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Packing Curves on Surfaces with Few Intersections

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Przytycki has shown that the size $\mathcal{N}_k(S)$ of a maximal collection of simple closed curves that pairwise intersect at most k times on a topological surface S grows at most as $|\chi(S)|^{k^2+k+1}$. In this article, we narrow Przytycki's bounds, obtaining

$$\mathcal{N}_k(S) = O\left(\frac{|\chi|^{3k}}{(\log|\chi|)^2}\right).$$

In particular, the size of a maximal 1-system grows sub-cubically in $|\chi(S)|$. The proof uses a circle packing argument of Aougab and Souto and a bound for the number of curves of length at most L on a hyperbolic surface.

When the genus g is fixed and the number of punctures n grows, we use a different argument to show

$$\mathcal{N}_k(S) \leq O(n^{2k+2}).$$

This may be improved when $k = 2$, and we obtain the sharp estimate $\mathcal{N}_2(S) = \Theta(n^3)$.

1 Introduction

Let $S = S_{g,n}$ be an oriented surface of genus g with n punctures, and set $\chi = \chi(S)$. Given $k \in \mathbb{N}$, a *k-system* of curves (respectively arcs) is a collection of pairwise non-homotopic

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simple closed curves (respectively properly embedded arcs) on S, no two of which intersect more than k times. Let

$$\mathcal{A}_k(S) = \max\{|\Gamma| : \Gamma \text{ is a } k\text{-system of arcs on } S\} \text{ and}$$
$$\mathcal{N}_k(S) = \max\{|\Gamma| : \Gamma \text{ is a } k\text{-system of curves on } S\}.$$

In 1996, Juvan *et al.* [5] showed that $\mathcal{N}_k(S)$ is always finite. The asymptotic study of $\mathcal{N}_k(S)$ as $|\chi| \to \infty$ was later popularized by Benson Farb and Chris Leininger, who vocally noticed that good bounds were unavailable even when $k = 1$.

In response, Malestein *et al.* [7] showed that when S is closed and any k is fixed, $\mathcal{N}_k(S)$ grows at least quadratically and at most exponentially in $|\chi|$. Observing that asymptotics are easier to find for arcs, and that the arc case informs the curve case, Przytycki [9] then showed that for fixed k

$$\mathcal{A}_k(S) = \Theta\left(|\chi|^{k+1}\right), \text{ and } \mathcal{N}_k(S) = O\left(|\chi|^{k^2+k+1}\right). \tag{1.1}$$

Aougab has shown [1] that *when S is fixed,* $\log(\mathcal{N}_k(S))$ grows at most linearly in k, which suggests that it might be possible to improve Przytycki's upper bound for curves. We show:

Theorem 1.1. Suppose S is a surface with Euler characteristic χ. Then as $|\chi| \to \infty$ we have

$$\mathcal{N}_k(S) \leq O\left(\frac{|\chi|^{3k}}{(\log|\chi|)^2}\right).$$ □

Note that when $k = 1$ and S is closed, this is a slight improvement of Przytycki's bound of $O(|\chi|^3)$. (For larger k, though, the improvement is significant.) The best known constructions of 1-systems show that $\mathcal{N}_1(S)$ grows at least quadratically in $|\chi|$. Thus, it is of course natural to ask whether there exists an $\epsilon > 0$ so that $\mathcal{N}_1(S) = O(|\chi|^{3-\epsilon})$. Indeed, one might expect in general that $\mathcal{N}_k(S) = \Theta(\mathcal{A}_k(S)) = \Theta(|\chi|^{k+1})$, but currently this is out of reach.

The proof of Theorem 1.1, which appears in Section 3, is short enough that there is no need for a detailed summary here. Nonetheless, we point out three main components. The first is (a trivial adaptation of) an argument of Aougab and Souto [2, Theorem 1.2], in which a circle packing argument is used to find a hyperbolic structure on which a given k-system can be realized length-efficiently. The second component is Theorem 2.1, which improves Przytycki's translation of the bound for arcs to a bound for curves.

