Analysis of Random Mobility Models with Sense and Avoid Protocols for UAS Capacity Management

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With the increasing number of unmanned aerials systems (UASs) in the National Airspace System, UAS traffic management (UTM), and in particular UAS capacity management, becomes crucial to maintain airspace safety. Unlike traditional aircraft that have predefined flight plans, UASs may have highly flexible, variable and uncertain movement patterns, and hence the capacity management problem becomes much more challenging. In this paper, we equipt the Random Direction (RD) Random Mobility Model (RMM) and its variants in a 1-D space with a sense-and-stop protocol, in order to capture the highly variable UAS mobility patterns subject to the separation safety constraint. We further develop theoretical analysis on critical statistics such as node distribution, inter-vehicle distance distribution, and collision probability. We also analyze the effectiveness of the sense-and-stop protocol subject to different randomness levels of the enhanced RD RMM. This study provides us insights of the capacity limits of an airspace that has dense UAS operations. Finally, simulation studies are developed to verify the theoretical results.

I. Introduction

Applications of unmanned aircraft system (UAS) span goods delivery, infrastructure surveillance, emergency response, and precision agriculture. With the increase of UASs in the National Airspace System (NAS), UAS Traffic Management (UTM) becomes crucial to maintain airspace safety. Pioneering research studies on UTM have been led by the National Aeronautics and Space Administration (NASA) [1, 2]. NASA takes an "incremental approach" to UTM, by first primarily focusing on sparsely populated areas and on providing information resources for the safety of the NAS. Yet little is known about the theoretical capacity limitations of the NAS subject to the existence of dense UAS operations.

In this paper, we aim to study collision probability and capacity of a dense UAS airspace, by taking a formal theoretical approach. Such study provides us valuable insights on the design of effective and safe UTM solutions and regulations. Traditional airspace capacity concepts were developed for commercial flights of pre-determined flight plans and rather deterministic flight trajectories [3–5]. They cannot be directly applied to UTM, considering the highly flexible, variable and uncertain movement patterns of UASs. Related to this direction, a capacity concept for UTM was proposed in Paper [6], based on the assumption that all UASs follow unified flow directions. In Paper [7, 8], a phase-transition-based capacity concept was developed based on simulation studies of randomly generated source-to-destination UAS trajectories in a two-dimensional (2-D) airspace. In this paper, we develop a tractable UAS mobility model that serves as the foundation to analyze airspace capacity limitations for UTM.

Random mobility models (RMMs) have long been used to capture the random movement patterns of mobile agents and to analyze communication and networking performance of mobile networks [9, 10]. Examples include Random Direction (RD), Random Walk (RW), Guass-Markov (GM), and Smooth Turn (ST) [10–14]. Our perspective here is that these well-established RMMs can serve as promising stochastic modeling frameworks for UTM, as they capture the highly variable, flexible, and uncertain UAS mobility patterns associated with many UAS missions [10]. However all of these existing studies on RMMs assume agents to move independently, which does not hold for UAS traffic where the safety constraint is enforced. In particular, a UAS should follow certain sense and avoid (S&A) protocols and maintain a safe separation distance among neighboring UASs.

The main contribution of this paper is on the enhancement of RMMs with S&A protocols, and on the analysis of critical statistical performance for such enhanced RMMs of inter-vehicle dependency, such as node distributions and inter-vehicle distance distributions. Inter-vehicle distance distribution leads to the analysis of collision probability, which is a key performance statistic for S&A protocols, airspace capacity, and other UTM concepts. In this paper, as a

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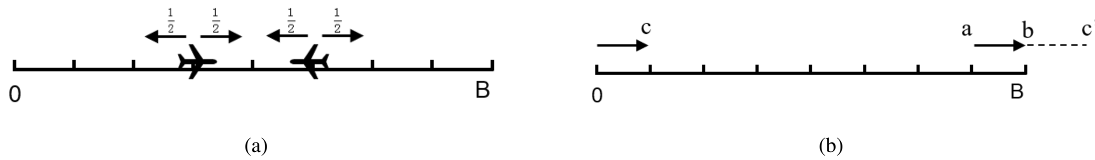


Fig. 1 (a) Illustration of the basic RD RMM in a 1-D airspace. (b) Illustration of the wrap-around boundary model for a 1-D airspace.

first attempt, we enhance the RD RMM with a sense-and-stop (S&S) protocol, and analyze effectiveness of the protocol subject to different randomness levels of the enhanced RD RMM.

The remainder of this paper is organized as follows. In Section II, we analyze statistics including node distribution and inter-vehicle distance distribution for the basic 1-D RD model. The analysis leads to the stationary collision probabilities for a pair of UASs and then an arbitrary number of UASs. In Section III, the analysis is extended to the case where a S&S protocol is applied. In Section IV, the enhanced RD RMM is modified to capture different randomness levels, and the analysis follows on the effect of randomness to collision probability. Section V includes the simulation study, and Section VI concludes the paper.

II. Analysis of the Basic RD RMM Model in a 1-D Airspace

In this section, the basic RD RMM without S&A protocols is described in a 1-D airspace. The stationary node distribution and inter-vehicle cyclic distance distribution between a pair of UASs are derived in closed forms. The concepts of collision probability for a pair of UASs and then an arbitrary numbers of UASs are developed based on the inter-vehicle distance distribution, respectively. The analysis in this section lays the foundation for the comparative analysis of S&A protocols in later sections.

A. Description of the RD RMM without S&A in $[0, B)$

In the basic RD RMM without S&A protocols, each UAS navigates independently in a bounded 1-D region $[0, B)$ (as shown in Figure 1(a)). At every time instance $1, 2, ..., k$, UAS i randomly chooses a heading direction $\Theta_i[k]$ from two values -1 (meaning going left) and 1 (meaning going right) with equal probabilities, and then moves along that direction until the next time instance. The heading speed V is assumed to be a constant, and each UAS is allowed to change direction at every time instance. $X_i[k]$ represents the x position of UAS i at the time instance k.

