ and

$\alpha > 0$ is a scaling parameter. In the limit $\alpha \to 0$, the on-line input-estimation performance (MMSE) approaches zero error, i.e. $P_{w-on}(q) \to 0$. Likewise, the on-line state estimation performance approaches zero error for all projection vectors $\mathbf{c}$, i.e. $P_{z-on}(q) \to 0$.

Proof: Since the network graph has a unique path between each pair of vertices, it necessarily has a unique path from the source vertex s to the measurement vertex q. Thus, From Theorem 5 in [16], the transfer function $H_q(s)$ is minimum phase. The result follows from Lemma 1. □

Several similar results can be developed by leveraging other graph-theoretic analyses of minimum-phase and non-minimum-phase dynamics in the diffusive network model, further details are omitted.

Remark: Since estimation of stationary processes is considered, a Wiener filtering approach is sufficient. However, the estimator and its error can equivalently be obtained using the steady-state Kalman filter (for on-line estimation) and smoother (for off-line estimation). Thus, the results on the estimator performance can be interpreted as graph-theoretic insights into the solution of the algebraic Riccati equation.

V. Conclusions

Off-line and on-line estimation of diffusive network dynamics from ambient measurements has been studied. The performance of stochastic-input and state estimators have been characterized from a graph-theoretic perspective. For off-line estimators, the performance (MMSE) was shown to exhibit a direct dependence on the network topology, with the estimation quality degrading for sequential network cuts away from the noise source. Meanwhile, for on-line estimation, graph-theoretic analyses were developed in the low-measurement-noise limit, using relationships between estimation performance and phase properties of network transfer functions. These results suggest that it may be possible to develop simple topological rubrics for evaluating estimators, as well as effective sensor placement schemes, for the special class of diffusive network models.

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