The third component is a bound due to Buser for the number of primitive curves of length at most L on a hyperbolic surface homeomorphic to S (see Lemma 3.2). For the reader's interest, we make comments on sharpness and possible improvements to the above argument in Section 5.

We should highlight that our argument uses Przytycki's bound $\mathcal{A}_k(S) = \Theta\left(|\chi|^{k+1}\right)$ essentially, via the proof of Theorem 2.1. It turns out that we can exploit his bounds even more effectively when the number of punctures is large in comparison to the genus. Here, the proof is inductive, where we relate $\mathcal{N}_k(S_{g,n})$ to $\mathcal{N}_k(S_{g,n-1})$ by projecting a k-system on $S_{g,n}$ to one on $S_{g,n-1}$ by filling in the puncture. We note that Malestein *et al.* [7, Theorem 1.2] used a similar inductive argument in the $k = 1$ case to show that $\mathcal{N}_1(S_{g,n}) = \mathcal{N}_1(S_{g,0}) + Cg \cdot n$.

Theorem 1.2. There is a constant $C = C(k)$ such that

$$\mathcal{N}_k(S_{g,n}) \leq \mathcal{N}_k(S_{g,0}) + C(g+n)^{2k+2}.$$

And in fact, for $k = 2$ we have

$$\mathcal{N}_2(S_{g,n}) \leq \mathcal{N}_2(S_{g,0}) + C(g+n)^3.$$ □

When k is even, Przytycki's construction [9, Example 4.1] of large k-systems of arcs can be tweaked in a straightforward manner to produce the lower bound $\approx (g+n)^{k+1}/(k+1)^{k+1}$ for $\mathcal{N}_k(S)$. For completeness we recreate this construction in the salient case below:

Corollary 1.3. When g is fixed and $n \to \infty$, we have $\mathcal{N}_2(S_{g,n}) = \Theta(n^3)$. □

Proof. For the lower bound, consider three columns of $\approx n/3$ punctures each, placed along the vertical lines $x = -1, 0$, and 1 on the Riemann sphere, and fix the pair of "starting" and "ending" points $(\pm 2, 0)$. Each "vertical" column of punctures cuts a vertical segment into $\approx n/3 - 1$ segments between punctures. For any choice of such an interval from the three vertical columns, form a polygonal path from the starting to the ending point that passes through the chosen three intervals. Appending this arc with the portion of the completed real axis that passes through ∞ and connects the endpoints, we obtain $\approx n^3$ closed curves, every pair of which intersects at most twice. Moreover, it is easy to see that they are distinct: for any pair of such curves, there is an arc that intersects one of them essentially, and not the other. This construction evidently can be performed on any surface $S_{g,n}$, and the upper bound is provided by Theorem 1.2. ■

2 Degree bounds for the intersection graph

Let $\mathcal{I}(\Gamma)$ denote the *intersection graph* of a curve system Γ, whose vertices are in 1-1 correspondence with the curves in Γ, and where two vertices are connected by an edge exactly when the corresponding curves intersect essentially on the surface S.

In [9], Przytycki's estimate for $\mathcal{N}_k(S)$ is a corollary of his estimate for $\mathcal{A}_k(S)$. The idea is as follows, say when $k = 1$. If Γ is a 1-system, cut S open along a curve $\gamma \in \Gamma$. Any curve in Γ intersecting γ becomes an arc on the new surface S'. The number of homotopy classes of such arcs is bounded above by $\mathcal{A}_1(S')$, and at most two curves in Γ correspond to the same homotopy class of arc on S'. It follows that the degree of γ in $\mathcal{I}(\Gamma)$ is at most $2 \cdot \mathcal{A}_1(S')$. To finish, note that the total number of vertices in $\mathcal{I}(\Gamma)$ is at most the sums of the degrees of the vertices in a maximal independent subset of $\mathcal{I}(\Gamma)$, and any independent set (i.e., a set of disjoint simple closed curves on S) has size at most linear in χ.