To avoid the border effect, the wrap-around boundary model, which is widely adopted for large-size regions, is used here [11]. When a UAS hits one end of the region, it wraps around and enters the other end with the same velocity and direction (see Figure 1(b)). The inter-vehicle cyclic distance $D_{i,j}[k]$ between two UASs i and j at time k is captured as [15]:

$$D_{i,j}[k] = \min(|X_i[k] - X_j[k]|, B - |X_i[k] - X_j[k]|) \tag{1}$$

B. Stationary location and distance distributions

Theorem 1. *N UASs move independently according to the basic RD RMM in a 1-D airspace $[0, B)$ subject to the wrap-around boundary model. The stationary joint node distribution is uniform, regardless of the initial joint node distribution.*

Proof. Notice that each UAS i chooses its heading direction randomly at every time instance. The stochastic process can be captured by a Markov chain, which is composed of all states of UAS i, $\hat{S}_i[k] = (X_i[k], \Theta_i[k])$. It was proved in Paper [11] that when the agents move independently following the basic RD RMM in a 1-D space, the stationary node distribution is uniform in the limit of large time. Following a similar process, the stationary node distribution for UAS i can be found as

$$\lim_{k\to\infty} P(X_i[k] = x, \Theta_i[k] = \theta) = \frac{1}{2B} \tag{2}$$

For the N UASs case, the stationary node distribution is also uniform. This is because when N UASs move independently, the joint node distribution is the multiplication of N individual node distributions. □

In the next theorem, the stationary inter-vehicle distance distribution between a pair of UASs is studied, when each UAS moves according to the basic RD RMM in a 1-D airspace $[0, B)$ subject to the wrap-around boundary model.

Theorem 2. *N UASs move independently according to the basic RD RMM in a 1-D airspace* $[0, B)$ *subject to the wrap-around boundary model. The stationary probabilistic distribution of the cyclic inter-vehicle distance,* $D_{i,j}[k]$*, denoted as* $P_{D_{i,j}}(d)$*, is*

$$P_{D_{i,j}}(d) = \lim_{k\to\infty} P(D_{i,j}[k] = d) = \frac{2}{B} \tag{3}$$

Proof. The inter-vehicle distance between a pair of UASs is defined in Equation (1). Therefore,

$$\begin{aligned} D_{i,j}[k] &= \begin{cases} |X_i[k] - X_j[k]| & |X_i[k] - X_j[k]| \leqslant \frac{B}{2} \\ B - |X_i[k] - X_j[k]| & |X_i[k] - X_j[k]| > \frac{B}{2} \end{cases} \\ &= \begin{cases} |D_E[k]| & |D_E[k]| \leqslant \frac{B}{2} \\ B - |D_E[k]| & |D_E[k]| > \frac{B}{2} \end{cases} \end{aligned} \tag{4}$$

where $D_E[k]$ is the Euclidean distance between the UASs i and j, e.g. $D_E[k] = X_i[k] - X_j[k]$. Let us then find the stationary distribution for $|D_E[k]|$ first. According to Theorem 1, we have

$$\lim_{k\to\infty} P(X_i[k] = x) = \frac{1}{B} \qquad (0 \leqslant x < B) \tag{5}$$

As UASs i and j move independently, the limiting distribution of $|D_E[k]|$ can be derived from Equation (5) as follows [16]:

$$\begin{aligned} \lim_{k\to\infty} P(|D_E[k]| = d) &= 2 \lim_{k\to\infty} \sum_{x=0}^{B-d} P(X_i[k] = x + d, X_j[k] = x) \\ &= 2 \lim_{k\to\infty} \sum_{x=0}^{B-d} P(X_i[k] = x + d) P(X_j[k] = x) \\ &= \frac{2(B-d)}{B^2} \qquad (0 \leqslant d < B) \end{aligned} \tag{6}$$

Then utilizing the relation between cyclic distance and Euclidean distance (see Equation (4)), the stationary probability distribution for $D_{i,j}[k]$ can be derived from Equation (6) as:

$$\begin{aligned} \lim_{k\to\infty} P(D_{i,j}[k] = d) &= \lim_{k\to\infty} P(|D_E[k]| = d) + P(|D_E[k]| = B - d) \\ &= \frac{2}{B} \qquad (0 \leqslant d < \frac{B}{2}) \end{aligned} \tag{7}$$

□

C. Collision Probability Analysis

Denote the collision distance as d_c, where $0 \leq d_c < \frac{B}{2}$. In this section, we define collision probabilities for a pair of UASs and then for an arbitrary number of UASs. We also develop closed-form expressions for these probabilities based on the stationary inter-vehicle distance distribution derived in Section II-B.

Definition 1. Collision occurs for a pair of UASs i and j at time k, if $D_{i,j}[k] \leq d_c$. The stationary collision probability between the two UASs is defined as follows,

$$\lim_{k\to\infty} \hat{P}_{i,j}[k] = \lim_{k\to\infty} P(D_{i,j}[k] \leqslant d_c) \tag{8}$$

To facilitate the comparative analysis in the rest of this sequel, we use $\lim_{k\to\infty} \hat{P}^b_{2,i,j}[k]$ and $\lim_{k\to\infty} \hat{P}^s_{2,i,j}[k]$ to represent the stationary collision probabilities between a pair of UASs following the basic RD RMM and following the enhanced RD RMM equipped with the S&S protocol, respectively.

