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On the growth of L^2-invariants for sequences of lattices in Lie groups

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Abstract

We study the asymptotic behaviour of Betti numbers, twisted torsion and other spectral invariants of sequences of locally symmetric spaces. Our main results are uniform versions of the DeGeorge–Wallach Theorem, of a theorem of Delorme and various other limit multiplicity theorems.

A basic idea is to adapt the notion of Benjamini–Schramm convergence (BS-convergence), originally introduced for sequences of finite graphs of bounded degree, to sequences of Riemannian manifolds, and analyze the possible limits. We show that BS-convergence of locally symmetric spaces $\Gamma\backslash G/K$ implies convergence, in an appropriate sense, of the normalized relative Plancherel measures associated to $L^2(\Gamma\backslash G)$. This then yields convergence of normalized multiplicities of unitary representations, Betti numbers and other spectral invariants. On the other hand, when the corresponding Lie group G is simple and of real rank at least two, we prove that there is only one possible BS-limit; i.e., when the volume tends to infinity, locally symmetric spaces always BS-converge to their universal cover G/K. This leads to various general uniform results.

When restricting to arbitrary sequences of congruence covers of a fixed arithmetic manifold we prove a strong quantitative version of BS-convergence, which in turn implies upper estimates on the rate of convergence of normalized Betti numbers in the spirit of Sarnak–Xue.

An important role in our approach is played by the notion of Invariant Random Subgroups. For higher rank simple Lie groups G, we exploit rigidity theory and, in particular, the Nevo–Stück–Zimmer theorem and Kazhdan's property (T), to obtain a complete understanding of the space of IRS's of G.

Contents

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1. Introduction and statement of the main results

Let G be a connected center-free semi-simple Lie group without compact factors, $K \leq G$ a maximal compact subgroup and $X = G/K$ the associated Riemannian symmetric space. The main results of this paper concern the asymptotic of L^2-invariants of the spaces $\Gamma\backslash X$, where Γ varies over the space of lattices of G.

Most of our results rely on the notion of *Benjamini–Schramm convergence*, or BS-*convergence*, for sequences of locally symmetric spaces $\Gamma_n\backslash X$. We start by introducing a particularly transparent case: when $\Gamma_n\backslash X$ BS-converges to X.

1.1. *Definition.* Let (Γ_n) be a sequence of lattices in G. We say that the X-orbifolds $M_n = \Gamma_n\backslash X$ BS-*converge* to X if for every $R > 0$, the probability that the R-ball centered around a random point in M_n is isometric to the R-ball in X tends to 1 when $n \to \infty$; i.e., for every $R > 0$, we have

$$\lim_{n\to+\infty} \frac{\mathrm{vol}((M_n)_{<R})}{\mathrm{vol}(M_n)} = 0,$$

where $M_{<R} = \{x \in M : \mathrm{InjRad}_M(x) < R\}$ is the R-thin part of M.

A straightforward and well-studied example is when $\Gamma \leq G$ is a uniform lattice and $\Gamma_n \leq \Gamma$ is a chain of normal subgroups with trivial intersection; in this case, the R-thin part of $\Gamma_n\backslash X$ is empty for large enough n.

General BS-*convergence.* The definition above fits into a more general notion of convergence, adapted from that introduced by Benjamini and Schramm [19] for sequences of bounded degree graphs.

Consider the space $\mathcal{M}$ of pointed, proper metric spaces, endowed with the pointed Gromov–Hausdorff topology. Each $\Gamma_n\backslash X$ can be turned into a probability measure on $\mathcal{M}$ by choosing the basepoint at random with respect to volume; this measure is supported on pointed spaces isometric to $\Gamma_n\backslash X$. We say that $\Gamma_n\backslash X$ BS-*converges* if these measures weakly converge. The limit object is then a probability measure on $\mathcal{M}$. This perspective is elaborated on in Section 3.

Most of the results of this paper assume (or prove) BS-convergence to X. These results can often be extended to general BS-convergent sequences, but

they tend to get more technical and sometimes further assumptions are needed; they will appear in a sequel of this paper to be extracted from our original arXiv paper [2].

This definition of BS-convergence is very broad and works just as well for sequences of finite volume Riemannian manifolds. In our situation, the common ambient group G allows a useful algebraic reformulation of BS-convergence where probability measures on $\mathcal{M}$ are replaced by *invariant random subgroups* of G, i.e., G-invariant measures on the space of closed subgroups of G. This reformulation is what we use in most of the paper. This will be discussed at the end of the introduction and in Sections 2 and 3.

Uniform discreteness. A family of lattices (resp. the associated X-orbifolds) is *uniformly discrete* if there is an identity neighborhood in G that intersects trivially all of their conjugates. For torsion-free lattices Γ_n, this is equivalent to saying that there is a uniform lower bound for the injectivity radius of the manifolds $M_n = \Gamma_n\backslash X$. (So, in particular, a uniformly discrete family of lattices consists only of uniform lattices.)

Any family (M_n) of covers of a fixed compact orbifold is uniformly discrete. Margulis has conjectured [87, p. 322] (see also [60, §10]) that the family of all cocompact torsion-free arithmetic lattices in G is uniformly discrete. This is a weak form of the famous Lehmer conjecture on monic integral polynomials.

BS-*convergence and Plancherel measure.* Our first result says that BS-convergence to X implies a spectral convergence: namely, when (Γ_n) is uniformly discrete, the relative Plancherel measure of $\Gamma_n\backslash G$ will converge to the Plancherel measure of G in a strong sense.

For an irreducible unitary representation $\pi \in \widehat{G}$ and a uniform lattice Γ in G, let $\mathrm{m}(\pi, \Gamma)$ be the multiplicity of π in the right regular representation $L^2(\Gamma\backslash G)$. Define the relative Plancherel measure of $\Gamma\backslash G$ as the measure

$$\nu_\Gamma = \frac{1}{\mathrm{vol}(\Gamma\backslash G)} \sum_{\pi\in\widehat{G}} \mathrm{m}(\pi, \Gamma)\delta_\pi$$

on $\widehat{G}$. Finally denote by ν^G the Plancherel measure of the right regular representation $L^2(G)$. Recall that the support of ν^G is $\widehat{G}_{\mathrm{temp}}$ — the subset of the unitary dual $\widehat{G}$ that consists of tempered representations.

1.2. Theorem (Theorem 6.7). *Let (Γ_n) be a uniformly discrete sequence of lattices in G such that the spaces $\Gamma_n\backslash X$ BS-converge to X. Then for every quasi-compact ν^G-regular open subset $S \subset \widehat{G}$ or $S \subset \widehat{G}_{\mathrm{temp}}$, we have*

$$\nu_{\Gamma_n}(S) \to \nu^G(S).$$

Note that the Plancherel measure of G depends on a choice of a Haar measure on G as does $\mathrm{vol}(\Gamma\backslash G)$. We recall basic facts on the topology of $\widehat{G}$ in Section 6.

Let $d(\pi)$ be the "multiplicity" — or rather the formal degree — of π in the regular representation $L^2(G)$ with respect to the Plancherel measure of G. Thus, $d(\pi) = 0$ unless π is a discrete series representation. Theorem 1.2 implies the following.

1.3. Corollary. *Let (Γ_n) be a uniformly discrete sequence of lattices in G such that the spaces $\Gamma_n\backslash X$ BS-converge to X. Then for all $\pi \in \widehat{G}$, we have*

$$\frac{m(\pi, \Gamma_n)}{\mathrm{vol}(\Gamma_n\backslash G)} \to d(\pi).$$

In the special situation when (Γ_n) is a chain of normal subgroups with trivial intersection in some fixed cocompact lattice $\Gamma \leq G$, Corollary 1.3 is the classical theorem of DeGeorge and Wallach [64]. In that very same situation Theorem 1.2 is due to Delorme [47]. Since the pioneering work of DeGeorge and Wallach, "limit formulas" have been the subject of extensive studies. Two main directions of improvement have been considered.

The first direction is concerned with the extension of the theorems of DeGeorge–Wallach and Delorme to nonuniform lattices. In the case of the DeGeorge–Wallach theorem we refer to [45], [14], [38], [100], [106]. Note that these works were partially motivated by a question of Kazhdan [71] pertaining to his work on the field of definition of arithmetic varieties. The limit multiplicity problem for the entire unitary dual has been solved for the standard congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ by Sarnak in [102] (see also [69], [46]) but is still open in general. A partial result for certain normal towers of congruence arithmetic lattices defined by groups of $\mathbb{Q}$-rank one has been shown in [46]. Very recently important progress has been made by Finis, Lapid and Müller [55] who can deal with groups of arbitrary rank. In these works the authors usually deal with towers of normal subgroups.

A second direction is to extend the theorems of DeGeorge–Wallach and Delorme to more general sequences of (uniform) lattices. This has been addressed in some of the above mentioned works for certain (nonprincipal) congruence subgroups of a fixed lattice, such as $\Gamma_0(N)$; see also [70] for another example. Theorem 1.2 is the first example where one can deal with sequences of noncommensurable lattices.

The classical theorem of DeGeorge and Wallach implies a corresponding statement on the approximation of L^2-Betti numbers by normalized Betti numbers of finite covers (see also Donnelly [49]). Theorem 1.2 implies the following uniform version of it.

1.4. Corollary. *Let $(\Gamma_n)_{n\geq 1}$ be a uniformly discrete sequence of uniform lattices in G such that $\Gamma_n\backslash X$ BS-converges to X. Then for every $k \leq \dim(X)$, we have*

$$\frac{b_k(\Gamma_n)}{\mathrm{vol}(\Gamma_n\backslash X)} \to \beta_k^{(2)}(X).$$

In the corollary, $b_k(\Gamma_n)$ is the k-th Betti number of the (virtually torsion-free) group Γ_n,[1] and

$$\beta_k^{(2)}(X) = \begin{cases} \frac{\chi(X^d)}{\mathrm{vol}(X^d)} & k = \frac{1}{2}\dim X, \\ 0 & \text{otherwise} \end{cases}$$

is the k-th *L^2-Betti number of* X, where X^d is the compact dual of X equipped with the Riemannian metric induced by the Killing form on $\mathrm{Lie}(G)$. We refer the reader to Section 6.24 for an analytic definition of $\beta_k^{(2)}(X)$. By [11] and [94], the Euler characteristic $\chi(X^d)$ is nonzero exactly when the *fundamental rank*

$$\delta(G) = \mathbb{C}\text{-rank}(G) - \mathbb{C}\text{-rank}(K)$$

of G is zero. Alternatively, it follows from the equality of the Euler characteristic and its L^2-analogue that in the middle dimension, $\beta_k^{(2)}(X) \neq 0$ if and only if the Euler characteristic of some (or, equivalently, every) closed X-manifold is nonzero.

Uniform BS*-convergence in higher rank.* In the higher rank case we have the following remarkable phenomenon, which gives a surprisingly strong result when combined with Theorem 1.2. Note that in the following result we do not restrict to the case where the Γ_n are cocompact and, in particular, we do not assume uniform discreteness.

1.5. Theorem (Corollary 4.7). *Suppose that G has property* (T) *and real rank at least two. Let $\Gamma_n \leq G$ be any sequence of pairwise nonconjugate irreducible lattices in G. Then $\Gamma_n\backslash X$* BS*-converges to X.*

1.6. Corollary. *If in addition to the conditions of Theorem 1.5 we have that (Γ_n) is uniformly discrete (in particular, cocompact), then for every quasi-compact ν^G-regular subset $S \subset \widehat{G}$, we have*

$$\nu_{\Gamma_n}(S) \to \nu^G(S)$$

and, in particular,

$$\frac{m(\pi, \Gamma_n)}{\mathrm{vol}(\Gamma_n\backslash X)} \to d(\pi)$$

for any $\pi \in \widehat{G}$. And even more particularly, we have

$$\frac{b_k(\Gamma_n)}{\mathrm{vol}(\Gamma_n\backslash X)} \to \beta_k^{(2)}(X)$$

for every $k \leq \dim(X)$.

[1] The group Γ_n being virtually torsion-free, the orbifold $\Gamma_n\backslash X$ is finitely covered by a manifold whose Γ_n-invariant rational k-th cohomology group coincides with the rational k-th orbifold cohomology of $\Gamma_n\backslash X$ and is of finite rank $b_k(\Gamma_n)$; in particular, if Γ_n is torsion-free, then $b_k(\Gamma_n)$ is the k-th Betti number of $\Gamma_n\backslash X$.

Here is a particular example to illustrate the strength of Corollary 1.6:

1.7. *Example.* Let $n \geq 3$, let Γ be a cocompact lattice in $\mathrm{SL}_n(\mathbb{R})$, and let $\Gamma_m \leq \Gamma$ be a sequence of distinct, finite index subgroups of Γ. Then for all k,

$$\frac{b_k(\Gamma_m)}{[\Gamma : \Gamma_m]} \to 0.$$

Even in this example, where all the lattices fall in one commensurability class, we do not see a proof that avoids using Theorem 1.5.

It is easy to see that the analogue of Corollary 1.6 — and Therefore, of Theorem 1.5 — is false for some rank one symmetric spaces. For instance, suppose M is a closed hyperbolic d-manifold and $\pi_1(M)$ surjects onto the free group of rank two. Then finite covers of M corresponding to subgroups of $\mathbb{Z} * \mathbb{Z}$ have first Betti numbers that grow linearly with the volume. However, for $d \neq 2$, there will be sublinear growth of the first Betti number in any sequence of covers corresponding to a chain of finite index normal subgroups of $\pi_1(M)$ with trivial intersection, e.g., by the DeGeorge–Wallach theorem.

Removing the injectivity radius condition for hyperbolic manifolds. If $\mathrm{rank}(X) \geq 2$ or if X is the symmetric space corresponding to $\mathrm{Sp}(d,1)$ or F_4^{-20}, then all irreducible X-manifolds are arithmetic, by Margulis's Arithmeticity [87, Th. 1.10, p. 298] and the Corlette–Gromov–Schoen Theorem [39], [67], respectively. For $\mathrm{SU}(d,1)$, there are few known examples of nonarithmetic manifolds for $d = 2, 3$, and it is likely that most manifolds are arithmetic. According to Margulis' conjecture it is Therefore, natural to expect that if X is not isometric to some real hyperbolic space $\mathbb{H}^d$ ($d \geq 2$), then the family of all irreducible compact X-manifolds is uniformly discrete. On the other hand, it is shown in [9], [22], [16] that for every $d \geq 2$, there are compact hyperbolic manifolds of dimension d with arbitrarily small closed geodesics. Still, a careful estimate of the norm of the heat kernel in the thin part of rank one manifolds (see Section 7) allows us to prove the following.

1.8. Theorem (Theorem 7.13). *Let $M_n = \Gamma_n \backslash \mathbb{H}^d$ be a sequence of compact hyperbolic d-manifolds that* BS*-converges to $\mathbb{H}^d$. Then for every $k \leq d$,*

$$\lim_{n \to +\infty} \frac{b_k(M_n)}{\mathrm{vol}(M_n)} = \beta_k^{(2)}(\mathbb{H}^d).$$

Note that for $X = \mathbb{H}^2$, the hyperbolic plane, Theorem 7.13 is a consequence of the Gauss–Bonnet theorem, even under the weak assumption that only $\mathrm{vol}(M_n) \to \infty$, without requiring BS-convergence. In general there are many sequences of hyperbolic manifolds that BS-converge to $\mathbb{H}^d$, but where the global injectivity radius is not bounded below. A typical example is given by Brock–Dunfield [31], and while these are (intentionally) integer homology

spheres, similar examples can be constructed where the only control on the first Betti numbers is through Theorem 1.8.

The idea of our argument for Theorem 1.8 also gives an alternative proof, in the real hyperbolic case, of the classical theorem of Gromov that Betti numbers are linearly bounded by volume [12, Th. 2]. We were not able to perform the same analysis in the higher rank case. However, assuming the Margulis conjecture, our result for higher rank symmetric spaces (Corollary 1.6) is much stronger than Gromov's linear bound.[2]

Explicit estimates for congruence covers. When restricted to congruence covers of a given arithmetic *hyperbolic* manifold, Gromov conjectured that the k-th Betti number should be bounded above by a constant times n^α, where n is the index of the cover and

$$\alpha = \frac{2k}{d-1}, \quad 0 \le k \le [(d-1)/2];$$

see Sarnak and Xue [103]. Cossutta and Marshall [40] and Bergeron, Millson and Moeglin [23] proved an even better (and sharp) bound for principal congruence covers of level a power of a prime and small degree $k < d/3$. Our next result is a weak form of Gromov's conjecture. While we cannot approach the precise constant suggested by Gromov, we do obtain a very general result that applies to all semi-simple Lie groups and general congruence (not just principal) subgroups.

1.9. Theorem (Theorem 6.14). *Let G be a semi-simple Lie group, and let $\Gamma \le G$ be a uniform arithmetic subgroup. Let $\pi \in \widehat{G}$ be a nontempered irreducible representation. Then there are constants $\alpha > 0$ and $C < \infty$ such that for every congruence subgroup $\Delta \le \Gamma$,*

$$m(\pi, \Delta) \le C \cdot [\Gamma : \Delta]^{1-\alpha}.$$

As a consequence we obtain the following.

1.10. Corollary. *Let G and Γ be as in Theorem 1.9. Suppose that*

$$|k - \frac{1}{2} \dim X| > \delta(G).$$

Then there exist constants $\alpha > 0$ and C such that for every congruence subgroup $\Delta \le \Gamma$, we have

$$b_k(\Delta) \le C \cdot \mathrm{vol}(\Delta \backslash X)^{1-\alpha}.$$

[2]Recall however that Gromov's theorem applies in the much broader setup of Hadamard spaces with bounded curvature and no Euclidian factors, which we do not consider in this paper.

Theorem 1.9 is a consequence of the following result, which is of independent interest.

1.11. Theorem (Theorem 5.6). *Let $\mathbf{G}$ be a k-simple simply connected algebraic group defined over a number field k. Let $\mathcal{O}$ be the ring of integers in k. There exist a finite index center-free subgroup $\Gamma \subset \mathbf{G}(\mathcal{O})$ and positive constants ϵ and C (depending only on Γ and some fixed word metric on it) with the following property.*

Let $g \in \Gamma - \{1\}$, and let H be a congruence subgroup of index N in Γ. Then g fixes at most $e^{Cl(g)}N^{1-\epsilon}$ points in the action of Γ on the right cosets $H\backslash\Gamma$ by multiplication. Here $l(g)$ is the length of g with respect to the fixed word metric of Γ.

Theorem 1.11 leads to the following effective version (for subgroups of a fixed lattice) of Theorem 1.5; this allows us to prove Theorem 1.9. Implicit here is an effective proof (again, for subgroups of a fixed lattice) of the second part of Corollary 1.6.

1.12. Theorem (Theorem 5.2). *Let $\Gamma_0 \subset G$ be a cocompact arithmetic lattice. Then there exist positive constants c and μ depending only on Γ_0, such that for any congruence subgroup $\Gamma \subset \Gamma_0$ and any $R > 1$, we have*

$$\mathrm{vol}((\Gamma\backslash X)_{<R}) \leq e^{cR}\mathrm{vol}(\Gamma\backslash X)^{1-\mu}.$$

Growth of Reidemeister torsion. When the fundamental rank $\delta(G)$ is positive, the symmetric space X is L^2-acyclic. It is then natural to investigate a secondary invariant such as the L^2-torsion of X; see [85], [24]. This is known to be nonvanishing if and only if $\delta(G) = 1$, e.g., in the case $G = \mathrm{SL}_2(\mathbb{C})$. We study L^2-torsion for BS-convergent sequences in Section 8; see, in particular, Theorem 8.4.

In this introduction, we stress the particular case of compact *orientable* hyperbolic 3-manifolds. Given such an M we denote by α_{can} the discrete faithful $\mathrm{SL}_2(\mathbb{C})$-representation of $\pi_1 M$. The corresponding twisted chain complex

$$C_*(\widetilde{M}) \otimes_{\mathbb{Z}[\pi_1 M]} \mathbb{C}^2$$

is acyclic [96] and it follows that the corresponding Reidemeister torsion

$$\tau(M, \alpha_{\mathrm{can}}) \in \mathbb{R}^*$$

is well defined. The following result is a consequence of Theorem 8.4.

1.13. Theorem. *Let $(M_n)_n$ be a uniformly discrete sequence of orientable compact hyperbolic 3-manifolds that* BS*-converges toward $\mathbb{H}^3$. Then*

$$\lim_{n\to+\infty} \frac{1}{\mathrm{vol}(M_n)} \log|\tau(M_n, \alpha_{\mathrm{can}})| = -\frac{11}{12\pi}.$$

The role of IRS. An important tool in our project is the notion of an *invariant random subgroup* (IRS). An IRS is a conjugacy invariant probability measure on the space Sub_G of closed subgroups of G. We refer the reader to [6], [29], [112], [61], [62] for other recent works that make use of this notion.

Any lattice $\Gamma \leq G$ defines an IRS μ_Γ supported on the conjugacy class Γ^G. It turns out (see Theorem 2.9) that if G is a connected simple Lie group, then any nonatomic IRS is supported on discrete subgroups (hereafter called a discrete IRS). Every discrete IRS gives rise to a probability measure on the space of rooted metric spaces $\mathcal{M}$ mentioned above, and one can relate weak* convergence of IRS's to weak* convergence of measures on $\mathcal{M}$. See Section 3 for details.

Denote by μ_G and μ_{Id} the atomic measures supported on $\{G\}$ and $\{\mathrm{Id}_G\}$ respectively. The following is a variant of Theorem 1.5 stated in the language of IRS's.

1.14. Theorem (Theorems 4.2 and 4.4). *Let G be a connected, center-free higher rank simple Lie group. Then*

- *the ergodic* IRS*'s are exactly μ_G, μ_{Id} and μ_Γ where Γ is a lattice in G;*
- *the set of ergodic* IRS*'s is compact and its only accumulation point is μ_{Id}.*

The first part of Theorem 1.14 is a consequence of the Nevo–Stück–Zimmer rigidity theorem [109], [91].

The picture is much wilder in rank one. For example, starting with a lattice $\Gamma \leq G$ and an infinite index normal subgroup $\Delta \lhd \Gamma$, one can induce the measure on $\Gamma\backslash G$ to an ergodic IRS supported on the conjugacy class Δ^G. More generally, any IRS in Γ can be induced to an IRS in G. We investigate these constructions and more exotic ones in a sequel of this paper to be extracted from our original arXiv paper [2]. In particular, we define their spectral measure and their L^2-Betti numbers and we prove spectral convergence along sequences that BS-converge toward a nontrivial IRS.

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2. **Invariant Random Subgroups**

Let G be a locally compact second countable group. We denote by Sub_G the set of closed subgroups of G. There exists a natural topology on Sub_G, the *Chabauty topology* [34], that is generated by open sets of the form

(1) $\mathcal{O}_1(K) = \{H \in \mathrm{Sub}_G : H \cap K = \emptyset\}$ for $K \subset G$ compact, and
(2) $\mathcal{O}_2(U) = \{H \in \mathrm{Sub}_G : H \cap U \neq \emptyset\}$ for $U \subset G$ open.

Alternatively, a sequence $(H_n)_{n\geq 0}$ in Sub_G converges to $H \in \mathrm{Sub}_G$ if and only if

(1) for every $x \in H$, there exists a sequence $(x_n) \in G^{\mathbb{N}}$ such that $x_n \in H_n$ and $x_n \to x$ in G;
(2) for every strictly increasing sequence of integers $(n_k)_{k\geq 0}$ and for any converging sequence $x_{n_k} \to x$ such that $x_{n_k} \in H_{n_k}$, we have $x \in H$.

The Chabauty topology is compact, separable and metrizable [17, Lemma E.1.1]. While the proof of metrizability referenced uses Urysohn's theorem, one can also write down an explicit metric. For instance, when G is compact, the Chabauty topology is induced by the Hausdorff metric on $C(G)$. In the noncompact case, one can metrize it by integrating up the Hausdorff metrics on all R-balls around a fixed base point; see [4]. We refer the reader to [68] (and also to [62]) for more information on the topology of Chabauty spaces.

Note. We will not always require G to be locally compact. In this case, the Chabauty topology is defined as above, but it will not always be compact.

Here is an easy exercise in the definitions that we will use in Section 3.

2.1. Lemma. *Let G be a connected Lie group, and suppose that (Γ_n) is a sequence in Sub_G with $\Gamma_n \cap U = \{\mathrm{id}\}$ for some fixed neighborhood U of the identity* id *in G. If (Γ_n) converges toward a group H in Sub_G, then H is discrete. Moreover, if all the Γ_n's are torsion free, so is H.*

Proof. If H is not discrete, then it intersects U. So, $\mathcal{O}_2(U \smallsetminus \{\mathrm{id}\})$ is a neighborhood of H that does not contain any Γ_n. If g is a nontrivial element in H with $g^k = \mathrm{id}$, then there is a sequence $\gamma_n \in \Gamma_n$ with $\gamma_n \to g$. Therefore, $\gamma_n^k \to \mathrm{id}$, so by uniform discreteness we have $\gamma_n^k = \mathrm{id}$ for large n, implying that Γ_n has torsion. □

2.2. Proposition. *Let G be a connected Lie group. Then G is an isolated point in Sub_G if and only if G is topologically perfect.*

Recall that a topological group is *topologically perfect* if its commutator subgroup is dense. So, in particular, the proposition implies that if G is connected and semi-simple, then G is an isolated point in Sub_G.

Proof. Suppose that G is topologically perfect. Then by [30, Th. 2.1] there exist $d = \dim(G)$ open sets $\Omega_1, \dots, \Omega_d \subset G$ such that for every choice of d elements $g_i \in \Omega_i$, $i = 1, \dots, d$, the subgroup $\langle g_1, \dots, g_d \rangle$ is dense in G. Therefore, if $H \in \mathrm{Sub}_G$ intersects each of the Ω_i, then $H = G$, and thus $\cap_{i=1}^d \mathcal{O}_2(\Omega_i) = \{G\}$ is open.

Conversely, if G is not topologically perfect, then it surjects on the circle S^1. Let H_n be the pre-image in G of the cyclic group of order n in S^1. Then clearly we have that H_n converges to G. □

We also recall the following well-known fact, which follows from classical work of Kuranishi [75] and Toyama [111]. See also Theorem 4.1.7 in [110].

