A BDDC ALGORITHM FOR THE STOKES PROBLEM WITH WEAK GALERKIN DISCRETIZATIONS

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Abstract. The BDDC (balancing domain decomposition by constraints) methods have been applied successfully to solve the large sparse linear algebraic systems arising from conforming finite element discretizations of second order elliptic and Stokes problems. In this paper, the Stokes equations are discretized using the weak Galerkin method, a newly developed nonconforming finite element method. A BDDC algorithm is designed to solve the linear system such obtained. Edge/face velocity interface average and mean subdomain pressure are selected for the coarse problem. The condition number bounds of the BDDC preconditioned operator are analyzed, and the same rate of convergence is obtained as for conforming finite element methods. Numerical experiments are conducted to verify the theoretical results.

Key words. Discontinuous Galerkin, HDG, weak Galerkin, domain decomposition, BDDC, Stokes, Saddle point problems, benign subspace

AMS subject classifications. 65F10, 65N30, 65N55

1. Introduction.

Numerical solution of saddle point problems using non overlapping domain decomposition methods have long been an active area of research; see, e.g., [28, 15, 11, 10, 18, 29, 30, 16, 33, 17, 34, 27]. The Balancing Domain Decomposition by Constraints (BDDC) algorithm is an advanced variant of the non-overlapping domain decomposition technique. It was first introduced by Dohrmann [5], and the theoretical analysis was later given by Mandel and Dohrmann [20]. In this theoretical development, optimal condition number bound was obtained for the BBDC operators proposed for symmetric positive definite systems. Nonetheless, the variational form of the incompressible Stokes problem is a saddle point problem [3], and the discretization by finite element methods lead to symmetric indefinite matrices. Thus, the conventional theory usually fails to apply. In the first attempt to apply BDDC to the incompressible Stokes problem by Li and Widlund [18], the approach via benign spaces was used to reduce the Stokes system to a symmetric positive definite problem, and optimal convergence result was obtained as for the elliptic case. However, this method was proposed and analyzed with discontinuous pressure approximation, and there is a big class of mixed finite element spaces featuring continuous pressure, e.g., the Taylor-Hood finite elements. Later, Li and Tu proposed a class of non-overlapping domain decomposition algorithms for continuous finite element pressure space, which were proved and numerically verified to be scalable [16, 33, 17, 34]. Earlier, Šístek et al. applied a parallel BDDC pre-conditioner based on the corner constraints to the Stokes flow using Taylor-Hood finite element [41]. They numerically demonstrated the promising speedup property of their BDDC pre-conditioner as applied to benchmark test problems of real-life relevance, even though optimal scalability was not achieved.

As the property of the discretized system to be solved is dependent on the numerical methods used, the BDDC algorithms have been extended to the second order elliptic problem with mixed and hybrid formulations, hybridizable discontinuous

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Galerkin (HDG) methods [29, 30, 35]. In this study, we design BDDC pre-conditioners for trending non-conforming finite element methods, in particular, the weak Galerkin (WG) methods. The WG methods are a class of nonconforming finite element methods, which were first introduced for second order elliptic problems by Wang and Ye [36]. The idea of WG is to introduce weak functions and their weak derivatives as distributions, which can be approximated by polynomials of different degrees on different support. For example, for second order elliptic problems, weak functions have the form of $v = \{v_0, v_b\}$, where v_0 is defined inside each element and v_b is defined on the boundary of the element. v_0 and v_b can both be approximated by polynomials. The gradient operator is approximated by a *weak gradient* operator, which is further approximated by polynomials. These weakly defined functions and derivatives make the WG methods highly flexible in terms of approximating functions and finite element partition of the domain. The same *weak* concepts have been extended to other differential operators such as *divergence* and *curl,* which appears in applications like Stokes [38] and Maxwell [25] equations respectively.

As most finite element methods, the WG methods result in a large number of degrees of freedom and therefore require solving large linear systems with condition number deteriorating with the refinement of the mesh. Efficient fast solvers for the resulting linear system are necessary. However, relatively few attempts on designing fast solvers for the WG methods can be found in the literature; see [4]. An effective implementation of WG methods is to reduce the unknown variables to those associated with element boundaries through a Schur-complement approach. It can be further reduced to the subdomain interface. The subdomain interface problem can then be solved using the conjugate gradient method preconditioned with a BDDC algorithm. It is necessary to impose edge or face average constraints across the interface. By a change of variable [19, 14], the primal constraint on edge or face average can be converted to an explicit variable. The reduced system for the primal variables will be the coarse problem to solve. The BDDC preconditioner can be built based on such designed coarse problem, and thus be used as a preconditioner for the conjugate gradient method.

In a recent study [35], the authors proved the condition number bound of the BDDC preconditioned operator arising from elliptic problems with hybridizable discontinuous Galerkin (HDG) discretizations. In this paper, a BDDC algorithm is further developed for weak Galerkin discretization with reduced polynomial basis functions. As in [35], we first establish the connection between the hybridized Raviart-Thomas(RT) method and the WG discretization and obtain the condition number estimate of the BDDC algorithm applied to the elliptic problem with the WG discretization. We then consider the BDDC algorithms for the saddle point problem arising from the WG discretization for the incompressible Stokes problem. In [26], a similar saddle point problem is obtained by the HDG discretization for incompressible Stokes flow, where the resulting system is solved by an augmented Lagrange approach. An additional time dependent problem is introduced and solved by a backward-Euler method. Here, we solve the saddle point problem from WG discretization directly using the BDDC methods. To the best of our knowledge, this is the first attempt for fast solvers applied to the Stokes problems with this type nonconforming discretization. There are many works on preconditioning the saddle point problems resulting from mixed finite element discretizations, such as [1, 2]. In those works, the original saddle point problems are reformulated to positive definite problems under specially defined inner products. In this paper, a *benign subspace* idea is used as in [18, 29, 40]. In the *benign subspace* approach, the positive definite system is obtained by carefully choosing the coarse components in the BDDC preconditioner. Therefore, the formulation, implementation, and rate of convergence of the BDDC algorithm for Stokes are very similar to those for the positive definite systems arising from the elliptic problems.

We prove that the condition number bound for the Stokes problem with the WG discretization is as good as for the conforming discretization. We note that the WG discretization has been extended to polytopal meshes, [37, 24, 23, 38]. With the development of the domain decomposition methods for irregular subdomain shapes, [6, 13, 7, 39, 8, 9], we believe that the BDDC algorithms proposed in this paper can be extended to polytopal meshes as well. But we will restrict ourselves to the standard finite element triangulation here and leave the complete analysis and numerical verification for more general polytopal meshes in the future study.

The rest of the paper is organized as follows. In Section 2, we introduce some notations for relevant Hilbert spaces. In Section 3, we introduce the Stokes problem and its weak Galerkin discretization. In Section 4, we reduce the linear system to an interface problem. Then, we introduce the BDDC preconditioner for the interface problem in Section 5, and give some auxiliary results in Section 6. In Section 7, we provide an estimate for the condition number of the BDDC preconditioned system. Finally, results from numerical experiments are presented and discussed in Section 8.

2. Notations for some relevant Spaces.

Let $\Omega \in \mathbb{R}^d$ $(d = 2,\, 3)$ be a bounded open set with Lipschitz continuous boundary. The Sobolev space $H^k(\Omega)$ for any integer $k \geq 1$ is a Hilbert space with inner product

$$(u,\, v)_{H^k(\Omega)} = \sum_{|\alpha| \leq k} (D^\alpha u,\, D^\alpha v)_{L^2(\Omega)}\,,$$

where the multi-index notation for derivatives

$$D^\alpha u = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} u = \frac{\partial^{|\alpha|} u}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n}$$

with $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Correspondingly, we can define the induced norm $\|\cdot\|_{H^k(\Omega)}$

$$\|u\|_{H^k(\Omega)}^2 = (u,\, u)_{H^k(\Omega)} = \sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^2\, dx,$$

and a semi-norm

$$|u|_{H^k(\Omega)}^2 = \sum_{|\alpha| = k} \int_\Omega |D^\alpha u|^2\, dx.$$

The space $H^0(\Omega)$ coincides with $L^2(\Omega)$, which is the space of square integrable functions on Ω, i.e.,

$$L^2(\Omega) = \left\{ u \,:\, \int_\Omega |u|^2\, dx < \infty \right\}.$$

The inner product and induced norm of $L^2(\Omega)$ are given by:

$$(u,\, v)_{L^2(\Omega)} = \int_\Omega uv dx; \qquad \|u\|_{L^2(\Omega)}^2 = \int_\Omega |u|^2\, dx.$$

We define the subspace of $L^2(\Omega)$ with zero mean to be

$$L_0^2(\Omega) = \left\{ u \,:\, u \in L^2(\Omega), \int_\Omega u dx = 0 \right\}.$$

Let H_Ω be the diameter of Ω. We have the following scaled norm for the Sobolev space $H^1(\Omega)$:

$$\|u\|_{s,H^1(\Omega)}^2 = \frac{1}{H_\Omega^2} \|u\|_{L^2(\Omega)}^2 + |u|_{H^1(\Omega)}^2 = \frac{1}{H_\Omega^2} \|u\|_{L^2(\Omega)}^2 + \int_\Omega |\nabla u|^2 \, dx,$$

with

$$\nabla = grad = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n} \right).$$

The subspace of $H^1(\Omega)$ with vanishing boundary values is denoted by

$$H_0^1(\Omega) = \left\{ v \in H^1(\Omega) : \; v = 0 \, \text{on} \, \partial\Omega \right\}.$$

The spaces introduced above can be extended to spaces of vector-valued functions in a straightforward way.