Theorem 3. *A pair of UASs i and j move independently according to the basic RD RMM in a 1-D airspace* $[0, B)$ *subject to the wrap-around boundary model. The stationary collision probability between the two UASs is*

$$\lim_{k\to\infty} \hat{P}^b_{2,i,j}[k] = \frac{2(d_c + 1)}{B} \tag{9}$$

Proof. According to Definition 1, the stationary collision probability between two UASs can be derived by directly summing all limiting probabilities $\lim_{k\to\infty} P(D_{i,j}[k] \leqslant d_c)$ in Equation (3) as

$$\lim_{k\to\infty} \hat{P}^b_{2,i,j}[k] = \sum_{d=0}^{d_c} \lim_{k\to\infty} P(D_{i,j}[k] = d) = \frac{2(d_c + 1)}{B} \tag{10}$$

□

To simplify the notations, we use $\hat{P}^b_{2,i,j}$ to denote the stationary collision probability $\lim_{k\to\infty} \hat{P}^b_{2,i,j}[k]$. All probabilities are studied in the limit of large time, and hence $\lim_{k\to\infty}$ and $[k]$ are omitted when such abbreviations do not cause confusion. Next, we generalize the above definition and theorem to the case where an arbitrary number of UASs are involved.

Definition 2. Collision occurs for N UASs at time k, if and only if there exists at least one pair of UASs satisfying $D_{i,j}[k] \leqslant d_c$. The stationary collision probability for the N UASs is defined as

$$\lim_{k\to\infty} \hat{P}_N[k] = \lim_{k\to\infty} P(\exists D_{i,j}[k] \leqslant d_c, i, j \in [1, N], i \neq j) \tag{11}$$

Similarly, we use $\lim_{k\to\infty} \hat{P}^b_N[k]$ and $\lim_{k\to\infty} \hat{P}^s_N[k]$ to represent the stationary collision probabilities for N UASs following the basic RD RMM and following the enhanced RD RMM equipped with the S&S protocol, respectively.

Theorem 4. *N UASs move independently according to the basic RD RMM in a 1-D airspace* $[0, B)$ *subject to the wrap-around boundary model. The stationary collision probability among the N UASs is*

$$\hat{P}^b_N = 1 - \left(1 - \frac{2(d_c + 1)}{B}\right)^{\frac{N(N-1)}{2}} \tag{12}$$

Proof. As the N UASs move independently, the stationary inter-vehicle distance between each pair of UASs, $D_{i,j}[k]$, is also independent. There are a total of $\frac{N(N-1)}{2}$ indepedent UAS pairs (i, j) satisfying $i, j \in [1, N], i \neq j$. Hence,

$$\begin{aligned} \hat{P}^b_N &= P(\exists D_{i,j} \leqslant d_c, i, j \in [1, N], i \neq j) \\ &= 1 - P(\forall D_{i,j} > d_c, i, j \in [1, N], i \neq j) \\ &= 1 - (1 - \hat{P}^b_{2,i,j})^{\frac{N(N-1)}{2}} \end{aligned} \tag{13}$$

Substituting Equation (9) into Equation (13), Equation (12) is derived naturally. □

III. Analysis of the Enhanced RD RMM Equipped with the S&S Protocol

In this section, we enhance the basic RD RMM with a simple S&A rule for the safety purpose. We implement a simplified version of the "right-of-way" rules described by FAA [17]. In particular, if UASs of the same category are converging at the same altitude, the UAS to the relative right has the right-of-way.

The enhanced RD RMM equipped with the S&S protocol is described as follows, with the sensing distance (also called observing distance) denoted as d_o: (a) when the relative distance between two UASs is larger than the sensing distance, i.e., $D_{i,j}[k] > d_o$, each UAS moves independently according to the basic RD RMM; b) when the relative distance is within the sensing distance, $D_{i,j}[k] \leqslant d_o$, the vehicle to the relative left stops, and the other vehicle maintains the original RD RMM until $D_{i,j}[k] > d_o$.

A. Stationary Location Distribution

The stationary location distribution of UASs are studied first when they move following the enhanced RD RMM equipped with the S&S protocol.

Theorem 5. *N UASs move according to the enhanced RD RMM equipped with the S&S protocol in a 1-D airspace $[0, B)$ subject to the wrap-around boundary model. The stationary location distribution for each UAS i is uniform, regardless of the initial distribution.*

Proof. We first consider the case when $N = 2$. To find the stationary location distribution, we introduce a set $\mathcal{S}_s[k]$ to hold all UAS location pairs that satisfy one of the following two conditions: 1) $D_{i,j}[k] \leq d_o$ and that UAS i is to the relative right of UAS j, and 2) $D_{i,j}[k] > d_o$. Denote the complement set of $\mathcal{S}_s[k]$ as $\bar{\mathcal{S}}_s[k]$, the stationary location distribution can be further written as:

$$\lim_{k\to\infty} P(X_i[k] = x_i) = \lim_{k\to\infty} P(X_i[k] = x_i|\bar{\mathcal{S}}_s[k])P(\bar{\mathcal{S}}_s[k]) + \lim_{k\to\infty} P(X_i[k] = x_i|\mathcal{S}_s[k])P(\mathcal{S}_s[k]) \tag{14}$$

To prove the uniform distribution, we need to show that $\lim_{k\to\infty} P(X_i[k] = x_i) = \frac{1}{B}$ is satisfied. We hence only need to show that $\lim_{k\to\infty} P(X_i[k] = x_i|\mathcal{S}_s[k]) = \frac{1}{B}$ and $\lim_{k\to\infty} P(X_i[k] = x_i|\bar{\mathcal{S}}_s[k]) = \frac{1}{B}$. The first statement is straightforward because UAS i moves independently following the basic RD RMM in this case. According to Theorem 1, $\lim_{k\to\infty} P(X_i[k] = x_i|\mathcal{S}_s[k]) = \frac{1}{B}$ holds. For the second statement, since UAS i stops at time k in this case, $\lim_{k\to\infty} P(X_i[k] = x_i|\bar{\mathcal{S}}_s[k]) = \frac{1}{B}$ should be equal to the conditional probability right before it stops. As the UAS i is moving randomly at that time step, $\lim_{k\to\infty} P(X_i[k] = x_i|\bar{\mathcal{S}}_s[k]) = \frac{1}{B}$ also holds according to the proof of the first statement. The proof for N UASs follows a similar argument. □

The above theorem shows that the S&S protocol does not remove the uniform stationary location distribution for each UAS. Next, we study how the S&S protocol affects the inter-vehicle distance distribution between UASs.