2.3. PROPOSITION. *Let G be a Lie group, and let $(\Gamma_n)_{n\geq 1}$ be a sequence of discrete subgroups in G that converges to a subgroup H. Then the connected component H° of H is nilpotent.*

We now come to the central definition of this section.

2.4. *Definition.* Let G be a topological group. An *invariant random subgroup* (IRS) of G is a G-invariant Borel probability measure on Sub_G.

Here, G acts on Sub_G by conjugation. The name IRS has been coined in [6]. We consider the set

$$\mathrm{IRS}(G) = \mathrm{Prob}(\mathrm{Sub}_G)^{\mathrm{inv}}$$

of invariant random subgroups of G endowed with the weak* (or, vaguely speaking, the weak) topology. When G is locally compact, as Sub_G is compact and the G-action is continuous, it follows from Riesz' representation theorem and Alaoglu's theorem that the space of invariant random subgroups of G is also compact.

IRS's arise naturally when dealing with nonfree actions, as the stabilizer of a random point in a probability measure preserving action is an IRS. More precisely, when G acts by measure preserving transformations on a countably separated probability space (Ω, ν), the push forward of ν under the stabilizer map[3]

$$\mathrm{stab} : \Omega \longrightarrow \mathrm{Sub}_G, \quad \mathrm{stab}(x) = G_x$$

is an IRS in G. We say that the IRS is *induced* from the probability measure preserving action.

As an example, suppose that H is a closed subgroup of G such that G/H admits a finite G-invariant measure; for instance, H could be a lattice in G. In this case, we scale the measure on G/H to a probability measure and denote by μ_H the invariant random subgroup induced by the left action of G on G/H.

This construction can be further generalized. Let H be a closed subgroup in G, and let $N = N_G(H)$ be its normalizer in G. Suppose that G/N admits a left G-invariant probability measure. Consider the map $G \to \mathrm{Sub}_G$ given by $g \mapsto gHg^{-1}$. The push-forward measure on Sub_G is a G-invariant measure supported on the conjugates of H, which we denote μ_H. This notation conflicts with that in the previous paragraph; indeed, if both G/H and G/N admit an invariant probability measure, then we have two definitions of μ_H. However, both these constructions give rise to the same measure.

[3]It is a result of Varadarajan that the stabilizers are closed subgroups; see [115, Cor. 2.1.20].

Invariant random subgroups can also be constructed as products. Let H_1, H_2 be commuting subgroups of G, and assume that H_1, H_2 and the product H_1H_2 are all closed in G. If μ_1, μ_2 are invariant random subgroups of G that are supported on $\mathrm{Sub}_{H_1}, \mathrm{Sub}_{H_2}$, we can push forward the product measure on $\mathrm{Sub}_{H_1} \times \mathrm{Sub}_{H_2}$ using

$$\mathrm{Sub}_{H_1} \times \mathrm{Sub}_{H_2} \to \mathrm{Sub}_G, \quad (K_1, K_2) \mapsto K_1 \times K_2$$

to a measure on Sub_G, which we denote $\mu_1 \otimes \mu_2$. It is easy to check that this measure is G-invariant.

Finally, suppose $\Gamma < G$ is a lattice and μ is an IRS of the discrete group Γ. The IRS of G *induced* from μ is the random subgroup of G obtained by taking a random conjugate of Γ and then a μ-random subgroup in this conjugate (which is well defined because of the invariance of μ). Formally, the natural map

$$G \times \mathrm{Sub}_\Gamma \ni (g, \Lambda) \longmapsto g\Lambda g^{-1} \in \mathrm{Sub}_G$$

factors through the quotient of $G \times \mathrm{Sub}_\Gamma$ by the Γ-action $(g, \Lambda)\gamma = (g\gamma, \gamma^{-1}\Lambda\gamma)$. This quotient has a natural G-invariant probability measure, and we define our IRS to be the push forward of this measure by the factored map $(G \times \mathrm{Sub}_\Gamma)/\Gamma \to \mathrm{Sub}_G$. This is a particularly important construction when $G = \mathrm{SO}(n, 1)$, in which case lattices have many IRS's; see [2].

2.5. IRS's *via stabilizers.* The following theorem shows that, when G is locally compact, every IRS is induced from a probability measure preserving action.

2.6. Theorem. *Let G be a locally compact second countable group, and let $\mu \in \mathrm{IRS}(G)$. Then μ is induced from some probability measure preserving action of G.*

When G is countable and discrete, this was proven in [6, Prop. 13]. The reader should note that we do not use this result anywhere in this paper, although it could be used to give a slightly shorter proof of Theorems 4.1 and 4.3. However, we consider it of independent interest.

Proof. The coset space of G, written Cos_G, is the set

$$\mathrm{Cos}_G = \{Hg \,:\, H \in \mathrm{Sub}_G,\, g \in G\},$$

equipped with the Fell topology of closed subsets of G. Then G acts on Cos_G, both from the left via $Hg \overset{k}{\mapsto} kHg$ and from the right via $Hg \overset{k}{\mapsto} Hgk$. With respect to the left action, the stabilizer map is the natural projection

$$\mathrm{stab} : \mathrm{Cos}_G \longrightarrow \mathrm{Sub}_G, \quad Hg \longmapsto H,$$

where the fiber above $H \in \mathrm{Sub}_G$ is the coset space $H\backslash G$. Note that as usual for a stabilizer map, the G-action permutes the fibers and descends to the conjugation action of G on Sub_G.

By [26, Th. 3.1], for almost every subgroup H in the support of μ, there is a right G-invariant measure ν_H on $H\backslash G$, and one can choose the map $H \mapsto \nu_H$ to be Borel.[4] So, integrating against μ creates a measure ν on Cos_G:

$$\nu = \int_{\mathrm{Sub}_G} \nu_H \, d\mu.$$

The left and right G-actions on Cos_G commute, so the left action of g pushes forward ν_H to a measure on $gHg^{-1}\backslash G$ that is again right G-invariant. By uniqueness, we have $g_*\nu_H = \lambda\nu_{gHg^{-1}}$ for some $\lambda \in \mathbb{R}$. Combining this with the conjugation invariance of μ, the left G-action preserves the *measure class* of ν.

Suppose for a moment that each ν_H is finite, i.e., that μ-a.e. $H \in \mathrm{Sub}_G$ has co-finite volume. Then after scaling, we can take each ν_H to be a probability measure. The ν_H are then permuted by the left G-action, which implies that ν (and not just its measure class) is left G-invariant. Furthermore, ν is then a probability measure that pushes forward to μ under stab. So, the theorem follows.

In general, it is enough to prove the theorem when μ is ergodic, so we can break into cases depending on whether for μ-a.e. $H \in \mathrm{Sub}_G$, we have

(1) ν_H is finite (in which case we are done),
(2) ν_H is infinite and $H\backslash G$ is discrete,
(3) ν_H is infinite and $H\backslash G$ is nondiscrete.

Of the latter two cases, (3) is the more difficult, so we will focus on that and mention (2) again at the end. Assuming (3), the two problems are that only the measure class of ν is G-invariant and that ν is not a probability measure. To deal with the first issue, we replace the action $G \curvearrowright \mathrm{Cos}_G$ with its *Maharam extension*

$$G \curvearrowright \mathrm{Cos}_G \times \mathbb{R}, \quad (Hg, t) \overset{k}{\longmapsto} \left(kHg, \frac{d\nu}{dk_*(\nu)}(Hg) \cdot t \right),$$

which preserves the measure $\nu \times \ell$ on $\mathrm{Cos}_G \times \mathbb{R}$, where ℓ is Lebesgue measure. Note that the stabilizer map for the Maharam extension is just the projection

$$\mathrm{Cos}_G \times \mathbb{R} \longrightarrow \mathrm{Sub}_G, \quad (Hg, t) \longmapsto H,$$

for since H acts trivially on $H\backslash G$, it preserves ν_H, and combining this with the conjugation invariance of μ gives that $\frac{d\nu}{dh_*(\nu)}(Hg) = 1$ whenever $h \in H$.

[4] Here, we regard the ν_H as measures on Cos_G, so the parametrization is Borel when for every Borel $B \subset \mathrm{Cos}_G$, the map $H \longmapsto \nu_H(B)$ is Borel.

We now have a G-invariant measure $\nu \times \ell$ and a disintegration

$$\nu \times \ell = \int_{\mathrm{Sub}_G} \nu_H \times \ell \, d\mu$$

with respect to the stabilizer map. If the fiber measures $\nu_H \times \ell$ were probability measures, we would be done as before, but they are not. So, the idea is to replace each fiber $H\backslash G \times \mathbb{R}$ with an associated Poisson process.

One way to make this rigorous is as follows. Both $\mathrm{Cos}_G \times \mathbb{R}$ and $\mathrm{Sub}_G \times \mathbb{R}_+$ are Polish spaces that map onto Sub_G, such that the fiber measures

$$\nu \times \ell \text{ on } H\backslash G \times \mathbb{R}, \quad \text{and} \quad \ell \text{ on } \{H\} \times \mathbb{R}_+$$

have no atoms, so a result of Rokhlin [101, p. 41] gives a measure isomorphism

$$\phi : \mathrm{Cos}_G \times \mathbb{R} \longrightarrow \mathrm{Sub}_G \times \mathbb{R}_+$$

that commutes with the projections to Sub_G.[5] Conjugating the G-action on $\mathrm{Cos}_G \times \mathbb{R}$ by ϕ, we have a measurable G-action on $\mathrm{Sub}_G \times \mathbb{R}_+$ such that

(a) the fibers $\{H\} \times \mathbb{R}_+$ are permuted by conjugating H;
(b) $g \in G$ pushes forward the Lebesgue measure on $\{H\} \times \mathbb{R}_+$ to the Lebesgue measure on $\{gHg^{-1}\} \times \mathbb{R}_+$;
(c) the stabilizer of $(H, t) \in \mathrm{Sub}_G \times \mathbb{R}_+$ is H.

Let $\mathcal{S}(\mathbb{R}_+)$ be the set of all countable subsets of $\mathbb{R}_+$, and let π be the Poisson process on $\mathbb{R}_+$ that we regard as a probability measure on $\mathcal{S}(\mathbb{R}_+)$. (We refer the reader to [41], [44] for details.) There is an induced action $G \curvearrowright \mathrm{Sub}_G \times \mathcal{S}(\mathbb{R}_+)$, which (using (b) above) preserves the *probability measure* $\mu \times \pi$.

By (b) and (c), the quotient group $N_G(H)/H$ acts freely on the fiber $\{H\} \times \mathbb{R}_+ \subset \mathrm{Sub}_G \times \mathbb{R}_+$, preserving the Lebesgue measure, so by Lemma 2.7 below the induced action on $\{H\} \times \mathcal{S}(\mathbb{R}_+)$ is essentially free. In other words, the G-stabilizer map

$$\mathrm{stab} : \mathrm{Sub}_G \times \mathcal{S}(\mathbb{R}_+) \longrightarrow \mathrm{Sub}_G$$

is the projection onto the first coordinate. Hence, $\mathrm{stab}_*(\mu \times \ell) = \mu$, and we are done.

Finally, a quick remark about the proof of (2). Here, there is no need for the Maharam extension since if one defines the ν_H to be the appropriate counting measures, then they will be permuted by the G-action. The rest of

[5] In the cited reference, Rokhlin's theorem is only for probability measures, not arbitrary σ-finite measures. However, one can always scale a σ-finite measure with a Borel function to become a probability measure, and it is not hard to then see that his theorem extends to the σ-finite case.

the proof proceeds in the same way, with all references to $\mathbb{R}_+$ replaced by $\mathbb{Z}$ and the Poisson process replaced by i.i.d. Benoulli measures. □

As promised, here is the lemma we used in the proof above.

2.7. Lemma. *Suppose that a group G acts measurably and freely on $\mathbb{R}_+$, preserving Lebesgue measure ℓ. Then G acts essentially freely on the space $\mathcal{S}(\mathbb{R}_+)$ of countable subsets of $\mathbb{R}_+$, with respect to the Poisson process π.*

This is folklore, but we include a brief proof for completeness, since we are not aware of a reference.

Proof. On the contrary, suppose that there is a positive π-probability that an element $D \in \mathcal{S}(\mathbb{R}_+)$ has nontrivial stabilizer. Then there is some interval $(0, n)$ such that there is a positive π-probability that for $D \in S(\mathbb{R}_+)$, the intersection $D \cap (0, n)$ contains elements $x_1, \dots, x_4$ with

$$g(x_1) = x_2,\ g(x_3) = x_4 \quad \text{for some } g \in G. \tag{2.7.1}$$

The intersection $D \cap (0, n)$ is almost surely finite, and after conditioning on cardinality k, the points of D are distributed within $(0, n)$ according to the Lebesgue measure on $(0, n)^k$, cf. [44, Ex. 7.1(a)]:

$$\text{Prob}\left(\begin{smallmatrix}\text{for } (x_1,\dots,x_k)\in(0,n)^k, \text{ we have } D\cap(0,n)=\{x_1,\dots,x_k\},\\ \text{given that } D\cap(0,n) \text{ has } k \text{ elements}\end{smallmatrix}\right) = d\ell^k(x_1, \dots, x_k).$$

So, in particular, there is a positive probability that $(x_1, \dots, x_4) \in (0, n)^4$ satisfies (2.7.1). But by freeness of the action, for such $(x_1, \dots, x_4)$, the last coordinate is determined by the first three. So, applying Fubini's theorem on $(0, n)^4 = (0, n)^3 \times (0, n)$, we have a contradiction. □

2.8. IRS*'s in Lie groups.* From now on, unless explicitly mentioned otherwise, we will assume that G is a connected Lie group.

The following is a variant of the classical Borel density theorem.

2.9. Theorem (Borel's density theorem). *If G is simple (with trivial center), then every* IRS *with no atoms is supported on discrete Zariski dense subgroups of G.*

A subgroup Γ of G is *Zariski dense* if the only closed subgroup $H < G$ that contains Γ and has finitely many connected components is G itself. This coincides with the algebraic definition of Zariski density when G has the structure of a real algebraic group. One recovers the classical theorem of Borel [28] when μ is the IRS μ_H associated with a closed subgroup of finite co-volume $H \leq G$.

Although the proof could be rearranged in a way that avoids and hence reproves the classical Borel density theorem, we will make use of Borel's result in the proof of the following.

2.10. Lemma. *The only* IRS *supported on the set of finite subgroups of* G *is the Dirac measure* $\delta_{\{\mathrm{id}\}}$.

Proof. Let μ be an ergodic IRS supported on finite subgroups of G. Since G has only countably many conjugacy classes of finite subgroups, μ is supported on a single conjugacy class, say F^G for some appropriate finite subgroup $F \leq G$. Thus μ induces a finite G-invariant probability measure on the homogenous space $G/N_G(F)$. Thus $N_G(F)$ is of finite co-volume in G. By the classical Borel density theorem, $N_G(F)$ is Zariski dense. Since F is finite, $N_G(F)$ is algebraic and hence $N_G(F) = G$. As G is connected and F is discrete, we deduce that F is central in G. Finally, since G has a trivial center, $F = \{\mathrm{id}\}$. □

Proof of Theorem 2.9. Associating to a closed subgroup $H < G$ either

(1) the Lie algebra of the identity component of H, or
(2) the Lie algebra of the identity component of the Zariski closure of H,

an IRS induces two $\mathrm{Ad}(G)$-invariant measures, μ_1 and μ_2, on the Grasmannian manifold of the Lie algebra $\mathfrak{g}$ of G. Note that both (1) and (2) are measurable as maps from Sub_G to the Grassmannian (see [63] for details). As follows from Furstenberg's proof of the Borel density theorem (see [59]), every such measure is supported on $\{\{0\}, \mathfrak{g}\}$.

The μ_1-mass of $\mathfrak{g}$ is exactly the mass that the given IRS gives to the atom G, which is by assumption 0. Thus μ_1 is the Dirac measure supported on $\{0\}$, which is equivalent to the statement that our IRS is supported on discrete subgroups.

On the other hand, any Zariski closed discrete subgroup of G is finite. Therefore, $\mu_2(\{0\})$ is the amount of mass that our IRS gives to finite subgroups of G. By Lemma 2.10, this must be 0. Therefore, μ_2 is the Dirac measure supported on $\mathfrak{g}$, which implies that our IRS is supported on Zariski dense subgroups. □

Remark. For a connected semi-simple group Lie G (which is neither necessarily simple nor center-free), an elaboration of the argument above provides the following. Given an ergodic IRS μ in G, there are normal subgroups $N_1, N_2 \lhd G$, with $N_1 \leq N_2$ such that for μ-a.e. $H \in \mathrm{Sub}_G$, the identity connected component H° is N_1 and the Zariski closure $\overline{H}^Z$ is N_2. For details, see [63], where the analog result is established for groups over general local fields.

3. **Large injectivity radius and** BS**-convergence**

Let G be a semi-simple Lie group, and let $X = G/K$ be an associated Riemannian symmetric space. An X*-orbifold* is a Riemannian orbifold obtained

as a quotient $\Gamma\backslash X$ for some discrete subgroup $\Gamma < G$. Our goal is to understand the topology of Sub_G geometrically through these quotient orbifolds and then to promote this to an understanding of discrete IRS's as random pointed X-orbifolds.

To begin with, let us understand the geometric meaning of Chabauty convergence to the identity. The *injectivity radius* of an X-orbifold $M = \Gamma\backslash X$ at $x \in M$ is

$$\mathrm{InjRad}_{\Gamma_n\backslash X}(x) = \frac{1}{2}\min\{d(\tilde{x}, \gamma\tilde{x}) \mid \gamma \in \Gamma_n - \{\mathrm{id}\}\},$$

where $\tilde{x} \in X$ is any lift of x. We then have

3.1. Lemma. *A sequence of subgroups $\Gamma_n < G$ converges to $\{\mathrm{id}\}$ in the Chabauty topology if and only if* $\mathrm{InjRad}_{\Gamma_n\backslash X}([\mathrm{id}]) \to \infty$.

Here, $[\mathrm{id}]$ is the projection of the identity in G.

Proof. It suffices to show that the subsets

$$U_R = \{H \in \mathrm{Sub}_G \mid \nexists \gamma \in H \smallsetminus \{\mathrm{id}\} \text{ with } d_X([\mathrm{id}], \gamma[\mathrm{id}]) \leq R\} \subset \mathrm{Sub}_G,$$

with $R \in (0, \infty)$, form a basis of open sets around $\{\mathrm{id}\} \in \mathrm{Sub}_G$.

To show that U_R is open, consider a sequence (H_n) of subgroups in Sub_G that do not belong to U_R; i.e., there are elements γ_n in $H_n \setminus \{\mathrm{id}\}$ with $d_X([\mathrm{id}], \gamma[\mathrm{id}]) \leq R$. Passing to a subsequence, we may suppose that the sequence (γ_n) converges towards some element $\gamma \in G$. Using the exponential map and replacing each γ_n by an appropriate power, we may furthermore assume that $\gamma \neq \mathrm{id}$. Then γ translates $[\mathrm{id}]$ by at most R, and appears in any Chabauty limit H of (H_n), so any such limit is also outside U_R.

We can prove that the U_R form a basis by comparing them with the basic open sets $O_1(K), O_2(U)$ defined in Section 2. First, suppose $K \subset G$ is compact and $\mathrm{id} \notin K$. Choosing R larger than the $[\mathrm{id}]$-translation distance of all $\gamma \in K$, we have $U_R \subset \mathcal{O}_1(K)$. And if $U \subset G$ is open and $\mathrm{id} \in U$, then $\mathcal{O}_2(U) = \mathrm{Sub}_G$, and hence contains U_R for all R. □

On the level of IRS's, we have

3.2. Proposition. *A sequence of* IRS*'s (μ_n) of G converges weakly to μ_{id} if and only if for every $R > 0$, the μ_n-probability that for a subgroup $\Gamma \in \mathrm{Sub}_G$ we have* $\mathrm{InjRad}_{\Gamma\backslash X}([\mathrm{id}]) \leq R$ *tends to zero as $n \to \infty$.*

Proof. With the notation of the proof of Lemma 3.1, $\mu_{\Gamma_n} \to \mu_{\mathrm{id}}$ if and only if

$$\lim_{n\to\infty} \mu_{\Gamma_n}(U_R) = 1 \quad \text{for all} \quad R > 0.$$ □

And for lattice IRS's, this can be rewritten as follows.

3.3. Corollary. *Suppose that (Γ_n) is a sequence of lattices in G. Then (μ_{Γ_n}) converges weakly to μ_{id} if for every $R > 0$, we have*

$$\lim_{n\to\infty} \mathcal{P}_n\{x \in \Gamma_n\backslash X \mid \mathrm{InjRad}_{\Gamma_n\backslash X}(x) \le R\} = 0, \tag{3.3.1}$$

where $\mathcal{P}_n$ is the normalized Riemannian measure on $\Gamma_n\backslash X$.

Proof. Consider the G-invariant probability measure $\hat{\mu}$ on $\Gamma_n\backslash G$. Pushing forward this measure to Sub_G by the stabilizer map $\Gamma_n g \mapsto g^{-1}\Gamma_n g$ gives μ_{Γ_n}, while pushing it forward under the projection $\Gamma_n\backslash G \to \Gamma_n\backslash X$ gives $\mathcal{P}_n$. Therefore,

$$\begin{aligned}\mathcal{P}_n &\left\{x \in \Gamma_n\backslash X \mid \mathrm{InjRad}_{\Gamma_n\backslash X}(x) \le R\right\} \\ &= \hat{\mu}\left\{\Gamma_n g \in \Gamma_n\backslash G \mid \mathrm{InjRad}_{\Gamma_n\backslash X}([g]) \le R\right\} \\ &= \mu\left\{g^{-1}\Gamma_n g \in \mathrm{Sub}_G \mid \mathrm{InjRad}_{g^{-1}\Gamma_n g\backslash X}([\mathrm{id}]) \le R\right\},\end{aligned}$$

where the last line uses that $d_X([g], \gamma[g]) = d_X([\mathrm{id}], g^{-1}\gamma g[\mathrm{id}])$. So, the corollary follows from Proposition 3.2. □

In Section 4, we will show that any sequence of irreducible lattice IRS's in a center-free higher rank semi-simple Lie group with property (T) weakly converges to μ_{id}. Using Corollary 3.3, we will see in Section 6 how to deduce asymptotics for Betti numbers and representation multiplicities, as discussed in the introduction.

As in the introduction, we say a sequence of X-orbifolds *Benjamini–Schramm converges* to X when (3.3.1) holds for all R. More generally, the theory of invariant random subgroups of Lie groups can be recast in a geometric context, where weak convergence is replaced by a suitable generalized BS-convergence. This interpretation is inspired by a program in graph theory, e.g., [5], [8], [10], [18], that was popularized by Benjamini–Schramm [19]. We will briefly discuss the graph theory and then explain how to translate to the continuous setting.

3.4. *Graphs and* IRS*'s of discrete groups.* All the material here is well known; for more information, we refer the reader to [7], [10], [26].

Let $\mathcal{G}$ be the space of all isomorphism types of rooted graphs (X, p), where

$$d\big((X,p),(Y,q)\big) = \inf\left\{\frac{1}{R} \;\middle|\; B_X(p,R) \cong B_Y(q,R)\right\},$$

so two rooted graphs are close if balls of large radius around their base points are isomorphic. We consider the set $\mathrm{Prob}(\mathcal{G})$ of all Borel probability measures μ on $\mathcal{G}$ with the topology of weak* convergence.

One way to understand weak* convergence is as follows. For each $R > 0$ and each finite rooted graph $B = (B, p)$, let $P_{R,B}(\mu)$ be the probability that

the R-ball around the root of a μ-randomly chosen $(X,p) \in \mathcal{G}$ is isomorphic to (B,p). Then

$$\mu_i \to \mu \text{ weakly} \iff P_{R,B}(\mu_i) \to P_{R,B}(\mu) \ \ \forall R>0, B=(B,p). \tag{3.4.1}$$

Here, the condition $B_X(p,R) \cong (B,p)$ determines a basic open set for the topology of $\mathcal{G}$, whose μ-measure is $P_{R,B}(\mu)$. Equation (3.4.1) follows since these sets are also closed.

Any finite graph X determines an element $\mu_X \in \mathrm{Prob}(\mathcal{G})$, by choosing the root uniformly from the vertices of X. One says that a sequence (X_i) of finite graphs *Benjamini–Schramm converges* to a measure $\mu \in \mathrm{Prob}(\mathcal{G})$ if $\mu_{X_i} \to \mu$ weakly. In light of (3.4.1), a Benjamini–Schramm limit captures, for large i, the limiting statistics of the isomorphism types of all R-balls in X_i.

Now let $G = \langle S \rangle$ be a finitely generated group. A subgroup $H < G$ determines a rooted *Schreier graph*, written $\mathrm{Sch}_S(H\backslash G)$, where vertices are cosets Hg, the root is H, and where an edge labeled $s \in S$ joins Hg to Hgs. Adding edge labels, we have a space $\mathcal{G}_S$ of isomorphism types of rooted S-labeled graphs, and the map

$$\mathrm{Sub}_G \longrightarrow \mathcal{G}_S, \quad H \longmapsto \mathrm{Sch}_S(H\backslash G)$$

is a homeomorphism onto its image; see [7]. So, an IRS μ of G determines a measure $\mathrm{Sch}_S(\mu)$ on $\mathcal{G}_S$, and the induced map

$$\{\text{IRS's of } G\} \longrightarrow \mathrm{Prob}(\mathcal{G}), \quad \mu \longmapsto \mathrm{Sch}_S(\mu)$$

is a weak* homeomorphism onto its image. Therefore, the study of IRS's of G is equivalent to the study of (certain) random S-labeled rooted graphs.

Passing to the continuous setting, we would like to study discrete IRS's Γ of a Lie group G as random pointed quotients $\Gamma\backslash X$. There are a number of settings in which one can develop such a theory, but the following is the most general.