Also, we recall that the space $H(\text{div}; \Omega)$ is defined as the set of vector-valued functions on Ω such that both the functions and their divergence are square integrable; i.e.,

$$H(\text{div}; \Omega) = \left\{ \mathbf{v} \,:\, \mathbf{v} \in \left[L^2(\Omega) \right]^d, \, \nabla \cdot \mathbf{v} \in L^2(\Omega) \right\}.$$

The scaled norm in $H(\text{div}; \Omega)$ is defined by

$$\|\mathbf{v}\|_{s,H(\text{div};\Omega)}^2 = \frac{1}{H_\Omega^2} \|\mathbf{v}\|_{L^2(\Omega)}^2 + \|\nabla \cdot \mathbf{v}\|_{L^2(\Omega)}^2.$$

3. A Stokes problem and its weak Galerkin Discretization.

We consider the primary velocity-pressure formulation for the Stokes problem on a bounded polygonal domain Ω, in two dimensions ($d = 2$), or three dimensions ($d = 3$), with a Dirichlet boundary condition:

$$\left\{ \begin{aligned} -\triangle u + \nabla p = f \quad & \text{in} \quad \Omega, \\ \nabla \cdot u = 0 \quad & \text{in} \quad \Omega, \\ u = g \quad & \text{on} \quad \partial\Omega, \end{aligned} \right. \tag{3.1}$$

where $f \in \left[L^2(\Omega) \right]^d$, and $g \in \left[H^{1/2}(\partial\Omega) \right]^d$. Without loss of generality, we assume that $g = 0$. The weak form in the primary velocity-pressure formulation for the Stokes problem seeks $u \in \left[H_0^1(\Omega) \right]^d$ and $p \in L_0^2(\Omega)$ such that

$$\left\{ \begin{array}{rcll} (\nabla u, \, \nabla v) - (\nabla \cdot v, \, p) & = & (f, \, v) & \forall v \in \left[H_0^1(\Omega) \right]^d, \\ (\nabla \cdot u, \, q) & = & 0 & \forall q \in L_0^2(\Omega). \end{array} \right. \tag{3.2}$$

The idea of weak Galerkin finite element scheme [38] is to substitute the standard function and differential operators with the weakly defined counterparts. A weak function over the domain D is defined as $v = \{v_0, v_b\}$ such that $v_0 \in L^2(D)$ and $v_b \in L^2(\partial D)$. The v_0 part represents the value of v in the interior of D, while the v_b part represents the value of v on the boundary of D. Note that v_b does not bind itself with v_0 from the definition. In essence, weak functions relax the continuity property of the standard functions, thus to offer more flexibility in terms of variable representation. Following the notation in [38], we denote by $\mathcal{V}(D)$ the space of weak functions over the domain D

$$\mathcal{V}(D) = \left\{v = \{v_0, v_b\} : v_0 \in L^2(D), v_b \in L^2(\partial D)\right\},$$

and the relevant vector-valued weak function space by

$$[\mathcal{V}(D)]^d = \left\{\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} : \mathbf{v}_0 \in \left[L^2(D)\right]^d, \mathbf{v}_b \in \left[L^2(\partial D)\right]^d\right\},$$

and

$$[V(D)]^d = \left\{\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} : \mathbf{v}_0 \in \left[L^2(D)\right]^d, \mathbf{v}_b \cdot \mathbf{n} \in L^2(\partial D)\right\}.$$

The space of weak gradient or divergence operators will be defined as the dual space of appropriate Hilbert space, in similar manner as the dual of $\left[L^2(D)\right]^d$ can be identified with itself by using the L^2 inner product as the action of the linear functionals. The following two definitions are [38, Definitions 2.1 and 2.3]:

Definition 1. *For any $\mathbf{v} \in [\mathcal{V}(D)]^d$, the weak gradient of $\mathbf{v}$ is defined as the linear functional $\nabla_w \mathbf{v}$ in the dual space of $[H^1(D)]^d$ whose action on each $\mathbf{q} \in [H^1(D)]^{d\times d}$ is given by*

$$(\nabla_w \mathbf{v}, \mathbf{q})_D = -(\mathbf{v}_0, \nabla \cdot \mathbf{q})_D + \langle \mathbf{v}_b, \mathbf{q} \cdot \mathbf{n}\rangle_{\partial D},$$

where $\mathbf{n}$ is the outward normal direction to ∂D.

Definition 2. *For any $\mathbf{v} \in [V(D)]^d$, the weak divergence of $\mathbf{v}$ is defined as the linear functional $\nabla_w \cdot \mathbf{v}$ in the dual space of $H^1(D)$ whose action on each $\varphi \in H^1(D)$ is given by*

$$(\nabla_w \cdot \mathbf{v}, \varphi)_D = -(\mathbf{v}_0, \nabla \varphi)_D + \langle \mathbf{v}_b \cdot \mathbf{n}, \varphi\rangle_{\partial D},$$

where $\mathbf{n}$ is the outward normal direction to ∂D.

Now, we introduce the weak Galerkin finite element discretization for (3.1) as in [38]. First, we introduce the mesh of the domain, then we will define discontinuous weak Galerkin finite element spaces over the mesh. Let $\mathcal{T}_h$ be a shape-regular and quasi-uniform triangulation of Ω, and the element in $\mathcal{T}_h$ denoted by K. For any $K \in \mathcal{T}_h$, we denote by h_K the diameter of K with $h = max_{K\in\mathcal{T}_h} h_K$. Define $\mathcal{F}_h$ be the set of edges/faces of elements $K \in \mathcal{T}_h$. $\mathcal{F}_h^i$ and $\mathcal{F}_h^\partial$ are subsets of $\mathcal{F}_h$, which consists of domain interior and boundary edges, respectively. For any domain D, let $P_k(D)$ be the space of polynomials of degree $\leq k$ on D. Define the weak Galerkin finite element spaces for the velocity variable associated with $\mathcal{T}_h$ as follows:

$$V_k = \left\{v = \{v_0, v_b\} : \{v_0, v_b\}|_K \in [P_k(K)]^d \times [P_{k-1}(e)]^d, \forall K \in \mathcal{T}_h, e \subset \partial K\right\}.$$

Note that a function $v \in V_k$ has a single value v_b on each edge $e \in \mathcal{F}_h$. The subspace of V_k with vanishing boundary values on $\partial\Omega$ is denoted by

$$V_k^0 = \{v = \{v_0, v_b\} \in V_k : v_b = 0 \,\text{on}\, \partial\Omega\}.$$

We denote a relevant matrix polynomial function space by

$$\mathbf{Q}_{k-1} = \left\{ \mathbf{v} : \mathbf{v}|_K \in [P_{k-1}(K)]^{d\times d}, \forall K \in \mathcal{T}_h \right\}.$$

For the pressure variable, define the following finite element space

$$W_{k-1} = \left\{ q : q \in L_0^2(\Omega), \, q|_K \in P_{k-1}(K) \right\}.$$

Denote the discrete weak gradient operator by $\nabla_{w,k-1}$, and the discrete weak divergence operator by $(\nabla_{w,k-1}\cdot)$, respectively. On the finite element space V_k, they are defined as follows: for $v = \{v_0, v_b\} \in V_k$, on each element $K \in \mathcal{T}_h$, $\nabla_{w,k-1} v \mid_K \in [P_{k-1}(K)]^{d\times d}$ and $\nabla_{w,k-1} \cdot v \mid_K \in P_{k-1}(K)$ are the unique solutions of the following equations, respectively,

$$(\nabla_{w,k-1} v \mid_K, \mathbf{q})_K = -(v_{0,K}, \nabla \cdot \mathbf{q})_K + \langle v_{b,K}, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial K}, \qquad \forall \mathbf{q} \in [P_{k-1}(K)]^{d\times d},$$

$$(\nabla_{w,k-1} \cdot v \mid_K, \varphi)_K = -(v_{0,K}, \nabla\varphi)_K + \langle v_{b,K}\cdot\mathbf{n}, \varphi\rangle_{\partial K}, \qquad \forall \varphi \in P_{k-1}(K),$$

where $v_{0,K}$ and $v_{b,K}$ are the restrictions of v_0 and v_b to K, respectively, $(u, w)_K = \int_K uwdx$, and $\langle u, w\rangle_{\partial K} = \int_{\partial K} uwds$. To simplify the notation, we shall drop the subscript $k-1$ in the notation $\nabla_{w,k-1}$ and $(\nabla_{w,k-1}\cdot)$ for the discrete weak gradient and the discrete weak divergence operators. We denote the L^2 inner product over the triangulation as a summation over each element of the triangulation, for example,$(\nabla_w u, \nabla_w w)_{\mathcal{T}_h} = \sum_{K\in\mathcal{T}_h} (\nabla_w u, \nabla_w w)_K$, $(\nabla_w \cdot v, q)_{\mathcal{T}_h} = \sum_{K\in\mathcal{T}_h} (\nabla_w \cdot v, q)_K$.

Let Q_0 be the L^2 projection from $\left[L^2(K)\right]^d$ onto $[P_k(K)]^d$, and Q_b be the L^2 projection from $\left[L^2(e)\right]^d$ onto $[P_{k-1}(e)]^d$, for $e \in \mathcal{F}_h$. We write the corresponding projection operator for the weak function as $Q_h = \{Q_0, Q_b\}$. Next, we define three bilinear forms as below

(3.3)
$$\begin{array}{rcl} s(v, w) &=& \sum_{K\in\mathcal{T}_h} h_K^{-1} \langle Q_b v_0 - v_b, Q_b w_0 - w_b\rangle_{\partial K}, \\ a(v, w) &=& (\nabla_w v, \nabla_w w)_{\mathcal{T}_h} + s(v, w), \\ b(v, q) &=& (\nabla_w \cdot v, q)_{\mathcal{T}_h}. \end{array}$$

The discrete problem resulting from the WG discretization can then be written as: find $u_h = \{u_0, u_b\} \in V_k^0$ and $p_h \in W_{k-1}$ such that

$$\left\{ \begin{array}{rcll} a(u_h, v) - b(v, p_h) &=& (f, v_0), & \forall v = \{v_0, v_b\} \in V_k^0, \\ b(u_h, q) &=& 0, & \forall q \in W_{k-1}. \end{array} \right.$$

We introduce the following operators: $A : V_k^0 \to V_k^0$, $B : V_k^0 \to W_{k-1}$, by

(3.4)
$$(Au_h, v) = a(u_h, v), \; (Bu_h, q) = -b(u_h, q).$$

Using these operators, the matrix form of the weak Galerkin scheme can be represented as

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u_h \\ p_h \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}.$$

At element level, for each K, given the edge component v_b of the velocity and the pressure p, the interior component v_0 of the velocity can be uniquely determined. Namely, v_0 can be eliminated in each element independently. We thus obtain the reduced system of v_b and p only with considerable smaller size but different sparsity pattern (denser than the full system) as below

$$\begin{bmatrix} A_{uu} & B_{pu}^T \\ B_{pu} & C_{pp} \end{bmatrix} \begin{bmatrix} u_{h,b} \\ p_h \end{bmatrix} = \begin{bmatrix} f_{u_b} \\ f_p \end{bmatrix}.$$

Throughout the rest of the paper, we will work with the reduced system such obtained.