To prove Theorem 1.1 in the closed case, we will need the following sharper upper bound for the degree of a vertex in the intersection graph of a k-system.

Theorem 2.1. Suppose that Γ is a k-system on a surface S, and $\gamma \in \Gamma$. Then the degree of γ in the graph $\mathcal{I}(\Gamma)$ is at most $C \cdot |\chi|^{3k-1}$, for some universal $C = C(k)$. □

When $k = 1$, this bound agrees with the bound $2 \cdot \mathcal{A}_1(S') = \Theta(|\chi|^2)$ one gets with Przytycki's argument above and (1.1). In general, though, his argument gives $C \cdot |\chi|^{k(k+1)}$, so Theorem 2.1 is quite a bit stronger. The improvement arises by adopting a slightly different perspective. Instead of cutting open S along a curve to produce an arc system on a surface of smaller complexity, one can introduce punctures to S and "slide" curves to arcs to arrive at an arc system on a surface S' of slightly larger complexity, and then apply Przytycki's bounds for $\mathcal{A}_k(S')$.

Proof. Let Γ be an arbitrary k-system on S with $|\chi(S)| = t$ and let $\gamma \in \Gamma$. We will show that the set of curves $\mathcal{I}(\gamma)$ consisting of elements of Γ intersecting γ non-trivially, has size $O(t^{3k-1})$. Begin by choosing a minimal position realization of $\mathcal{I}(\gamma) \cup \gamma$ with no triple points, and pick an orientation and a basepoint x on γ. Then the intersections of γ with the curves in $\mathcal{I}(\gamma)$ are ordered according to when they appear when one traverses γ in the given direction starting at x.

Let S_x be the surface obtained by puncturing S at x. If $\alpha \in \mathcal{I}(\gamma)$, we produce an arc $\tilde{\alpha}$ on S_x as follows. Let y be the first intersection point of α and γ, with respect to the order above. Isotope α by pushing y along γ to x, in the direction opposite to the orientation. This gives an arc $\tilde{\alpha}$ on S_x,

$$\tilde{\alpha} = \gamma_\alpha^{-1} * \alpha * \gamma_\alpha,$$

where γ_α is the directed sub-arc of γ from x to y. Since y was the *first* intersection point along γ from x, the arc $\tilde{\alpha}$ can be perturbed to be simple (unperturbed, it tracks γ_α twice.).

We claim that when $\alpha, \rho \in \mathcal{I}(\gamma)$, we have $\iota(\tilde{\alpha}, \tilde{\rho}) \leq 3k-2$. Suppose that with respect to the order above, γ intersects α before it intersects ρ. Then no new intersections with ρ are created when α is replaced by $\tilde{\alpha}$. Moreover, when pushing ρ along γ to create $\tilde{\rho}$, one may encounter at most $k-1$ strands of $\tilde{\alpha}$. This follows from the fact that $\iota(\alpha, \gamma) \leq k$ and that the first intersection point between α and γ has been pushed all the way to x in the construction of $\tilde{\alpha}$, leaving at most $k-1$ strands. Each such strand will contribute to two intersection points between $\tilde{\alpha}$ and $\tilde{\rho}$, and as ρ, α both belong to a k-system, we had $\iota(\alpha, \rho) \leq k$ to begin with. Thus $\iota(\tilde{\alpha}, \tilde{\gamma}) \leq 2(k-1) + k = 3k-2$, as desired.

If $\tilde{\alpha}, \tilde{\rho}$ were homotopic on S_x then α, ρ would be homotopic on S, so we have constructed a $(3k-2)$-system of arcs on S_x with size equal to $|\mathcal{I}(\gamma)|$. By Przytycki's upper bound (1.1) for arc systems, $|\mathcal{I}(\gamma)| = O(t^{3k-1})$. This completes the proof of Theorem 2.1. ■

3 Proof of Theorem 1.1

Let $\Gamma = \{\gamma_1, \ldots, \gamma_N\}$ be a k-system of curves on a closed surface S with $|\chi(S)| = t$. We begin the proof of Theorem 1.1 by using the following result of Aougab and Souto.