3.5. *The Gromov–Hausdorff space.* Consider the set

$$\mathcal{M} = \big\{\text{proper, pointed length spaces } (X,x_0)\big\}/\text{pointed isometry}.$$

In [66], Gromov defined a notion of convergence of pointed metric spaces using a generalization of the Hausdorff metric. Following a variant given in [33, §3.2], two pointed metric spaces (X,x_0) and (X',x_0') are (ε,R)-*related*, written

$$(X,x_0) \sim_{\varepsilon,R} (X',x_0'),$$

if there are compact subspaces $K \subset X$ and $K' \subset X'$ containing the basepoints and a relation $\sim$ between K and K' that satisfies the following properties:

(1) $B_X(x_0,R) \subset K$ and $B_{X'}(x_0',R) \subset K'$;
(2) $x_0 \sim x_0'$;
(3) for each $x \in K$, there exists $x' \in K'$ such that $x \sim x'$;

(4) for each $x' \in K'$, there exists $x \in K$ such that $x \sim x'$; and
(5) if $x \sim x'$ and $y \sim y'$, then $|d_X(x,y) - d_{X'}(x',y')| \leq \varepsilon$.

This defined a (*pointed*) *Gromov–Hausdorff topology* on $\mathcal{M}$: a basis of neighborhoods of (X, x_0) is defined by considering for each $\varepsilon > 0$ and $R > 0$ the set of proper, pointed length spaces that are (ε, R)-related to (X, x_0).

It is well known that this topology is separable and completely metrizable, i.e., Polish; a distance between (X, x_0) and (X', x_0') can be defined by taking the infimal $\epsilon = \epsilon_R$ such that (X, x_0) and (X', x_0') are (ϵ, R)-related and then setting

$$d\big((X, x_0), (X', y_0)\big) = \sum_{R=1}^{\infty} \frac{\min\{\epsilon_R, 1\}}{2^R}.$$

Note that the space of rooted graphs $\mathcal{G}$ from the previous section embeds in $\mathcal{M}$, once we declare all edges in a graph to have unit length.

Suppose that G is a semi-simple Lie group, and let $X = G/K$ be an associated Riemannian symmetric space. For simplicity, we will deal with X-*manifolds* $\Gamma\backslash X$ rather than X-orbifolds in the rest of the section; i.e., we will assume that our discrete Γ is torsion free. As a geometric analogue of Lemma 3.1, we have

3.6. Proposition. *A sequence of pointed X-manifolds (M_i, p_i) converges in the Gromov–Hausdorff topology to X if and only if* $\mathrm{InjRad}_{M_i}(p_i) \to \infty$.

Here, any base point for a space with a transitive isometry group, like X, gives the same element of $\mathcal{M}$, so we drop the base point from the notation. The difficulty in proving Proposition 3.6 is that Gromov–Hausdorff convergence is metric, not topological, and the homeomorphism type may change drastically in a Gromov–Hausdorff limit. However, the following lemma shows that when the curvature and its derivatives are bounded, this is not the case.

3.7. Lemma. *Suppose that (M_i, p_i) is a sequence of complete Riemannian d-manifolds and that the covariant derivatives of the curvature tensors R_i satisfy*

$$|\nabla^k R_i| < C_k < \infty$$

for some fixed sequence (C_k) independent of i and of the point in M_i. If (M_i, p_i) converges to a Riemannian d-manifold (M, p) in the Gromov–Hausdorff topology, then the convergence is smooth and

$$\mathrm{InjRad}_{M_i}(p_i) \longrightarrow \mathrm{InjRad}_M(p).$$

Here, tensor norms are induced by the associated Riemannian metrics. We say that $(M_i, p_i) \longrightarrow (M, p)$ *smoothly* if there is a sequence of embeddings

$$\phi_i : B(p, R_i) \longrightarrow M_i$$

with $R_i \to \infty$ and $\phi_i(p) = p_i$, such that $\phi_i^* g_i \to g$ in the C^∞-topology, where g_i, g are the Riemannian metrics on M_i, M.

The authors would like to thank Igor Belegradek for a very helpful conversation related to the proof below.

Proof of Lemma 3.7. First, suppose that $\mathrm{InjRad}_{M_i}(p_i) \to 0$. As the sectional curvatures of M_i are bounded, a result of Cheeger–Gromov–Taylor [37, Th. 4.7] implies that $\mathrm{vol}_{M_i}(p_i, 1) \to 0$ as well. The Gromov–Hausdorff limit then has Hausdorff dimension at most $d-1$; this dates back to work of Gromov in the 1970's, but a citation for a more general result is [36, Th. 3.1]. In our case, though, the limit is a Riemannian d-manifold, so we have a contradiction.

So, there is a lower bound on the injectivity radii $\mathrm{InjRad}_{M_i}(p_i)$. From this and the bounds on the derivatives of curvature, it is well known that the convergence is smooth; see [78, Th. 4.1]. The continuity of injectivity radius is then a result of Erlich [52]. □

Proof of Proposition 3.6. Pick a base point $p \in X$. If the injectivity radius at p_i goes to infinity, then there are radii $R_i \to \infty$ such that the ball $B_{M_i}(p_i, R_i)$ is isometric to $B_X(p, R_i)$. This isometry gives a $(0, R_i)$-relation.

Conversely, suppose (M_i, p_i) converges in the Gromov–Hausdorff topology to X. The hypotheses of Lemma 3.7 are satisfied, since for every i and at every point in M_i, we have $|\nabla^k R_i| = |\nabla^k R_X|$, which we can take as our C_k. So, the injectivity radius at p_i goes to infinity. □

Benjamini–Schramm convergence of X-manifolds to X can now be reinterpreted using weak convergence of measures on $\mathcal{M}$. Note that every complete finite volume Riemannian orbifold M produces a probability measure μ_M on $\mathcal{M}$: one pushes forward the normalized Riemannian measure of M under the natural map $M \longrightarrow \mathcal{M}$, where $x \longmapsto (M, x)$. Also, we denote the atomic measure on $X \in \mathcal{M}$ by μ_X.

As an immediate consequence of Corollary 3.3 and Proposition 3.6, we have

3.8. Corollary. *For X-manifolds $M_i = \Gamma_i \backslash X$, the following are equivalent*:

(1) *the* IRS*'s μ_{Γ_i} weakly converge to μ_{id}*;
(2) *for every $R > 0$, $\lim_{i\to\infty} \mathcal{P}_i\{x \in \Gamma_i\backslash X \mid \mathrm{InjRad}_{\Gamma_i\backslash X}(x) \le R\} = 0$*;
(3) *the measures μ_{M_i} on $\mathcal{M}$ weakly converge to μ_X.*

3.9. *General Benjamini–Schramm convergence.* To reinterpret weak convergence of more general IRS's geometrically, we need to add frames to our space $\mathcal{M}$, similarly to how we added S-labels to rooted graphs in Section 3.4.

A *frame* for a Riemannian manifold M is an orthonormal basis f for some tangent space $T_{\pi(f)}M$, where $\pi(f) \in M$. We let

$$\mathcal{MF}^d = \big\{\text{framed Riemannian } d\text{-manifolds } (M, f)\big\}/\text{framed isometry}.$$

A *framed* (ϵ, R)*-relation* between (M, f) and (N, f') is an (ϵ, R)-relation between the pointed manifolds $(M, \pi(f))$ and $(N, \pi(f'))$ with the additional assumption

$$\exp_f(v) \sim \exp_{f'}(v) \text{ when } v \in B_{\mathbb{R}^d}(0, R). \tag{3.9.1}$$

Here, $\exp_f : \mathbb{R}^d \longrightarrow M$ is the Riemannian exponential map associated to f. If a framed (ϵ, R)-relation exists, we write $(M, f) \sim_{\epsilon,R} (N, f')$. As in the pointed case, framed (ϵ, R)-relations induce a (*framed*) *Gromov–Hausdorff topology* on the set $\mathcal{MF}^d$, and this topology is again Polish.

If G is a semi-simple Lie group and $X = G/K$ is a symmetric space, let

$$\mathrm{Sub}_G^{\mathrm{dtf}} = \{\text{discrete, torsion-free } \Gamma < G\} \subset \mathrm{Sub}_G.$$

Fixing an orthonormal frame f for X and setting $d = \dim X$, we have a map

$$\Phi : \mathrm{Sub}_G^{\mathrm{dtf}} \longrightarrow \mathcal{MF}^d, \quad \Gamma \overset{\Phi}{\longmapsto} (\Gamma\backslash X, [f]).$$

3.10. Proposition. *The map* Φ *is a homeomorphism onto its image.*

Proof. As $\mathcal{MF}^d$ is Hausdorff, it suffices to show that Φ is a continuous, proper injection. Injectivity is clear, since an isometry $(\Gamma\backslash X, [f]) \longrightarrow (\Gamma'\backslash X, [f])$ lifts to a (Γ, Γ')-equivariant isometry $X \longrightarrow X$ fixing the base frame f, which must then be the identity, implying $\Gamma = \Gamma'$. For properness, Lemma 3.7 implies that on any compact subset $K \subset \mathcal{MF}^d$, there is some $\epsilon > 0$ such that

$$\mathrm{InjRad}_M(p) \geq \epsilon \quad \text{for all } (M, p) \in K.$$

So, $\Phi^{-1}(K)$ is a family of uniformly discrete, torsion-free subgroups of G. Lemma 2.1 implies that $\Phi^{-1}(K)$ is precompact in $\mathrm{Sub}_G^{\mathrm{dtf}}$, so it suffices to check that Φ is continuous, since then the preimage $\Phi^{-1}(K)$ will be closed, hence compact.

Suppose that $\Gamma_i \to \Gamma_\infty$ in Sub_G, write $M_i = \Gamma_i\backslash X$, and let the projection of the frame $f \in X$ be $f_i \in T_{p_i}M_i$. Fixing $R > 0$, we define a relation $\sim$ between the balls $B_{M_i}(p_i, R)$ and $B_{M_\infty}(p_\infty, R)$ as in (3.9.1), via

$$\exp_{f_i}(v) \sim \exp_{f_\infty}(v) \text{ when } v \in B_{\mathbb{R}^d}(0, R).$$

Fixing $\epsilon > 0$, we want to show that $\sim$ is an (ϵ, R)-relation for large i. As conditions (1)–(4) are immediate, the point is to prove (5), i.e., that for large i,

$$\Big| d_{M_i}(\exp_{f_i}(v), \exp_{f_i}(w)) - d_{M_\infty}(\exp_{f_\infty}(v), \exp_{f_\infty}(w)) \Big| \leq \epsilon \tag{3.10.1}$$

for all $v, w \in B_{\mathbb{R}^d}(0,R)$. If not, then after passing to a subsequence, there are sequences v_i, w_i that violate this inequality for all i. Passing to another subsequence, we can assume that $(v_i, w_i) \to (v_\infty, w_\infty)$ in $\overline{B_{\mathbb{R}^d}(0,R)} \times \overline{B_{\mathbb{R}^d}(0,R)}$.

Now $\exp_{f_i}(v_i)$ is the projection under $X \longrightarrow M_i$ of the point $\exp_f(v_i) \in X$, and similarly for w_i and $i = \infty$. So for $i = 1, 2, \ldots, \infty$, there are $g_i \in \Gamma_i$ with

$$d_{M_i}(\exp_{f_i}(v_i), \exp_{f_i}(w_i)) = d_X(g_i \exp_f(v_i), \exp_f(w_i)).$$

That is, $g_i \exp_f(v_i)$ is the closest point in the Γ_i-orbit of $\exp_f(v_i)$ to $\exp_f(w_i)$.

Passing to a subsequence, we may assume that $g_i \to g$ in G, since all the g_i translate a point inside $B_X(\pi(f), R)$ a distance at most, say, $10R$. Then by Chabauty convergence, we have $g \in \Gamma_\infty$, so

$$\begin{aligned}\lim_{i\to\infty} d_{M_i}(\exp_{f_i}(v_i), \exp_{f_i}(w_i)) &= \lim_{i\to\infty} d_X(g_i \exp_f(v_i), \exp_f(w_i)) \\ &= d_X(g \exp_f(v_\infty), \exp_f(w_\infty)) \\ &\geq d_{M_\infty}(\exp_{f_\infty}(v_\infty), \exp_{f_\infty}(w_\infty)).\end{aligned}$$

On the other hand, Chabauty convergence also provides a sequence $x_i \in \Gamma_i$ with $x_i \to g_\infty$. So,

$$\begin{aligned}\lim_{i\to\infty} d_{M_i}(\exp_{f_i}(v_i), \exp_{f_i}(w_i)) &\leq \lim_{i\to\infty} d_X(x_i \exp_f(v_i), \exp_f(w_i)) \\ &= d_X(g_\infty \exp_f(v_\infty), \exp_f(w_\infty)) \\ &= d_{M_\infty}(\exp_{f_\infty}(v_\infty), \exp_{f_\infty}(w_\infty)).\end{aligned}$$

This contradicts the fact that the v_i, w_i were chosen to violate (3.10.1). □

An IRS μ of G is *discrete* or *torsion free* if μ-a.e. closed subgroup of G is discrete, or torsion free. As a consequence of Proposition 3.10, we have

3.11. Corollary. *The following map is a homeomorphism onto its image*:

$$\Phi_* : \{\textit{discrete, torsion-free} \text{ IRS's } \textit{of } G\} \longrightarrow \mathrm{Prob}(\mathcal{MF}^d)$$

So, weak convergence of (discrete, torsion-free) IRS's can be viewed as weak convergence of measures on the Gromov–Hausdorff space of framed X-manifolds.

Corollary 3.11 captures most of the interesting topology of the space of all IRS's of G. When G is a simple Lie group, for instance, Theorem 2.9 and Proposition 2.2 imply that every IRS of G is supported on discrete subgroups, except for a possible atom on the isolated subgroup $\{G\} \in \mathrm{Sub}_G$. So, the space of all IRS's is a cone on the space of discrete IRS's.

While we have chosen to simplify the argument by considering manifolds instead of orbifolds, the corollary is still true if one replaces framed manifolds by framed orbifolds and drops the torsion-free hypothesis. For simple Lie groups G, a quick and dirty argument is as follows. One proves as in Proposition 3.10 that the map Φ in a continuous injection, and hence a Borel

isomorphism onto its image,[6] so Φ_* is a continuous injection. Since G is simple, the argument in the previous paragraph shows that the space of discrete IRS's is compact, and any continuous injection from a compact space into a Hausdorff space is a homeomorphism onto its image.

4. **IRS's in higher rank**

As in the previous section, suppose that G is a center free semi-simple Lie group without compact factors, and let $X = G/K$ be the associated Riemannian symmetric space. We say an IRS is *irreducible* if every simple factor acts ergodically. When G has higher rank and Kazhdan's property (T), we prove the following strong result using the Nevo–Stück–Zimmer rigidity theorem (see below).

4.1. Theorem. *Let G be a center-free semi-simple Lie group of real rank at least two and with Kazhdan's property* (T). *Let μ be a nonatomic irreducible* IRS *of G. Then $\mu = \mu_\Gamma$ for some irreducible lattice $\Gamma \leq G$.*

Recall that a simple Lie group of $\mathbb{R}$-rank at least two has property (T) by Kazhdan's theorem [15, §1.6] and a rank-one simple Lie group has property (T) if and only if it is locally isomorphic to $\mathrm{Sp}(n,1)$, $n \geq 2$ or $F_{4(-20)}$ by Kostant's result [15, §3.3]. A semi-simple Lie group has property (T) if and only if all its simple factors have (T). By the arithmeticity theorems of Margulis and Corlette–Gromov–Schoen [86], [39], [67], if G has property (T), then all its lattices are arithmetic.

When all the simple factors of G are of real rank at least two, one can furthermore classify all the ergodic invariant random subgroups of G as follows.

4.2. Theorem. *Let G be a connected semi-simple Lie group without center, and suppose that each simple factor of G has $\mathbb{R}$-rank at least two. Then every ergodic invariant random subgroup is either*

(1) *μ_N for a normal subgroup N in G;*
(2) *μ_Λ for a lattice Λ in a normal subgroup M of G; or*
(3) *products of the previous two measures, where N and M commute.*

Explicitly, if μ is an ergodic invariant random subgroup, then there are commuting normal subgroups N, M in G and a lattice Λ in M such that $\mu = \mu_N \otimes \mu_\Lambda = \mu_{N\times\Lambda}$.

We shall prove Theorems 4.2 and 4.1 by making use of the following fundamental result of Nevo, Stück and Zimmer, which is a particular case of [109, Th. 4.3].

[6] Any continuous injection between standard Borel spaces is a Borel isomorphism [72, Th. 15.6], and Sub_G and $\mathcal{MF}^d$ are Polish spaces.

4.3. THEOREM (Nevo–Stück–Zimmer). *Let G be a connected semi-simple Lie group without center, such that each simple factor of G has $\mathbb{R}$-rank at least 2 or is isomorphic to $\mathrm{Sp}(n,1)$, $n \geq 2$ or $F_{4(-20)}$. Suppose that G, as well as every rank one factor of G, acts ergodically and faithfully preserving a probability measure on a space X. Then there is a normal subgroup $N \lhd G$ and a lattice $\Gamma < N$ such that for almost every $x \in X$, the stabilizer of x is conjugate to Γ.*

Let us mention that some new results in the spirit of Theorem 4.3 were established recently in [42] and [80].

Before we start the proofs, we would like to mention that the following could be simplified a bit by appealing to Theorem 2.6. In particular, one could avoid referencing the Margulis normal subgroup theorem. However, we have chosen to give an independent proof, as it is not that much longer.

Proof of Theorems 4.1 and 4.2. Let us assume that G has $\mathbb{R}$-rank at least two and Kazhdan's property (T). At various points in the proof we will mention how the assumptions of Theorem 4.2 imply a stronger conclusion. In the following, let μ be an ergodic invariant random subgroup of G.

Suppose first that the action of G is faithful. By 4.3 we obtain a normal subgroup N and a lattice $\Gamma < N$ such that the stabilizer, i.e., the normalizer, of a μ-random subgroup is conjugated to Γ. We claim that $N = G$. Indeed, the direct complement M of N in G normalizes every conjugate of any subgroup of Γ. Hence M fixes almost every point in Sub_G and as the action is faithful, M is trivial.

Next we claim that if Λ is a subgroup of G whose normalizer is Γ, then $[\Gamma : \Lambda] < \infty$. Recall the Margulis Normal Subgroup Theorem: *a normal subgroup of an irreducible lattice in a semi-simple Lie group with $\mathbb{R}$-rank ≥ 2 is either central or is a lattice.* In our cases, the assumptions of Theorem 4.1 impl that Γ is irreducible, but the assumptions of Theorem 4.2 do not. However, by [98, Th. 5.22], there is a decomposition of G as a product of normal subgroups $\prod G_i$ such that $\Gamma_i := \Gamma \cap G_i$ is an irreducible lattice in G_i and $\prod \Gamma_i$ has finite index in Γ. Note that by the assumptions of 4.2, $\mathbb{R}\text{-rank}(G_i) \geq 2$ for every i. Moreover, the projection of Λ to each G_i cannot be trivial since Γ is the full normalizer of Λ. By considering the commutator $[\Gamma_i, \Lambda]$ one deduces that $\Lambda_i := \Lambda \cap G_i$ is nontrivial for every i. By the normal subgroup theorem, Λ_i is of finite index in Γ_i as the latter is center free. Therefore, $\prod \Lambda_i$ and hence also Λ is a lattice in $G = \prod G_i$.

We have shown that a μ-random subgroup in Sub_G is a lattice. It is proved in [109] that the action of G on the subset of lattices in Sub_G is tame; i.e., the Borel structure on the orbit space is countably separable. In particular, an ergodic measure supported on this subset must be supported on a single orbit. Thus $\mu = \mu_\Lambda$ for some lattice Λ. In particular, this finishes the proof of Theorem 4.1.

We now finish the remaining cases of Theorem 4.2 when the action is not faithful. Let N be the kernel of this action. If $N = G$, then μ is supported on a normal subgroup of G, and we are done. Otherwise, take a direct complement M of N such that $G \simeq N \times M$.

We note that a subgroup normalized by N has a certain decomposition as a direct product. To this end, suppose that a subgroup $H \in \mathrm{Sub}_G$ is normalized by a simple factor L of N. By simplicity, either H contains L or $L \cap H = 1$. In the latter case, L and H commute, and thus the projection of H to L is central, and hence trivial. It follows that if H is normalized by N, then it decomposes as $H = H_N \times H_M$, where $H_N := H \cap N$ is a product of simple factors in N and $H_M := H \cap M$.

As there are finitely many possibilities for H_N, this factor of the decomposition is independent of H, by ergodicity. That is, there exists a normal subgroup $L \leq N$ such that $H = L \times (H \cap M)$ for almost every $H \in \mathrm{Sub}_G$. Thus, $\mu = \mu_L \otimes \mu'$, where μ' is an invariant ergodic measure supported on the image of Sub_M in Sub_G. Since M acts faithfully and ergodically on (Sub_M, μ'), we deduce from the previous case that $\mu' = \mu_\Lambda$ for a lattice Λ in M. Finally, it is easy to check that $\mu = \mu_L \otimes \mu_\Lambda = \mu_{L\times\Lambda}$. This completes the proof of Theorem 4.2. □

The proof of the uniform approximation results in the higher rank case relies on the following.

4.4. Theorem. *Let G be a center-free semi-simple Lie group of $\mathbb{R}$-rank at least two with Kazhdan's property* (T). *Then μ_{id} is the only accumulation point of the set*

$$\{\mu_\Gamma \mid \Gamma \text{ is an irreducible lattice in } G\}.$$

We will make use of the following result.

4.5. Theorem (Glasner–Weiss [65]). *Let X be a compact topological space, and let G be a topological group with property* (T) *acting continuously on X. Let (μ_n) be a sequence of G-invariant Borel probability measures on X that weakly converges to μ_∞. If the measures μ_n are ergodic, then the limit measure μ_∞ is ergodic.*

Proof of Theorem 4.4. Fix a sequence Γ_n of distinct irreducible lattices in G such that $\mu_n := \mu_{\Gamma_n}$ weakly converges and let μ_∞ be the limit measure. An important point here is that μ_∞ is ergodic with respect to the action of every simple factor of G. By our assumption, if N is any simple factor of G, then it has property (T). Therefore, by Theorem 4.5, N acts ergodically on μ_∞. Combining this with Theorem 4.1, we deduce that either $\mu_\infty = \mu_N$ for a normal subgroup $N \leq G$, or $\mu_\infty = \mu_\Lambda$ for an irreducible lattice $\Lambda < G$.

Let us start by ruling out the possibility that $\mu_\infty = \mu_N$ for any connected normal subgroup of positive dimension. Since N is not nilpotent, by

Proposition 2.3 there is a neighborhood U of N in Sub_G that does not contain any lattice. Thus, if μ_n weakly converges to μ_N, we would have

$$0 = \liminf_{n\to\infty} \mu_n(U) \geq \mu_N(U) = 1,$$

which is absurd.

Next, we exclude the case that $\mu_\infty = \mu_\Gamma$ for a lattice Γ in G. By our assumption G has property (T). Therefore, by a theorem of Leuzinger [79] there is a uniform lower bound for $\lambda_1(\Gamma_n\backslash X)$, the first nonzero eigenvalue of the Laplacian operator on $\Gamma_n\backslash X$. Furthermore, since (μ_n) is not eventually constant, the co-volumes of Γ_n must tend towards infinity by Wang's Finiteness Theorem [114, 8.1]. Theorem 4.4 then follows from the following lemma.

4.6. Lemma. *Let G be a semi-simple Lie group, let X be its associated Riemannian symmetric space and let Γ_n be a sequence of lattices in G where the co-volume of Γ_n tends to infinity and $\inf \lambda_1(\Gamma_n\backslash X) > 0$. Then the set $\{\mu_{\Gamma_n}\}$ is discrete.*

Proof. For simplicity, we will first describe the proof when all the Γ_n are torsion-free, and afterward we will indicate the modifications necessary to deal with torsion.

Assume that after passing to a subsequence, μ_{Γ_n} weakly converges to μ_Γ for some lattice Γ in G. As these measures are supported on the conjugates of their defining lattices, after conjugations and passing to a further subsequence we can assume that Γ_n converges to Γ in the Chabauty topology. By Proposition 3.10 and Lemma 3.7, this implies that after a suitable choice of base points the manifolds $Y_n = \Gamma_n\backslash X$ converge to $Y = \Gamma\backslash X$ in the pointed smooth topology.

If Y is compact, then the sequence (Γ_n) is eventually constant, contradicting the fact that the co-volumes tend to infinity. Otherwise, for every $\delta > 0$, there is a codimension-zero submanifold $B_\delta \subset Y$ with

$$\frac{\mathrm{vol}(Y)}{2} \leq \mathrm{vol}(B_\delta) \leq \mathrm{vol}(Y) \quad \text{and} \quad \mathrm{vol}^{d-1}(\partial B_\delta) < \delta.$$

This implies that for large n, there is a subset $B_n \subset Y_n$ such that

$$\frac{\mathrm{vol}(Y)}{4} \leq \mathrm{vol}(B_n) \leq 2\mathrm{vol}(Y) \quad \text{and} \quad \mathrm{vol}^{d-1}(\partial B_n) < 2\delta,$$

where if $d = \dim X$, then vol^{d-1} is $(d-1)$-dimensional volume. As $\mathrm{vol}(Y_n) \to \infty$, we have $\mathrm{vol}(Y_n \setminus B_n) \to \infty$ as well. This implies that for large n, the *Cheeger constant*

$$h(Y_n) := \inf_{B\subset Y_n} \frac{\mathrm{vol}^{d-1}(\partial B)}{\min\{\mathrm{vol}(B), \mathrm{vol}(Y_n \setminus B)} \leq \frac{8\delta}{\mathrm{vol}(Y)},$$

where the infimum is over codimension-zero submanifolds of Y_n. As δ was arbitrary, this implies that $h(Y_n) \to 0$. An inequality of Buser [32] then implies that $\lambda_1(Y_n) \to 0$, contradicting the uniform spectral gap condition.