4. Reduced Subdomain Interface Problem. We decompose Ω into N non overlapping subdomain Ω_i with diameters H_i, $i = 1, \ldots, N$, and set $H = \max_i H_i$. We assume that each subdomain is a union of shape-regular coarse triangles and that the number of such elements forming an individual sudomain is uniformly bounded. We define edges/faces as open sets shared by two subdomains. Two nodes belong to the same face when they are associated with the same pair of subdomains. Let Γ be the interface between the subdomains. The set of the interface nodes $\mathit{\Gamma}_h$ is defined as $\mathit{\Gamma}_h := (\cup_{i \neq j} \partial\Omega_{i,h} \cap \partial\Omega_{j,h}) \setminus \partial\Omega_h$, where $\partial\Omega_{i,h}$ is the set of nodes on $\partial\Omega_i$ and $\partial\Omega_h$ is that of $\partial\Omega$. We assume the triangulation of each subdomain is quasi-uniform.

We decompose the discrete velocity and pressure spaces V_k and W_{k-1} into:

$$V_k = V_I \oplus \widehat{V}_\Gamma, \qquad W_{k-1} = W_I \oplus W_0.$$

Here, V_I and W_I are products of subdomain interior velocity spaces $V_I^{(i)}$ and subdomain interior pressure spaces $W_I^{(i)}$, respectively; i.e.,

$$V_I = \prod_{i=1}^{N} V_I^{(i)}, \qquad W_I = \prod_{i=1}^{N} W_I^{(i)}.$$

The elements of $V_I^{(i)}$ are supported in the subdomain Ω_i and vanishes on its interface $\mathit{\Gamma}_i$, while the elements of $W_I^{(i)}$ are restrictions of the pressure variables to Ω_i which satisfy $\int_{\Omega_i} p_I^{(i)} = 0$. $\widehat{V}_\Gamma$ is the subspace of edge functions on $\mathit{\Gamma}$, and W_0 is the subspace of W with constant values $p_0^{(i)}$ in the subdomain Ω_i that satisfy $\sum_{i=1}^{N} p_0^{(i)} m(\Omega_i) = 0$, where $m(\Omega_i)$ is the measure of the subdomain Ω_i.

We denote the space of interface edge velocity variables of the subdomain Ω_i by $V_\Gamma^{(i)}$, and the associated product space by $V_\Gamma = \prod_{i=1}^{N} V_\Gamma^{(i)}$; generally edge functions in V_Γ are discontinuous across the interface. We define the restriction operators $R_\Gamma^{(i)} : \widehat{V}_\Gamma \to V_\Gamma^{(i)}$ to be an operator which maps functions in the continuous global interface edge variable space $\widehat{V}_\Gamma$ to the subdomain component space $V_\Gamma^{(i)}$. Also, $R_\Gamma : \widehat{V}_\Gamma \to V_\Gamma$ is the direct sum of $R_\Gamma^{(i)}$. We denote the spaces of the right-hand-side interior load vectors f_I and interface load vectors f_Γ by F_I and F_Γ, respectively. Similar notation conventions apply to the spaces $\widetilde{F}_\Gamma$, $\widehat{F}_\Gamma$, $\widehat{F}_\Pi$, $F_\Delta^{(i)}$, $F_\Gamma^{(i)}$, and F_0. We will use them throughout this paper without further explanation.

With the decomposition of the solution space, the global Stokes problem can be written as follows: find $(u_I,\, p_I,\, u_\Gamma,\, p_0) \in \left(V_I, W_I, \widehat{V}_\Gamma, W_0\right)$ such that

$$\begin{bmatrix} A_{II} & B_{II}^T & \widehat{A}_{\Gamma I}^T & 0 \\ B_{II} & C_{II} & \widehat{B}_{\Gamma I}^T & 0 \\ \widehat{A}_{\Gamma I} & \widehat{B}_{\Gamma I} & \widehat{A}_{\Gamma\Gamma} & \widehat{B}_{0\Gamma}^T \\ 0 & 0 & \widehat{B}_{0\Gamma} & 0 \end{bmatrix} \begin{bmatrix} u_I \\ p_I \\ u_\Gamma \\ p_0 \end{bmatrix} = \begin{bmatrix} f_I \\ f_{p_I} \\ f_\Gamma \\ 0 \end{bmatrix}. \tag{4.1}$$

The lower left block in (4.1) is zero, because the bilinear form $b(u_h, \varphi)$ does not explicitly relate to u_I and p_I for any $u_h \in V_k^0$ and $\varphi \in W_0$. The leading two-by-two block of the matrix above can be rewritten into a block diagonal form with each block corresponding to an independent subdomain problem. And the global problem can be assembled from the subdomain problems, defined as below

$$\begin{bmatrix} A_{II}^{(i)} & B_{II}^{(i)T} & \widehat{A}_{\Gamma I}^{(i)T} & 0 \\ B_{II}^{(i)} & C_{II}^{(i)} & \widehat{B}_{\Gamma I}^{(i)T} & 0 \\ \widehat{A}_{\Gamma I}^{(i)} & \widehat{B}_{\Gamma I}^{(i)} & \widehat{A}_{\Gamma\Gamma}^{(i)} & \widehat{B}_{0\Gamma}^{(i)T} \\ 0 & 0 & \widehat{B}_{0\Gamma}^{(i)} & 0 \end{bmatrix} \begin{bmatrix} u_I^{(i)} \\ p_I^{(i)} \\ u_\Gamma^{(i)} \\ p_0^{(i)} \end{bmatrix} = \begin{bmatrix} f_I^{(i)} \\ f_{p_I}^{(i)} \\ f_\Gamma^{(i)} \\ 0 \end{bmatrix}. \tag{4.2}$$

We can eliminate the subdomain interior variables $u_I^{(i)}$ and $p_I^{(i)}$ in each subdomain independently, and assemble the global interface problem from the subdomain interface problems.

Definition 3. *(Schur complement of the Stokes problem) Define the subdomain Schur complement $S_\Gamma^{(i)}$ for the Stokes problem as follows: given $u_\Gamma^{(i)} \in V_\Gamma^{(i)}$, determine $S_\Gamma^{(i)} u_\Gamma^{(i)} \in F_\Gamma^{(i)}$ such that*

$$\begin{bmatrix} A_{II}^{(i)} & B_{II}^{(i)T} & A_{\Gamma I}^{(i)T} \\ B_{II}^{(i)} & C_{II}^{(i)} & B_{\Gamma I}^{(i)T} \\ A_{\Gamma I}^{(i)} & B_{\Gamma I}^{(i)} & A_{\Gamma\Gamma}^{(i)} \end{bmatrix} \begin{bmatrix} u_I^{(i)} \\ p_I^{(i)} \\ u_\Gamma^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \\ S_\Gamma^{(i)} u_\Gamma^{(i)} \end{bmatrix}. \tag{4.3}$$

The global interface problem can then be written as: to find $(u_\Gamma,\, p_0) \in \left(\widehat{V}_\Gamma, W_0\right)$, such that

$$\widehat{S} \begin{bmatrix} u_\Gamma \\ p_0 \end{bmatrix} = \begin{bmatrix} g_\Gamma \\ 0 \end{bmatrix}. \tag{4.4}$$

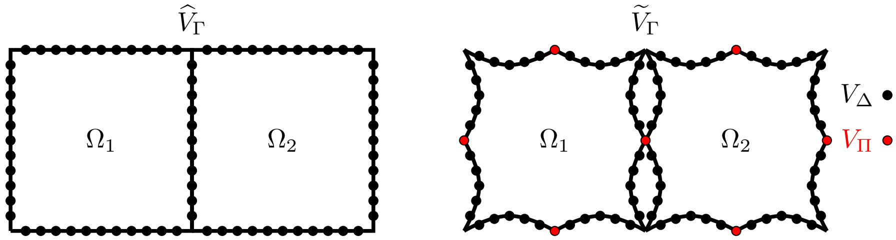
Here the global interface matrix $\widehat{S}$ is defined as

$$\widehat{S} = \begin{bmatrix} \widehat{S}_\Gamma & \widehat{B}_{0\Gamma}^T \\ \widehat{B}_{0\Gamma} & 0 \end{bmatrix},$$

where $\widehat{S}_\Gamma = \sum_{i=1}^N R_\Gamma^{(i)T} S_\Gamma^{(i)} R_\Gamma^{(i)}$, $\widehat{B}_{0\Gamma} = \sum_{i=1}^N B_{0\Gamma}^{(i)} R_\Gamma^{(i)}$, and

$$g_\Gamma = \textstyle\sum_{i=1}^N R_\Gamma^{(i)T} \left\{ f_\Gamma^{(i)} - \begin{bmatrix} A_{\Gamma I}^{(i)} & B_{I\Gamma}^{(i)T} \end{bmatrix} \begin{bmatrix} A_{II}^{(i)} & B_{II}^{(i)T} \\ B_{II}^{(i)} & C_{II}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} f_I^{(i)} \\ f_{p_I}^{(i)} \end{bmatrix} \right\}.$$

FIG. 1. *Left: $\widehat{V}_\Gamma$ for the subdomains Ω_1 and Ω_2; Right: $\widetilde{V}_\Gamma$ for the subdomains Ω_1 and Ω_2, where the midpoint of each edge is chosen as the primal unknown. The red dots are the primal variable V_Π and the black dots are the dual variable V_Δ.*



The operator $\widehat{S}_\Gamma$ is symmetric positive definite but $\widehat{S}$ is symmetric indefinite. In what follows, we will propose a BDDC preconditioner, and show that the preconditioned operator is positive definite when restricted to a proper subspace. A preconditioned conjugate gradient method can then be used to solve the global interface problem.

5. The BDDC Preconditioner. The BDDC (Balancing Domain Decomposition by Constraints) algorithm is a variant of the two-level Neumann-Neumann type preconditioner. It was introduced and analyzed by Dohrmann, Mandel, and Tezaur [5, 20, 21] for standard finite element discretization of elliptic problems. The BBDC preconditioner consists of local solvers for the subdomain problems and the artistically designed global coarse-level problem. The coarse level problem is assembled from primal variables, such as edge/face averages across the subdomain interface on which the continuity constraints are enforced. In contrast to earlier versions of balancing Neumann-Neumann methods, the BDDC methods do not need to solve singular systems and the algorithms demonstrate good scalability for parallel computation.

In order to introduce the BDDC preconditioner, we first introduce a partially assembled interface space $\widetilde{V}_\Gamma$ by

$$\widetilde{V}_\Gamma = \widehat{V}_\Pi \oplus V_\Delta = \widehat{V}_\Pi \oplus \left(\prod_{i=1}^{N} V_\Delta^{(i)} \right).$$

Here, $\widehat{V}_\Pi$ is the continuous, coarse level, primal interface edge velocity space. The variables in this space are called the primal unknowns, and each primal unknown is shared by the adjacent subdomains. The remaining interface velocity variables live in the complimentary dual space V_Δ . This space is the direct sum of the $V_\Delta^{(i)}$, which are spanned by basis functions with vanishing value at the primal degrees of freedom. The functions in V_Δ are generally discontinuous, see Figure 1. Thus, in the space $\widetilde{V}_\Gamma$, we relax the continuity constraints across the interface at the dual variables but retain the continuity at the primal variables, which makes all the component linear systems in the preconditioner nonsingular.