B. Stationary Inter-vehicle Distance Distribution

When the S&S algorithm is used, the inter-vehicle distance distribution is not uniformly distributed any more. In the next theorem, we analyze the stationary inter-vehicle distance distribution.

Theorem 6. *Two UASs move according to the enhanced RD RMM equipped with the S&S protocol in a 1-D airspace $[0, B)$ subject to the wrap-around boundary model. The probability distribution of the cyclic inter-vehicle distance can be described as follows,*

$$P(d) = \begin{cases} \frac{2}{2d_o+B} & d = 0 \text{ or } d_o \\ \frac{4}{2d_o+B} & 1 \leqslant d < d_o, \text{ or } d_o + 1 \leqslant d < \frac{B}{2} \text{ and } d \text{ is odd} \\ 0 & d_o + 1 \leqslant d < \frac{B}{2} \text{ and } d \text{ is even} \end{cases} \tag{15}$$

Proof. Define a Markov chain, with states being the inter-vehicle distance ($S[k] = D_{i,j}[k]$). The Markov transition matrix can be captured as follows according to the mobility model.

$$
P = \begin{array}{c}
0 \\ 1 \\ 2 \\ \vdots \\ d_o \\ d_o+1 \\ d_o+3 \\ \vdots \\ \frac{B}{2}-3 \\ \frac{B}{2}-1
\end{array}
\left[\begin{array}{cccccc:cccccc}
0 & 1 & 0 & & & & & & & & & \\
\frac{1}{2} & 0 & \frac{1}{2} & & & & & & & & & \\
 & \frac{1}{2} & 0 & \frac{1}{2} & & & & & & & & \\
 & & & \ddots & & & & & & & & \\
 & & & & \frac{1}{2} & 0 & \frac{1}{2} & & & & & \\
\hdashline
 & & & & & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & & & & \\
 & & & & & & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & & & \\
 & & & & & & & & & \ddots & & \\
 & & & & & & & & & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\
 & & & & & & & & & 0 & \frac{1}{4} & \frac{3}{4}
\end{array}\right]
$$

Denote the stationary inter-vehicle distance distribution as $S = [s_1, s_2, ..., s_{d_o+1+\frac{B-d_o}{2}}]$. The distribution can then be found according to the equation below.

$$SP = S \tag{16}$$

Substituting the Markov transition matrix P into Equation (16), the stationary inter-vehicle distance distribution in Equation (15) is thus derived. □

C. Collision Probability Analysis

The next two theorems show the closed-form stationary collision probabilities for a pair of UASs and for an arbitrary number of UASs, when the UASs move according to the enhanced RD RMM equipped with the S&S protocol in a 1-D airspace.

Theorem 7. *A pair of UASs i and j move according to the enhanced RD RMM equipped with the S&S protocol in a 1-D airspace $[0, B)$ subject to the wrap-around boundary model. The stationary collision probability between the two UASs is*

$$\hat{P}^s_{2,i,j} = \frac{4d_c + 2}{2d_o + B} \tag{17}$$

Proof. From Definition 1, collision between a pair of UASs occurs when the inter-vehicle distance is less than or equal to the sensing distance d_o. The definition described in Equation (8) and the inter-vehicle distance distribution (Equation (15)) naturally lead to the stationary collision probability shown in Equation (17). □

Theorem 8. *N UASs move according to the enhanced RD RMM equipped with the S&S protocol in a 1-D airspace $[0, B)$ subject to the wrap-around boundary model. The upper bound of the stationary collision probability between the N UASs is*

$$\hat{P}^s_N \leqslant 1 - (1 - \frac{4d_c + 2}{2d_o + B})^{\frac{N(N-1)}{2}} \tag{18}$$

The equality is satisfied if and only if $N = 2$.

Proof. According to Definition 2, collision occurs between N UASs, if and only if there exists at lease one pair of UASs satisfying $D_{i,j}[k] \leqslant d_c$. When UASs move independently, the collision probability is described in Equation (13). When the S&S protocol is enforced, the independence among UASs is violated, and the collision probability between a pair of UASs i and j when N ($N \geqslant 3$) UASs move in the airspace (denoted as $\hat{P}^s_{N,i,j}$) is no longer equal to the collision probability between them when they move alone in the airspace ($\hat{P}^s_{2,i,j}$). Here we analyze how other UASs affect the collision probability for the UAS pair i and j, and derive the relationship between the two stationary collision probabilities $\hat{P}^s_{N,i,j}$ and $\hat{P}^s_{2,i,j}$.