Morally, the proof for orbifolds is the same, but we cannot rely on smooth convergence because Lemma 3.7 applies only to manifolds. However, one can proceed as follows. Choose a large metric ball $B \subset X$ whose projection to Y is nearly full measure, but where ∂B projects to have small $(d-1)$-dimensional volume. Fixing $\epsilon > 0$, we can verify that the projection of ∂B has small volume by choosing a small number of ϵ-balls whose Γ-translates cover an $\epsilon/2$-neighborhood of ∂B. For large n, the projection of B to Y_n will still have volume bounded below. In X, a neighborhood of the boundary ∂B will still be covered by the Γ_n-translates of our ϵ-balls above, so its projection in Y_n has small volume. This is enough to force the first eigenvalue $\lambda_1(\Gamma_n\backslash X) \to 0$; compare with [25, Prop. 2.1]. □

In summary, we have shown that the only possible accumulation point of the set

$$\{\mu_\Gamma \mid \Gamma \text{ is a lattice in } G\}$$

is μ_{id}. On the other hand, μ_{id} is clearly an accumulation point. For instance, if $\Gamma < G$ is any lattice, then by residually finiteness, there is a chain of finite index normal subgroups $\Gamma_n < \Gamma$ with trivial intersection. Then $\Gamma_n\backslash X$ BS-converges to X, and by Corollary 3.3, $\mu_{\Gamma_n} \to \mu_{\mathrm{id}}$. Hence we have proved Theorem 4.4. □

4.7. Corollary. *Let G be a center-free semi-simple Lie group with $\mathbb{R}$-rank at least two and Kazhdan's property* (T), *and let $(\Gamma_n)_{n\geq 0}$ be a sequence of irreducible lattices in G where the co-volume of Γ_n tends to infinity. Then $\Gamma_n\backslash X$* BS-*converges to X.*

As a consequence we have

4.8. Corollary. *Let G and X be as above. Then for every $r > 0$ and for every sequence of X-orbifolds M_n with* $\mathrm{vol}(M_n) \to \infty$, *one has*

$$\frac{\mathrm{vol}((M_n)_{<r})}{\mathrm{vol}(M_n)} \to 0.$$

In Section 6 we will see how to use Corollary 4.8 to obtain estimates on the growth of Betti numbers. In particular, if $(\Gamma_n)_{n\geq 0}$ is a uniformly discrete sequence of nonconjugate lattices in a higher rank, center-free simple Lie group, then the hypotheses of Theorem 6.7 and Corollaries 6.9 and 6.25 follow from Corollary 4.7. In particular, the convergence of volume-normalized Betti numbers (Corollary 1.6) follows.

5. Benjamini–Schramm convergence for congruence lattices

Let G be a semi-simple real simple Lie group, $X = G/K$ its associated symmetric space, and let $\Gamma_0 \subset G$ be a uniform irreducible arithmetic lattice. We will assume that Γ_0 is torsion free so, in particular, Γ_0 intersects the center $Z(G)$ of G at the identity. There exists a k-simple, simply connected algebraic group $\mathbf{G}$ defined over a totally real number field k such that Γ_0 is commensurable with $\mathbf{G}(\mathcal{O}_S)$, the group of S-integral points of $\mathbf{G}$. The *principal congruence subgroups* of Γ_0,

$$\Gamma_0(N) = \{\gamma \in \Gamma_0 \cap \mathbf{G}(\mathcal{O}_S) : \gamma \equiv \mathrm{id} \bmod N\},$$

obviously form a BS-convergent sequence of lattices in G. One may even quantify this observation:

5.1. Lemma. *There are constants $a > 0$ and b, depending on Γ, such that for all $N \geq 1$,*

$$\mathrm{InjRad}(\Gamma_0(N)) \geq a \log \mathrm{vol}(\Gamma_0(N)\backslash X) + b.$$

Here we denote by $\mathrm{InjRad}(\Gamma)$ the infimum over $x \in \Gamma\backslash X$ of the local injectivity radii $\mathrm{InjRad}_{\Gamma\backslash X}(x)$.

The conclusion of Lemma 5.1 does not hold for general congruence lattices (i.e., lattices that contain a principal congruence subgroup), as shown in the following example.

Example. Let $\mathbf{H}$ be a proper k-subgroup of $\mathbf{G}$ that contains a semi-simple element of $\mathbf{G}$. Consider the congruence subgroups of Γ_0:

$$\{\gamma \in \Gamma_0 \cap \mathbf{G}(\mathcal{O}_S) : \gamma \in \mathbf{H}(\mathcal{O}_S) \bmod N\}.$$

These form a sequence of cocompact lattices in G whose volumes tend to infinity but whose (minimal) injectivity radius remains bounded. (In fact it becomes stationary.)

It nevertheless remains true that any sequence of distinct congruence subgroups of Γ_0 locally converges toward the trivial group.

The main result of this section is to prove the following quantified version of the above statement.

5.2. Theorem. *There exist positive constants c and δ depending only on Γ_0 (and G), such that for any congruence subgroup $\Gamma \subset \Gamma_0$ and any $R > 1$, we have*

$$\mathrm{vol}((\Gamma\backslash X)_{<R}) \leq e^{cR}\mathrm{vol}(\Gamma\backslash X)^{1-\delta}.$$

The proof of this theorem is given in the rest of the section.[7]

[7]Note that recently corresponding results in special cases were obtained in [56], [81], [99] for congruence lattices that do not admit a common ambient lattice.

We note that if Γ is an arithmetic lattice that has the Strong Approximation Property below, then one can prove uniform BS-convergence for finite index subgroups of Γ without having to use Theorem 5.2.

An arithmetic lattice Γ in a Lie group G has the *Strong Approximation Property* (SAP) if, for every Zariski dense subgroup H of Γ, the closure of H in the congruence completion of Γ is open. This is equivalent to the statement that for every Zariski dense subgroup H, there exists $M > 0$, such that for any congruence subgroup K of Γ, we have $|\Gamma : HK| \leq M$. For Γ with SAP, one can prove uniform BS-convergence for congruence subgroups of Γ without having to use Theorem 5.2:

5.3. Theorem. *Let Γ be an arithmetic lattice with* SAP, *and assume that Γ has trivial center. Then for any sequence of congruence subgroups Γ_n of Γ with $[\Gamma : \Gamma_n] \to \infty$, we have $\mu_{\Gamma_n} \to \mu_1$.*

SAP has been proved by Nori [92] and Matthews, Vaserstein and Weisfeiler [88] for arithmetic lattices in simple, connected, simply connected groups G. So,

5.4. Corollary. *Let Γ be an arithmetic lattice in a simple connected Lie group G and assume that Γ has trivial center. Then for any sequence of congruence subgroups Γ_n of Γ with $[\Gamma : \Gamma_n] \to \infty$, we have $\mu_{\Gamma_n} \to \mu_1$.*

Proof. The group Γ is commensurable with an arithmetic lattice Γ_0 in the universal cover of G. By replacing Γ with Γ_0 and using [92] and [88] we can Therefore, reduce the corollary to the previous theorem. □

Proof of Theorem 5.3. Assume, by contradiction, that μ_{Γ_n} does not converge to μ_1. By passing to a subsequence, we can assume that $\mu_{\Gamma_n} \to \mu$ where $\mu \neq \mu_1$. Let K be the μ-random subgroup of Γ. Since $[\Gamma : \Gamma_n] \to \infty$, K has infinite index in Γ a.s. We shall prove that $K = 1$ a.s., to reach a contradiction.

Let us say that a subgroup H of Γ is hyperclosed if it can be obtained as an ascending union $H = \bigcup_{k=1}^{\infty} J_k$ of congruence closed subgroups $J_k \subset \Gamma$. We claim that a Zariski dense hyperclosed subgroup H of Γ has finite index in Γ. Indeed, by a standard dimension argument, there exists some k such that J_k is already Zariski dense. By SAP the congruence closure of J_k (which equals J_k) has finite index in Γ and so does H.

Now given a sequence of congruence subgroups H_n of Γ, in the Chabauty topology, we have

$$\lim H_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} H_n.$$

Since the intersection of congruence closed subgroups is congruence closed, we get that $\lim H_n$ is hyperclosed. The set of possible Chabauty limits of

congruence subgroups is compact in the Chabauty topology, so we obtain that μ is supported on hyperclosed subgroups. Applying SAP on hyperclosed subgroups and that K has infinite index in Γ a.s. gives us that K is not Zariski dense in Γ a.s. By Theorem 2.9, this implies that $K = 1$ a.s., a contradiction. □

5.5. *Proof of Theorem 5.2.* We first reduce the proof to the case where Γ_0 is a finite index subgroup of $\mathbf{G}(\mathcal{O}_S)$. In fact we will show that if Γ_0 and Δ_0 are two arithmetic commensurable torsion-free lattices, then the conclusion of Theorem 5.2 holds for the congruence subgroups of Γ_0 if and only if it holds for the congruence subgroup of Δ_0 provided we change the constant c.

By considering $\Gamma_0 \cap \Delta_0$ inside Γ_0 and inside Δ_0, we see that it is sufficient to prove the case when Δ_0 is a finite index subgroup of Γ_0.

Let $\Gamma \subset \Gamma_0$ be a congruence subgroup, and denote by M the corresponding X-manifold $\Gamma\backslash X$. Set $\Delta = \Gamma \cap \Delta_0$ and $M' = \Delta\backslash X$. Let $p : M' \to M$ be the covering map and $m := [\Gamma_0 : \Delta_0]$. Then $[\Gamma : \Delta] \leq m$ and p is of degree at most m so that for any $x \in M$ and $x' \in M'$ with $p(x') = x$, we have

$$\frac{\mathrm{InjRad}_{M'}(x')}{m} \leq \mathrm{InjRad}_M(x) \leq \mathrm{InjRad}_{M'}(x').$$

In particular,

$$\mathrm{Vol}(M_{<R}) \leq \frac{\mathrm{Vol}(M'_{<mR})}{[\Gamma : \Gamma']} \leq \mathrm{Vol}(M_{<mR}).$$

In turn, $\mathrm{vol}(M') \leq \mathrm{vol}(M)m$. So if we have the inequality

$$\mathrm{vol}(M'_{<mR}) \leq e^{cmR}\mathrm{vol}(M')^{1-\delta}$$

for some c and δ, then by changing the constant c (to $cm + (1-\delta)\log m$) we obtain the corresponding inequality

$$\mathrm{vol}(M_{<R}) \leq e^{cmR+(1-\delta)\log m}\mathrm{vol}(M)^{1-\delta}.$$

The other direction is even easier: starting with a congruence subgroup Δ containing $\Delta_0(N)$ for some integer N, we put $\Gamma := \Delta\Gamma_0(N)$, a congruence subgroup in Γ, and observe that $\Delta = \Gamma \cap \Delta_0$. The inequality for $\mathrm{vol}((\Gamma\backslash X)_{<R})$ easily gives a corresponding inequality for $\mathrm{vol}((\Delta\backslash X)_{<R})$.

So it is sufficient to prove Theorem 5.2 in the case when Γ_0 is any given finite index subgroup of $\mathbf{G}(\mathcal{O}_S)$.

We will first prove the following combinatorial version of Theorem 5.2.

5.6. Theorem. *Let $\mathbf{G}$ be a k-simple simply connected algebraic group defined over a number field k. For a finite set of valuations S of $\mathcal{O}$, including all archimedian ones, let $\mathcal{O}_S$ be the ring of S-integers in k. There exist some finite index center-free subgroup $\Gamma \subset \mathbf{G}(\mathcal{O}_S)$ and some positive constants ϵ and C (depending only on Γ and some fixed word metric on it) with the following property.*

Let $g \in \Gamma - \{1\}$, and let H be a congruence subgroup of index N in Γ. Then g fixes at most $e^{Cl(g)}N^{1-\epsilon}$ points in the action of Γ on the right cosets $H\backslash\Gamma$ by multiplication. Here $l(g)$ is the length of g with respect to the fixed word metric of Γ.

The proof of this theorem that appeared in our original 2012 arXiv preprint had a mistake in Proposition 5.14 below, a correct proof of which has since been given by Finis–Lapid [53], who also obtain explicit bounds in Theorem 5.6. Here we give a self-contained proof based on the theory of p-adic analytic groups, which avoids the algebraic geometry arguments in [53]. We only need the basic Lemma 5.17 on the solutions of polynomial congruences and well-known results on the fixity of permutation actions of simple groups of Lie type. A careful examination of all the steps of the proof can lead to an explicit value of ϵ, at least for Chevalley groups, on the order of magnitude $(\dim \mathbf{G})^{-[k:\mathbb{Q}]\dim \mathbf{G}}$, which seems however very far from optimal.

We postpone the proof of Theorem 5.6 and first show how it implies Theorem 5.2.

5.7. *Proof of Theorem 5.2.* According to Section 5.5 we may assume that Γ_0 is the finite index subgroup of $\mathbf{G}(\mathcal{O}_S)$ given by Theorem 5.6. Let $\Gamma \subset \Gamma_0$ be any congruence subgroup.

Let $\Omega \subset X$ be a compact fundamental domain for the action of Γ_0 on X, and let $p : M = \Gamma\backslash X \to M_0 = \Gamma_0\backslash X$ be the covering map. We will identify the elements of M (resp. M_0) with the orbits of Γ (resp. Γ_0) in X.

Suppose that $y \in M$ has $\mathrm{InjRad}_M(y) < R$. Let x be a lift of y in X, i.e., $y = \Gamma x \in \Gamma\backslash X$. We have that $d(x, \gamma x) < 2R$ for some $\gamma \in \Gamma$.

Now let g be the unique element of Γ_0 such that $g^{-1}x = x_0 \in \Omega$. We have

$$d(x, \gamma x) = d(x_0, g^{-1}\gamma x) = d(x_0, \gamma^g x_0) < 2R,$$

where $\gamma^g = g^{-1}\gamma g$. Since γ^g moves the point x_0 of Ω to a point of distance at most $2R$ from it and since Ω is compact, it follows that $l(\gamma^g) < C'R$ for some constant C' depending only on the choice of Ω and a generating set (fixed by the choice of the word metric in Theorem 5.6) of Γ_0.

Now, given the element $x_0 \in \Omega$ and a nontrivial $\gamma_0 \in \Gamma_0$ with $l(\gamma_0) < C'R$, suppose that for some $x = gx_0 \in X$ ($g \in \Gamma_0$), there exists $\gamma \in \Gamma$ with $d(x, \gamma x) < 2R$ and $\gamma^g = \gamma_0$. Then $g^{-1}\gamma g = \gamma_0$ so that $\Gamma g = \Gamma g\gamma_0$. The number of Γ-equivalence classes of points $x = gx_0$ in Γx_0 as above giving rise to the same γ_0 is Therefore, equal to the cardinal of the set $\mathrm{fix}(\gamma_0, \Gamma\backslash\Gamma_0)$ of fixed points of γ_0 acting on $\Gamma\backslash\Gamma_0$.

Therefore,

$$\mathrm{Vol}(M_{<R}) \leq \mathrm{Vol}(\Omega) \sum_{0<l(\gamma_0)<C'R} |\mathrm{fix}(\gamma_0, \Gamma\backslash\Gamma_0)|.$$

In turn, by Theorem 5.6,

$$|\mathrm{fix}(\gamma_0, \Gamma\backslash\Gamma_0)| \leq e^{Cl(\gamma_0)}[\Gamma_0 : \Gamma]^{1-\epsilon}$$

and there are at most $e^{C''R}$ elements γ_0 with $0 < l(\gamma_0) < C'R$ that combine to give the desired bound for large enough c. □

5.8. The proof of Theorem 5.6 takes up the rest of this section. We can consider $\mathbf{G}(k)$ as the rational points of a restriction of scalars of an absolutely simple group defined over a larger field. Moreover, the respective groups of integral points and their congruence topologies are compatible. So by enlarging the field k if necessary we may assume from the start that $\mathbf{G}$ is absolutely simple.

Take a prime ideal $\mathfrak{P}$ of $\mathcal{O}$, and let p be the rational prime such that $\mathfrak{P}|p$. Let $r_\mathfrak{P}$ be the ramification index of $\mathfrak{P}$, and let $w_\mathfrak{P}$ be its residue degree, i.e., $|\mathcal{O} : \mathfrak{P}| = p^{w_\mathfrak{P}}$. We have $p = \prod_{\mathfrak{P}|p} \mathfrak{P}^{r_\mathfrak{P}}$ and $[k : \mathbb{Q}] = \sum_{\mathfrak{P}|p} r_\mathfrak{P} w_\mathfrak{P}$.

From now on we will denote by $\mathfrak{P}$ a prime ideal of $\mathcal{O}_S$ dividing a rational prime p. We will denote by $|-|_\mathfrak{P}$ the $\mathfrak{P}$-adic valuation on k defined by $|x|_\mathfrak{P} = p^{-w_\mathfrak{P} n}$ for $x \in \mathfrak{P}^n\backslash\mathfrak{P}^{n+1}$. Denote by $k_\mathfrak{P}$ and $\mathcal{O}_\mathfrak{P}$ the completions of k and $\mathcal{O}_S$ with respect to this valuation. We have $[k_\mathfrak{P} : \mathbb{Q}_p] = r_\mathfrak{P} w_\mathfrak{P}$. Let $G_\mathfrak{P} = \mathbf{G}(\mathcal{O}_\mathfrak{P})$ be the congruence completion of $\mathbf{G}(\mathcal{O}_S)$ with respect to the $\mathfrak{P}$-adic topology. For $m \geq 1$, let $G_\mathfrak{P}(m)$ be the principal congruence subgroup mod $\mathfrak{P}^{r_\mathfrak{P} m}$, i.e., the matrices in $G_\mathfrak{P}$ that are congruent to the identity mod $\mathfrak{P}^{r_\mathfrak{P} m}$. (The presence of $r_\mathfrak{P}$ in the exponent is to ensure that $(G_\mathfrak{P}(i))_{i=1}^\infty$ is the Frattini series of the p-adic analytic group $G_\mathfrak{P}(1)$.) The dimension of $G_\mathfrak{P}$ as an analytic group over $\mathbb{Q}_p$ is $r_\mathfrak{P} w_\mathfrak{P} \dim \mathbf{G} = [k_\mathfrak{P} : \mathbb{Q}_p] \dim \mathbf{G}$. Note that all but finitely many of the prime ideals $\mathfrak{P}$ are unramified, i.e., $r_\mathfrak{P} = 1$. For almost all unramified prime ideals $\mathfrak{P}$, the quotient $S_\mathfrak{P} := G_\mathfrak{P}/G_\mathfrak{P}(1)$ is the reduction $\mathbf{G}(\mathcal{O}_S/\mathfrak{P})$ of $\mathbf{G}$ mod $\mathfrak{P}$. Since $\mathbf{G}$ is absolutely simple and simply connected, it follows that $S_\mathfrak{P}$ is generated by its unipotent elements and is Therefore, a finite quasi-simple group $S_\mathfrak{P}$ of Lie type over the field $\mathcal{O}_S/\mathfrak{P}$; see Proposition 7.5 of [95]. (A perfect group is called quasi-simple if it is simple modulo its center.) Moreover, the Frattini subgroup $\Phi(G_\mathfrak{P})$ contains $G_\mathfrak{P}(1)$; see [83, Lemma 16.4.5]. Let us call these prime ideals *good* and all other prime ideals *bad*. Let Z be the finite center of $\mathbf{G}$.

For a rational prime p, define $G_p := \prod_{\mathfrak{P}|p, \mathfrak{P}\notin S} G_\mathfrak{P}$, and for $m \in \mathbb{N}$, let $G_p(m) := \prod_{\mathfrak{P}|p, \mathfrak{P}\notin S} G_\mathfrak{P}(m)$. That is, $G_p(m)$ consists of the elements of G_p congruent to 1 mod p^m. The group G_p is a semi-simple p-adic analytic group of dimension at most $D := [k : \mathbb{Q}] \dim \mathbb{G}$. (The dimension of G_p is exactly D when the set S avoids any ideal divisors of p.) The level of an open subgroup $H \leq G_p$ is defined to be the integer n, such that H contains $G_p(n)$ but not $G_p(n-1)$.

Let $P = \{\mathfrak{P}_1, \dots, \mathfrak{P}_r\}$ be a finite set of prime ideals of $\mathcal{O}_S$ including all the bad primes such that the principal congruence subgroup

$$\Gamma = \{g \in \mathbf{G}(\mathcal{O}_S) : g \equiv \mathrm{id} \bmod \mathfrak{P}_1 \cdots \mathfrak{P}_r\}$$

intersects Z trivially.

We will show that Γ satisfies the conclusion of Theorem 5.6. From now on we fix an element $g \in \Gamma \backslash \{1\}$.

5.9. The congruence subgroups of $\mathbf{G}(\mathcal{O}_S)$ correspond to open subgroups of its congruence completion

$$\widehat{G} = \prod_{p \text{ prime}} G_p = \prod_{\mathfrak{P} \nmid S} G_{\mathfrak{P}}.$$

The strategy is to reduce the fixity estimate of Theorem 5.6 to an analogous problem inside each local p-adic factor G_p. This reduction follows easily from the detailed knowledge of the subgroups and representations of the simple factors $S_{\mathfrak{P}}$ for different prime ideals and, in particular, the elementary fact that $S_{\mathfrak{P}}$ has no proper subgroups of index less than p. We then solve the local problem (Proposition 5.14) using the natural coordinate system of uniform subgroups of G_p. To formulate the local estimate we need some further notation.

Let L be the Lie algebra of $\mathbf{G}$, which is a simple Lie algebra defined over k. Let $L_{\mathfrak{P}}$ be the $\mathcal{O}_{\mathfrak{P}}$-Lie ring that corresponds to the uniform pro-p group $G_{\mathfrak{P}}(1)$, so $L_{\mathfrak{P}}$ is an $\mathcal{O}_{\mathfrak{P}}$-lattice of the $k_{\mathfrak{P}}$-Lie algebra $L \otimes_k k_{\mathfrak{P}}$. There is a free $\mathcal{O}_S$-lattice L_0 of L such that for almost all prime ideals $\mathfrak{P}$, we have $L_{\mathfrak{P}} = L_0 \otimes_{\mathcal{O}_S} \mathfrak{P}O_{\mathfrak{P}}$. (The two lattices are open in $L \otimes_k k_{\mathfrak{P}}$ and so commensurable for all $\mathfrak{P}$.) Let $\mathrm{Ad}_{\mathfrak{P}} : G_{\mathfrak{P}} \to \mathrm{End}(L_{\mathfrak{P}})$ be the adjoint representation on $L_{\mathfrak{P}}$. Define $n_{\mathfrak{P}}(g)$ to be the largest integer m such that $\mathrm{Ad}_{\mathfrak{P}}(g) \equiv 1 \bmod \mathfrak{P}^m$.

Note that since $g \notin Z$ by our choice of Γ, the integers $n_{\mathfrak{P}}(g)$ exist.

Define $n_p(g) = \max_{\mathfrak{P}|p} n_{\mathfrak{P}}(g)$ (where the maximum is over all prime ideals $\mathfrak{P}$ of $\mathcal{O}_S$ dividing p). For completeness, we set $n_p(g) = 0$ for those primes p that are units in $\mathcal{O}_S$. For example when $\mathbf{G}(\mathcal{O}_S) = \mathrm{SL}_n(\mathbb{Z})$, then $L_p = \mathfrak{sl}_n(\mathbb{Z}_p)$, in which case $n_p(g)$ is the largest integer m such that g is congruent to a scalar matrix modulo p^m.

Since g is not in the center of G, we have $\mathrm{Ad}(g) - 1 \neq 0$ as a linear endomorphism of L. Choose any nonzero matrix coefficient $\beta \in k$ of the matrix of $\mathrm{Ad}(g) - 1$ with respect to a fixed $\mathcal{O}_S$-basis of L_0. For any prime p that is not a unit of $\mathcal{O}_S$, by the definition of $n_p(g)$ there is a prime ideal $\mathfrak{P}$ of $\mathcal{O}$ outside S that divides p and such that $|\beta|_{\mathfrak{P}} \leq |\mathcal{O} : \mathfrak{P}|^{n_p(g)} \leq p^{-n_p(g)}$. (Here, we may need to change β by a multiplicative constant to take account for the finitely many prime ideals such that $L_{\mathfrak{P}} \neq L_0 \otimes_{\mathcal{O}_S} \mathfrak{P}O_{\mathfrak{P}}$.)

Hence $\prod_{\mathfrak{P}\notin S}|\beta|_{\mathfrak{P}} \le \prod_p p^{-n_p(g)}$. On the other hand, for any valuation v, the coefficient $|\beta|_v$ is bounded above exponentially by the word length $l(g)$. Putting it all together we conclude that $\prod_{v\in S}|\beta|_v \le e^{C_1 l(g)}$ for some constant C_1 depending only on Γ (and the basis of L_0 and the word metric on Γ). Note that

$$\prod_v |\beta|_v = 1,$$

where the product runs over all valuations of $\mathcal{O}$. Therefore, all but finitely many of the $n_p(g)$'s are zero and (compare with Lemma 5.1)

$$\prod_p p^{n_p(g)} \le e^{C_1 l(g)}. \tag{5.9.1}$$

Given a congruence subgroup Γ' of Γ, we want to compute the *fixity ratio*

$$\alpha(g, \Gamma'\backslash\Gamma) = \frac{|\mathrm{fix}(g, \Gamma'\backslash\Gamma)|}{[\Gamma : \Gamma']}$$

of g acting the right cosets of Γ' in Γ by multiplication.

The congruence completion of $\mathbf{G}(\mathcal{O}_S)$ is $\tilde{G} := \prod_p G_p$. Let $\bar{\Gamma}$ be the closure of Γ in $\tilde{G}$, and let H be the closure of Γ'. We have now reduced our claim to showing that there exists some positive constants C and δ depending only on Γ such that

$$\alpha(g, H\backslash\bar{\Gamma}) \le e^{Cl(g)}[\bar{\Gamma} : H]^{-\delta}. \tag{5.9.2}$$

5.10. *Reduction to the local case.* We have $\bar{\Gamma} = \prod_p \Gamma_p$ is a product of its projections Γ_p onto G_p. Moreover, $\Gamma_p = G_p$ for almost all p and always $\Gamma_p \ge G_p(1)$. Let H_p be the projection of H onto the factor Γ_p of $\bar{\Gamma}$. We have that $H \le \prod_p H_p$ and so

$$\alpha(g, H\backslash\tilde{\Gamma}) \le \alpha\Big(g, \prod_p H_p\backslash\tilde{\Gamma}\Big) = \prod_p \alpha(g, H_p\backslash\Gamma_p). \tag{5.10.1}$$

Since H contains G_p for almost all primes p, the product above is equal to a finite product.