We need to introduce several restriction, extension, and scaling operators between different spaces. $\overline{R}_\Gamma^{(i)} : \widetilde{V}_\Gamma \to V_\Gamma^{(i)}$ restricts functions in the space $\widetilde{V}_\Gamma$ to the components $V_\Gamma^{(i)}$ of the subdomain Ω_i. $\overline{R}_\Gamma : \widetilde{V}_\Gamma \to V_\Gamma$ is the direct sum of $\overline{R}_\Gamma^{(i)}$. $R_\Delta^{(i)} : \widehat{V}_\Gamma \to V_\Delta^{(i)}$ maps the functions from $\widehat{V}_\Gamma$ to $V_\Delta^{(i)}$, its dual subdomain components. $R_{\Gamma\Pi} : \widehat{V}_\Gamma \to \widehat{V}_\Pi$ is a restriction operator from $\widehat{V}_\Gamma$ to its subspace $\widehat{V}_\Pi$. $\widetilde{R}_\Gamma : \widehat{V}_\Gamma \to \widetilde{V}_\Gamma$ is the direct sum of $R_{\Gamma\Pi}$ and $R_{\Delta}^{(i)}$. We define the positive scaling factor $\delta_i^{\dagger}(x)$ as follows:

$$\delta_i^{\dagger}(x) = \frac{1}{card\,(\mathcal{I}_x)}, \qquad x \in \partial\Omega_{i,h} \cap \Gamma_h,$$

where $\mathcal{I}_x$ is the set of indices of the subdomains that have x on their boundaries, and $card\,(\mathcal{I}_x)$ counts the number of the subdomain boundaries to which x belongs. It is clear that $\delta_i^{\dagger}(x)$'s provide a partition of unity, i.e., $\sum_{i\in\mathcal{I}_x} \delta_i^{\dagger}(x) = 1$, for any $x \in \Gamma_h$. We note that $\delta_i^{\dagger}(x)$ is constant on each edge. Multiplying each row of $R_{\Delta}^{(i)}$ with the scaling factor gives us $R_{D,\Delta}^{(i)}$. The scaled operators $\widetilde{R}_{D,\Gamma}$ is the direct sum of $R_{\Gamma\Pi}$ and $R_{D,\Delta}^{(i)}$.

The partially assembled Schur complement $\widetilde{S}_{\Gamma}$, defined on the interface velocity space $\widetilde{V}_{\Gamma}$, can be represented as follows: given $u_{\Gamma} \in \widetilde{V}_{\Gamma}$, $\widetilde{S}_{\Gamma} u_{\Gamma} \in \widetilde{F}_{\Gamma}$ satisfies

$$\begin{bmatrix} A_{II}^{(1)} & B_{II}^{(1)T} & A_{\Delta I}^{(1)T} & \cdots & \widetilde{A}_{\Pi I}^{(1)T} \\ B_{II}^{(1)} & C_{II}^{(1)} & B_{\Delta I}^{(1)T} & \cdots & \widetilde{B}_{\Pi I}^{(1)T} \\ A_{\Delta I}^{(1)} & B_{\Delta I}^{(1)} & A_{\Delta\Delta}^{(1)} & \cdots & \widetilde{A}_{\Pi\Delta}^{(1)T} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \widetilde{A}_{\Pi I}^{(1)} & \widetilde{B}_{\Pi I}^{(1)} & \widetilde{A}_{\Pi\Delta}^{(1)} & \cdots & \widetilde{A}_{\Pi\Pi} \end{bmatrix} \begin{bmatrix} u_I^{(1)} \\ p_I^{(1)} \\ u_{\Delta}^{(1)} \\ \vdots \\ u_{\Pi} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \\ \left(\widetilde{S}_{\Gamma} u_{\Gamma}\right)_{\Delta}^{(1)} \\ \vdots \\ \left(\widetilde{S}_{\Gamma} u_{\Gamma}\right)_{\Pi} \end{bmatrix}.$$

Here, $\widetilde{A}_{\Pi\Pi} = \sum_{i=1}^{N} R_{\Pi}^{(i)T} A_{\Pi\Pi}^{(i)} R_{\Pi}^{(i)}$, $\widetilde{A}_{\Pi I}^{(i)} = R_{\Pi}^{(i)T} A_{\Pi I}^{(i)}$, $\widetilde{A}_{\Pi\Delta}^{(i)} = R_{\Pi}^{(i)T} A_{\Pi\Delta}^{(i)}$, and $\widetilde{B}_{\Pi I}^{(i)} = R_{\Pi}^{(i)T} B_{\Pi I}^{(i)}$.

Based on this definition, we can also obtain $\widetilde{S}_{\Gamma}$ from subdomain Schur complements $S_{\Gamma}^{(i)}$ by assembling with respect to the global degrees of freedom of the primal interface velocities, i.e.,

$$\widetilde{S}_{\Gamma} = \overline{R}_{\Gamma}^{T} S_{\Gamma} \overline{R}_{\Gamma}. \tag{5.1}$$

Here, we denote the direct sum of $S_{\Gamma}^{(i)}$ by S_{Γ}. The global interface Schur operator $\widehat{S}_{\Gamma}$ on the continuous interface velocity space $\widehat{V}_{\Gamma}$ can be obtained by further assembling with respect to the dual interface variables, i.e.,

$$\widehat{S}_{\Gamma} = \widetilde{R}_{\Gamma}^{T} \widetilde{S}_{\Gamma} \widetilde{R}_{\Gamma} = R_{\Gamma}^{T} S_{\Gamma} R_{\Gamma}. \tag{5.2}$$

We note that, for any $x_{\Gamma} \in \widetilde{V}_{\Gamma}$ with $x_{\Gamma}^{T} \widetilde{S}_{\Gamma} x_{\Gamma} = 0$, x_{Γ} has to be a constant on each subdomain. Due to the continuity of the primal components of x_{Γ} and the Dirichlet boundary condtion of (3.1), x_{Γ} has to be zero and therefore $\widetilde{S}_{\Gamma}$ is symmetric positive definite.

Correspondingly, we define an operator $\widetilde{B}_{0\Gamma}$, which maps the partially assembled interface velocity space $\widetilde{V}_{\Gamma}$ into F_0, the space of right-hand sides corresponding to W_0. $\widetilde{B}_{0\Gamma}$ can be obtained from the subdomain operators $B_{0\Gamma}^{(i)}$ by assembling with respect to the primal interface velocity part, i.e., $\widetilde{B}_{0\Gamma} = \sum_{i=1}^{N} B_{0\Gamma}^{(i)} \overline{R}_{\Gamma}^{(i)}$. Similarly, the operator $\widehat{B}_{0\Gamma}$ can be obtained from the partially assembled operator $\widetilde{B}_{0\Gamma}$ by further assembling with respect to the dual interface velocity variables on the subdomain interfaces, i.e., $\widehat{B}_{0\Gamma} = \widetilde{B}_{0\Gamma} \widetilde{R}_{\Gamma}$. By the definition, we have that the $\widetilde{B}_{0\Gamma}$ has a full row rank since $\widehat{B}_{0\Gamma}$ does.

Let

$$\widetilde{R}_D = \begin{bmatrix} \widetilde{R}_{D,\Gamma} & \\ & I \end{bmatrix}, \qquad \widetilde{S} = \begin{bmatrix} \widetilde{S}_\Gamma & \widetilde{B}_{0\Gamma}^T \\ \widetilde{B}_{0\Gamma} & 0 \end{bmatrix}. \tag{5.3}$$

Due to the positive definiteness of $\widetilde{S}_\Gamma$ and the full row rank of $\widetilde{B}_{0\Gamma}$, $\widetilde{S}$ is invertible and we can define the preconditioner for solving the global interface Stokes problem as

$$M^{-1} = \widetilde{R}_D^T \widetilde{S}^{-1} \widetilde{R}_D.$$

Note that $\widetilde{R}_{D,\Gamma}$ is of full rank and that the preconditioner is nonsingular. The preconditioned BDDC algorithm is then of the form: to find $(u_\Gamma, p_0) \in \left(\widehat{V}_\Gamma, W_0\right)$, such that

$$M^{-1}\widehat{S} \begin{bmatrix} u_\Gamma \\ p_0 \end{bmatrix} = M^{-1} \begin{bmatrix} g_\Gamma \\ 0 \end{bmatrix}. \tag{5.4}$$

We require that $\int_{\partial\Omega_i} u_\Delta^{(i)} \cdot \mathbf{n}_i = 0$, for all the dual interface velocity variables $u_\Delta^{(i)} \in V_\Delta^{(i)}$, with $\mathbf{n}_i$ the unit outward normal of $\partial\Omega_i$; see [18, 29]. We will refer to this assumption as the divergence free constraint for the dual velocity variables. When the Conjugate Gradient (CG) method is used to solve the preconditioned system (5.4), the divergence free constraint can ensure the CG iterations will be in a special subspace where the preconditioned operator is positive definite and therefore the CG method can be applied. In order to satisfy this constraint, we choose the primal variables which are spanned by subdomain interface edge/face basis functions with constant values on these edges/faces for two/three dimensions. We change the variables so that the degree of freedom of each primal constraint is explicit; see [19, 14]. The dual space is correspondingly spanned by the remaining interface degrees of freedom with zero average values over the interface edge/face. This constraint is critical to the design of the preconditioner, as we will see more details in Section 6.

At the end of this section, we discuss the implementation of the preconditioner. The main operation is the product of $\widetilde{S}^{-1}$ with a vector, which requires solving a coarse problem related to the primal variables we choose and independent subdomain Stokes problems with Neumann type boundary conditions. The size of the coarse problem will increase with the increasing of the number of the subdomains and it can be a bottleneck of the algorithm. The multilevel extension of the algorithms can be explored as in [32, 31, 22].

6. Some Auxiliary Results.

We adopt the convention that C denotes a generic constant independent of the mesh size h and subdomain size H. In general, its value may vary at different instances.

First, we list two useful results. For shape regular partition $\mathcal{T}_h$ as detailed in [38], the following trace and inverse inequalities hold; see[37].