Consider three UASs ($N = 3$), i, j, and l, that move in the airspace $[0, B)$. The collision probability for UASs i and j in this case is denoted as $\hat{P}^s_{3,i,j}$. To find the relationship between $\hat{P}^s_{3,i,j}$ and $\hat{P}^s_{2,i,j}$, we introduce a set $\mathcal{S}_c[k]$ to hold all the UAS inter-vehicle distances that satisfy any of the following three conditions: 1) $D_{i,l}[k] > d_o$ and $D_{j,l}[k] > d_o$, 2) $(D_{i,l}[k] \leqslant d_o) \cap (X_i[k] > X_l[k]) \cap (D_{j,l}[k] > d_o)$, or $(D_{j,l}[k] \leqslant d_o) \cap (X_j[k] > X_l[k]) \cap (D_{i,l}[k] > d_o)$, and 3) $(D_{i,l}[k] \leqslant d_o) \cap (D_{j,l}[k] \leqslant d_o) \cap (X_i[k] > X_l[k]) \cap (X_j[k] > X_l[k])$. $\bar{\mathcal{S}}_c[k]$ is the complement of $\mathcal{S}_c[k]$. The limiting collision probability between the UAS pair i and j can be written as

$$\begin{aligned}
&\lim_{k\to\infty} \hat{P}^s_{3,i,j}[k] = \lim_{k\to\infty} P(D_{i,j}[k] \leqslant d_c) \\
&= \lim_{k\to\infty} P(D_{i,j}[k] \leqslant d_c | \mathcal{S}_c[k]) P(\mathcal{S}_c[k]) + \lim_{k\to\infty} P(D_{i,j}[k] \leqslant d_c | \bar{\mathcal{S}}_c[k]) P(\bar{\mathcal{S}}_c[k])
\end{aligned} \tag{19}$$

The first conditional probability $\lim_{k\to\infty} P(D_{i,j}[k] \leqslant d_c | \mathcal{S}_c[k])$ equals $\lim_{k\to\infty} \hat{P}^s_{2,i,j}[k]$, since the UAS l does not change the movements of UAS i and j under the condition $\mathcal{S}_c[k]$. For the second conditional probability, UAS i or j stops under the condition $\bar{\mathcal{S}}_c[k]$, and the other moves according to the basic RD RMM. For any position (x_0, y_0) that UAS i or j stops at, the inter-vehicle distance between i and j is $D_{i.j}[k] = min(X_i[k] - x_0, B - |X_i[k] - x_0|)$ or $D_{i.j}[k] = min(X_j[k] - x_0, B - |X_j[k] - x_0|)$. Since $X_i[k]$ and $X_j[k]$ are both uniformly distributed in the limit of large time according to Theorem 1, $\lim_{k\to\infty} D_{i.j}[k]$ can be easily proved to be also uniformly distributed in $[0, \frac{B}{2})$, following a similar argument as in the proof of Theorem 2. Therefore, the second conditional probability $\lim_{k\to\infty} P(D_{i,j}[k] \leqslant d_c | \bar{\mathcal{S}}_c[k])$ equals the stationary collision probability between a pair of UASs that follow the basic RD RMM $\lim_{k\to\infty} \hat{P}^b_{2,i,j}[k]$. Therefore, Equation (19) can be further written as

$$\begin{aligned}
\lim_{k\to\infty} \hat{P}_{3,i,j}[k] &= \lim_{k\to\infty} \hat{P}^s_{2,i,j}[k] P(\mathcal{S}_c[k]) + \lim_{k\to\infty} \hat{P}^b_{2,i,j}[k] P(\bar{\mathcal{S}}_c[k]) \\
&< \lim_{k\to\infty} \hat{P}^s_{2,i,j}[k]
\end{aligned} \tag{20}$$

The last inequality holds because $\lim_{k\to\infty} \hat{P}^s_{2,i,j}[k] > \lim_{k\to\infty} \hat{P}^b_{2,i,j}[k]$, through a comparison between Equations (9) and (17).

Following a similar process, the relation between $\lim_{k\to\infty} \hat{P}^s_{N,i,j}[k]$ and $\lim_{k\to\infty} \hat{P}^s_{2,i,j}[k]$ can also be derived as

$$\lim_{k\to\infty} \hat{P}^s_{N,i,j}[k] < \lim_{k\to\infty} \hat{P}^s_{2,i,j}[k] \tag{21}$$

Equations (11) and (21) lead to the upper bound for the stationary collision probability between N UASs, as shown in Equation (18). □

The comparison between the two collision probabilities (Equations (9) and (17)) shows a surprising result. The collision probability for the basic RD RMM without the S&S protocol is less than that for the enhanced RD RMM with the S&S protocol. This implies that the widely used S&S protocol leads to increased collision probability between UAS pairs. This suppressing result is caused by the highly random UAS patterns. In particular, in our RD RMM, UASs are allowed to change directions at every time instance to mimic the high random UAS mobility. The result suggests that the S&S is not effective for highly random UAS mobilities. In order to understand the effect of randomness levels to the performance of S&S protocols, in the next section, we extend the analysis to the RD RMM with a stochastic travel time.

IV. Analysis of the Basic and Enhanced RD RMMs with a Stochastic Travel Time

In Sections II and III, we assume that the travel time (i.e., the time duration for a vehicle to hold its current heading direction) is unity. In this section, we enhance the RD RMM with a stochastic travel time $\tau(T_I) = T_{I+1} - T_I$ that maintains its current direction, where T_I is the time instance to update the heading direction. Here, $\tau(T_I)$ is a uniform distributed integer with the probability distribution function $P(\tau(T_I)) = \frac{1}{\tau_{\max}-1}$, where $\tau_{\max}$ is the maximum value of $\tau(T_I)$. A larger $\tau_{\max}$ indicates a longer expected travel time of the current direction. We use this formulation to study the effect of randomness levels to the statistical properties of the basic and enhanced RD RMMs.

A. Analysis of the Basic RD RMM with a Stochastic Travel Time

We first study if the stochastic travel time affects properties of the basic RD RMM.