Let $x_p = [\Gamma_p : H_p]$. Clearly $N = [\Gamma : \Gamma'] = [\bar{\Gamma} : H] \ge \prod_p x_p$, where again we have that $x_p = 1$ for almost all primes p.

5.11. Lemma. *There is a positive constant δ_1, such that*

$$N^{\delta_1} \le \prod_p x_p.$$

Proof. Let $\Delta = \prod_p G_p(1) \le \tilde{\Gamma}$. Each $G_p(1)$ is a pro-p group, and hence $H \cap \Delta$ is the direct product of its projections onto the Sylow pro-p factors $G_p(1)$ of Δ. Since

$$N = [\tilde{\Gamma} : H\Delta][\Delta : H \cap \Delta],$$

it is enough to prove the special case when $H = H\Delta$; i.e., $H \geq \Delta$. Then $H/\Delta \leq \Gamma/\Delta = \prod_{\text{good } \mathfrak{P}} S_{\mathfrak{P}}$ where by the choice of Γ the product runs over the good prime ideals $\mathfrak{P}$ of $\mathcal{O}_S$. Thus each $S_{\mathfrak{P}}$ is a finite quasi-simple group of Lie type of bounded rank of characteristic p. Let $S_p = \prod_{\text{good } \mathfrak{P}|p} S_{\mathfrak{P}}$, and let us denote by $\mathrm{Lie}(p)$ the set of simple groups of Lie type over fields of characteristic p. We can replace H with its image in a finite product $\prod_{i=1}^{s} S_{p_i}$ such that H does not contain any of the factors S_{p_i}. Then H_{p_i} becomes the projection of H in the factor S_{p_i}. By the Larsen–Pink theorem [76], there is a function $f : \mathbb{N} \to \mathbb{N}$ such that a subgroup of $\mathrm{GL}(n, \mathbb{F}_{p^m})$ contains a subgroup of index at most $f(n)$ whose non-abelian composition factors are from $\mathrm{Lie}(p)$. In particular, if $q > f := f(\dim \mathbb{G})$ is a large prime, then the direct factors of S_q cannot occur as composition factors of any subgroup of any of the factors of S_p for $p \neq q$. It follows that if $q > f$ is a prime and $H_q = S_q$, i.e., $x_q = 1$, then actually $H \geq S_q$.

By enlarging the set P defining Γ to include all prime ideals dividing q for primes $q \leq f$, we can ensure that H_{p_i} is a proper subgroup of S_{p_i}. The group S_{p_i} is a product of finite quasi-simple groups from $\mathrm{Lie}(p_i)$ and, in particular, it is generated by elements of order p_i. Hence a proper subgroup of S_{p_i} must have index at least p_i and $x_{p_i} = [S_{p_i} : H_{p_1}] \geq p_i$. On the other hand, $|S_{p_i}| < p_i^{[k:\mathbb{Q}] \dim \mathbf{G}}$ and $N \leq \prod_{i=1}^{s} |S_{p_i}|$. So Lemma 5.11 follows with $\delta_1 = ([k : \mathbb{Q}] \dim \mathbf{G})^{-1} = D^{-1}$. □

5.12. *The local case.* Lemma 5.11 reduces the proof of (5.9.2) to its local counterpart (5.14.1) below at each prime p. Here we explain how to conclude the proof of Theorem 5.6 assuming the following.

5.13. Proposition. *There exist constants C_2 and $\delta_2 > 0$ depending only on Γ such that for all primes p,*

$$\alpha(g, H_p \backslash \tilde{\Gamma}) \leq p^{C_2 n_p(g)} x_p^{-\delta_2}. \tag{5.13.1}$$

Multiplying the inequalities(5.13.1) for all primes p and using (5.10.1), (5.9.1) and Lemma 5.11, we obtain

$$\alpha(g, H \backslash \tilde{\Gamma}) \leq \prod_p p^{C_2 n_p(g)} \Big(\prod_p x_p\Big)^{-\delta_2} \leq e^{C_1 C_2 l(g)} N^{-\delta_1 \delta_2}.$$

Theorem 5.6 follows. □

By considering the image of H_p in the local factor G_p, Proposition 5.13 is easily deduced from the following local bound.

5.14. Proposition. *There are constants $a, b, c > 0$ that depend on $\mathbf{G}$ but not on the prime p such that if H is an open subgroup of G_p of level n and $g \in G_p - Z(G_p)$, then the fixity proportion $\alpha(g, H \backslash G_p)$ of g on $H \backslash G_p$ is at most p^{-cn} provided $n > \max\{a, b n_p(g)\}$.*

Let us show how Proposition 5.14 implies the existence of C_2 and δ_2 such that

$$\alpha(g, H\backslash G_p) \leq p^{C_2 n_p(g)}[G_p : H]^{-\delta_2} \tag{5.14.1}$$

provided $g \in \Gamma\backslash\{1\}$. This inequality easily gives (5.13.1) by increasing C_2 to take into account the index $[G(\mathcal{O}_S) : \Gamma]$.

Since H contains $G_p(n)$ from the definition of n, we have $[G_p : H] \leq [G_p : G_p(n)] \leq p^{n[k:\mathbb{Q}]\dim \mathbf{G}} = p^{nD}$ (where $D = [k : \mathbb{Q}]\dim \mathbf{G}$). Therefore,

$$\alpha(g, H\backslash G_p) \leq [G_p : H]^{-c/D} \tag{5.14.2}$$

provided $n > \max\{a, bn_p(g)\}$.

It remains to prove (5.14.1) when $n \leq a$ or $n \leq bn_p(g)$. First we consider the case that $n_p(g) = 0$. Then by the choice of Γ we have that p is a product of good prime ideals.

Since $n_p(g) = 0$, it follows that g is not in the center of any of the factors of the semi-quasisimple group $S_p = G_p/G_p(1) = \prod_{\mathfrak{P}|p} S_{\mathfrak{P}}$. We also have $HG_p(1) < G_p$ because $G_p(1)$ is in the Frattini subgroup of G_p. Choose a maximal subgroup M of S_p containing the image of H. There are two possibilities:

(1) M is the preimage of a maximal subgroup $M_0 \leq S_{\mathfrak{P}}$ of one of the factors of S_p. The main result of [82] says that with finitely many exceptions any nontrivial element of a simple group S of Lie type over a field of size q has fixity at most $4/3q$ in any primitive action of S unless $S = \mathrm{PSL}(2, q)$ when it is at most $2/(q+1)$ for $q \geq 5$. The proof relies on a case-by-case analysis of the Aschbacher classification of the maximal subgroups of S. It follows that $\alpha(g, M\backslash S_p) \leq \alpha(g, M_0\backslash S_{\mathfrak{P}}) \leq 2/q$.

(2) M is the preimage of a diagonal subgroup T in a product $S \times S$ of two isomorphic factors of S_p. Now a direct computation shows that for $(y_1, y_2) \in S \times S$, the fixity ratio $\alpha((y_1, y_2), T\backslash S \times S)$ is nonzero only if y_1 and y_2 are conjugate, and if so, then is at most $|C_S(y_1)|/|S|$. Since we are assuming that y_1 is noncentral in S and S is generated by elements of order p, we have as before that $|C_S(y_1)|/|S| \leq 1/p$.

All together, we can deduce that

$$\alpha(g, H\backslash G_p) \leq \alpha(g, M\backslash S_p) \leq p^{-\epsilon_2},$$

for example, with $\epsilon_2 = 1/2$.

We set $\delta_2 = \min\{\frac{c}{D}, \frac{\epsilon_2}{aD}\}$ and note that when $n \leq a$, then

$$p^{-\epsilon_2} \leq p^{-a\delta_2 D} \leq [G_p : H]^{-\delta_2}.$$

To deal with the case $n_p(g) \geq 1$ we simply set $C_2 = (a + b)\delta_2 D$, which ensures that $p^{C_2 n_p(g)}[G_p : H]^{-\delta_2} \geq 1$ when $n \leq \max\{a, bn_p(g)\}$.

5.15. *Proof of Proposition 5.14.* From now on we fix a prime number p and denote $n_p(g)$ by n_p.

The congruence completion G_p is a p-adic analytic group of dimension $d = d(p) = \dim \mathbb{G}(\sum_{\mathfrak{P}|p,\mathfrak{P}\notin S}[k_{\mathfrak{P}} : \mathbb{Q}_p])$ over $\mathbb{Q}_p$, and we shall refer to [48] for standard results about these. Note that $d \le D = [k : \mathbb{Q}] \dim \mathbf{G}$. Recall that $G_p(i)$ is the kernel of $G_p \to G(\mathcal{O}_S/p^i)$. There exists a constant $c_0 \ge 1$ such that the congruence subgroup $G_p(c_0)$ is a uniform pro-p group for all primes p. In fact [83, Cor. 16.4.6] gives that $c_0 = 1$ for all except finitely many primes p. We set $c_1 = c_0$ unless $p = 2$, when we set $c_1 = c_0 + 1$. Define $U = G_p(c_1)$. So U is a uniform group of dimension d and, in addition, when $p = 2$, U is the Frattini subgroup of a uniform group. (We need this in order to apply Proposition 8.21 from [48] at a later point.) The series $(G_p(i))_{i\ge c_1}$ coincides with the Frattini series of U defined by $U_0 = U$ and $U_{i+1} = \Phi(U_i)$, i.e., $U_i = G_p(i + c_1) = \prod_{\mathfrak{P}|p} G_{\mathfrak{P}}(i + c_1)$ for all $i \ge 0$. For a prime ideal $\mathfrak{P}$ of $\mathcal{O}_S$ dividing p, we define $U_{\mathfrak{P},i} = G_{\mathfrak{P}}(i + c_1)$, so that $U_i = \prod_{\mathfrak{P}|p} U_{\mathfrak{P},i}$.

Let n be the level of H. From now on we denote $n' = n - c_1$, so $G_p(n) = U_{n'} \le H$ but $U_{n'-1} \not\le H$.

First observe that at the cost of increasing a and b and decreasing c, it is sufficient to prove Proposition 5.14 with n' in place of n.

Next we introduce coordinates of the second kind for U, which are more suitable for parametrizing its open subgroups, in the spirit of [104]. The properties of a uniform pro-p group U we state below can be found in [48, Chs. 4 and 8].

Suppose elements $e_1, \dots, e_d$ in U are given such that their images form a basis of the vector space $U/U_1 = U/\Phi(U)$. Then $e_1^{p^i}, \dots, e_d^{p^i}$ is a basis for U_i/U_{i+1} for each $i \in \mathbb{N}$.

Moreover, every element of U can be expressed uniquely in the form $\prod_{i=1}^d c_i$, where $c_i \in \overline{\langle e_i \rangle} \simeq \mathbb{Z}_p$. The following properties hold:

(1) The map $\mu : (\mathbb{Z}_p)^d \to U$ defined by $\mu(y_1, \dots, y_d) = e_1^{y_1} \cdots e_d^{y_d}$ is a homeomorphism. We say that the elements $e_1, \dots, e_d$ provide a *basis for coordinates μ of the second kind for U*.

(2) U_i is the image of $(p^i\mathbb{Z}_p)^d$ under μ.

(3) If we identify U with $(\mathbb{Z}_p)^d$ via μ, the group operations in U (including exponentiation $\exp_a(z) = a^z$, $z \in \mathbb{Z}_p$ for fixed $a \in U$) are given by a converging power series in $\mathbb{Z}_p[[x_1, \dots, x_d]]$ with $(x_1, \dots, x_d) \in (\mathbb{Z}_p)^d$. This follows from [48, Prop. 8.21]; inequality (5) in loc. cit., together with the formula for the power series g_i on page 192, implies that the coefficients of g_i are p-adic integers.

Suppose H is an open subgroup of G_p of level n. Put $H' = H \cap U$. Then there exists a basis $e_1, \dots, e_d$ for coordinates μ of the second kind for U with the following additional properties:

(4) For some integers $0 \le s_1 \le s_2 \le \cdots \le s_d$, we have $\mu(x_1, \ldots, x_d) \in H'$ if and only if $x_i \in p^{s_i}\mathbb{Z}_p$. Moreover, $H' \ge U_{s_d}$, $H' \not\ge U_{s_d-1}$ and so $n' = s_d$.
(5) For each $m \in \mathbb{N}$, the subgroup $H'U_m$ has the parametrization $H'U_m = \mu(p^{k_1}\mathbb{Z}_p, \ldots, p^{k_d}\mathbb{Z}_p)$, where $k_i = \min\{s_i, m\}$.

In the language of [104] the elements $e_1^{s_1}, \ldots, e_d^{s_d}$ are a good basis for H' in U.

Let us indicate how to find a good basis for an open subgroup H' such that $U \ge H' \ge U_l$. We find subsets $B_1, \ldots B_l$ of U inductively as follows.

B_0 is any subset of U that is a basis for the vector subspace $H'U_1/U_1$ in U/U_1. Having found $B_0, \ldots, B_{i-1}$, we choose B_i such that the set

$$\bigcup_{r=0}^{i} \{a^{p^{i-r}} \mid a \in B_r\}$$

is a basis for the subspace $(H'U_{i+1} \cap U_i)/U_{i+1}$ in U_i/U_{i+1}.

Then $\{e_1, \ldots e_d\} = B_0 \cup \cdots \cup B_l$. In addition, each $|B_i|$ is the number of the integers from $s_1, \ldots, s_d$ that are equal to i.

We will choose an integer $m \le n'$ with n'/m bounded, and for any element $w \in G_p$, we will find an upper bound for the proportion

$$\alpha_m(g, H, w) = \frac{|\{U_{n'}x \mid x \in U_m w,\ xgx^{-1} \in H\}|}{[U_m : U_{n'}]}.$$

The reason for focusing on fixed points of g on the cosets of each $U_{n'}\backslash U_m w$ separately as opposed to the whole space $U_{n'}\backslash G_p$ is that we will be able to express $\alpha_m(g, H, w)$ as the probability of solving polynomial congruences in $\mathbb{Z}_p$ of *bounded degree* in the coordinates $x_1, \ldots, x_d$.

If we prove that there are some constants a, b, c depending only on G, such that $\alpha_m(g, H, w) < p^{-n'c}$ for any $w \in G_p$ and $n' > \max\{a, bn_p\}$, it will follow that $\alpha(g, H\backslash G_p) \le p^{-n'c}$ and we will be done.

We may assume that the numerator of $\alpha_m(g, H, w)$ is not zero. (Otherwise the proportion is zero and we are done.) So we may assume that $wgw^{-1} \in H$ and by replacing g with wgw^{-1}, we are reduced to proving that for some constant c (to be specified later),

$$\beta_m(g, H) = \frac{|\{x \in U_{n'}\backslash U_m \mid [g, x] \in H\}|}{[U_m : U_{n'}]} \le p^{-n'c}$$

for all sufficiently large integers n'.

The main idea of the proof is to reduce the membership condition $[g, x] \in H$ to a power series congruence defined by a good basis for H and then the choice of m reduces this to a polynomial congruence of bounded degree whose solutions we then estimate with Lemma 5.17 below.

If $\mathbf{x} = \mu(x_1, \dots x_d) = \prod_{i=1}^d e_i^{x_i}$, then $\mu^{-1}([g, \mathbf{x}]) = (f_1, \dots, f_d)$, where $f_i \in \mathbb{Z}_p[[x_1, \dots, x_d]]$ are converging power series in $x_1, \dots, x_d \in \mathbb{Z}_p$. Indeed if $e_i^g = t_i$, then $[g, \mathbf{x}] = \mathbf{x}^{-g}\mathbf{x} = (\prod_{i=1}^d t_i^{x_i})^{-1} \prod_{i=1}^d e_i^{x_i}$ and we are composing the power series defining multiplication and exponentiation in the coordinate system μ.

Recall the definition of $n_p = n_p(g)$ as the largest integer l such that $Ad(g)_{\mathfrak{P}} \equiv 1 \bmod \mathfrak{P}^l$ for some prime ideal $\mathfrak{P}$ of $\mathcal{O}_S$ dividing p.

The following lemma is well known in the case $m = 1$ and is a consequence of the simplicity of the Lie algebras of the p-adic analytic groups $G_{\mathfrak{P}}$ and the fact that they are defined uniformly over k.

5.16. Lemma. *There exists a constant A independent of p, such that for any $m \geq 1$, $\langle g^{U_m} \rangle$ contains U_{Am+n_p}.*

Proof. Since U_m is equal to the direct product $\oplus_{\mathfrak{P}|p} U_{\mathfrak{P},m}$, it is sufficient to prove the lemma with $U_{\mathfrak{P},m}$ in place of U_m Note that

$$\langle g^{U_{\mathfrak{P},m}} \rangle \supseteq [g, U_{\mathfrak{P},m}, U_{\mathfrak{P},m}, \dots, U_{\mathfrak{P},m}]$$

so it is sufficient to show that $[g, U_{\mathfrak{P},m}, U_{\mathfrak{P},m}, \dots, U_{\mathfrak{P},m}]$ generates a group containing $U_{\mathfrak{P},Am+n_p}$ for some constant A.

The logarithm map establishes a bijection between the uniform group $U_{\mathfrak{P}}$ and its $\mathcal{O}_{\mathfrak{P}}$-Lie ring $L_{\mathfrak{P}}$ such that $p^i L_{\mathfrak{P}} = \mathfrak{P}^{r_{\mathfrak{P}} i} L_{\mathfrak{P}}$ corresponds to $U_{\mathfrak{P},i}$. Moreover, for any $j \in \mathbb{N}$, the graded Lie algebras

$$\oplus_{i \geq j} U_{\mathfrak{P},i}/U_{\mathfrak{P},i+j} \text{ and } \oplus_{i \geq j} p^i L_{\mathfrak{P}}/p^{i+j} L_{\mathfrak{P}}$$

are isomorphic. In particular, g acts nontrivially by conjugation on

$$U_{\mathfrak{P},m}/U_{\mathfrak{P},m+n_p+1}.$$

Hence $[g, U_{\mathfrak{P},m}] \not\subset U_{\mathfrak{P},m+n_p+1}$ and we can choose $y \in [g, U_{\mathfrak{P},m}]$ with $y \notin U_{\mathfrak{P},m+n_p+1}$. Now for almost all prime ideals $\mathfrak{P}$, we have that $L_{\mathfrak{P}}$ is isomorphic to $L_0 \otimes_{\mathcal{O}_S} \mathfrak{P}\mathcal{O}_{\mathfrak{P}}$, where L_0 is a fixed integral lattice of the k-Lie algebra of $\mathbf{G}$. For all $\mathfrak{P}$, the absolute simplicity of $\mathbf{G}$ gives that $L_1 = L_0 \otimes_{\mathcal{O}_S} k_{\mathfrak{P}} = L_{\mathfrak{P}} \otimes_{\mathcal{O}_{\mathfrak{P}}} k_{\mathfrak{P}}$ is a simple $k_{\mathfrak{P}}$-Lie algebra of dimension $\dim \mathbf{G}$. In particular, $[L_1, L_1] = L_1$ and the Jacobi identity gives $[T, L_1] \subseteq [[T, L_1], L_1]$ for any subset T of L_1. Here and below, for subsets X, Y of a given Lie algebra, we denote by $[X, Y]$ the vector space spanned by all Lie brackets $[x, y]$ with $x \in X$ and $y \in Y$, and for an integer $n \in \mathbb{N}$, we denote $[X,_n Y] := [\cdots [X, Y], Y], \dots, Y]$ (n times). It follows that for any $w \in L_1 - \{0\}$, the ascending sequence of subspaces $[w, L_1] \subset [w, L_1, L_1] \subset \cdots$ stabilizes at L_1 in at most $\dim L_1 = \dim \mathbf{G}$ steps and hence $[w,_{\dim \mathbf{G}} L_1] = L_1$.

We claim that there is a constant $A_{\mathfrak{P}}$ such that for any $w \in L_{\mathfrak{P}}$ outside $pL_{\mathfrak{P}}$,

$$p^{A_{\mathfrak{P}}} L_{\mathfrak{P}} \subset [w, {}_{\dim \mathbf{G}} L_{\mathfrak{P}}]. \tag{5.16.1}$$

Indeed if this is not true, there is a sequence $(w_k)_{k=1}^{\infty}$ with $w_k \in L_{\mathfrak{P}} - pL_{\mathfrak{P}}$ such that $p^k L_{\mathfrak{P}} \not\subset [w_k, {}_{\dim \mathbf{G}} L_{\mathfrak{P}}]$ for any $k \in \mathbb{N}$. By passing to a subsequence we may assume that w_k converge to some w_0 in the $\mathfrak{P}$-adic topology of $L_{\mathfrak{P}}$. Moreover, $w_0 \notin pL_{\mathfrak{P}}$ and, in particular, $w_0 \neq 0$. Now $[w_0, {}_{\dim \mathbf{G}} L_1] = L_1$ and so $p^m L_{\mathfrak{P}} \subset [w_0, {}_{\dim \mathbf{G}} L_{\mathfrak{P}}]$ for some $m \in \mathbb{N}$. Since $w_k \to w_0$, we have $w_k \equiv w_0 \bmod p^m L_{\mathfrak{P}}$ for almost all k, but this contradicts the choice of w_k when $k > m$. The claim follows.

In fact for almost all prime ideals, the reduction $L' := L_0/\mathfrak{P}L_0$ of the lattice L_0 mod $\mathfrak{P}$ is a simple Lie algebra over $\mathcal{O}_S/\mathfrak{P}$, and so $L' = [w, {}_{\dim \mathbf{G}} L']$ for any nonzero $w \in L'$. Since $L_{\mathfrak{P}} \simeq L_0 \otimes_{\mathcal{O}_S} \mathfrak{P}\mathcal{O}_{\mathfrak{P}}$ for almost all $\mathfrak{P}$, this gives that $A_{\mathfrak{P}}$ can be taken to be $\dim \mathbf{G}$ for those $\mathfrak{P}$. By further increasing $A_{\mathfrak{P}}$ at the remaining finitely many prime ideals, we conclude that (5.16.1) holds with a constant A_1 in place of $A_{\mathfrak{P}}$, which does not depend on p.

Applying (5.16.1) to the graded Lie algebra

$$\bigoplus_{i > A_1} U_{\mathfrak{P},i}/U_{\mathfrak{P},i+A_1+1} \simeq \bigoplus_{i \geq A_1} p^i L_{\mathfrak{P}}/p^{i+A_1+1} L_{\mathfrak{P}}$$

we see that $\langle [y, {}_{\dim \mathbf{G}} U_{\mathfrak{P},m}] \rangle$ contains $U_{\mathfrak{P},A_1+m\dim \mathbf{G}+m+n_p}$. The lemma follows with $A = \dim \mathbf{G} + A_1 + 1$. □

To illustrate the main steps of the proof we first consider the special case when the integers s_i associated to the good basis describing $H' = H \cap U$ are $s_1 = \cdots = s_{d-1} = 0$ and $s_d = n'$.

For $n' > 6A + 6n_p$, choose an integer m such that $\frac{n'}{3A} < m \leq \frac{n'-2n_p}{2A}$.

The membership of $x = \mu(x_1, \ldots, x_d) \in U_m$ is described by $x_i \in p^m \mathbb{Z}_p$, and so we can write $x_i = p^m y_i$ for some $y_i \in \mathbb{Z}_p$. We now claim that $f_d(p^m\mathbb{Z}_p, \ldots, p^m\mathbb{Z}_p) \not\subset p^{Am+n_p+1}\mathbb{Z}_p$ and, in particular, not all coefficients of $f_d(p^m y_1, \ldots, p^m y_d)$ are not divisible by p^{Am+n_p+1}.

Suppose for the sake of contradiction that

$$f_d(p^m\mathbb{Z}_p, \ldots, p^m\mathbb{Z}_p) \subset p^{Am+n_p+1}\mathbb{Z}_p.$$

Then $[g, U_m] \subset H' U_{Am+n_p+1}$. Since $g \in H$, it follows that $\langle g^{U_m} \rangle \leq H U_{Am+n_p+1}$, and so Lemma 5.16 gives $HU_{Am+n_p+1} \geq U_{Am+n_p}$. But since $U_{Am+n_p+1} = \Phi(U_{Am+n_p})$ we obtain $H \geq U_{Am+n_p}$. Since $m < (n' - n_p)/A$, it follows that $H \geq U_{n'-1}$, which contradicts the condition that H has level n. The claim follows.

We now estimate the number of elements $\mathbf{x} = \mu(p^m y_1, \dots, p^m y_d) \in U_m/U_{n'}$ such that $[g, \mathbf{x}] \in H'$. This is equivalent to the congruence

$$f_d(p^m y_1, \dots, p^m y_d) \equiv 0 \bmod p^{n'}, \quad y_i \in \mathbb{Z}_p/p^{n'-m}\mathbb{Z}_p.$$

On the other hand, since $m > n'/3A$, the above congruence is equivalent to a polynomial congruence of the form

$$p^l F(y_1, \dots, y_d) \equiv 0 \bmod p^{n'},$$

where the degree of F is at most $3A$ and the integer l is chosen such that some coefficient of F is coprime to p.

We showed that $f_d(p^m\mathbb{Z}_p, \dots, p^m\mathbb{Z}_p) \not\subset p^{Am+n_p+1}\mathbb{Z}_p$ and Therefore, $l \leq Am + n_p \leq n'/2$, where the last inequality following from $m < (n' - 2n_p)/2A$. So $y_1, \dots, y_d \in \mathbb{Z}_p$ are solutions to the congruence

$$F(y_1, \dots, y_d) \equiv 0 \bmod p^t,$$

where F is a polynomial in d variables of degree at most $3A$ and t is an integer with $n' > t \geq n'/2$. By Lemma 5.17 the number of solutions has proportion at most $(3A)^d(t+1)^{d-1}p^{-t/3A}$ in $(\mathbb{Z}_p/p^t\mathbb{Z}_p)^d$. Since A does not depend on p and we can find a constant a_1 such that if $n' > a_1$ we get that $(3A)^d(n'+1)^{d-1} < 2^{n'/12A} \leq p^{n'/12A}$. This gives $\beta_m(g, H) \leq p^{-n'/12A}$ whenever $n' > \max\{a_1, 12A, 12n_p\}$.