LEMMA 4. *(Trace Inequality) There exists a constant C such that*

$$\|g\|_e^2 \le C\left(h_K^{-1}\|g\|_K^2 + h_K\|\nabla g\|_K^2\right), \tag{6.1}$$

where $g \in H^1(K)$, and K is an element of $\mathcal{T}_h$ with e as an edge/face.

LEMMA 5. *(Inverse Inequality) There exists a constant $C = C(k)$ such that*

$$\|\nabla g\|_K \le C(k) h_K^{-1} \|g\|_K, \qquad \forall K \in \mathcal{T}_h \tag{6.2}$$

for any piecewise polynomial g of degree k on $\mathcal{T}_h$.

We collect a few results of the weak Galerkin finite element scheme, which will be used in our analysis of the BDDC preconditioner. Note that the discrete weak velocity function space V_k^0 is a normed linear space with a triple-bar norm given by [38, (4.1)]

$$|||v|||^2 = \sum_{K\in\mathcal{T}_h} \|\nabla_w v\|_K^2 + \sum_{K\in\mathcal{T}_h} h_K^{-1} \|Q_b v_0 - v_b\|_{\partial K}^2. \tag{6.3}$$

LEMMA 6. *For the weak Galerkin scheme described in Section 3, the following results hold:*

1. For any $v = \{v_0, v_b\} \in V_k$, we have $\sum_{T\in\mathcal{T}_h} \|\nabla v_0\|_T^2 \le C|||v|||^2$;
2. For any $v \in V_k^0$, $a(v, v) = |||v|||^2$;
3. For any $v, w \in V_k^0$, $|a(v, w)| \le |||v|||\,|||w|||$;
4. For any $v = \{v_0, v_b\} \in V_k^0$, $\rho \in W_{k-1}$, $|b(v, \rho)| \le C|||v|||\,\|\rho\|_{L^2}$;
5. For any $\rho \in W_{k-1}$, $sup_{v\in V_k^0} \frac{b(v,\rho)}{|||v|||_w} \ge \beta \|\rho\|_{L^2}$, where β is positive constant independent of the mesh size h.

Proof. The first result is in [38, Lemma A.2]; the second and third results give the coercivity and boundedness property of the bilinear form $a(\cdot, \cdot)$, which are proved in [38, Lemma 4.1]. The fourth result is the boundedness property of the bilinear form $b(\cdot, \cdot)$. This can be proved as follows.

$$\begin{aligned}
|b(v, \rho)| &= \left| \sum_{K\in\mathcal{T}_h} (\nabla_w \cdot v, \rho)_K \right| \\
&= \left| \sum_{K\in\mathcal{T}_h} \left(-(v_0, \nabla\rho)_K + \langle v_b \cdot n, \rho\rangle_{\partial K}\right) \right| \\
&= \left| \sum_{K\in\mathcal{T}_h} \left((\nabla\cdot v_0, \rho)_K - \langle (Q_b v_0 - v_b)\cdot n, \rho\rangle_{\partial K}\right) \right| \\
&\le C \left(\sum_{K\in\mathcal{T}_h} \|\nabla v_0\|_{L^2(K)}^2 \right)^{1/2} \left(\sum_{K\in\mathcal{T}_h} \|\rho\|_{L^2(K)}^2 \right)^{1/2} \\
&\quad + C \left(\sum_{K\in\mathcal{T}_h} h_K^{-1} \|v_b - Q_b v_0\|_{L^2(\partial K)}^2 \right)^{1/2} \left(\sum_{K\in\mathcal{T}_h} h_k \|\rho\|_{L^2(\partial K)}^2 \right)^{1/2} \\
&\le C|||v|||\,\|\rho\|_{L^2},
\end{aligned}$$

where we use the definition of weak divergence for the second equality, and integration by parts for the third equality. We use the Cauchy-Schwarz inequality for the fourth inequality. Part (1) of Lemma 6, the definition of the triple-bar norm (6.3), and the trace inequality (6.1) and the inverse inequality (6.2) for the last inequality.

The last result is the discrete inf-sup condition, which is proved in [38, Lemma 4.3]. These results also hold for the subdomain Ω_i. If follows that the weak Galerkin scheme is well-posed for the global interface problem and local subdomain problems.∎

We introduce several conceptual tools which will be useful in our analysis of the BDDC preconditioner.

Definition 7. *(Schur complement of the subdomain elliptic problem) The subdomain Schur complement for the elliptic problem, denoted by $S^{(i)}_{\Gamma,E}$, is defined as follows: given $u^{(i)}_{\Gamma} \in V^{(i)}_{\Gamma}$, determine $S^{(i)}_{\Gamma,E} u^{(i)}_{\Gamma} \in F^{(i)}_{\Gamma}$ such that*

$$A^{(i)} \begin{bmatrix} u^{(i)}_I \\ u^{(i)}_{\Gamma} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ S^{(i)}_{\Gamma,E} u^{(i)}_{\Gamma} \end{bmatrix},$$

where $A^{(i)} = \begin{bmatrix} A^{(i)}_{II} & A^{(i)T}_{\Gamma I} \\ A^{(i)}_{\Gamma I} & A^{(i)}_{\Gamma\Gamma} \end{bmatrix}$.

Since the subdomain elliptic problem $A^{(i)}$ is symmetric positive definite [36], the Schur complement $S^{(i)}_{\Gamma,E}$ is also symmetric positive definite by the inertia of Schur complements [19]. Thus, we can define the norm

$$\left| u^{(i)} \right|^2_{A^{(i)}} = u^{(i)T} A^{(i)} u^{(i)} = a\left(u^{(i)},\, u^{(i)}\right), \text{ for all } u^{(i)} \in V^{(i)},$$

and

$$\left| u^{(i)}_{\Gamma} \right|^2_{S^{(i)}_{\Gamma,E}} = u^{(i)T}_{\Gamma} S^{(i)}_{\Gamma,E} u^{(i)}_{\Gamma}, \text{ for all } u^{(i)}_{\Gamma} \in V^{(i)}_{\Gamma}.$$

Similarly, the subdomain Schur complements for the Stokes problems, defined in (4.3), are symmetric, positive semi-definite [18]. They are singular for any floating subdomains, by which we mean the boundary of the subdomain does not intersect with the global domain boundary $\partial\Omega$. Thus, we can define the $S^{(i)}_{\Gamma}-$ seminorms by

$$\left| u^{(i)}_{\Gamma} \right|^2_{S^{(i)}_{\Gamma}} = u^{(i)T}_{\Gamma} S^{(i)}_{\Gamma} u^{(i)}_{\Gamma}, \text{ for all } u^{(i)}_{\Gamma} \in V^{(i)}_{\Gamma}.$$

It follows that

$$|u_{\Gamma}|^2_{S_{\Gamma}} = u^T_{\Gamma} S_{\Gamma} u_{\Gamma} = \sum_{i=1}^{N} \left| u^{(i)}_{\Gamma} \right|^2_{S^{(i)}_{\Gamma}}.$$

The fully and partially assembled global interface velocity operators $\widehat{S}_{\Gamma}$ and $\widetilde{S}_{\Gamma}$, given in (5.2) and (5.1), are both symmetric, positive definite because of the Dirichlet boundary conditions on $\partial\Omega$ and the adequacy of the primal continuity constraints for the divergence free condition. In similar way as before, we define the $\widehat{S}_{\Gamma}-$ and $\widetilde{S}_{\Gamma}-$norms on the spaces $\widehat{V}_{\Gamma}$ and $\widetilde{V}_{\Gamma}$, respectively, as below.

$$\|u_{\Gamma}\|^2_{\widehat{S}_{\Gamma}} = u^T_{\Gamma} \widehat{S}_{\Gamma} u_{\Gamma} = u^T_{\Gamma} R^T_{\Gamma} S_{\Gamma} R_{\Gamma} u_{\Gamma} = |R_{\Gamma} u_{\Gamma}|^2_{S_{\Gamma}}, \qquad \forall u_{\Gamma} \in \widehat{V}_{\Gamma},$$

$$\|u_{\Gamma}\|^2_{\widetilde{S}_{\Gamma}} = u^T_{\Gamma} \widetilde{S}_{\Gamma} u_{\Gamma} = u^T_{\Gamma} \overline{R}^T_{\Gamma} S_{\Gamma} \overline{R}_{\Gamma} u_{\Gamma} = \left|\overline{R}_{\Gamma} u_{\Gamma}\right|^2_{S_{\Gamma}}, \qquad \forall u_{\Gamma} \in \widetilde{V}_{\Gamma}.$$

The global interface operator $\widehat{S}$ and $\widetilde{S}$, introduced in (4.4) and (5.3), are symmetric indefinite on the space $\widehat{V}_{\Gamma} \times W_0$ and $\widetilde{V}_{\Gamma} \times W_0$, respectively. However, when restricted to the proper subspaces, these operators can be positive semidefinite, and we can thus define a $\widehat{S}-$ and $\widetilde{S}-$seminorms on these subspaces. We call such subspaces as the benign subspaces, and denote them by $\widehat{V}_{\Gamma,B} \times W_0$ and $\widetilde{V}_{\Gamma,B} \times W_0$, respectively. Specifically, they can be defined as follows.

Definition 8. *(Benign subspaces)*

$$\widehat{V}_{\Gamma,B} = \left\{ u_\Gamma \in \widehat{V}_\Gamma | \widehat{B}_{0\Gamma} u_\Gamma = 0 \right\} \quad \textit{and} \quad \widetilde{V}_{\Gamma,B} = \left\{ u_\Gamma \in \widetilde{V}_\Gamma | \widetilde{B}_{0\Gamma} u_\Gamma = 0 \right\}.$$

If follows that we can define

$$|u|^2_{\widehat{S}} = u^T \widehat{S} u, \qquad \forall u = (u_\Gamma,\, p_0) \in \widehat{V}_{\Gamma,B} \times W_0,$$

$$|u|^2_{\widetilde{S}} = u^T \widetilde{S} u, \qquad \forall u = (u_\Gamma,\, p_0) \in \widetilde{V}_{\Gamma,B} \times W_0.$$

We can show by direct computation that the following facts hold.

$$|u|^2_{\widehat{S}} = \|u_\Gamma\|^2_{\widehat{S}_\Gamma}, \qquad \forall u = (u_\Gamma,\, p_0) \in \widehat{V}_{\Gamma,B} \times W_0,$$

$$|u|^2_{\widetilde{S}} = \|u_\Gamma\|^2_{\widetilde{S}_\Gamma}, \qquad \forall u = (u_\Gamma,\, p_0) \in \widetilde{V}_{\Gamma,B} \times W_0.$$

We denote the null space of the $\widehat{S}-$seminorm operator on the space $\widehat{V}_{\Gamma,B} \times W_0$ by $\widehat{Z}$. It is easy to see that this space is comprised of elements $u = (0,\, p_0) \in \widehat{V}_{\Gamma,B} \times W_0$.