Theorem 9. *N UASs move independently in a 1-D airspace* $[0, B)$ *subject to the wrap-around boundary model. The movement follows the basic RD RMM with a uniformly distributed travel time between* $(0, \tau_{max}]$*. The joint stationary location distribution and inter-vehicle distance distribution are both uniform in the limit of large* k*, regardless of the initial joint distribution.*

Proof. The location distribution is uniform according to the analysis in Paper[13] Proposition 3.2. The proof for the uniformity of the inter-vehicle distance distribution follows a similar process as in the proof of Theorem 2, and hence is omitted here.

Since the inter-vehicle distance distribution remains the same as described in Theorem 2, the collision probabilities for a pair of UASs and for an arbitrary number of UASs can also be described by Equations (9) and (17) according to Theorems 3 and 4, respectively. □

B. Analysis of the Enhanced RD RMM with a Stochastic Travel Time

In this section, we study how the stochastic travel time affects properties of the enhanced RD RMM equipped with the S&S protocol. Theorem 10 is concerned with the stationary location distribution, and Theorem 11 is concerned with the stationary inter-vehicle distribution.

Theorem 10. *N UASs move in a 1-D airspace* $[0, B)$ *subject to the wrap-around boundary model. The movement follows the enhanced RD RMM equipped with the S&S protocol and a uniformly distributed travel time between* $(0, \tau_{max}]$*. The stationary location distribution for each UAS is uniform, regardless of the initial distributions.*

$$\lim_{k\to\infty} P(X_i[k] = x_i) = \frac{1}{B} \tag{22}$$

Proof. Since $\lim_{k\to\infty} P(X_i[k] = x_i | \mathcal{S}_s[k]) = \frac{1}{B}$ holds according to Theorem 9, the proof follows a similar process as in the proof of Theorem 5, and thus is omitted here. □

Theorem 11. *Two UASs move in a 1-D airspace* $[0, B)$ *subject to the wrap-around boundary model. The movement follows the enhanced RD RMM equipped with the S&S protocol and a uniformly distributed travel time between* $(0, \tau_{max}]$*. The stationary probability distribution of the inter-vehicle distance is*

$$\begin{aligned} P_{D_{i,j}}(d) &= \lim_{k\to\infty} P(D_{i,j}[k] = d) \\ &= \begin{cases} \frac{1}{B} & d = 0 \text{ or } d_o \\ \frac{2}{B} & 1 \leqslant d < d_o \\ \frac{4}{B} & d_o + 1 \leqslant d < \frac{B}{2}, \text{ and } d \text{ is odd} \\ 0 & d_o + 1 \leqslant d < \frac{B}{2}, \text{ and } d \text{ is even} \end{cases} \end{aligned} \tag{23}$$

when $\tau_{\max} \to \infty$*.*

Proof. First, let us illustrate the periodicity, τ_p, of the travel time . The Markov transition matrix with unit travel time $\tau(T_I) = 1$ is described in Theorem 6. Following a similar process, the Markov transition matrix with other travel times

($\tau(T_I) = 2, 3, \ldots$) can also be listed according to their corresponding mobility models.

$$P_{\tau(T_I)=2} = \begin{array}{r} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ d_o \\ d_o+1 \\ d_o+3 \\ \vdots \\ \frac{B}{2}-3 \\ \frac{B}{2}-1 \end{array} \left[\begin{array}{ccccccccc|cccccc} 0 & 0 & 1 & 0 & 0 & & & & & & & & & & \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & & & & & & & & & & \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & & & & & & & & & & \\ & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & & & & & & & & & \\ & & & & \ddots & & & & & & & & & & \\ & & & & & & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & & & & \\ \hline & & & & & & & \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} & & & \\ & & & & & & & & \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} & & \\ & & & & & & & & & & & & \ddots & & \\ & & & & & & & & & & \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} \\ & & & & & & & & & & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{array} \right]$$

$$P_{\tau(T_I)=3} = \begin{array}{r} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \vdots \\ d_o \\ d_o+1 \\ d_o+3 \\ \vdots \\ \frac{B}{2}-5 \\ \frac{B}{2}-3 \\ \frac{B}{2}-1 \end{array} \left[\begin{array}{ccccccccc|cccccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & & & & & & & & \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & & & & & & & & \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & & & & & & & & \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & & & & & & & & \\ & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & & & & & & & \\ & & & & \ddots & & & & & & & & & & \\ & & & & & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & & & \\ \hline & & & & & & \frac{1}{4} & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{4} & & \\ & & & & & & & \frac{1}{4} & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{4} & \\ & & & & & & & & & & & & \ddots & & \\ & & & & & & & & \frac{1}{4} & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{4} \\ & & & & & & & & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{3}{4} & 0 \\ & & & & & & & & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} \end{array} \right]$$

It can be easily found that when $\tau(T_I) = B + d_o - 1$, the transition matrix for this Markov process is the same as the transition matrix when $\tau(T_I) = 1$, e.g., $P_{\tau(T_I)=B+d_o-1} = P_{\tau(T_I)=1}$. Similarly, the following equalities hold: $P_{\tau(T_I)=B+d_o} = P_{\tau(T_I)=2}$, $P_{\tau(T_I)=B+d_o+1} = P_{\tau(T_I)=3}$, ... This indicates that the travel time $\tau(T_I)$ is periodic for its corresponding Markov process, e.g., $P_{\tau(T_I)=j} = P_{\tau(T_I)=n\tau_p+j}$, with n is an integer. Hence the period is $\tau_p = B + d_o - 1$.