Proposition 3.1. [2, Theorem 1.2] There exists a hyperbolic structure X on S such that the geodesic realization of Γ on X has total length

$$\ell_X(\Gamma) \leq 4\sqrt{2t \cdot \iota(\Gamma, \Gamma)}\ ,$$

where $\iota(\Gamma, \Gamma)$ is the total geometric self-intersection number of Γ. □

The idea behind Proposition 3.1 is as follows. By Koebe's discrete uniformization theorem, there exists a hyperbolic structure X on S so that the union of all the curves in Γ can be realized inside the dual graph of a circle packing on X. The sum of the areas of these circles is at most $2\pi t$, by Gauss– Bonnet. Using the Cauchy–Schwarz inequality, this translates into an upper bound on the sum of the radii of the circles, which bounds the length of Γ. This argument is carried out for self-intersecting curves in [2], but it applies verbatim to our setting of curve systems.

Now since Γ is a k-system, we have

$$|\iota(\Gamma,\Gamma)| \leq k \sum_{i=1}^{N} \deg_{\mathcal{I}(\Gamma)}(\gamma_i) \leq k \cdot C \cdot t^{3k-1} \cdot N, \tag{3.1}$$

where $\deg_{\mathcal{I}(\Gamma)}$ is the degree of γ in the intersection graph $\mathcal{I}(\Gamma)$, that is, the number of curves in Γ that γ intersects, and $C = C(k)$ is the constant from Theorem 2.1.

Using Proposition 3.1, the *average length* of a curve from Γ is then

$$\frac{\ell_X(\Gamma)}{N} \leq C\sqrt{\frac{t^{3k}}{N}} =: L,$$

for some new $C = C(k)$. By Markov's inequality, at least half of $\ell_X(\gamma_1), \ldots, \ell_X(\gamma_N)$ are less than or equal to twice the average length, so by we obtain $N/2$ curves of length at most $2L$.

To finish, we employ a bound of Buser for the number of primitive geodesics on a hyperbolic surface that uses L and $|\chi|$ efficiently (see Section 5 below for more detail and comments on sharpness):

Lemma 3.2. [3, Lemma 6.4.4] There is an absolute constant $C > 0$ so that the number of primitive closed geodesics on X of length at most L is less than $C\,|\chi|\,\mathrm{e}^L$. □

We immediately conclude that

$$N \leq 2 \cdot \#\left\{\text{closed geodesics } \gamma \text{ on } X : \ell_X(\gamma) \leq 2L\right\} \leq C^{\sqrt{t^{3k}/N}}t\ , \tag{3.2}$$

for some new constant $C = C(k)$.

If we suppose that $N \neq O\left(\frac{t^{3k}}{(\log t)^2}\right)$, then we can find arbitrarily large t for which there is a k-system with $N \geq \left(\frac{\log C}{\log t}\right)^2 t^{3k}$ curves. Then (3.2) says

$$\left(\frac{\log C}{\log t}\right)^2 t^{3k} \leq N \leq C^{\sqrt{t^{3k}/N}} \cdot t \leq C^{\sqrt{t^{3k}/\left(\left(\frac{\log C}{\log t}\right)^2 t^{3k}\right)}} \cdot t = t^2\ ,$$

a contradiction when $k > 0$ and t is large. This concludes the proof of Theorem 1.1. ■