The following lemma is crucial to the analysis of the preconditioned BDDC operator. The proof can be found in [18, 29].

Lemma 9. *Under the divergence free constraint for the dual interface velocities, introduced in Section 5, we have $\widetilde{R}_D^T u \in \widehat{V}_{\Gamma,B} \times W_0$ for any $u \in \widetilde{V}_{\Gamma,B} \times W_0$.*

With the choice of the primal velocity continuity constraints of the BDDC algorithm, the preconditioned BDDC operator $M^{-1}\widehat{S}$ is positive definite on the quotient space, and correspondingly, we can use the preconditioned conjugate gradient method when the iterations are restricted to the quotient space. The design of the BDDC preconditioner and the result from Lemma 9 guarantee that the iterations of the preconditioned conjugate gradient method will stay in the quotient subspace if the initialization lies in the quotient subspace [18].

Next we introduce two important extension operators for the trace over the subdomain boundary.

Definition 10. *(Discrete harmonic extension) The discrete harmonic extension of $\gamma \in V_\Gamma^{(i)}$ over the subdomain Ω_i, denoted by $\mathcal{H}(\gamma) : V_\Gamma^{(i)} \to V^{(i)}$, satisfies the following:*

$$\begin{cases} a\left(\mathcal{H}(\gamma),\, v\right) = 0, \qquad \forall v = \{v_0,\, v_b\} \in V_k^0\left(\Omega_i\right), \\ \mathcal{H}(\gamma)\,|_{\partial\Omega_i} = \gamma. \end{cases}$$

The bilinear form $a(\cdot,\, \cdot)$ is defined in (3.3).

Definition 11. *(Discrete Stokes extension) The discrete Stokes extension of $\gamma \in V_\Gamma^{(i)}$ over the subdomain Ω_i, denoted by $\mathcal{S}(\gamma) : V_\Gamma^{(i)} \to V^{(i)}$, satisfies the following:*

$$\begin{cases} a\left(\mathcal{S}(\gamma),\, v\right) - b\left(v,\, \mathcal{P}(\gamma)\right) = 0, \qquad \forall v = \{v_0,\, v_b\} \in V_k^0\left(\Omega_i\right), \\ b\left(\mathcal{S}(\gamma),\, q\right) = 0, \qquad \forall q \in W_{k-1}\left(\Omega_i\right), \\ \mathcal{S}(\gamma)\,|_{\partial\Omega_i} = \gamma, \end{cases}$$

where $\mathcal{P}(\gamma)$ is the corresponding pressure extension with zero mean value living in the space $W_{k-1}(\Omega_i)$. The bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are defined in (3.3).

The connection between the discrete harmonic/Stokes extensions and the Schur complements of the corresponding linear systems can be revealed as follows.

REMARK 1. *By definition, it is clear that*

$$\left|u_\Gamma^{(i)}\right|^2_{S^{(i)}_{\Gamma,E}} = \left|\mathcal{H}\left(u_\Gamma^{(i)}\right)\right|^2_{A^{(i)}} = \inf_{u^{(i)}\in V^{(i)},\, u^{(i)}|_{\partial\Omega_i}=u_\Gamma^{(i)}} \left|u^{(i)}\right|^2_{A^{(i)}},$$

and that

$$\left|u_\Gamma^{(i)}\right|^2_{S^{(i)}_{\Gamma}} = \left|\mathcal{S}\left(u_\Gamma^{(i)}\right)\right|^2_{A^{(i)}} = \inf_{u^{(i)}\in V^{(i)},\, u^{(i)}|_{\partial\Omega_i}=u_\Gamma^{(i)},\, B^{(i)}u^{(i)}=0} \left|u^{(i)}\right|^2_{A^{(i)}}.$$

For the same edge velocities $u_\Gamma^{(i)}$ over the subdomain boundary $\partial\Omega_i$, we have

$$\left|\mathcal{H}\left(u_\Gamma^{(i)}\right)\right|^2_{A^{(i)}} \le \left|\mathcal{S}\left(u_\Gamma^{(i)}\right)\right|^2_{A^{(i)}},$$

since the infimum over a larger set is smaller. It follows that

$$\left|u_\Gamma^{(i)}\right|^2_{S^{(i)}_{\Gamma,E}} \le \left|u_\Gamma^{(i)}\right|^2_{S^{(i)}_{\Gamma}}.$$

Next, we prove the connection between the edge velocity seminorms defined by the Schur complements of the elliptic and Stokes problems for the same subdomain. Similar proof for the conforming discretizations can be found in [2].

LEMMA 12. *For any $u_\Gamma^{(i)} \in V_\Gamma^{(i)}$, we have*

$$C\frac{\beta^2}{(1+\beta)^2}\left|u_\Gamma^{(i)}\right|^2_{S^{(i)}_{\Gamma}} \le \left|u_\Gamma^{(i)}\right|^2_{S^{(i)}_{\Gamma,E}} \le \left|u_\Gamma^{(i)}\right|^2_{S^{(i)}_{\Gamma}},$$

where β is the inf-sup stability constant defined in Lemma 6.

Proof. The second inequality directly follow from the *Remark*.

We prove the first inequality as follows. Denote the discrete harmonic and Stokes extension of $u_\Gamma^{(i)} \in V_\Gamma^{(i)}$ by $\mathcal{H}\left(u_\Gamma^{(i)}\right)$ and $\mathcal{S}\left(u_\Gamma^{(i)}\right)$, respectively. Using $v = \mathcal{S}\left(u_\Gamma^{(i)}\right) - \mathcal{H}\left(u_\Gamma^{(i)}\right)$ as the test function in Definition 11, we have

$$a\left(\mathcal{S}\left(u_\Gamma^{(i)}\right), \mathcal{S}\left(u_\Gamma^{(i)}\right) - \mathcal{H}\left(u_\Gamma^{(i)}\right)\right) - b\left(\mathcal{S}\left(u_\Gamma^{(i)}\right) - \mathcal{H}\left(u_\Gamma^{(i)}\right), \rho\right) = 0,$$

where ρ is the corresponding pressure extension with zero mean value living in the space $W_{k-1}(\Omega_i)$.

Since $b\left(\mathcal{S}\left(u_\Gamma^{(i)}\right), \rho\right) = 0$, it follows that

$$a\left(\mathcal{S}\left(u_\Gamma^{(i)}\right), \mathcal{S}\left(u_\Gamma^{(i)}\right)\right) = a\left(\mathcal{S}\left(u_\Gamma^{(i)}\right), \mathcal{H}\left(u_\Gamma^{(i)}\right)\right) + b\left(\mathcal{H}\left(u_\Gamma^{(i)}\right), \rho\right).$$

By the part (4) in Lemma 6, we have

$$\left|\mathcal{S}\left(u_\Gamma^{(i)}\right)\right|^2_{A^{(i)}} \le \left|\mathcal{S}\left(u_\Gamma^{(i)}\right)\right|_{A^{(i)}} \left|\mathcal{H}^{(i)}\left(u_\Gamma^{(i)}\right)\right|_{A^{(i)}} + C\left|\mathcal{H}\left(u_\Gamma^{(i)}\right)\right|_{A^{(i)}} \|\rho\|_{L^2(\Omega_i)} \tag{6.4}$$

By the inf-sup condition (the part (5) in Lemma 6) ,

$$\| \rho \|_{L^2(\Omega_i)}^2 \le \beta^{-2} \sup_{v\in V_k^0(\Omega_i)} \frac{b(v,\rho)^2}{|||v|||^2}$$

$$= \beta^{-2} \sup_{v\in V_k^0(\Omega_i)} \frac{a\left(\mathcal{S}\left(u_\Gamma^{(i)}\right), v\right)^2}{|||v|||^2} \tag{6.5}$$

$$\le \beta^{-2} \left|\left|\left|\mathcal{S}\left(u_\Gamma^{(i)}\right)\right|\right|\right|^2 = \beta^{-2}\left|\mathcal{S}\left(u_\Gamma^{(i)}\right)\right|_{A^{(i)}}^2,$$

where we have used Definition 11 for the second equality and the parts (2) and (3) in Lemma 6 for the last inequality.

Substituting (6.5) into (6.4), we have

$$\left|\mathcal{S}\left(u_\Gamma^{(i)}\right)\right|_{A^{(i)}}^2 \le \left|\mathcal{S}\left(u_\Gamma^{(i)}\right)\right|_{A^{(i)}} \left|\mathcal{H}\left(u_\Gamma^{(i)}\right)\right|_{A^{(i)}} + C\beta^{-1}\left|\mathcal{S}\left(u_\Gamma^{(i)}\right)\right|_{A^{(i)}} \left|\mathcal{H}\left(u_\Gamma^{(i)}\right)\right|_{A^{(i)}}$$
$$\le C\frac{1+\beta}{\beta}\left|\mathcal{S}\left(u_\Gamma^{(i)}\right)\right|_{A^{(i)}} \left|\mathcal{H}\left(u_\Gamma^{(i)}\right)\right|_{A^{(i)}}.$$

It follows that

$$C\tfrac{\beta^2}{(1+\beta)^2}\left|u_\Gamma^{(i)}\right|_{S_{\Gamma,S}^{(i)}}^2 = C\tfrac{\beta^2}{(1+\beta)^2}\left|\mathcal{S}\left(u_\Gamma^{(i)}\right)\right|_{A^{(i)}}^2 \le \left|\mathcal{H}\left(u_\Gamma^{(i)}\right)\right|_{A^{(i)}}^2 = \left|u_\Gamma^{(i)}\right|_{S_{\Gamma,E}^{(i)}}^2. \qquad \square$$

In order to prove the condition number bounds for the BDDC preconditioner, we define an averaging operator for the Stokes problem, denoted by E_D, which maps $\widetilde{V}_\Gamma \times W_0$, with generally discontinuous interface velocities, to the same space with continuous interface velocities. Specifically, for any $u = (u_\Gamma, p_0) \in \widetilde{V}_\Gamma \times W_0$, $E_D\left[u_\Gamma, p_0\right]^T \in \widetilde{V}_\Gamma \times W_0$, where

$$E_D = \widetilde{R}\widetilde{R}_D^T = \begin{bmatrix}\widetilde{R}_\Gamma & \\ & I\end{bmatrix}\begin{bmatrix}\widetilde{R}_{D,\Gamma}^T & \\ & I\end{bmatrix} = \begin{bmatrix}E_{D,\Gamma} \\ I\end{bmatrix},$$

and $E_{D,\Gamma} = \widetilde{R}_\Gamma \widetilde{R}_{D,\Gamma}^T$ is the interface averaging operator for the velocities across the interface Γ. The operator $E_{D,\Gamma}$ computes a weighted average for the edge velocity across the subdomain interface Γ, and then distributes the average back to the original degree of freedoms on the interface.