Next we sketch the structure of the rest of the proof, and leave out the full proof which is lengthy and may not be of particular interest to this audience. We first construct a Markov process with states $S[k] = D_{i,j}[k]$ and stochastic travel time $\tau(T_I)$. $\tau(T_I)$ follows a uniform distribution with the density function $P(\tau(T_I)) = \frac{1}{\tau_{\max}-1}$, where $\tau_{\max}$ is an integer multiple of τ_p, i.e., $\tau_{\max} = n\tau_p$. Then utilizing the periodicity of $\tau(T_I)$, we find the probability transition matrix for the Markov chain defined at the time sequence T_I, namely $S[T_I]$. Next, based on analysis of the transition matrix, the invariant distribution of $S[T_I]$ is found. Finally, the limiting probability distribution of the Markov process $S[k]$ is derived based on the Palm Formula [11, 18] as a function of n, and hence the limiting expression when $n \to \infty$ is derived in a closed form as shown in Equation (23). □

C. Collision Probability Analysis

Based on the inter-vehicle distance distribution described in Theorem 11, the collision probabilities for a pair of UASs and for an arbitrary number of UASs are analyzed in the following theorems.

Theorem 12. *A pair of UASs i and j move in a 1-D airspace* $[0, B)$ *subject to the wrap-around boundary model. The movement follows the enhanced RD RMM equipped with the S&S protocol and a uniformly distributed travel time. The stationary collision probability for the two UASs (denoted as* $\hat{P}^{s_2}_{2,i,j}$*) is*

$$\hat{P}^{s_2}_{2,i,j} = \frac{2d_c + 1}{B} \quad (24)$$

Proof. The theorem can be easily proved following a similar process as in the proof of Theorem 3, and thus is omitted here. □

Theorem 13. *N UASs move in a 1-D airspace* $[0, B)$ *subject to the wrap-around boundary model. The movement follows the enhanced RD RMM equipped with the S&S protocol and a uniformly distributed travel time. The lower bound of the stationary collision probability for the N UASs (denoted as* $\hat{P}^{s_2}_{N}$*) is*

$$\hat{P}^{s_2}_{N} \geqslant 1 - (1 - \frac{2d_c + 1}{B})^{\frac{N(N-1)}{2}} \quad (25)$$

Proof. The proof follows a similar process as in the proof of Theorem 8. Notice that the relation $\hat{P}^{s_2}_{2,i,j} < \hat{P}^{b}_{2,i,j}$ holds through a comparison between Equations (9) and (24). Hence "≤" in Equation (20) should be replaced by "≥" here. The lower bound for the stationary collision probability between N UASs can then be derived as shown in Equation (25), following a similar process as in the proof of Theorem 8. □

The statistical analysis above for the RD RMMs equipped with a stochastic travel time also provides us useful insights. The comparison between Equations (24) and (9) suggests that the collision probability is reduced when the S&S protocol with uniformly distributed travel time is applied. The comparison between Equations (24) and (17) also suggested that the extension of travel time also decreases the collision probability. The "randomness" level plays a important role for the collision probability. Less randomness leads to less collision probability for the S&S protocol.

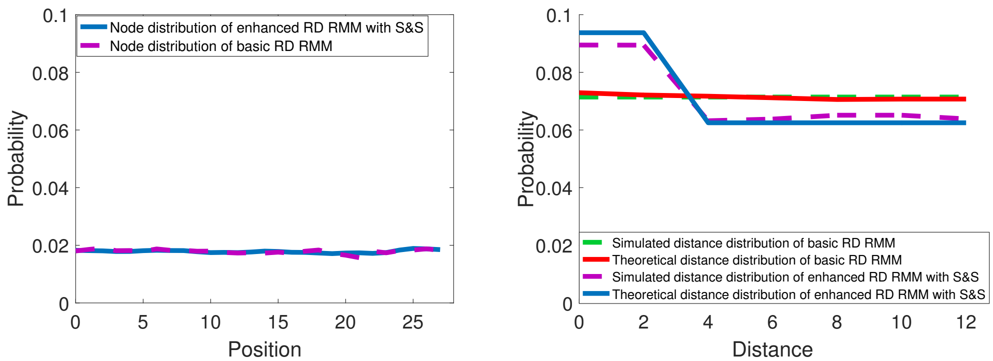
V. Simulation Studies

In this section, we conduct simulation studies to illustrate and verify the results in this paper. First, we simulate two UASs moving in a bounded airspace $[0, 28)m$ according to the basic RD RMM. At every time instance $1s, 2s, 3s, \ldots$, UASs choose a heading direction between -1 and 1 uniformly. The velocity is set as $1m/s$, and nodes are randomly distributed initially. The stationary location distribution and inter-vehicle distance distribution are approximated by counting the number of aircraft in each state over a long period. The two distributions are plotted in Figures 2(a) and 2(b), respectively. Clearly, the stationary distributions of location and inter-vehicle distance are both uniform when UASs move according to the basic RD RMM.

We then simulate the two UASs moving according to the enhanced RD RMM equipped with the S&S protocol. The sensing distance d_o is set as $2m$, and the collision distance d_c is $1m$. The stationary location distribution and inter-vehicle distance distribution are plotted also in Figures 2(a) and 2(b), respectively, for comparison. The figures verify the results that: 1) the S&S protocol does not change the uniformity of the stationary location distribution, as stated in Theorem 5; and 2) the S&S protocol removes the uniformity of the inter-vehicle distance distribution, and the stationary distance distribution matches with the theoretical results shown in Theorem 6.