Remark 3.3. We recall an observation of Malestein *et al.*: it is evident that the size of an independent set of $\mathcal{I}(\Gamma)$ is bounded above by $3t/2$, and an application of Turán's theorem to the complementary graph of $\mathcal{I}(\Gamma)$ implies

$$N \leq 3t/2 \cdot (D+1)\ ,$$

where D is the average degree of $\mathcal{I}(\Gamma)$ [7, Theorem 1.5]. Coarsely, $N = O(t \cdot D)$. Running through the above argument with $|\iota(\Gamma, \Gamma)| = DN/2$ replacing (3.1), one obtains

$$N = O\left(\frac{t \cdot D}{(\log t)^2}\right)$$

for maximal k-systems Γ. □

4 The Proof of Theorem 1.2

Let $S_{g,n}$ be an orientable surface with genus g and n punctures. We first show

$$\mathcal{N}_k(S_{g,n}) \le \mathcal{N}_k(S_{g,n-1}) + \mathcal{A}_{k-1}(S_{g,n}) + 2\mathcal{A}_{2k}(S_{g,n}) \le \mathcal{N}_k(S_{g,n-1}) + C(g+n)^{2k+1} ,$$

where $C = C(k)$ is a constant coming from Przytycki's [9] bounds $\mathcal{A}_{2k}(S) = \Theta(|\chi|^{2k+1})$. Applying this step iteratively, we may conclude

$$\mathcal{N}_k(S_{g,n}) \le \mathcal{N}_k(S_{g,0}) + \sum_{i=1}^{n} C(g+i)^{2k+1} \le \mathcal{N}_k(S_{g,0}) + C(g+n)^{2k+2} .$$

So, let Γ be a k-system on $S_{g,n}$. Fix a minimal position realization of Γ, and choose arbitrarily a puncture p of $S_{g,n}$. Project each curve in Γ to a curve on $S_{g,n-1}$ by filling in the puncture p.

We first bound the number of curves that have inessential projection to S_{n-1}. Each such curve c bounds a twice-punctured disk, where one of the punctures is p. In other words, c is the boundary of a regular neighbourhood of an arc α_c from p to some other puncture. If c, d both have inessential projections, then $\iota(c, d) = 4\iota(\alpha_c, \alpha_d) + \epsilon$, where $\epsilon = 4$ or $\epsilon = 2$, depending on whether c, d share the same second puncture or not. In particular, $\iota(\alpha_c, \alpha_d) \le k - 1$, so the set of all α_c is a $(k-1)$-system of arcs. Therefore, the number of curves c with inessential projection is bounded above by $\mathcal{A}_{k-1}(S_{g,n})$.

Remove all curves from Γ that have inessential projection to $S_{g,n-1}$, and continue to denote this by Γ. It suffices to show that now

$$|\Gamma| \le \mathcal{N}_k(S_{g,n-1}) + 2\mathcal{A}_{2k}(S_{g,n}).$$

We would like to relate $|\Gamma|$ to $\mathcal{N}_k(S_{g,n-1})$ by saying that the projection of Γ to $S_{g,n-1}$ is a k-system. The problem is that curves can become homotopic after projection, so the size of the new k-system can be smaller. Let $\mathcal{G} \subset \Gamma$ be a maximal subset of "good" curves

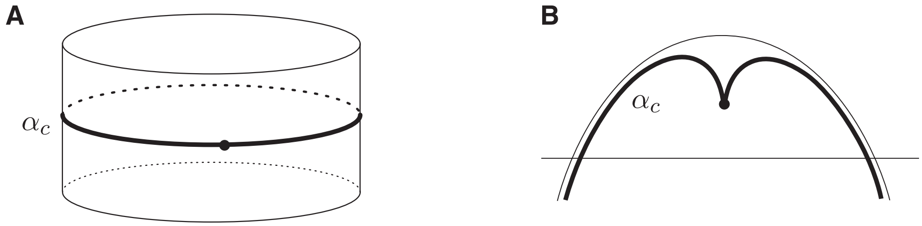

Fig. 1. The region A_c cobounded by c and g_c, and the bold arc α_c. (A) c is of type (1). (B) c is of type (2).