To facilitate further analysis, we introduce a useful norm as defined in [12]:

$$|||\lambda|||_{h,D}^* = \left(\sum_{K\in\mathcal{T}_h, K\subseteq\overline{D}} \frac{1}{h}\|\lambda - m_K(\lambda)\|_{\partial K}^2\right)^{1/2}, \tag{6.6}$$

where

$$m_K(\lambda) = \frac{1}{|\partial K|}\int_{\partial K} \lambda ds.$$

Denote $|||\lambda|||_h^* = |||\lambda|||_{h,\Omega}^*$.

Define the local lifting operators $\boldsymbol{\mathcal{Q}}(\cdot)$ and $\mathcal{U}(\cdot)$ for the weak Galerkin (WG) method as below: given λ on ∂K,

$$(\boldsymbol{\mathcal{Q}}\lambda, \boldsymbol{r})_K + (\mathcal{U}\mu, \nabla\cdot\boldsymbol{r})_K = \langle \lambda, \boldsymbol{r}\cdot\boldsymbol{n}\rangle_{\partial K} \quad \text{for all } \boldsymbol{r}\in [P_{k-1}(K)]^{d\times d}, \tag{6.7a}$$

$$-(w, \nabla\cdot\boldsymbol{\mathcal{Q}}\lambda)_K + \langle h_K^{-1}(Q_b\mathcal{U}\lambda - \lambda), Q_b w\rangle_{\partial K} = 0 \quad \text{for all } w\in [P_k(K)]^d. \tag{6.7b}$$

Let $u = (u_0, u_b) = (\mathcal{U}\lambda, \lambda)$. We have $\boldsymbol{\mathcal{Q}}\lambda = \nabla_w u$ and can obtain a reduced norm of λ using the norm from the WG bilinear form as given in (6.3) as

$$(6.8) \qquad |||\lambda|||_D^2 = \textstyle\sum_{K\in\mathcal{T}_h, K\subseteq\overline{D}} \|\nabla_w u\|_K^2 + \sum_{K\in\mathcal{T}_h, K\subseteq\overline{D}} h_K^{-1} \|Q_b u_0 - u_b\|_{\partial K}^2.$$

Denote $|||\lambda||| = |||\lambda|||_\Omega$

We will show the equivalence between the triple-bar norm defined above in (6.6) and (6.8). To denote the triple-bar norm defined over an element K, we add a subscript K to it. Note that similar strategy was used to prove the equivalence between the norm generated by the bilinear form from a hybridized mixed method and triple-bar norm (6.6) in [12].

LEMMA 13. *The function $|||\lambda|||_K$ is zero on $K \in \mathcal{T}_h$ if and only if λ is constant on ∂K.*

Proof. Assume that $|||\lambda|||_K = 0$ on K. It follows that

$$0 = (\nabla_w u, \nabla_w u) + h_K^{-1}\langle Q_b\mathcal{U}\lambda - \lambda, Q_b\mathcal{U}\lambda - \lambda\rangle_{\partial K},$$

where $u = \{\mathcal{U}\lambda, \lambda\}$, and $\nabla_w u = \boldsymbol{\mathcal{Q}}\lambda$. This implies that $\nabla_w u = 0$ on element K and $Q_b\mathcal{U}\lambda = \lambda$ on ∂K. Further, we have from the definition of the discrete weak gradient operator or the lifting operator $\boldsymbol{\mathcal{Q}}$ given in (6.7b) that for any $\tau \in [P_{k-1}(K)]^n$,

$$\begin{aligned}
0 &= (\nabla_w u, \tau)_K \\
&= -(\mathcal{U}\lambda, \nabla\cdot\tau)_K + \langle\lambda, \tau\cdot n\rangle_{\partial K} \\
&= (\nabla\mathcal{U}\lambda, \tau)_K - \langle\mathcal{U}\lambda - \lambda, \tau\cdot n\rangle_{\partial K} \\
&= (\nabla\mathcal{U}\lambda, \tau)_K - \langle Q_b\mathcal{U}\lambda - \lambda, \tau\cdot n\rangle_{\partial K} \\
&= (\nabla\mathcal{U}\lambda, \tau)_K.
\end{aligned}$$

Let $\tau = \nabla\mathcal{U}\lambda$. Then we have $\nabla\mathcal{U}\lambda = 0$ on K. It follows that $\mathcal{U}\lambda = const.$ on K. Thus, $Q_b\mathcal{U}\lambda = const.$ on ∂K. Since $Q_b\mathcal{U}\lambda = \lambda$ on ∂K, we have $\lambda = const.$ Note that similar argument as above was provided in [38] to prove that (6.3) gives a norm.

Conversely, assume λ is a constant on ∂K. Substituting the ordered pair (r, w) in (6.7) with $(\boldsymbol{\mathcal{Q}}\lambda, \mathcal{U}\lambda)$ and adding up, we obtain

$$|||\lambda|||_K^2 = \langle\lambda,\ \boldsymbol{\mathcal{Q}}\lambda\cdot n\rangle_{\partial K} - h_K^{-1}\langle Q_b\mathcal{U}\lambda - \lambda,\ \lambda\rangle_{\partial K}.$$

Let $w = \lambda$ be the test function in (6.7b). Since λ is constant, $\lambda = Q_b\lambda$. It follows from (6.7b) that

$$-\langle\lambda,\ \boldsymbol{\mathcal{Q}}\lambda\cdot n\rangle_{\partial K} + h_K^{-1}\langle Q_b\mathcal{U}\lambda - \lambda,\ \lambda\rangle_{\partial K} = 0. \qquad \square$$

Therefore, $|||\lambda|||_K = 0$.

LEMMA 14. *Let $M_h = \{v_b : v = \{v_0, v_b\} \in V_k^0\}$. For all $\lambda \in M_h$,*

$$c|||\lambda|||_h^{*,2} \le |||\lambda|||^2 \le C|||\lambda|||_h^{*,2}.$$

Proof. First, we prove the lower bound. By Lemma 13, $|||\lambda|||_K = 0$ implies that λ is constant on ∂K. Similarly as in [12], by a scaling argument, it can be shown that

$$|||\lambda|||_K \ge \frac{c}{|\partial K|^{1/2}} \inf_{\kappa\in\mathbb{R}} \|\lambda - \kappa\|_{\partial K} = \frac{c}{|\partial K|^{1/2}} \|\lambda - m_K(\lambda)\|_{\partial K} = c|||\lambda|||_{h,K}^*,$$

for some constant c independent of λ.
Next, we prove the upper bound. Let $r = \boldsymbol{\mathcal{Q}}\lambda$, and $w = \mathcal{U}\lambda$. Plugging the ordered pair $(r,\, w)$ into (6.7), and adding up, we obtain

$$\begin{aligned}
|||\lambda|||_K^2 &= \langle \lambda,\ \boldsymbol{\mathcal{Q}}\lambda \cdot n\rangle_{\partial K} - h_K^{-1}\langle Q_b\mathcal{U}\lambda - \lambda,\ \lambda\rangle_{\partial K}\\
&= \langle \lambda,\ \boldsymbol{\mathcal{Q}}\lambda \cdot n - h_K^{-1}(Q_b\mathcal{U}\lambda - \lambda)\rangle_{\partial K}\\
&= \langle \lambda - m_K(\lambda),\ \boldsymbol{\mathcal{Q}}\lambda \cdot n - h_K^{-1}(Q_b\mathcal{U}\lambda - \lambda)\rangle_{\partial K}\\
&\le \frac{C}{|\partial K|^{1/2}}\, \|\lambda - m_K(\lambda)\|_{\partial K}\, |||\lambda|||_K\\
&= C|||\lambda|||_{h,K}^{*}|||\lambda|||_K,
\end{aligned}$$

where we have used (6.7b) for the third equality, the trace inequality (6.1) and inverse inequality (6.2) for the second-to-last inequality. It follows that

$$c|||\lambda|||_{h,K}^{*,2} \le |||\lambda|||_K^2 \le C|||\lambda|||_{h,K}^{*,2}.$$

Summing up over all elements in $\mathcal{T}_h$, we obtain

$$c|||\lambda|||_h^{*,2} \le |||\lambda|||^2 \le C|||\lambda|||_h^{*,2}. \qquad \square$$

Based on the equivalence of norms in Lemma 14, similar to the proof of [35, Lemma 5], we can obtain that the interface averaging operator $E_{D,\Gamma}$ satisfies the following bound:

LEMMA 15. *For any $w_\Gamma \in \widetilde{V}_\Gamma$,*

$$|E_{D,\Gamma}w_\Gamma|_{\widetilde{S}_{\Gamma,\,E}}^2 \le C\left(1 + \log\frac{H}{h}\right)^2 |w_\Gamma|_{\widetilde{S}_{\Gamma,\,E}}^2,$$

where C is a positive constant independent of the domain size H, and mesh size h.

Now, we are in a position to prove the bound of the averaging operator E_D for the Stokes problem.

LEMMA 16. *There exists a positive constant C, which is independent of H and h, such that*

$$|E_Dw|_{\widetilde{S}}^2 \le C\left(\frac{1+\beta}{\beta}\right)^2\left(1 + \log\frac{H}{h}\right)^2 |w|_{\widetilde{S}}^2 \qquad \forall w = (w_\Gamma,\, q_0) \in \widetilde{V}_{\Gamma,\,B} \times W_0,$$

where β is the inf-sup stability constant.

Proof. For any vector $w = (w_\Gamma,\, q_0) \in \widetilde{V}_{\Gamma,\,B} \times W_0$, by Lemma 9, $\widetilde{R}_D^T w \in \widehat{V}_{\Gamma,B} \times W_0$. Thus, $E_Dw = \widetilde{R}\widetilde{R}_D^T w \in \widetilde{V}_{\Gamma,\,B} \times W_0$.

From the definition of the $\widetilde{S}$-seminorm, we have $|E_Dw|_{\widetilde{S}}^2 = \|E_{D,\Gamma}w_\Gamma\|_{\widetilde{S}_\Gamma}^2 = |\bar{R}_\Gamma\,(E_{D,\Gamma}w_\Gamma)\,|_{S_\Gamma}^2$.