This concludes the proof in the special case when H can be described with a good basis with integers $s_1 = \cdots = s_{d-1} = 0$ and $s_d = n'$.

We now prove the general case when the integers s_i associated to a good basis of $H' = H \cap U$ are $0 \leq s_1 \leq \cdots \leq s_d = n'$. Set $s_0 = 0$, and let $\epsilon = (3A)^{-D} \leq (3A)^{-d}$.

Since $s_d = n' > 0$ and $s_0 = 0$, there exists an integer $i \geq 1$ such that $s_i > 3As_{i-1}$. Let $1 \leq i_0 \leq n$ be the largest such integer. Since $s_j/s_{j-1} \leq 3A$ for all $d \geq j > i_0$ and $s_d = n'$, we have $s_{i_0}/s_j \geq \epsilon$ for all $j \geq i_0$ and, in particular, $s_{i_0} \geq n'\epsilon$. Assuming $s_{i_0} > 6n_p + 6A$ (which is the case provided $n' > 6(n_p+A)\epsilon^{-1}$), choose an integer m such that $s_{i_0}/3A < m \leq (s_{i_0} - 2n_p)/2A$. Consider the power series $f, \dots, f_d$ in $\mathbb{Z}_p[[x_1, \dots, x_d]]$ defined by $\mu(f_1, \dots, f_d) = [g, \mathbf{x}]$, $\mathbf{x} = \mu(x_1, \dots, x_d) \in U$.

The condition $[g, \mathbf{x}] \in H'$ is equivalent to

$$f_j(x_1, \dots, x_d) \equiv 0 \bmod p^{s_j}. \ \forall j = 1, \dots, d.$$

Now take $\mathbf{x} \in U_m$ i.e., $x_j \in p^m\mathbb{Z}_p$. By setting $x_k = p^m y_k$, $k = 1, \dots, d$, define p^{l_j} to be the largest power of p dividing all the coefficients of the power series $z_j(y_1, \dots, y_d) := f_j(p^m y_1, \dots, p^m y_d)$.

Now we claim that $l_j < Am + n_p + 1$ for some $j \geq i_0$. Suppose for the sake of contradiction that $l_j \geq Am + n_p + 1$ for all $j \geq i_0$. This means that $p^{Am+n_p+1} | f_j(x_1, \dots, x_d)$ for all $j \geq i_0$. At the same time, since $\mathbf{x} \in U_m$,

$[g, \mathbf{x}] \in U_m$ and so $p^m | f_j$ for all j. Since $m > s_{i_0}/3A$ and $s_{i_0-1} < s_{i_0}/3A$, it follows that $m > s_{i_0-1}$ and so $m > s_j$ for all $j \leq i_0 - 1$. Altogether we have $p^{\min\{s_j, Am+n_p+1\}} | f_j$ for all $j = 1, \dots, d$ and so $[g, U_m] \subseteq H' U_{Am+n_p+1}$. Lemma 5.16 now gives $U_{Am+n_p} \leq H U_{Am+n_p+1}$ and hence $H \geq U_{Am+n_p}$. This is a contradiction since $Am + n_p \leq s_{i_0}/2 < n'$. The claim follows.

Therefore, $l_j \leq Am + n_p \leq s_{i_0}/2$ for some $j \geq i_0$.

We want to estimate the number of cosets $\mathbf{x}U_{n'}$ in $U_m/U_{n'}$ with $[g, \mathbf{x}] \in H'$. Let $\mathbf{x} = \mu(p^m y_1, \dots, p^m y_d) \in U_m$. We will estimate the number of solutions in $(y_1, \dots, y_d) \in (\mathbb{Z}_p/p^{n'-m}\mathbb{Z}_p)^d$ to the congruence

$$f_j(p^m y_1, \dots, p^m y_d) \equiv \mod p^{s_j}.$$

Since $s_j \geq s_{i_0}$, we must have that $p^{s_{i_0}} | f_j(p^m y_1, \dots, p^m y_d)$. Since $m > s_{i_0}/3A$, the last congruence implies a polynomial congruence

$$p^{l_j} F_j(y_1, \dots, y_d) \equiv 0 \bmod p^{s_{i_0}},$$

where $\deg F_j \leq s_{i_0}/m < 3A$ and the polynomial F_j is not divisible by p. We proved $l_j \leq s_{i_0}/2$ and, in particular,

$$F_j(y_1, \dots, y_d) \equiv 0 \bmod p^t,$$

where the integer t satisfies $n' > t \geq s_{i_0}/2$. Recall also that $s_{i_0} \geq n'\epsilon$. By Lemma 5.17 below the proportion of solutions in $(\mathbb{Z}_p/p^{n'-m}\mathbb{Z}_p)^d$ to the last congruence is at most

$$(3A)^d (t+1)^{d-1} p^{-t/\deg F_j} < (3An')^d p^{-n'\epsilon/6A}.$$

So if $n' > 6(A + n_p)\epsilon^{-1}$ is in addition sufficiently large in terms of A and D so that $(3An')^D < p^{n'\epsilon/12A}$, we see that $\beta_m(g, H) < p^{-n'\epsilon/12A}$. Proposition 5.14 follows with $c = \epsilon/12A = (3A)^{-D-1}/4$.

The following is proved in [53, Lemma A.9] with a slightly stronger bound, but we include a proof here for completeness.

5.17. Lemma. *Let $f \in \mathbb{Z}_p[x_1, \dots, x_d]$ be a polynomial of degree r, with at least one coefficient that is not divisible by p. For any $n \in \mathbb{N}$, the proportion of solutions to $f \equiv 0 \bmod p^n$ in $(\mathbb{Z}_p/p^n\mathbb{Z}_p)^k$ is at most $r^d(n+1)^{d-1}p^{-n/r}$.*

Proof. The case $d = 1$ can be found in [108, Cor. 2], which proves a stronger bound involving the discriminant of f. In particular, the bound $rp^{n/r}$ we require is the inequality (44) there without any condition on the discriminant.

To prove the lemma in general we argue by induction on d and assume it holds for $d-1$. Write $f = g_0 x_d^m + g_1 x_d^{m-1} + \cdots + g_m$, where $g_i \in \mathbb{Z}_p[x_1, \dots, x_{d-1}]$. At least one of the g_i, say g_j, is not divisible by p and $\deg g_j \leq r$.

For $0 \leq s \leq n$, let X_s be the set of tuples $(x_1, \dots, x_{d-1}) \in (\mathbb{Z}_p/p^n\mathbb{Z}_p)^{d-1}$ such that p^s is the greatest power of p dividing all of $g_0, \dots, g_m$ evaluated at

$(x_1, \dots, x_{d-1})$. By the induction hypothesis applied to g_j we may assume

$$|X_s| \leq r^{d-1}(s+1)^{d-2}p^{n(d-1)-s/r}.$$

For a given $(d-1)$-tuple in X_s, the number of choices for $x_d \in \mathbb{Z}_p/p^n\mathbb{Z}_p$ such that $p^n|f$ is at most

$$mp^{n-\frac{n-s}{m}} \leq rp^{n-\frac{n-s}{r}}$$

by case $d=1$ of the lemma applied to $\frac{g_0}{p^s}x_d^m + \cdots + \frac{g_m}{p^s}$ and $n-s$ in place of n.

Putting everything together, the number of solutions to $p^n|f$ is at most

$$\sum_{s=0}^{n} |X_s| r p^{n-\frac{n-s}{r}} \leq r^d \sum_{s=0}^{n} (s+1)^{d-2} p^{nd-\frac{s}{r}-\frac{n-s}{r}} \leq r^d (n+1)^{d-1} p^{nd-\frac{n}{r}}.$$

The last inequality following from the crude estimate $1+2^{d-2}+\cdots+(n+1)^{d-2} \leq (n+1)^{d-1}$, the lemma follows. □

6. **Spectral approximation for locally convergent sequences of lattices**

Let G be a connected center free semi-simple Lie group. We let $\widehat{G}$ be the unitary dual of G, i.e., the set of equivalence classes of irreducible unitary representations of G, endowed with the Fell topology; see, e.g., [20, §2.2]. We fix once and for all a Haar measure on G.

Let $\phi \in C_c^\infty(G)$. If $\pi \in \widehat{G}$, then

$$\pi(\phi) := \int_G \phi(g)\pi(g)dg : \mathcal{H}_\pi \to \mathcal{H}_\pi$$

is a bounded operator of trace class. We denote by

$$\widehat{\phi} : \pi \mapsto \text{trace } \pi(\phi)$$

the (scalar) Fourier transform on $\widehat{G}$.

6.1. *Topology of* $\widehat{G}$. As a topological space, $\widehat{G}$ is not separated. It is somewhat easier to work with the set $\Theta(G)$ of *infinitesimal characters* of G, i.e., the set of characters of the center $Z(\mathfrak{g})$ of the universal enveloping algebra of G.

Fix MAN a minimal parabolic subgroup of G and a corresponding real vector space

$$\mathfrak{h}_0 = i\mathfrak{b}_0 \oplus \mathfrak{a}_0,$$

where $\mathfrak{b}_0$ is a Cartan subalgebra of the compact Lie group $K \cap M$. The space $\mathfrak{h}_0$ can be identified with a split Cartan subalgebra of a split inner form of G. In particular, the complex Weyl group W of G acts on $\mathfrak{h}_0$. We fix a positive definite, W-invariant inner product $(\cdot,\cdot)$ on $\mathfrak{h}_0$.

The infinitesimal character of an irreducible representation $\pi \in \widehat{G}$ is represented by a W-orbit θ_π in the complex dual space $\mathfrak{h}^*$ of $\mathfrak{h}_0$. It satisfies

$$\pi(zf) = \langle h(z), \theta_\pi \rangle \pi(f), \quad (z \in Z(\mathfrak{g}),\ f \in C_c^\infty(G)),$$

where $h : Z(\mathfrak{g}) \to S(\mathfrak{h})^W$ is the isomorphism of Harish-Chandra, from $Z(\mathfrak{g})$ onto the algebra of W-invariant polynomial on $\mathfrak{h}^*$.

The map

$$p : \widehat{G} \to \Theta(G) \tag{6.1.1}$$

that maps $\pi \in \widehat{G}$ onto its infinitesimal character θ_π is continuous with respect to the Fell topology. See [105, Lemma 3.4] for a more precise description of the topological space $\widehat{G}$ with respect to this map.

The Plancherel measure ν^G is a positive Borel measure on $\widehat{G}$. Note that ν^G depends on a choice of a Haar measure on G: if the Haar measure is multiplied by a scalar c, then ν^G is multiplied by c^{-1}. Denote by $\mathcal{B}_c(\widehat{G})$ the space of bounded ν^G-measurable functions f on $\widehat{G}$ such that the support of f has compact image in the space of infinitesimal character via the map p defined in (6.1.1).

6.2. *Definition.* Let $\widetilde{\mathcal{F}}(\widehat{G})$ be the space of functions $f \in \mathcal{B}_c(\widehat{G})$ such that for every Levi subgroup L of G and every discrete series σ of L, the function

$$\chi \mapsto f(\mathrm{ind}_L^G(\sigma \otimes \chi))$$

on "unramified" unitary characters of L (see [105, §3]) has the property that its discontinuous points are contained in a measure zero set. Here by definition $f(\mathrm{ind}_L^G(\sigma \otimes \chi))$ is the sum of $f(\sigma')$ as σ' runs over the irreducible subquotients of the (normalized) induced representation $\mathrm{ind}_L^G(\sigma \otimes \chi)$ with multiplicity. (Any such subquotient σ' is unitary.)

For any $\phi \in C_c^\infty(G)$, the function $\widehat{\phi}$ belongs to $\widetilde{\mathcal{F}}(\widehat{G})$.

6.3. *Definition.* Let $\mathcal{F}(\widehat{G})$ be the subspace

$$\{\widehat{\phi} : \phi \in C_c^\infty(G)\} \subset \widetilde{\mathcal{F}}(\widehat{G}).$$

Remark. There are many functions in $\widetilde{\mathcal{F}}(\widehat{G})$ that do not belong to $\mathcal{F}(\widehat{G})$: any characteristic function of a ν^G-regular open subset $S \subset \widehat{G}$ or $S \subset \widehat{G}_{\mathrm{temp}}$ belongs to $\widetilde{\mathcal{F}}(\widehat{G})$; see [105, Lemma 7.2].

It is much easier to work with continuous linear forms on $\mathcal{F}(\widehat{G})$ than with Borel measures on $\widehat{G}$. This is possible thanks to the following fundamental density principle due to Sauvageot [105, Th. 7.3(b)]; see also [107, App. A] for some corrections.

6.4. Proposition. *Let $f \in \widetilde{\mathcal{F}}(\widehat{G})$. For every positive ε, there exist $\phi, \psi \in C_c^\infty(G)$ such that for every $\pi \in \widehat{G}$, we have*

$$|f(\pi) - \widehat{\phi}(\pi)| \leq \widehat{\psi}(\pi) \text{ and } \nu^G(\widehat{\psi}) \leq \varepsilon.$$

In other words the Plancherel measure ν^G is completely determined by the continuous linear I^G form that it defines on $\mathcal{F}(\widehat{G})$. Granted this proposition we shall work with continuous linear forms on $\mathcal{F}(\widehat{G})$.

6.5. *The measure associated to a uniform lattice.* Let Γ be a uniform lattice in G. We denote by ρ_Γ the quasi-regular representation of G in the space $L^2(\Gamma\backslash G)$. Then ρ_Γ is a direct sum of representations $\pi \in \widehat{G}$ occuring with finite multiplicities $\mathrm{m}(\pi, \Gamma)$. The measure

$$\nu_\Gamma = \frac{1}{\mathrm{vol}(\Gamma\backslash G)} \sum_{\pi \in \widehat{G}} \mathrm{m}(\pi, \Gamma)\delta_\pi$$

is, up to the factor $\mathrm{vol}(\Gamma\backslash G)^{-1}$, the Plancherel measure of $L^2(\Gamma\backslash G)$. This measure defines a continuous linear form I_Γ on $\mathcal{F}(\widehat{G})$. Here again, as the spectrum of ρ_Γ is discrete, the measure ν_Γ is determined by I_Γ and, if $\phi \in C_c^\infty(G)$, we have

$$\begin{aligned} \mathrm{trace}\ \rho_\Gamma(\phi) &= \sum_{\pi \in \widehat{G}} \mathrm{m}(\pi, \Gamma)\mathrm{trace}\ \pi(\phi) \\ &= \mathrm{vol}(\Gamma\backslash G) I_\Gamma(\widehat{\phi}). \end{aligned}$$

On the other hand, given $f \in L^2(\Gamma\backslash G)$, we have[8]

$$\begin{aligned} (\rho_\Gamma(\phi)f)(x) &= \int_G \phi(y) f(xy) dy \\ &= \int_G \phi(x^{-1}y) f(y) dy \\ &= \int_{\Gamma\backslash G} \left(\sum_{\gamma \in \Gamma} \phi(x^{-1}\gamma y) \right) f(y) dy. \end{aligned}$$

It follows that the kernel of $\rho_\Gamma(\phi)$ is

$$K_\Gamma^\phi(x, y) = \sum_{\gamma \in \Gamma} \phi(x^{-1}\gamma y), \quad (x, y \in \Gamma\backslash G). \tag{6.5.1}$$

The sum over Γ is finite for any x and y, since it may be taken over the intersection of the discrete group Γ with the compact subset $x\mathrm{supp}(\phi)y^{-1} \subset G$.

[8] Here and below we shall often abusively identify functions on $\Gamma\backslash G$ and Γ-invariant functions on G. Similarly we often use the same notation (x or y) for an element in $\Gamma\backslash G$ and for a choice of a representative of this element in G.

We conclude that

$$\begin{aligned}\nu_\Gamma(\widehat{\phi}) &= \frac{1}{\mathrm{vol}(\Gamma\backslash G)}\int_{\Gamma\backslash G} K_\Gamma^\phi(x,x)dx \\ &= \int_{\Lambda\in \mathrm{Sub}_G} K_\Lambda^\phi(\mathrm{id},\mathrm{id})d\mu_\Gamma(\Lambda).\end{aligned} \tag{6.5.2}$$

Here for any discrete subgroup $\Lambda \in \mathrm{Sub}_G$ and any $(x,y) \in G$, we denote by $K_\Lambda^\phi(x,y)$ the sum

$$K_\Lambda^\phi(x,y) = \sum_{\lambda\in\Lambda} \phi(x^{-1}\lambda y).$$

The latter equality of (6.5.2) then follows from the fact that $K^\phi_{g\Lambda g^{-1}}(x,y) = K^\phi_\Lambda(g^{-1}x, g^{-1}y)$. Note that $\Lambda \mapsto K^\phi_\Lambda(\mathrm{id},\mathrm{id})$ defines a continuous function on the support of μ_Γ.

6.6. *Definition.* We say that a discrete IRS μ or a sequence $\mu_1, \mu_2, \dots$ of discrete IRS's of G is *uniformly discrete* if there exists some positive ε such that

$$\forall \Lambda \in \overline{\cup_{n=1}^\infty \mathrm{supp}(\mu_n)}, \quad \Lambda \cap \mathrm{B}_G(\mathrm{id},\varepsilon) = \{\mathrm{id}\}.$$

We shall sometimes specify ε by saying that a sequence of IRS or a single IRS is ε-discrete.

Example. Let $(\Gamma_n)_{n\geq 1}$ be a uniformly discrete sequence of uniform lattices in G. Then the sequence $(\mu_{\Gamma_n})_{n\geq 1}$ is uniformly discrete.

6.7. Theorem. *Let $(\Gamma_n)_{n\geq 1}$ be a uniformly discrete sequence of uniform lattices in G such that $\Gamma_n\backslash X$ BS-converges to X. Then for every relatively compact ν^G-regular open subset $S \subset \widehat{G}$ or $S \subset \widehat{G}_{\mathrm{temp}}$, the sequence of measures $(\nu_{\Gamma_n})_{n\geq 1}$ is such that*

$$\nu_{\Gamma_n}(S) \to \nu^G(S).$$

Proof. Set $\nu_n = \nu_{\Gamma_n}$ and $I_n = I_{\Gamma_n}$. Let $\phi \in C_c^\infty(G)$. We shall first prove that

$$\lim_{n\to+\infty} I_n(\widehat{\phi}) = I^G(\widehat{\phi}). \tag{6.7.1}$$

We will make use of the following general lemma.

6.8. Lemma. *Let μ_n be a uniformly discrete sequence of* IRS*'s. Then there exist an open neighborhood of the identity $U \subset G$ and a compact subset $K \subset G$ such that, setting $U^* := U \setminus \{\mathrm{id}\}$, the open sets $\mathcal{O}_1(K)$ and $\mathcal{O}_2(U^*)$ in Sub_G are* disjoint, *every nondiscrete subgroup $H \in \mathrm{Sub}_G$ is contained in $\mathcal{O}_2(U^*)$, and*

$$\overline{\bigcup_n \mathrm{supp}(\mu_n)} \subset \mathcal{O}_1(K).$$

Proof. Let d be a left invariant metric on G, and let δ be small enough so that the corresponding δ-ball around id has no nontrivial subgroups. Since the sequence $(\Gamma_n)_{n\geq 1}$ is uniformly discrete, there exists some $\varepsilon < \delta$ such that

$$\Lambda \in \overline{\bigcup_n \mathrm{supp}(\mu_n)} \Rightarrow \overline{B}_G(\mathrm{id}, \varepsilon) \cap \Lambda = \{\mathrm{id}\}.$$

Let U be the open ball $B_G(\mathrm{id}, \varepsilon)$, and let K be the compact set that is the closed ϵ ball minus the open $\epsilon/2$ ball around id. Recall the following definitions:

$$\mathcal{O}_1(K) = \{H \in \mathrm{Sub}_G : H \cap K = \emptyset\}$$

and

$$\mathcal{O}_2(U^*) = \{H \in \mathrm{Sub}_G : H \cap U^* \neq \emptyset\}.$$

These are open subsets of Sub_G. Every nondiscrete subgroup $H \in \mathrm{Sub}_G$ is obviously contained in $\mathcal{O}_2(U^*)$ and

$$\overline{\bigcup_n \mathrm{supp}(\mu_n)} \subset \mathcal{O}_1(K).$$

Let us now prove that $\mathcal{O}_1(K)$ and $\mathcal{O}_2(U^*)$ are disjoint: suppose by way of contradiction that their intersection contains some subgroup $H < G$. Then the intersection of H with the closed $\varepsilon/2$ ball around id contains a nontrivial element h. Since $\overline{B}_G(\mathrm{id}, \varepsilon/2)$ does not contain nontrivial subgroups, the cyclic group $\langle h \rangle$ is not entirely contained into $\overline{B}_G(\mathrm{id}, \varepsilon/2)$. Let h^k be the first nontrivial power that does not belong to $\overline{B}_G(\mathrm{id}, \varepsilon/2)$. Since both $d(\mathrm{id}, h)$ and $d(\mathrm{id}, h^{k-1})$ are $\leq \varepsilon/2$ and since the metric d is left invariant, we conclude that we have $d(\mathrm{id}, h^k) \leq \varepsilon$. Therefore, h^k belongs to K, a contradiction. □

We shall apply Lemma 6.8 to the sequence $\mu_n = \mu_{\Gamma_n}$. Let V be an open symmetric neighborhood of the identity in G such that $V^2 \subset U$. The G-translates of V form an open covering of G from which we may extract a finite collection $g_1V, \ldots, g_kV$ that covers the *compact* support of ϕ. Every $\Lambda \notin \mathcal{O}_2(U^*)$ intersects each g_iV along at most one element. It follows that the function

$$\Lambda \mapsto \sum_{\lambda \in \Lambda} \phi(\lambda)$$

is well defined, continuous and uniformly bounded (by $k||\phi||_\infty$) on $\mathrm{Sub}_G \setminus \mathcal{O}_2(U^*)$. Tietze's Extension Theorem then allows to extend this function to a compactly supported continuous function F_ϕ on Sub_G such that

$$F_\phi(\Lambda) = \begin{cases} \sum_{\lambda \in \Lambda} \phi(\lambda) & \text{if } \Lambda \in \overline{\bigcup_n \mathrm{supp}(\mu_n)}, \\ 0 & \text{if } \Lambda \text{ is not discrete.} \end{cases}$$

Since by hypothesis the sequence μ_n converges weakly toward μ_{id}, we get that

$$I_n(\widehat{\phi}) = \int_{\mathrm{Sub}_G} F_\phi d\mu_n \to \int_{\mathrm{Sub}_G} F_\phi d\mu_{\mathrm{id}}.$$

The limit is equal to $\phi(\mathrm{id})$ which, according to the Plancherel formula proved by Harish-Chandra, is equal to $I^G(\widehat{\phi})$. This proves (6.7.1).

To conclude the proof of Theorem 6.7 we recall that the linear form I_n determines the Borel measure ν_n on $\widehat{G}$ and that it similarly follows from Proposition 6.4 (Sauvageot's density principle) that the linear form I^G determines the Plancherel measure of G. The theorem easily follows. Indeed, let $S \subset \widehat{G}$, or $S \subset \widehat{G}_{\mathrm{temp}}$, be a relatively compact open subset that is regular with respect to the Plancherel measure of G (i.e., $\nu^G(S) = \nu^G(\overline{S})$). Let ε be a positive real number. By the density principle, there exist $\phi, \psi \in C_c^\infty(G)$ such that

$$|1_S - \widehat{\phi}| \leq \widehat{\psi} \text{ and } \nu^G(\widehat{\psi}) \leq \varepsilon.$$

We conclude that

$$\begin{aligned} |\nu_{\Gamma_n}(S) - \nu^G(S)| &\leq \nu_{\Gamma_n}(\widehat{\psi}) + |\nu_{\Gamma_n}(\widehat{\phi}) - \nu^G(\widehat{\phi})| + \nu^G(\widehat{\psi}) \\ &\leq |I_n(\widehat{\psi}) - I^G(\widehat{\psi})| + 2I^G(\widehat{\psi}) + |I_n(\widehat{\phi}) - I^G(\widehat{\phi})| \\ &\leq 4\varepsilon \end{aligned}$$

for sufficiently large n. □

Theorem 6.7 implies the following.

6.9. Corollary (Pointwise convergence). *Let $(\Gamma_n)_{n\geq 1}$ be a uniformly discrete sequence of uniform lattices in G such that $\Gamma_n\backslash X$ BS-converges to X. Then*

$$\lim_{n\to+\infty} \nu_{\Gamma_n}(\{\pi\}) = \nu^G(\{\pi\})$$

for every $\pi \in \widehat{G}$.

Note that $d(\pi) := \nu^G(\{\pi\})$ is 0 unless π is square integrable (i.e., is a discrete series), in which case it is the formal degree of π; see [64, Th. 6.2].