whose projections are not homotopic, and let $\mathcal{B} = \Gamma \setminus \mathcal{G}$ be the complementary set of "bad" curves. As $|\mathcal{G}| \le \mathcal{N}_k(S_{g,n-1})$, it suffices to prove

$$|\mathcal{B}| \le 2\mathcal{A}_{2k}(S_{g,n}).$$

Let $c \in \mathcal{B}$ be a bad curve. There is then a unique good curve g_c such that the projections of c and g_c to $S_{g,n-1}$ are homotopic. It follows that on $S_{g,n}$ the curves c and g_c either:

(1) are disjoint and bound an annulus A_c punctured by p,
(2) intersect, and there are arcs of c and g_c that bound a bigon A_c punctured by p.

We refer to bad curves where the former holds as *Type (1)* and the other as *Type (2)*. In both cases, define an arc α_c on $S_{g,n}$ by connecting p to c with an arc β in A_c, and then setting

$$\alpha_c = \beta^{-1} * c * \beta.$$

Both endpoints of α_c are at p, and α_c is well defined up to homotopy. See Figure 1.

The desired conclusion follows immediately from the following lemma.

Lemma 4.1. The set of homotopy classes $\{[\alpha_c] : c \in \mathcal{B}\}$ is a $2k$-system of arcs of size at least $|\mathcal{B}|/2$. □

Proof. Fix a pair $c, d \in \mathcal{B}$, the corresponding curves $g_c, g_d \in \mathcal{G}$, the arcs α_c, α_d, and the regions A_c, A_d.

We start by assuming that c and d are of type (1). The component of the intersection $A_c \cap A_d$ containing p is evidently a disk, with at least one side bounded by an

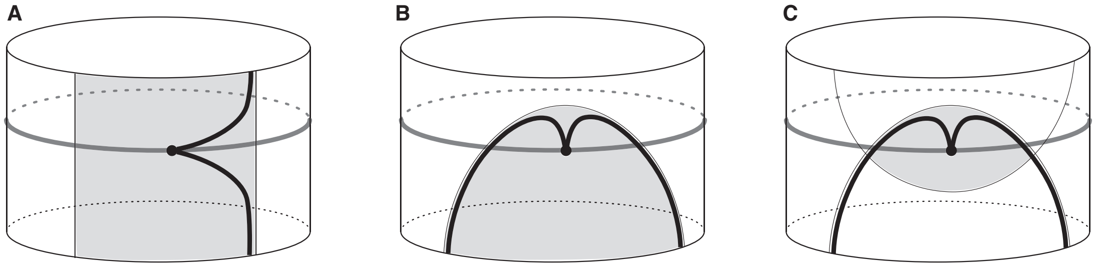

Fig. 2. Possibilities for $A_c \cap A_d$ when c is of type (1), with α_c shaded and α_d in bold.

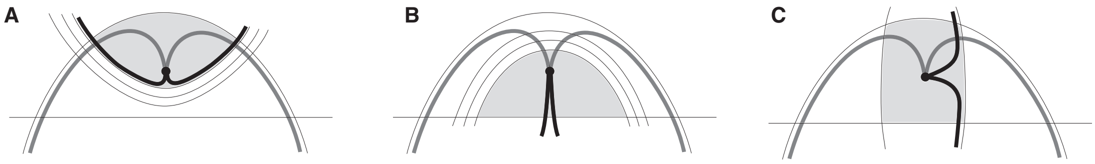

Fig. 3. Possibilities for $A_c \cap A_d$ when c, d are of type (2), with α_c shaded and α_d bold.

arc of either d or g_d. If this arc does not have its endpoints on the same component of ∂A_c, we are in the setting of Figure 2A; if it does we are in the setting of Figure 2B. In the latter case, whichever of d or g_d is on the boundary of the bigon pictured in Figure 2B, the contributions to the intersection number with c coming from any *other* components of the intersection with A_c are unchanged. Because the two pictured intersections can be homotoped away, we conclude $\iota([\alpha_c], [\alpha_d]) \leq k - 2$. Similarly, in Figure 2A, there is a realization of α_d pictured that has at least one fewer intersection point with α_c than c has with d, so that $\iota(\alpha_c, \alpha_d) \leq \iota(c, d) - 1 \leq k - 1$.