Noting that $S_\Gamma = diag(S_\Gamma^{(i)})$, and applying Lemma 12 to each subdomain, we have

$|\bar{R}_\Gamma\,(E_{D,\Gamma}w_\Gamma)\,|_{S_\Gamma}^2 \le C\left(\frac{1+\beta}{\beta}\right)^2 |\bar{R}_\Gamma\,(E_{D,\Gamma}w_\Gamma)\,|_{S_{\Gamma,E}}^2$

Further, we have

$$|\bar{R}_\Gamma (E_{D,\Gamma} w_\Gamma)|^2_{S_{\Gamma,E}} = |E_{D,\Gamma} w_\Gamma|^2_{\widetilde{S}_{\Gamma,E}} \le C\left(1+\log\frac{H}{h}\right)^2 |w_\Gamma|^2_{\widetilde{S}_{\Gamma,E}}$$
$$\le C\left(1+\log\frac{H}{h}\right)^2 |w_\Gamma|^2_{\widetilde{S}_\Gamma}.$$

Combining these inequalities, we have
$|E_D w|^2_{\widetilde{S}} \le C\left(\frac{1+\beta}{\beta}\right)^2 \left(1+\log\frac{H}{h}\right)|w_\Gamma|^2_{\widetilde{S}_\Gamma} = C\left(\frac{1+\beta}{\beta}\right)^2\left(1+\log\frac{H}{h}\right)^2 |w|^2_{\widetilde{S}}$. ◻

7. Condition number estimate for the BDDC preconditioner.

We are now ready to formulate and prove our main results. It follows by proving the lower and upper bound for $u^T M^{-1}\widehat{S}u$. See similar proof in [18].

THEOREM 17. *Assume the divergence free constraint holds for the interface velocities. The preconditioned operator $M^{-1}\widehat{S}$ is symmetric, positive definite with respect to the bilinear form $\langle\cdot,\cdot\rangle_{\widehat{S}}$ on the space $\widehat{V}_{\Gamma,B}\times W_0$. Its eigenvalues are bounded from below by 1 and from above by $C\frac{(1+\beta)^2}{\beta^2}\left(1+\log\frac{H}{h}\right)^2$, where C is a constant which is independent of the domain size H, and the mesh size h, and β is the inf-sup stability constant.*

Proof. It is sufficient to prove that for any $u=(u_\Gamma,\, p_0)\in\widehat{V}_{\Gamma,B}\times W_0$, with $u_\Gamma\neq 0$,
$\langle u,\, u\rangle_{\widehat{S}} \le \left\langle u,\, M^{-1}\widehat{S}u\right\rangle_{\widehat{S}} \le C\left(\frac{1+\beta}{\beta}\right)^2\left(1+\log\left(\frac{H}{h}\right)\right)^2\langle u,\, u\rangle_{\widehat{S}}$.

In what follows, we prove the lower and upper bound for $\left\langle u,\, M^{-1}\widehat{S}u\right\rangle_{\widehat{S}}$ respectively.

Let $\widetilde{u}=\widetilde{S}^{-1}\widetilde{R}_D\widehat{S}u$. Obviously, $\widetilde{u}\in\widetilde{V}_{\Gamma,B}\times W_0$.

Note that $\widetilde{R}^T\widetilde{R}_D=\widetilde{R}_D^T\widetilde{R}=I$. The details for the proof of the lower bound go as follows:

$$\langle u,\, u\rangle_{\widehat{S}} = u^T\widehat{S}\widetilde{R}_D^T\widetilde{R}u = u^T\widehat{S}\widetilde{R}_D^T\widetilde{S}^{-1}\widetilde{S}\widetilde{R}u = \left\langle\widetilde{u},\widetilde{R}u\right\rangle_{\widetilde{S}}$$
$$\le \langle\widetilde{u},\widetilde{u}\rangle^{1/2}_{\widetilde{S}}\left\langle\widetilde{R}u,\widetilde{R}u\right\rangle^{1/2}_{\widetilde{S}} = \langle\widetilde{u},\widetilde{u}\rangle^{1/2}_{\widetilde{S}}\langle u,u\rangle^{1/2}_{\widehat{S}}.$$

Thus, we obtain $\langle u,\, u\rangle_{\widehat{S}}\le\langle\widetilde{u},\widetilde{u}\rangle_{\widetilde{S}}$ by canceling a common factor and squaring on both sides.

Since
$\langle\widetilde{u},\widetilde{u}\rangle_{\widetilde{S}} = u^T\widehat{S}\widetilde{R}_D^T\widetilde{S}^{-1}\widetilde{S}\widetilde{S}^{-1}\widetilde{R}_D\widehat{S}u = \left\langle u,\,\widetilde{R}_D^T\widetilde{S}^{-1}\widetilde{R}_D\widehat{S}u\right\rangle_{\widehat{S}} = \left\langle u,\, M^{-1}\widehat{S}u\right\rangle_{\widehat{S}}$,
we have $\langle u,\, u\rangle_{\widehat{S}}\le\left\langle u,\, M^{-1}\widehat{S}u\right\rangle_{\widehat{S}}$.

Next, we prove the upper bound.

Since $M^{-1}=\widetilde{R}_D^T\widetilde{S}^{-1}\widetilde{R}_D$, we have $\widetilde{R}_D^T\widetilde{u}=M^{-1}\widehat{S}u$.

By using Lemma 16 and the fact that $\widehat{S}=\widetilde{R}^T\widetilde{S}\widetilde{R}$, we obtain

$$\left\langle M^{-1}\widehat{S}u,\, M^{-1}\widehat{S}u\right\rangle_{\widehat{S}} = \left\langle\widetilde{R}_D^T\widetilde{u},\,\widetilde{R}_D^T\widetilde{u}\right\rangle_{\widehat{S}} = \left\langle\widetilde{R}\widetilde{R}_D^T\widetilde{u},\,\widetilde{R}\widetilde{R}_D^T\widetilde{u}\right\rangle_{\widetilde{S}} = |E_D\widetilde{u}|^2_{\widetilde{S}}$$
$$\le C\left(\frac{1+\beta}{\beta}\right)^2\left(1+\log\frac{H}{h}\right)^2|\widetilde{u}|^2_{\widetilde{S}}$$
$$\le C\left(\frac{1+\beta}{\beta}\right)^2\left(1+\log\frac{H}{h}\right)^2\left\langle u,\, M^{-1}\widehat{S}u\right\rangle_{\widehat{S}}$$

TABLE 1
Condition number estimates and iteration counts for the BDDC preconditioned operator with changing subdomains numbers. $\frac{H}{h} = 8$*, and* $k = 1$*.*

Number of Subdomains	Iterations	Condition number
4×4	11	4.12
8×8	13	5.01
16×16	13	4.90
24×24	13	5.05
32×32	12	4.94

TABLE 2
Condition number estimates and iteration counts for the BDDC preconditioned operator with changing subdomains numbers. $\frac{H}{h} = 8$*, and* $k = 2$*.*

Number of Subdomains	Iterations	Condition number
4×4	13	7.37
8×8	17	9.24
16×16	20	9.89
24×24	20	10.29
32×32	19	10.26

Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\left\langle u,\, M^{-1}\widehat{S}u\right\rangle_{\widehat{S}} &\le \langle u,\, u\rangle_{\widehat{S}}^{1/2}\left\langle M^{-1}\widehat{S}u,\, M^{-1}\widehat{S}u\right\rangle_{\widehat{S}}^{1/2}\\ &\le C\frac{1+\beta}{\beta}\left(1+\log\frac{H}{h}\right)\langle u,\, u\rangle_{\widehat{S}}^{1/2}\left\langle u,\, M^{-1}\widehat{S}u\right\rangle_{\widehat{S}}^{1/2}.\end{aligned}$$

This gives $\left\langle u,\, M^{-1}\widehat{S}u\right\rangle_{\widehat{S}} \le C\left(\frac{1+\beta}{\beta}\right)^2\left(1+\log\frac{H}{h}\right)^2\langle u,\, u\rangle_{\widehat{S}}$. The upper bound of the eigenvalues thus follows. ∎

8. Numerical Experiments. In this section, we will report some numerical results for the BDDC algorithm proposed for the weak Galerkin discretization of the Stokes problem. We used the BDDC algorithm to solve the model problem (3.1) on the square domain $\Omega = [0,\, 1]^2$ with zero Dirichlet boundary condition. The analytical solution of the test problem is given by

$$u = \begin{bmatrix} \sin^3(\pi x)\sin^2(\pi y)\cos(\pi y)\\ -\sin^2(\pi x)\sin^3(\pi y)\cos(\pi x)\end{bmatrix} \quad\text{and}\quad p = x^2 - y^2.$$

We decompose the unit square into $N\times N$ subdomains with side length $H = 1/N$. Each subdomain has a characteristic mesh size h. Both the first order ($k = 1$) and second order ($k = 2$) weak Galerkin methods are used to discretize the model equations. The BDDC preconditioned conjugate gradient iterations are stopped when the l_2-norm of the residual has been reduced by a factor of 10^6.

In the first set of experiments, we fix the size of the subdomain problem to be $\frac{H}{h} = 8$. Table 1 and 2 show the iteration counts and the estimates of the condition numbers for the BDDC preconditioned operator with changing subdomain numbers for $k = 1$ and $k = 2$, respectively. The condition numbers are found to be independent of the number of subdomains. As the second set of experiment, instead of fixing the

TABLE 3

Condition number estimates and iteration counts for the BDDC preconditioned operator with changing subdomain problem size. 8×8 subdomains, and $k = 1$.

$\frac{H}{h}$	Iterations	Condition number
4	9	2.49
8	13	5.01
16	15	7.48
24	18	9.12
32	19	10.37

TABLE 4

Condition number estimates and iteration counts for the BDDC preconditioned operator with changing subdomain problem size. 8×8 subdomains, and $k = 2$.

$\frac{H}{h}$	Iterations	Condition number
4	14	5.87
8	17	9.24
16	21	12.47
24	23	15.33
32	23	16.09

size of the subdomain problems, we fix the subdomain partition to be 8×8, and allow the subdomain problem size to vary. The condition number is found to increase logarithmically with the subdomain problem size. Table 3 and 4 demonstrate results for the second set of experiments for $k = 1$ and $k = 2$, respectively.

To conclude, we have carried out a series of experiments to obtain iteration counts and condition number estimates. The experimental results prove to be consistent with the theory. That is the condition number bound of the BDDC preconditioned system is of the form $C\frac{(1+\beta)^2}{\beta^2}\left(1 + \log\frac{H}{h}\right)^2$, where H and h are the diameters of the subdomains and elements, respectively. Possible future work will be to explore the order of the basis functions effects on C.

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