6.10. *An alternative proof of Corollary* 6.9. Here we propose a proof of Corollary 6.9 in the spirit of DeGeorge–Wallach [64] and Savin [106] that avoids the intricate analysis of [105]. We first prove that

$$\begin{aligned} \limsup_{n\to\infty} \frac{\mathrm{m}(\pi,\Gamma_n)}{\mathrm{vol}(\Gamma_n\backslash G)} &= \limsup_{n\to\infty} \nu_n(\{\pi\}) \\ &\leq \nu^G(\{\pi\}) = d(\pi). \end{aligned} \tag{6.10.1}$$

Let $\phi \in C_c^\infty(G)$. We first note that

$$\begin{aligned} \frac{\mathrm{m}(\pi,\Gamma)}{\mathrm{vol}(\Gamma\backslash G)} ||\pi(\phi)||^2_{\mathrm{H-S}} &\leq \frac{||\rho_\Gamma(\phi)||^2_{\mathrm{H-S}}}{\mathrm{vol}(\Gamma\backslash G)} \\ &\leq \frac{\mathrm{trace}\ \rho_\Gamma(\phi * \widetilde{\phi})}{\mathrm{vol}(\Gamma\backslash G)} \\ &\leq \widehat{\nu_\Gamma(\phi * \widetilde{\phi})}. \end{aligned} \tag{6.10.2}$$

Remark. We have

$$\widehat{\nu^G(\phi * \widetilde{\phi})} = (\phi * \widetilde{\phi})(1) = ||\phi||^2.$$

Note that

$$||\pi(\phi)||^2_{\mathrm{H-S}} \geq |\langle \pi(\phi)v, v\rangle|^2 = \left| \int_G \phi(g)\langle \pi(g)v, v\rangle dg \right|^2,$$

where v is any unit vector in the Hilbert space associated with π. It is Therefore, tempting to apply (6.10.2) with $\phi(g) = \phi_r(g) := \chi_r(g)\langle \pi(g)v, v\rangle$, where χ_r is the characteristic function of $G_r = KA_r^+K$, $A_r^+ = \{a \in A^+ : a = \exp(H), ||H|| \leq r\}$ for some metric $||\cdot||$ on the Lie algebra of the Cartan subgroup A. The function ϕ_r is not smooth. However it is a limit in L^2 of smooth functions with support in G_r and (6.10.2) still holds. Similarly, under the hypotheses of Corollary 6.9, equation (6.7.1) applies to $\phi_r * \widetilde{\phi}_r$. We Therefore, conclude from the remark above that we have

$$\limsup_{n\to+\infty} \frac{\mathrm{m}(\pi, \Gamma_n)}{\mathrm{vol}(\Gamma_n\backslash G)} \leq \frac{1}{||\phi_r||^2}. \tag{6.10.3}$$

As r tends to infinity, $1/||\phi_r||^2$ tends to 0 if π is not square integrable and tends to $d(\pi)$ if π is a discrete series. Inequality (6.10.3) Therefore, implies (6.10.1).

6.11. Now fix π a discrete series representation of G. The set

$$\widehat{G}(\pi) = \{\omega \in \widehat{G} : \theta_\omega = \theta_\pi\}$$

is finite. Computing the $G(\pi)$-part of the Euler characteristic, DeGeorge and Wallach [64, Cor. 5.3] proved the following.

6.12. Proposition. *Given a discrete series representation π of G, there are constants $c(\omega)$, $\omega \in \widehat{G}(\pi)$ with $c(\omega) = 1$ whenever ω is a discrete series representation, such that*

$$\sum_{\omega\in\widehat{G}(\pi)} c(\omega)\frac{\mathrm{m}(\omega, \Gamma)}{\mathrm{vol}(\Gamma\backslash G)} = \sum_{\omega\in\widehat{G}(\pi)} d(\omega).$$

Note that when ω is not a discrete series, then the limit multiplicity is 0 by (6.10.1). Proposition 6.12 and (6.10.1) therefore imply Corollary 6.9. □

6.13. *Sequences of congruence lattices.* Now we fix a uniform irreducible arithmetic lattice $\Gamma \subset G$ as Γ_0 in Theorem 5.2. We also fix $\pi \in \widehat{G}$ a *non-tempered* representation; i.e., π is not weakly contained in $L^2(G)$. In this setting we prove the following.

6.14. THEOREM. *Let* $(\Gamma_n)_{n\geq 1}$ *be* any *infinite sequence of distinct congruence subgroups of* Γ. *Then there exists* $\alpha = \alpha(G,\Gamma,\pi) > 0$ *such that*

$$\mathrm{m}(\pi,\Gamma_n) \ll \mathrm{vol}(\Gamma_n\backslash G)^{1-\alpha}.$$

Proof. This follows the same lines as Section 6.10: Let (τ, V_τ) be the lowest K-type of π, as defined by Vogan [113], and let $v \in V_\tau$ be a highest weight vector. As in [106] we introduce

$$W_n = \mathrm{span}\left\{Tv : T \in \mathrm{Hom}_G(V_\tau, L^2(\Gamma_n\backslash G))\right\} \subset L^2(\Gamma_n\backslash G)$$

and

$$B_n(x) = \sup_{f\in W_n} \frac{|f(x)|^2}{||f||^2} \quad (x \in \Gamma_n\backslash G).$$

As in Section 6.10 we let

$$\phi_r(g) = \chi_r(g)\langle \pi(g)v, v\rangle \quad (g \in G,\ r > 0).$$

We will use the following two lemmas. The first goes back at least to Kazhdan's proof [71] of the so-called Kazhdan's inequality according to which along a residual tower the limsup of the normalized Betti numbers are bounded above by the corresponding L^2-Betti numbers; we include a proof of this first lemma for the reader's convenience. The second — due to Savin [106, Prop. 3] — is a reformulation of the basic identity of DeGeorge and Wallach.

6.15. LEMMA. *We have*

$$\int_{\Gamma_n\backslash G} B_n(x)dx = \mathrm{m}(\pi,\Gamma_n).$$

Proof. Let $f_1,\dots,f_m$ $(m = \mathrm{m}(\pi,\Gamma_n))$ be an orthonormal basis of W_n. The Cauchy-Schwarz inequality implies that $B_n(x) \leq \sum_{i=1}^m |f_i(x)|^2$. Now if we fix x, the function $F : y \mapsto \sum_i \overline{f_i(x)} f_i(y)$ belongs in W_n and we have

$$||F||^2 = F(x) = \sum_i |f_i(x)|^2.$$

It follows that for all x, we have

$$B_n(x) = \sum_{i=1}^m |f_i(x)|^2.$$

Integration over $\Gamma_n\backslash G$ gives the lemma. □

6.16. LEMMA. *We have*

$$\pi(\phi_r)v = ||\phi_r||^2 v.$$

Now let $f \in W_n$. It follows from Lemma 6.16 that

$$||\phi_r||^2 f(x) = \int_G \phi_r(g)f(xg)dg = \int_{\Gamma_n\backslash G} \sum_{\gamma\in\Gamma_n} \phi_r(x^{-1}\gamma g)f(g)dg.$$

By the Cauchy-Schwarz inequality, we have

$$||\phi_r||^2|f(x)| \le ||f|| \left(\int_{\Gamma_n\backslash G} \left| \sum_{\gamma\in\Gamma_n} \phi_r(x^{-1}\gamma g) \right|^2 dg \right)^{1/2}. \tag{6.16.1}$$

Given $x \in G$, we set

$$N_n(x;r) = \#\{\gamma \in \Gamma_n : \chi_r(x^{-1}\gamma x) \neq 0\}.$$

Theorem 5.2 implies the following.

6.17. Proposition. *There exist positive constants β, c such that for all n,*

$$\mathrm{vol}((\Gamma_n\backslash G)_{<c\log \mathrm{vol}(\Gamma_n\backslash G)}) \le \mathrm{vol}(\Gamma_n\backslash G)^{1-\beta}.$$

We now recall the following.

6.18. Lemma. *There exist constants $c_1, c_2 > 0$, depending only on G, such that for any $x, y \in X$,*

$$N(x;R) := |\{\gamma \in \Gamma : d(x,\gamma x) \le R\}| \le c_1 \mathrm{InjRad}_{\Gamma_n\backslash G}(x)^{-d} e^{c_2 R},$$

where d is the dimension of X.

Proof. Clearly, it suffices to prove this for $R \ge \mathrm{InjRad}_{M_n}(x)$. By definition,

$$B(x, \mathrm{InjRad}_{\Gamma_n\backslash G}(x)) \cap B(\gamma x, \mathrm{InjRad}_{M_n}(x)) = \emptyset$$

for all $\gamma \in \Gamma - \{\mathrm{id}\}$. This implies

$$\begin{aligned} N(x;R) \cdot \mathrm{vol}\, B(x, \mathrm{InjRad}_{\Gamma_n\backslash G}(x)) &\le \mathrm{vol}\, B(x, R + \mathrm{InjRad}_{\Gamma_n\backslash G}(x)) \\ &\le \mathrm{vol}\, B(x, 2R). \end{aligned}$$

Now, Knieper [74] shows that there exists a constant $c_2 = a(G)$ such that

$$\mathrm{vol}\, B(x,R) \approx R^{\frac{\mathrm{rank}_{\mathbb{R}}(G)-1}{2}} e^{c_2 R}$$

asymptotically as $R \to \infty$. This yields an upper bound for $\mathrm{vol}\, B(x, 2R)$.

On the other hand, since X has nonpositive curvature, the volume of a ball in X is bounded below by the volume of a ball with the same radius in d-dimensional Euclidean space. Hence

$$\mathrm{vol}\, B(x, \mathrm{InjRad}_{\Gamma_n\backslash G}(x)) \ge b \cdot \mathrm{InjRad}_{\Gamma_n\backslash G}(x)^d,$$

with a constant $b = b(d)$. The lemma follows. □

Remark. When $\mathrm{InjRad}_{\Gamma_n\backslash G}(x)$ and R are both sufficiently small, it is possible to attain better bounds in 6.18 by using the Margulis lemma; see the analysis in Section 7.

6.19. Replacing the constant c by some smaller positive constant we may assume:

$$cc_2 \leq \beta. \tag{6.19.1}$$

Here c_2 is the constant of Lemma 6.18. From this we conclude

6.20. Lemma. *There exists a positive constant C such that for all n,*

$$\int_{\Gamma_n\backslash G} N_n(x; 2c\log \mathrm{vol}(\Gamma_n\backslash G))dx \leq C\mathrm{vol}(\Gamma_n\backslash G).$$

Proof. We split the integral into two parts:

$$I_1 = \int_{\{x\in\Gamma_n\backslash G:\mathrm{InjRad}_{\Gamma_n\backslash G}(x)\leq c\log \mathrm{vol}(\Gamma_n\backslash G)\}} N_n(x; 2c\log \mathrm{vol}(\Gamma_n\backslash G))dx$$

and I_2. Since in I_2 the integrand is everywhere equal to 1, we have $I_2 \leq \mathrm{vol}(\Gamma_n\backslash G)$. As for I_1, we use Lemma 6.18 to get the bound:

$$N_n(x; 2c\log \mathrm{vol}(\Gamma_n\backslash G)) \leq c_1 \mathrm{InjRad}_{\Gamma_n\backslash G}(x)^{-d}\mathrm{vol}(\Gamma_n\backslash G)^{c_2 c}.$$

Since each lattice Γ_n is a subgroup of Γ, there exists a uniform (in n) lower bound on $\mathrm{InjRad}_{\Gamma_n\backslash G}(x)$. We Therefore, conclude from Proposition 6.17 and (6.19.1) that

$$I_1 \leq (\mathrm{const})\mathrm{vol}(\Gamma_n\backslash G)^{c_2 c+1-\beta} \leq (\mathrm{const})\mathrm{vol}(\Gamma_n\backslash G).$$

The lemma follows. □

6.21. Now taking $r = c\log \mathrm{vol}(\Gamma_n\backslash G)$ we note that for every $x \in \Gamma_n\backslash G$ and $g \in G$, the sum $\sum_{\gamma\in\Gamma_n} \phi_r(x^{-1}\gamma g)$ has at most $N_n(x; 2r)$ nonzero term. Therefore,

$$\left|\sum_{\gamma\in\Gamma_n} \phi_r(x^{-1}\gamma g)\right|^2 \leq N_n(x; 2r) \sum_{\gamma\in\Gamma_n} \left|\phi_r(x^{-1}\gamma g)\right|^2.$$

Moreover, since

$$\int_{\Gamma_n\backslash G} \sum_{\gamma\in\Gamma_n} \left|\phi_r(x^{-1}\gamma g)\right|^2 dg = ||\phi_r||^2,$$

it follows from (6.16.1) that for every $x \in \Gamma_n\backslash G$,

$$\frac{|f(x)|^2}{||f||^2} \leq \frac{N_n(x, 2r)}{||\phi_r||^2}. \tag{6.21.1}$$

Integrating (6.21.1) over $\Gamma_n\backslash G$ we conclude from Lemma 6.20 that

$$\mathrm{m}(\pi, \Gamma_n) \leq \frac{1}{||\phi_r||^2} \int_{\Gamma_n\backslash G} N_n(x; c\log \mathrm{vol}(\Gamma_n\backslash G))dx \leq C\frac{\mathrm{vol}(\Gamma_n\backslash G)}{||\phi_r||^2}.$$

We finally note that

$$||\phi_r||^2 = \int_{G_r} |\langle \pi(g)v, v\rangle|^2 dg$$

and

$$\mathrm{vol}(G_r) \geq \exp(\nu r) = \mathrm{vol}(\Gamma_n \backslash G)^{c\nu}$$

for some positive constant ν. Combining this last inequality with the asymptotics of the matrix coefficient $g \mapsto \langle \pi(g)v, v\rangle$ (see, e.g., [73] or [13, Cor. 3.18 and Lemma 4.4]), we conclude from the fact that π is nontempered that there exists some positive constant α such that

$$\frac{1}{||\phi_r||^2} \ll \mathrm{vol}(\Gamma_n \backslash G)^{-\alpha}.$$

The theorem follows. □

6.22. *Nonuniform lattices.* In the nonuniform case things get more complicated: there is continuous spectrum in $L^2(\Gamma\backslash G)$, and the integral

$$\int_{\Gamma\backslash G} K_\Gamma^\phi(x,x)dx$$

is divergent. We may nevertheless hope that, maybe under suitable conditions, Theorem 6.7 holds when replacing ν_Γ by the measure associated to the *discrete*, or to the *cuspidal*, spectrum of $L^2(\Gamma\backslash G)$. There are not yet such complete results even in the case of towers of coverings. We may however refer to the already mentioned work of Shin [107] and to the recent work of Finis, Lapid and Müller [55], [54] which, in particular, completely solves the problem for the case of principal congruence subgroups of $\mathrm{GL}(n)$.

6.23. *From representations to differential forms.* We conclude this section by relating the above results with the study of the spectrum of the Laplace operator.

Given a unitary representation τ of K we consider the following subset of $\widehat{G}$:

$$\widehat{G}_\tau = \{\pi \in \widehat{G} : \mathrm{Hom}_K(\tau, \mathcal{H}_\pi) \neq \{0\}\}.$$

Let τ_k $(k = 0, 1, \ldots)$ be the adjoint representation of K into $\wedge^k \mathfrak{p}$. Representations in $\widehat{G}_{\tau_k}$ are exactly the ones that correspond to k-differential forms on $X = G/K$. Our choice of Haar measure on G corresponds to a choice of a left invariant Riemannian metric on G. We denote by $\mathrm{vol}(K)$ the corresponding volume of K.

Let Γ be a lattice in G. First note that we have

$$\mathrm{vol}(\Gamma\backslash G) = \mathrm{vol}(\Gamma\backslash X)\mathrm{vol}(K). \tag{6.23.1}$$

Now let $C \in Z(\mathfrak{g})$ be the Casimir element. Set $\lambda_\pi = -\theta_\pi(C)$. Let $\pi \in \widehat{G}_{\tau_k}$ and $v \in \mathcal{H}_\pi$ be a nonzero vector in the K-type τ_k. Any element in

$$E_\pi^k(\Gamma\backslash G) := \mathrm{span}\left\{Tv : T \in \mathrm{Hom}_G(\mathcal{H}_\pi, L^2(\Gamma\backslash G)\right\} \subset L^2(\Gamma\backslash G)$$

defines a square-integrable k-differential form on $\Gamma\backslash G/K$ whose eigenvalue is λ_π. Conversely it follows from Matsushima's formula (see, e.g., [20, Th. 1.0.2]) that

$$E^k_\lambda(\Gamma\backslash G) = \bigoplus_{\substack{\pi\in\widehat{G}_{\tau_k}\\ \lambda_\pi=\lambda}} E^k_\pi(\Gamma\backslash G),$$

where $E^k_\lambda(\Gamma\backslash G)$ denotes the λ-eigenspace of the Laplace operator on square-integrable k-differential forms on $\Gamma\backslash X$.

We let $\Theta_k(G)$ be the image of $\widehat{G}_{\tau_k}$ by the map p in (6.1.1). Evaluation on the Casimir element Therefore, gives a map

$$\Theta_k(G) \to \mathbb{R}^+. \tag{6.23.2}$$

A Borel measure ν on $\widehat{G}$ induces a measure $p^*\nu$ on $\Theta(G)$ that we may restrict to a measure on $\Theta_k(G)$; we denote by ν^k the push-forward of the latter by the map (6.23.2) so that ν^k is a measure on $\mathbb{R}^+$.

6.24. Suppose that Γ is uniform. We have

$$\nu^k_\Gamma(\{\lambda\}) = \frac{1}{\mathrm{vol}(\Gamma\backslash G)}\dim E^k_\lambda(\Gamma\backslash G).$$

In particular,

$$\nu^k_\Gamma(\{0\}) = \frac{b_k(\Gamma)}{\mathrm{vol}(\Gamma\backslash G)},$$

where $b_k(\Gamma)$ is the k-th Betti number of Γ. Note that Γ being virtually torsion-free, $b_k(\Gamma)$ makes sense. If Γ is torsion-free, we have $b_k(\Gamma) = b_k(\Gamma\backslash X)$. Similarly $\nu^{G,k}$ is the spectral measure of the Laplace operator on square-integrable differential K-forms on X, and we define the k-th L^2-Betti number of the symmetric space $X = G/K$ as

$$\beta^{(2)}_k(X) = \nu^{G,k}(\{0\})\mathrm{vol}(K).$$

Note that it follows from (6.23.1) that $\mathrm{vol}(\Gamma\backslash X)\beta^{(2)}_k(X)$ is the usual k-th L^2-Betti number of Γ.

Theorem 6.7 implies the following two corollaries:

6.25. Corollary. *Let $(\Gamma_n)_{n\geq 1}$ be a uniformly discrete sequence of uniform lattices in G such that $\Gamma_n\backslash X$ BS-converges to X. Then for each k, the sequence of spectral measures $\nu^k_{\Gamma_n}$ converges weakly toward $\nu^{G,k}$.*

6.26. Corollary. *Let $(\Gamma_n)_{n\geq 1}$ be a uniformly discrete sequence of uniform lattices in G such that $\Gamma_n\backslash X$ BS-converges to X. Then*

$$\lim_{n\to\infty}\frac{b_k(\Gamma_n)}{\mathrm{vol}(\Gamma_n\backslash X)} = \beta^{(2)}_k(X)$$

for $0 \leq k \leq \dim(X)$.

6.27. We finally recall from [27, p. 98] that if $\pi \in \widehat{G}$ is such that $\pi \in \widehat{G}_{\tau_k}$ and $\lambda_\pi = 0$, we have

$$\pi \text{ is tempered } \Leftrightarrow k \in \left[\frac{1}{2}\dim X - e, \frac{1}{2}\dim X + e\right],$$

where $e = \frac{1}{2}(\mathrm{rank}_{\mathbb{C}} G - \mathrm{rank}_{\mathbb{C}}(K))$. Theorem 6.14 Therefore, implies the following.

6.28. Corollary. *Let $(\Gamma_n)_{n\geq 1}$ be a sequence of congruence lattices in a fixed rational form $\mathbf{G}(\mathbb{Q})$. Suppose that $\mathrm{vol}(\Gamma_n\backslash X) \to \infty$. Then there exists $\alpha = \alpha(\mathbf{G}) > 0$ such that for every $k \notin \left[\frac{1}{2}\dim X - e, \frac{1}{2}\dim X + e\right]$,*

$$b_k(\Gamma_n) \ll \mathrm{vol}(\Gamma_n\backslash X)^{1-\alpha}.$$

7. Heat kernel estimates and hyperbolic manifolds

As explained in the announcement [3], our original proof of Corollary 6.26 used the heat kernel following the original path of DeGeorge–Wallach and especially Donnelly [49]. Introducing the notion of BS-convergence allowed us to deal with more general sequences than our predecessors did. However, as in these classical works, this approach relies on heat kernel estimates that require a lower bound on the injectivity radius (our "uniformly discrete" assumption). One novel aspect of the current section is a fine study of heat kernel estimates in the thin parts of hyperbolic manifolds in dimension $d \geq 4$. This will allow us to get rid of the "uniform discreteness" assumption.

As in the preceding sections we let $X = G/K$ be the symmetric space associated to a connected center free semi-simple Lie group G.

7.1. *The heat kernel on forms.* We denote by $e^{-t\Delta_k^{(2)}}(x,y)$ the heat kernel on square-integrable k-forms on X. The corresponding bounded integral operator in $\mathrm{End}(\Omega^k_{(2)}(X))$ defined by

$$(e^{-t\Delta_k^{(2)}} f)(x) = \int_X e^{-t\Delta_k^{(2)}}(x,y) f(y)\, dy \quad \forall f \in \Omega^k_{(2)}(X)$$

is the fundamental solution of the heat equation (cf. [14]).

A standard result from local index theory (see, e.g., [24, Lemma 3.8]) implies

7.2. Lemma. *Let $m > 0$. There exists a positive constant $c = c(G,m)$ such that*

$$||e^{-t\Delta_k^{(2)}}(x,y)|| \leq c t^{-d/2} e^{-d(x,y)^2/5t}, \quad 0 < t \leq m.$$

Much of the content of the statement above is when $t \to 0$. Here, we are mostly interested in the case of fixed t, in which case Lemma 7.2 gives

constants c_1, c_2 depending only on G, t such that

$$||e^{-t\Delta_k^{(2)}}(x,y)|| \leq c_1 e^{-d(x,y)^2/c_2}. \tag{7.2.1}$$

Now let $M = \Gamma\backslash X$ be a compact X-manifold. Let Δ_k be the Laplacian on differentiable k-forms on M. It is a symmetric, positive definite, elliptic operator with pure point spectrum. Write $e^{-t\Delta_k}(x,y)$ $(x,y \in M)$ for the heat kernel on k-forms on M. Then for each positive t, we have

$$e^{-t\Delta_k}(x,y) = \sum_{\gamma\in\Gamma} (\gamma_{\tilde{y}})^* e^{-t\Delta_k^{(2)}}(\tilde{x}, \gamma\tilde{y}), \tag{7.2.2}$$

where $\tilde{x}, \tilde{y}$ are lifts of x, y to X and by $(\gamma_y)^*$, we mean pullback by the map $(x,y) \mapsto (\tilde{x}, \gamma\tilde{y})$. The sum converges absolutely and uniformly for $\tilde{x}, \tilde{y}$ in compacta; this follows from Lemmas 6.18 and 7.2. Given $x \in M$ and a lift $\tilde{x} \in X$, we set

$$f_t(x) = \|e^{-t\Delta_k}(x,x) - e^{-t\Delta_k^{(2)}}(\tilde{x},\tilde{x})\| = \left\| \sum_{\gamma\in\Gamma\backslash\{1\}} e^{-t\Delta_k^{(2)}}(\tilde{x}, \gamma\cdot\tilde{x}) \right\|. \tag{7.2.3}$$

Here, the middle part of the equation can be made well defined by identifying the tangent spaces of T_xM and $T_{\tilde{x}}X$. Let $\tilde{f}_t(\tilde{x}) = f_t(x)$, and note that $\tilde{f}_t$ is Γ-invariant. Recall that we denote by $\mathrm{InjRad}_M(x)$ the injectivity radius of M at x.

7.3. (L^2-)*Betti numbers.* The trace of the heat kernel $e^{-t\Delta_k^{(2)}}(x,x)$ on the diagonal is independent of $x \in X$, being G-invariant. We denote it by

$$\mathrm{Tr}\, e^{-t\Delta_k^{(2)}} := \mathrm{tr}\, e^{-t\Delta_k^{(2)}}(x,x).$$

It follows from Section 6.24 that

$$\beta_k^{(2)}(X) = \lim_{t\to\infty} \mathrm{Tr} e^{-t\Delta_k^{(2)}}.$$

It is equal to zero unless $\delta(G) = 0$ and $k = \frac{1}{2}\dim X$, in which case

$$\beta_k^{(2)}(X) = \frac{\chi(X^d)}{\mathrm{vol}(X^d)},$$

where X^d is the compact dual; see [93].

Recall also that the usual Betti numbers of M are given by

$$b_k(M) = \lim_{t\to\infty} \mathrm{Tr} e^{-t\Delta_k} = \lim_{t\to\infty} \int_M \mathrm{tr}\; e^{-t\Delta_k}(x,x)dx$$

and that since $\mathrm{Tr}\, e^{-t\Delta_k} = \sum_i e^{-t\lambda_i}$, where λ_i are the eigenvalues of Δ_k, the limit above is *monotone decreasing* in t.

7.4. Lemma. *Let $t > 0$ be a real number. There exists a constant $C = C(t, G)$ such that for any $x \in M$,*

$$f_t(x) \leq C \cdot \mathrm{InjRad}_M(x)^{-d}.$$

Proof. Let $x \in M$, and let $\tilde{x}$ be a lift of x to X. Then by the definition (7.2.3),

$$\begin{aligned} f_t(x) &\leq \sum_{\gamma \in \Gamma \setminus \{1\}} \|e^{-t\Delta_k^{(2)}}(\tilde{x}, \gamma \cdot \tilde{x})\|. \\ &\leq \sum_{\gamma \in \Gamma \setminus \{1\}} c e^{-d(x,\gamma\tilde{x})^2/5t}, \quad \text{by (7.2.1)} \\ &\leq \sum_{n=0}^{\infty} c e^{-n^2/5t} N(x, n+1), \end{aligned}$$

where in an overestimate, $N(x; n+1) = \#\{\gamma \in \Gamma \setminus \{1\} : d(\tilde{x}, \gamma\tilde{x}) \leq n+1\}$. But

$$N(x; n+1) \leq c_1 \mathrm{InjRad}_M(x)^{-d} e^{c_2(n+1)}$$

by Lemma 6.18 for some c_1, c_2 depending only on G. So for some $C = C(t, G)$,

$$f_t(x) \leq \sum_{n=0}^{\infty} c c_1 e^{-n^2/5t + c_2(n+1)} \mathrm{InjRad}_M(x)^{-d} \leq C \cdot \mathrm{InjRad}_M(x)^{-d}. \qquad \square$$

7.5. *Convergence of Betti numbers.* We now explain how to use the heat kernel estimates above to prove the following proposition, which implies Corollary 6.26.