The argument is the same when c is of type (1) but d is of type (2), but in this case any of Figure 2A, B, or C are possible. Moreover, there is a minor complication in Figure 2B: it is now possible that one of the boundary arcs of $A_c \cap A_d$ pictured is g_d, so that $d \cap A_c$ may consist of k arcs connecting distinct boundary components of A_c. Nonetheless, we still have that the pictured representative for α_d satisfies $\iota([\alpha_c], [\alpha_d]) \leq k$ in any of Figure 2A, B, or C.

Suppose now that both c and d are of type (2). The component of $A_c \cap A_d$ containing p is either as in Figure 3A or B, or (after homotoping A_d if necessary, leaving it in minimal position with A_c) it is as in Figure 3C. The worst of these cases is Figure 3B, where we see that $\iota(\alpha_c, \alpha_d) \leq 2k$.

Finally, note that c can be recovered as the boundary of a regular neighbourhood of α_c on $S_{g,n}$, a once-punctured annulus. Thus $\{\alpha_c : c \in \mathcal{B}\} \to \{[\alpha_c] : c \in \mathcal{B}\}$ is at most 2-to-1, as desired. ∎

When $k = 2$, note the useful fact that whenever two curves $\gamma, \eta \in \Gamma$ have homotopic projections in $S_{g,n-1}$, we must have that γ, η are disjoint: If not, there are subarcs γ_p, η_p of γ, η that bound a once-punctured bigon in S_n, punctured by p. The assumption that γ and η are homotopic after filling p implies that $\gamma \setminus \gamma_p$ and $\eta \setminus \eta_p$ must jointly bound another bigon on S_{n-1} in the complement of the point p, and this contradicts the assumption that γ and η are in minimal position. Thus, when $k = 2$, all bad curves $c \in \mathcal{B}$ are of type (1), and the proof of Lemma 4.1 demonstrates the stronger claim:

Lemma 4.2. When $k = 2$ the set of homotopy classes $\{[\alpha_c] : c \in \mathcal{B}\}$ is a 1-system of size at least $|\mathcal{B}|/2$. □

The bound $\mathcal{N}_2(S_{g,n}) \leq \mathcal{N}_2(S_{g,0}) + C(g+n)^3$ now follows using the same arguments as above.

Remark 4.3. This method is close to that of [7, Theorem 1.2], where they adopt the perspective that for $k = 1$ the size of $|\mathcal{B}|$ can be bounded by observing that $\{A_c : c \in \mathcal{B}\}$ is a collection of annuli which pairwise intersect essentially. □

5 Comments on Sharpness in the Proof of Theorem 1.1

A crucial step at the end of the proof of Theorem 1.1 involved Buser's Lemma 3.2, bounding the number of primitive closed geodesics on a hyperbolic surface roughly by $|\chi| e^L$. In fact, it is pertinent to note that similar uniform bounds may not be pushed much further. Recall Keen's Collar Lemma [4, Lemma 13.6] [6], which states that a pair of disjoint simple geodesics on a hyperbolic surface of lengths l_1 and l_2 have disjoint annular neighbourhoods of widths $w(l_1)$ and $w(l_2)$, respectively where

$$w(x) = \sinh^{-1}\left(\frac{1}{\sinh(x/2)}\right).$$

Lemma 5.1. There exists an absolute constant $C > 0$ so that, for all $L > 0$, there is a hyperbolic structure on S with at least $C\,|\chi|\,e^{\frac{L}{4}}$ simple closed geodesics of length at most L. □

Proof. Consider first the unique pants curve α in a pants decomposition of a four-holed sphere $F \subset S$, and moreover fix a choice of "zero twisting" so that there is a geodesic β which intersects α twice orthogonally. When a hyperbolic structure is chosen with the length of α given by r, by the Collar Lemma β has length at least $4\log(1/r)$. Evidently, if r is chosen much smaller than $e^{-L/4}$, there are *no* non-peripheral simple geodesics supported on F with length at most L other than α. We will choose instead $r \approx e^{-L/4}$ that will allow the construction of many other "short" simple geodesics.