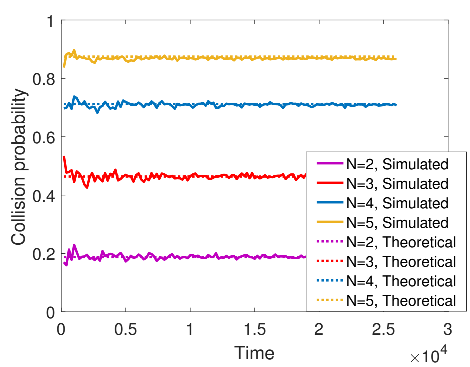
In addition, the collision probabilities are studied when N ($N = 2, 3, 4, 5$ respectively) UASs move following the enhanced RD RMM equipped with the S&S protocol. The results are plotted in Figure 3. The solid lines are simulated collision probabilities, and the dotted lines are theoretical upper bounds derived in Theorem 4. It can be seen from the figure that: 1) For each N, the collision probability converges in the limit of large time, which indicates the existence of stationary collision probabilities; 2) the stationary collision probability increases with the increase of the number of UASs, which is consistent with the common sense; 3) the theoretical upper bounds are tight as they are close to the simulated stationary collision probabilities.

Finally, we simulate the enhanced RD RMM equipped with the S&S protocol and a uniformly distributed travel time. Figure 4(a) shows the stationary inter-vehicle distance distribution when $\tau_{\max} \to \infty$. The uniformity of the distribution



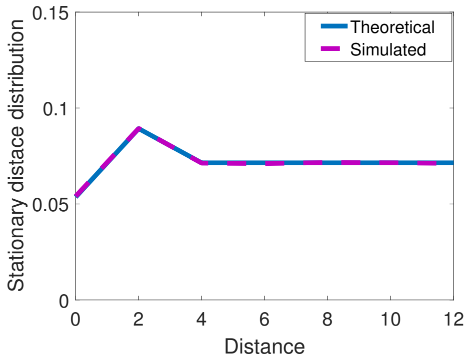
(a) (b)

Fig. 2 (a) Stationary location distribution. (b) Stationary inter-vehicle distance distribution.

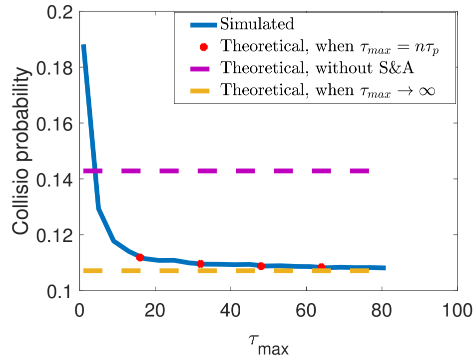


(a)

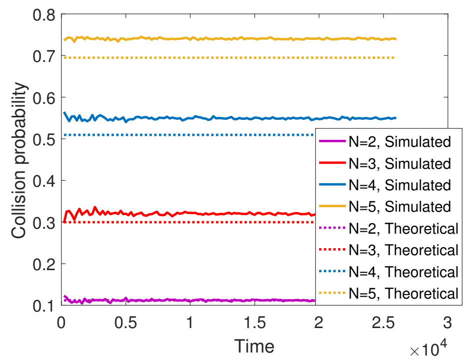
Fig. 3 Collision probabilities for a different number of UASs, when they move following the enhanced RD RMM equipped with the S&S protocol.



(a)



(b)



(c)

Fig. 4 (a) Stationary inter-vehicle distance distribution when the UASs move following the enhanced RD RMM and a uniform distributed travel time when $\tau_{\max} \to \infty$. (b) Stationary collision probabilities between a pair of UASs for different $\tau_{\max}$. (c) Collision probabilities for a different number of UASs when they move following the enhanced RD RMM and a uniformly distributed travel time where $\tau_{\max} = \tau_p$.

is violated due to the S&S protocol. Figure 4(b) shows the stationary collision probability between a pair of UASs versus $\tau_{\max}$. The blue solid line shows the simulated collision probability with different $\tau_{\max}$, and the red dots are the collision probabilities when $\tau_{\max} = n\tau_p$ $(n = 1, 2, 3, 4)$ derived analytically in the proof for Theorem 11. The yellow dash line shows the theoretical collision probability with $\tau_{\max} \to \infty$, and the purple dash line shows the collision probability for the basic RD RMM for a comparative study. It can be concluded from the two figures that: 1) the simulated stationary distance distribution matches with the theoretical one derived in Theorem 11; 2) the increase of $\tau_{\max}$ results in the decrease of the stationary collision probability, which indicates that less randomness leads to less collision probability; and 3) the S&S protocol is effective in reducing collision probability, when the maximum duration is greater than 6. Figure 4(c) shows the collision probabilities between N UASs ($N = 2, 3, 4, 5$ respectively). The solid lines are the simulated collision probabilities, and the dotted lines are the corresponding theoretical lower bounds derived in Theorem 13. We can see from the figure that: 1) collision probabilities are larger with the increase of the number of UASs; 2) compared with Figure 3, the stationary collision probability for each N is decreased with the extension of travel time.

VI. Conclusion

In this paper, we equipped the RD RMM with a S&S protocol as the foundation to analyze capacity limits for UTM in a 1-D airspace. We proved that the stationary location distribution remains uniform in the enhanced model, and the stationary inter-vehicle distance distribution is no more uniform due to the correlations introduced by the S&S protocol. We defined the stationary collision probability for a pair of UASs and then for an arbitrary number of UASs, based on the stationary inter-vehicle distance distribution. The closed-form stationary collision probability for the two-UASs case, and the upper bound for the N-UASs case are both found. After finding that the commonly used S&S protocol is ineffective, we change the constant travel time to a uniform random variable. The relation between the stationary collision probability and the maximum travel time is studied, suggesting that extending the travel time can reduce the collision probability of the S&S protocol. In the future work, we will study other mobility models and S&A protocols, examine the collision probability metric, and further explore the airspace capacity concept based on the stationary collision probabilities.

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