Consider the curve β_n formed by applying n half-Dehn twists around α to β. The hyperbolic length of β_n in a hyperbolic structure where α has length r is at most

$$n \cdot r + 4 \log\left(\frac{1}{r}\right) + C_0 ,$$

where C_0 is some uniform constant determined by the hyperbolic geometry of three-holed spheres with geodesic boundary.

Now let r be chosen as $e^{-\frac{L}{4}+C_0}$. The length of β_n is at most

$$n \cdot e^{-\frac{L}{4}+C_0} + L - 3C_0 ,$$

which is in turn less than L as long as n is at most $C_1 e^{L/4}$, where $C_1 = 3C_0 e^{-C_0}$.

Since there are some constant proportion of $|\chi(S)|$ many subsurfaces which are four-holed spheres, we have constructed $C \cdot e^{L/4} \cdot |\chi(S)|$ curves of length at most L, as desired. ■

Remark 5.2. The above construction in one-holed tori subsurfaces would produce $Ce^{L/2}$ curves of length at most L, with the difference attributable to the fact that one can build curves that only cross the collar once. On the other hand, planar surfaces have no one-holed tori subsurfaces, so this would only produce a better lower bound for surfaces with high genus. □

Remark 5.3. Lemma 5.1 is in contrast with the celebrated computations of Mirzakhani on the asymptotics of the number of simple closed geodesics of length at most L on a hyperbolic surface [8]; in the latter theorem the hyperbolic surface is fixed as L grows, whereas in the above lemma it is not. □

In our application of Lemma 3.2 in the proof of Theorem 1.1 (see (3.2)), it is essential that our upper bounds are of the form $|\chi|^r C^L$, for $r < 3$ and some constant $C > 0$. Remarkably, Lemma 3.2 ignores the assumption about the intersection data, and in light of the construction in Lemma 5.1 it seems likely that much stronger control should be possible.

In the most interesting case $k = 1$, we offer a different proof that makes essential use of the intersection data, and that moreover would apply in the context of (3.2). In the process, we obtain a slightly better bound than that of Lemma 3.2. On the other hand, one might view this second proof as evidence that our circle packing approach will not produce any new bounds much better than those we obtain.

Lemma 5.4. Let Γ be a 1-system of curves realized by geodesics on X, each of length at most L. Then

$$|\Gamma| \leq \pi \ (2g+1) \ |\chi| \ \frac{e^{L/2}}{L}. \qquad \square$$

Proof. For each $\gamma \in \Gamma$, let $A(\gamma)$ indicate the annulus $N_{\omega(\ell(\gamma,X))}(\gamma)$ whose existence in guaranteed by the Collar Lemma. Consider the map induced by inclusion

$$i : \bigsqcup_{\gamma} A(\gamma) \to X.$$

By the Collar Lemma, if $|i^{-1}(p)| = m$, then there are m curves from Γ that intersect. In particular, we obtain m curves pairwise intersecting exactly once. By [7], such a collection has $m \leq 2g+1$, so we conclude that i is at most $(2g+1)$-to-1. It is standard that the area of $A(\gamma)$ is $\ell(\gamma, X)\sinh(\omega(\gamma))$, which is at least $2Lexp(-L/2)$. It follows that

$$|\Gamma| \leq \pi \ (2g+1) \ |\chi| \ \frac{e^{L/2}}{L}. \qquad \blacksquare$$

Remark 5.5. In case $k = 1$, the bound from Lemma 3.2 is worse than that in Lemma 5.4 for values of L roughly larger than $\log |\chi|$. $\square$

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