7.6. Proposition. *Suppose (M_n) is a sequence of compact X-manifolds that* BS*-converges to X. Then we have* (1) $\implies$ (2) $\implies$ (3) $\implies$ (4), *where*

(1) *(M_n) is uniformly discrete;*

(2) $\dfrac{1}{\mathrm{vol}(M_n)} \displaystyle\int_{M_n} \mathrm{InjRad}_{M_n}(x)^{-d} dx \to 0;$

(3) $\displaystyle\lim_{n \to +\infty} \frac{b_k(M_n)}{\mathrm{vol}(M_n)} \leq \beta_k^{(2)}(X)$ *for* $k = 0, \ldots, \dim(X);$

(4) $\displaystyle\lim_{n \to +\infty} \frac{b_k(M_n)}{\mathrm{vol}(M_n)} = \beta_k^{(2)}(X)$ *for* $k = 0, \ldots, \dim(X).$

Proof. (1) $\implies$ (2). Since (M_n) is uniformly discrete, there is some $\epsilon > 0$ such that the injectivity radius $\mathrm{InjRad}_{M_n}(x) \geq \epsilon$ for all x, n. Fixing $R > 0$,

$$\lim_{n \to \infty} \frac{1}{\mathrm{vol}(M_n)} \int_{M_n} \mathrm{InjRad}_{M_n}(x)^{-d} dx \leq \lim_{n \to \infty} \left(R^{-d} + \frac{\mathrm{vol}(M_n)_{\leq R}}{\mathrm{vol}(M_n)} \epsilon^{-d} \right) = R^{-d},$$

by integrating separately over $(M_n)_{\leq R}$ and its complement, and then using BS-convergence. Sending $R \to \infty$ proves (2).

(2) $\Longrightarrow$ (3). Since $\beta_k^{(2)}(X) = \lim_{t\to\infty} \mathrm{Tr}(e^{-t\Delta_k^{(2)}}(\tilde{x},\tilde{x}))$, we may fix an arbitrarily small $\nu > 0$ and t large enough so that

$$\beta_k^{(2)}(X) < \mathrm{Tr}(e^{-t\Delta_k^{(2)}}(\tilde{x},\tilde{x})) + \nu.$$

Then since $b_k(M) \leq \int_M \mathrm{tr}\, e^{-t\Delta_k}(x,x)dx$ for each fixed t, we have

$$\begin{aligned}\frac{b_k(M_n)}{\mathrm{vol}(M_n)} - \beta_k^{(2)}(X) &\leq \frac{1}{\mathrm{vol}(M_n)} \int_{M_n} \mathrm{tr}\, e^{-t\Delta_k}(x,x) - \beta_k^{(2)}(X)dx \\ &\leq \frac{1}{\mathrm{vol}(M_n)} \int_{M_n} f_t^n(x)dx + \nu.\end{aligned}$$

Now it follows from the hypothesis of the proposition and Lemma 7.4 that for n large enough, the right-hand side is less than 2ν, so (3) follows.

(3) $\Longrightarrow$ (4). Unless $\dim(X)$ is even and $k = \dim(X)/2$, the equality in (4) is automatic since $\beta_k^{(2)}(X) = 0$. The equality when $k = \dim(X)/2$ follows since the Euler characteristic of M_n is the same as its L^2-analogue. □

7.7. *Heat kernel estimates in rank one.* We now establish some preliminary estimates on the difference $f_t(x)$ between the heat kernel on a rank one locally symmetric space M and the L^2 heat kernel in the universal cover. While we will apply these estimates only to real hyperbolic manifolds, we write them up more generally here, since we anticipate they will be useful in the future and the proof is not any simpler for $X = \mathbb{H}^n$.

Let $G = \mathbb{G}(\mathbb{R})$ be a connected adjoint simple real algebraic group of real rank one. We fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g} = \mathrm{Lie}(G)$ and let $K \leq G$ be the maximal compact subgroup of G corresponding to $\mathfrak{k}$.

Let $\tilde{x}_0 \in X = G/K$ be the point corresponding to K. Recall that $\mathfrak{p}$ is identified with the tangent space $T_{\tilde{x}_0}X$ and the Killing form on G induces an inner product on $\mathfrak{p}$ that determines the Riemannian structure on X. Fix an $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{k}$, and extend it to an $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{g}$ so that $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal. Finally, let $d = \dim(X)$ and let $s = \mathrm{rank}_{\mathbb{C}}(\mathbb{G})$ be the complex rank of $\mathbb{G}$; e.g., if $G = \mathrm{SO}(d,1)$, then $s = [\frac{d+1}{2}]$.

We wish to establish estimates on f_t within the "thin parts" of an X-manifold, i.e., parts where the injectivity radius is small. The geometry of thin parts is controlled by the classical Margulis lemma:

7.8. Theorem ([110, §4.1]). *There is a constant $\epsilon = \epsilon(X) > 0$ such that if Λ is a discrete, torsion-free subgroup of G consisting of semi-simple elements and is generated by $\{\gamma \in \Lambda : d(\gamma\cdot\tilde{x},\tilde{x}) < \epsilon\}$ for some $\tilde{x} \in X$, then Λ is cyclic. Moreover, there is a unique geodesic, the axis of Λ, on which it acts by translations.*

An important consequence of the Margulis lemma is the thick-thin decomposition which, in our case, says that the thin part

$$M_{<\epsilon} = \left\{ x \in M : \ \mathrm{InjRad}_M(x) < \frac{\epsilon}{2} \right\}$$

consists of finitely many connected components, each of which is a tubular neighborhood of a short closed geodesic.

For $\tilde{x} \in X$, we shall denote by $\Sigma_{\tilde{x},\epsilon}$ the set of elements in Γ that move $\tilde{x}$ by less than ϵ and by $\Gamma_{\tilde{x},\epsilon} = \langle \Sigma_{\tilde{x},\epsilon} \rangle$ the cyclic group it generates.

The following proposition gives an estimate on f_t in terms of the number of Γ-orbit points in a ball. It is easily deduced from the proof of Lemma 7.4, more precisely from both Lemma 7.2 and the fact that X has exponential growth.

7.9. Proposition. *Given $r > 0$, there is $D = D(r,t)$ such that for any $x \in X$,*

$$f_t(x) \leq D \cdot \mathrm{card}(\Gamma \cdot x \cap B(x,r)).$$

In view of Proposition 7.9, our goal is to estimate the number of orbit points in a given ball and deduce bounds on f_t. We will split this into two estimates: one that is better close to the geodesic core of the thin part, and one that is better at points far from the core.

7.10. Lemma (Near the core). *Let $x \in M$ be a point in an ϵ-thin tubular neighborhood of a short geodesic, and suppose that the length of that short geodesic is τ. Then $f_t(x) \leq C_1 \tau^{-1}$, for some constant $C_1 = C_1(X,t)$.*

Proof. Let x be such a point in M, and let $\tilde{x}$ be a lift of x to X. In view of Proposition 7.9 we should obtain an upper bound of the form $\mathrm{const} \cdot \tau^{-1}$ on the cardinality of the set

$$\mathcal{E} = \Gamma \cdot \tilde{x} \cap B(\tilde{x}, \epsilon) = \Gamma_{\tilde{x},\epsilon} \cdot \tilde{x} \cap B(\tilde{x}, \epsilon).$$

Let c be the axis of $\Gamma_{\tilde{x},\epsilon}$ and $\pi_c : X \to c$ be the nearest point projection. Since c is convex and X is nonpositively curved, π_c is 1-Lipschitz. Since $\Gamma_{\tilde{x},\epsilon}$ is torsion free and stabilizes c, it follows that the restriction of π_c to a $\Gamma_{\tilde{x},\epsilon}$-orbit is one-to-one and its image is again a $\Gamma_{\tilde{x},\epsilon}$-orbit. Moreover, since $\mathcal{E}$ has diameter $\leq 2\epsilon$, we deduce that $\pi_c(\mathcal{E})$ is contained in an interval of length 2ϵ in c. Thus $\mathrm{Card}(\mathcal{E}) \leq \frac{2\epsilon}{\tau}$. □

Lemma 7.10 gives a sufficiently good bound on $f_t(x)$ when x is close to a short geodesic. However when x is far from the geodesic, the injectivity radius $\mathrm{InjRad}_M(x)$ might be of several magnitude larger than the minimal displacement τ, and the result of 7.10 will not be enough for our purpose, so we should obtain a better estimate in terms of $\mathrm{InjRad}_M(x)$. At first glance one may expect that the number of orbit points in a ball is controlled by $\mathrm{InjRad}_M(x)^{-1}$

(or by $\mathrm{InjRad}_M(x)^{-r}$ in general when $r = \mathrm{rank}_{\mathbb{R}}(X)$). However the rotational parts of the isometries may make the orbit denser at certain distances from the submanifold of local minimal displacement. The true exponent is the absolute rank s:

7.11. Lemma (Far from the core). *If x lies in the ϵ-thin part of M, then*

$$f_t(x) \leq C_2 \mathrm{InjRad}_M(x)^{-s}$$

for some constant $C_2 = C_2(X, t)$.

Proof. Let $\delta > 0$ be sufficiently small so that for

$$U_{\tilde{x}_0} = \exp(\{X \in \mathfrak{g} : \|X\| \leq \delta\},$$

we have that $U^2_{\tilde{x}_0}$ forms a Zassenhaus neighborhood in G (see [98, Ch. XI] and [110, §4.1]). Here, $\tilde{x}_0 \in X$ is the point corresponding to $K \in G/K = X$.

We shall call $U_{\tilde{x}_0}$ the Zassenhaus neighborhood associated to $\tilde{x}_0$. Since G acts transitively, for any $\tilde{x} \in X$ we have some $g \in G$ such that $g \cdot \tilde{x}_0 = \tilde{x}$. Set $U_{\tilde{x}} = gU_{\tilde{x}_0}g^{-1}$ and $U^2_{\tilde{x}}$ as the Zassenhaus neighborhood associated to $\tilde{x}$. Since $U_{\tilde{x}_0}$ is invariant under conjugation by K, $U_{\tilde{x}}$ is well defined.

The orbit map $X \mapsto \exp(X) \cdot \tilde{x}_0$ restricted to $\{X \in \mathfrak{g} : \|X\| \leq \delta\}$ is α-bi-Lipschitz for some constant α and covers an open ball $B_X(\tilde{x}_0, \beta)$ for some $1 \geq \beta > 0$. It follows that if $V_1, \ldots, V_t \in \mathfrak{g}$ are of norm at most δ and $\{\exp(V_1) \cdot \tilde{x}_0, \ldots, \exp(V_t) \cdot \tilde{x}_0\}$ forms a ρ-discrete subset of X, then $\{V_1, \ldots, V_t\}$ is $\frac{\rho}{\alpha}$ discrete in $\mathfrak{g}$.

Now let $x \in M_{\leq \epsilon}$ be the point in question, and let $\tilde{x} \in X$ be a lift of x. We may suppose that $\mathrm{InjRad}_M(x) < \beta$. Let

$$m = \frac{\mu(U_{\tilde{x}} \cdot \{g \in G : d(g \cdot \tilde{x}, \tilde{x}) \leq 1\})}{\mu(U_{\tilde{x}})} + 1.$$

Note that m is independent of $\tilde{x}$. In the proof of the Margulis lemma given in [110, §4.1] it is shown that the Margulis constant ϵ can be chosen to be $1/m$ or smaller. Since we have defined m and β independently of ϵ, we may assume that $\epsilon \leq \frac{\beta}{2m}$. In that case, as follows easily from the argument in [110, §4.1], $N = \langle U^2_{\tilde{x}} \cap \Gamma_{\tilde{x},\epsilon} \rangle$ is a subgroup of index $\leq m$ in $\Gamma_{\tilde{x},\epsilon}$ and one can choose coset representatives within $\Sigma^m_{\tilde{x},\epsilon}$. In particular, it follows that

$$\mathrm{card}\left(\Gamma \cdot \tilde{x} \cap B\left(\tilde{x}, \frac{\beta}{2}\right)\right) \leq m \cdot \mathrm{card}(N \cdot \tilde{x} \cap B(\tilde{x}, \beta)).$$

Moreover, by the Zassenhaus–Kazhdan-Margulis theorem (see [98, Ch. XI]), $\log N$ spans a connected nilpotent Lie sub-algebra $\mathfrak{n}$ of the Lie algebra of the stabilizer $\mathrm{Stab}_G(c)$, where c is the axis of $\Gamma_{\tilde{x},\epsilon}$. Note that $\mathrm{Stab}_G(c)$ is isomorphic to a compact group times $\mathbb{R}^*$ and hence admits no unipotent elements. It follows that $\mathfrak{n}$ is abelian and semi-simple and its exponentiation $\exp(\mathfrak{n})$ is a torus

in G. In particular, $\dim \mathfrak{n} \leq s$. Finally, since $N \cdot \tilde{x} \cap B(\tilde{x}, \beta)$ is $\mathrm{InjRad}_M(x)/2$ discrete, we get that $\log(N)$ is $\mathrm{InjRad}_M(x)/(2\alpha)$ discrete in $\mathfrak{n}$. Thus

$$\mathrm{card}(N \cdot \tilde{x} \cap B(\tilde{x}, \beta)) \leq \mathrm{card}(\log(N) \cap B_{\mathfrak{g}}(0, \delta)) \leq C' \left(\frac{\mathrm{InjRad}_M(x)}{\alpha} \right)^s,$$

and the result follows from Proposition 7.9. □

7.12. *Real hyperbolic manifolds.* Given a symmetric space X of noncompact type, Margulis has conjectured that the set of arithmetic compact X-manifolds is uniformly discrete. If $\mathrm{rank}(X) \geq 2$ or if X is the symmetric space corresponding to $\mathrm{Sp}(d, 1)$ or F_4^{-20}, then all irreducible X-manifolds are arithmetic, by Margulis's Arithmeticity Theorem [86] and the Corlette–Gromov–Schoen Theorem [39], [67], respectively. For $\mathrm{SU}(d, 1)$ there are a few known examples of nonarithmetic manifolds for $d = 2, 3$, and it is likely that most manifolds are arithmetic. It is thus natural to conjecture that if X is not isometric to $\mathbb{H}^d$, then the family of all irreducible compact X-manifolds is uniformly discrete.

For $X = \mathbb{H}^d$, it is known that for all d, there are closed hyperbolic d-manifolds with arbitrarily short systoles, [9], [22], [16]. Our aim here is to prove a strong generalization of Corollary 6.26 for $\mathbb{H}^d$, not assuming uniform discreteness.

7.13. Theorem. *Let $M_n = \Gamma_n \backslash \mathbb{H}^d$ be a sequence of compact hyperbolic d-manifolds that* BS*-converges to $\mathbb{H}^d$. Then for every $k = 0, \ldots, d$,*

$$\lim_{n \to +\infty} \frac{b_k(M_n)}{\mathrm{vol}(M_n)} = \beta_k^{(2)}(\mathbb{H}^d).$$

Remark. The analog of Theorem 7.13 holds in the greater generality where $\mathbb{H}^d$ is replaced by a general rank one symmetric space. The proof of that, however, uses different techniques and is much longer. This result will appear in [1], where we will also treat higher rank symmetric spaces.

Note that for $X = \mathbb{H}^2$, the hyperbolic plane, Theorem 7.13 is a consequence of the Gauss–Bonnet theorem, even under the weak assumption that only $\mathrm{vol}(M_n) \to \infty$, without requiring BS-convergence. The cases $d = 3$ and $d \geq 4$ will be handled separately. When $d \geq 4$, Theorem 7.13 will follow from a fine analysis of the heat kernel in the thin parts of the M_n, which is of independent interest. In dimension 3, the analogous statements about the heat kernel are not true, as we will explain, but we can use a trick to reduce the calculation of Betti numbers to estimates on the heat kernel only over the thick part of M_n.

We start with the case $d \geq 4$. Fix $k \in \{1, \dots, d\}$, and let f_t be the function defined in 7.2.3. In view of Proposition 7.6, we need to prove

$$\lim_n \frac{1}{\mathrm{vol}(M_n)} \int_{M_n} f_t(x)\, dx \to 0. \tag{7.13.1}$$

The crucial technical tool is the following.

7.14. THEOREM. *Given $d \geq 4$ and $t > 0$, there is a constant $C = C(d,t)$ such that*

$$\int_{M_{\leq \epsilon/2}} f_t(x)\, dx \leq C_0 \cdot \mathrm{vol}(M_{\leq \epsilon})$$

for every compact hyperbolic d-manifold.

Assuming Theorem 7.14, let us prove (7.13.1). First, note that

$$\begin{aligned} \lim_n \frac{\int_{M_n} f_t(x)\, dx}{\mathrm{vol}(M_n)} &= \lim_n \frac{\int_{(M_n)_{\geq \epsilon}} f_t(x)\, dx}{\mathrm{vol}(M_n)} + \frac{\int_{(M_n)_{<\epsilon}} f_t(x)\, dx}{\mathrm{vol}((M_n)_{<\epsilon})} \cdot \frac{\mathrm{vol}((M_n)_{<\epsilon})}{\mathrm{vol}(M_n)} \\ &\leq \lim_n \frac{\int_{(M_n)_{\geq \epsilon}} c(t)\mathrm{InjRad}(x)^{-d}\, dx}{\mathrm{vol}(M_n)} + C_0 \cdot \lim_n \frac{\mathrm{vol}((M_n)_{<\epsilon})}{\mathrm{vol}(M_n)}. \end{aligned} \tag{7.14.1}$$

Here, $c(t)\mathrm{InjRad}(x)^{-d}$ comes from Lemma 7.4 and the C_0 is from Theorem 7.14. On the far right, BS-convergence $M_n \to X$ implies that the limit is zero. So, for any fixed $r \geq \epsilon$, splitting up the first term in (7.14.1) gives the upper bound

$$\begin{aligned} &\leq \lim_n \frac{\int_{(M_n)_{\geq \epsilon} \cap (M_n)_{<r}} c(t)\epsilon^{-d}\, dx}{\mathrm{vol}(M_n)} + \frac{\int_{(M_n)_{\geq r}} c(t) r^{-d}\, dx}{\mathrm{vol}(M_n)} \\ &\leq c(t)\epsilon^{-d} \lim_n \frac{\mathrm{vol}((M_n)_{<r})}{\mathrm{vol}(M_n)} + c(t) r^{-d} \\ &= c(t) r^{-d}, \end{aligned}$$

again by BS-convergence. Letting $r \to \infty$, this proves (7.13.1).

7.15. *The proof of Theorem 7.14.* We shall work in radial horospherical coordinates of the upper half space model of $\mathbb{H}^d$

$$\{(x_1, \dots, x_d \in \mathbb{R}^d) : x_d > 0\},\ ds^2 = \frac{\sum dx_i^2}{x_d^2}.$$

Consider the vertical geodesic $c = (0, \infty)$ and the horizontal (intrinsically Euclidean) horosphere $\mathbb{E}^{d-1}$ passing through c at $p = (0, \dots, 0, 1)$. We will consider the coordinates (r, θ) for points on $\mathbb{E}^{d-1}$ where r is the horospherical radial distance to p and θ is the direction. (Note that the hyperbolic distance of the point (r, θ) to p is roughly $\log r$.) We can extend these coordinates to the upper half space, letting (r, θ, a) denote the point $a \cdot x$ where x is the

point on $\mathbb{E}^{d-1}$ of coordinate (r,θ) and a is the isometric homothety corresponding to a multiplication by $a>0$ in $\mathbb{R}^n$. Let G_c be the stabilizer of c in G, $G_c \cong \mathrm{SO}(d-1)\times\mathbb{R}^{>0}$.

7.16. Lemma. *There are $R<\infty$ and $\alpha>1$ such that if $r_1,r_2>R$, then for any two points points $x_1=(r_1,\theta)$, $x_2=(r_2,\theta)$ at the same direction θ and any $g\in G_c$ for which $d_g(x_1),d_g(x_2)\le\epsilon$, we have*

$$\alpha^{-1}\frac{r_1}{r_2}<\frac{d_g(x_1)}{d_g(x_2)}<\alpha\frac{r_1}{r_2}.$$

Proof. Since the points x_1,x_2 are far from the invariant geodesic $c=(0,\infty)$ and have small g-displacement, the distances $d(g\cdot x_i,x_i)$ are approximated, up to a bounded multiplicative error, by the intrinsic Euclidean distance between the Euclidean projections of $g\cdot x_i$ and x_i to the horosphere $\mathbb{E}^{d-1}$. For the projections (considered with the intrinsic distance), however, the ratio in question is equal to $\frac{r_1}{r_2}$ by similarity of Euclidean triangles. □

Now let $M^\circ_{\le\epsilon}$ be a thin component that is a tubular neighborhood of a short geodesic, and let $\tilde{M}^\circ_{\le\epsilon}$ be a connected component of its pre-image in the upper half space. We may suppose that the short geodesic lifts to $c=(0,\infty)$. Suppose that the length of the short geodesic is τ. Note that $G_c=N_G(G_c)$, and hence it follows from the Margulis lemma that $\Gamma_{p,\epsilon}$ is contained in G_c. Choose a fundamental domain for $\Gamma_{p,\epsilon}$ in $\tilde{M}^\circ_{\le\epsilon}$ of the following form:

$$\mathcal{F}=\{(r,\theta,a): r\le\psi(\theta),\ 1\le a<e^\tau\},$$

where $\psi(\theta)$ is defined to be the radial horospherical distance for which at direction θ the minimal displacement is exactly ϵ, i.e.,

$$\min\{d_\gamma(x):\gamma\in\Gamma_{p,\epsilon}\setminus\{1\}\}=\min\{d_\gamma(x):\gamma\in\Gamma\setminus\{1\}\}=\epsilon$$

for $x\in\mathbb{E}^{d-1}$ of coordinates $(\psi(\theta),\theta)$.

7.17. Lemma. *Given $R>0$, there is some $\tilde{\tau}(R)>0$ such that if $\tau\le\tilde{\tau}$, then*

$$\psi(\theta)>R\quad\forall\theta.$$

Proof. Let $\alpha>0$ be sufficiently small so that any two horocylic rays $r_1(t)=(t,\theta_1,a)$ and $r_2(t)=(t,\theta_2,a)$ starting at an angle $\le\alpha$ stay at distance $\le\epsilon/2$ from each other when $t\le R$. Since $\mathrm{SO}(d-1)$ is compact, there is some $l\in\mathbb{N}$ such that for any $o\in\mathrm{SO}(d-1)$, there is $j=j(o)\le l$ such that $\angle(o^j(\hat{v}),\hat{v})<\alpha$ for every $\hat{v}\in\mathbb{R}^{d-1}$. Let $\lambda>0$ be small enough so that any two horocyclic rays orthogonal to c that start parallel to each other at distance $\le\lambda$ stay at distance $\le\epsilon/2$ for $t\le T$.

Take $\tau_T=\frac{\lambda}{l}$. If $g\in G_c$ is any isometry with displacement $\tau\le\tilde{\tau}$ and rotational part o, it is easy to see that $g^{j(o)}$ has translational part $\le\lambda$ on c

and rotational part $\leq \alpha$. Thus its displacement is $\leq \epsilon$ everywhere on the R neighborhood of c. □

We may fix $R > 0$ and assume $\tau \leq \tilde{\tau}(R)$. To estimate the integral of $f_t(x)$ over $\mathcal{F}$, we divide the domain into two parts, $\mathcal{F}_1 = \{0 \leq r \leq R\}$ and $\mathcal{F}_2 = \{R < r < \psi(\theta)\}$:

$$\int_{\mathcal{F}} f_t(x)dx = \int_{\mathcal{F}_1} f_t(x)dx + \int_{\mathcal{F}_2} f_t(x)dx.$$

The first integral can be bounded using Lemma 7.10:

$$\int_{\mathcal{F}_1} f_t(x)dx \leq \mathrm{vol}(\mathcal{F}_1) \cdot C_1\tau^{-1} \leq \tau \cdot \mathrm{vol}(B^{d-1}(R)) \cdot C_1\tau^{-1} = \mathrm{vol}(B^{d-1}(R)) \cdot C_1,$$

where $B^{d-1}(R)$ is a Euclidian $(d-1)$-ball of radius R. So the first integral is bounded by a constant. Recall that the volume of each thin component is bounded below by a constant since one can inject an $\frac{\epsilon}{2}$ ball tangent to the boundary of the component.

Let us estimate the second integral. Note that by Lemma 7.16 the $\Gamma_{p,\epsilon}$ minimal displacement at (r, θ) for $r > R$ is at least $\alpha^{-1}\frac{\epsilon r}{\psi(\theta)}$. Therefore, using Lemma 7.11 we deduce

$$\begin{aligned}\int_{\mathcal{F}_2} f_t(x)dx &\leq C_2\alpha^s \int_{\theta\in\mathbb{S}^{d-2}} \int_R^{\psi(\theta)} \left(\frac{\epsilon \cdot r}{\psi(\theta)}\right)^{-s} \cdot \tau \cdot r^{d-2} dr d\theta \\ &\leq \mathrm{Const} \cdot \tau \int_{S^{d-2}} \left(\psi(\theta)^s \int_0^{\psi(\theta)} r^{d-s-2} dr\right) d\theta.\end{aligned}$$

Here $s = \left[\frac{d+1}{2}\right]$, and since $d \geq 4$, we have $d - s - 2 \geq 0$. It follows that

$$\int_{\mathcal{F}_2} f_t(x)dx \leq \mathrm{Const} \int_{\mathbb{S}^{d-2}} \tau \cdot \psi(\theta) \cdot \psi(\theta)^{d-2} d\theta.$$

The point is that the last term is, up to a constant, the volume of the thin component. This concludes the proof of Theorem 7.14. □

7.18. *The case* $d = 3$. Equation (7.13.1) is false when $d = 3$, essentially since f_t has infinite integral when M has a cusp, and cusped manifolds can be approximated by closed manifolds using hyperbolic Dehn surgery.

To discuss this in more detail, suppose that M is a finite volume hyperbolic d-manifold. When M is noncompact, the heat operators on k-forms $e^{-t\Delta_k}$ are not of trace class. In fact, following [90, eq. (3.3)], we may endow M with a height function in the cusps. For Y big enough, the truncation $M(Y)$ of M at height Y is diffeomorphic to the so-called Borel–Serre compactification of M. Fixing t, it is a consequence of the Selberg trace formula (see, e.g., [58] for the case of functions and [90, eq. (5.5)] for the general case) that

$$\int_{M(Y)} \mathrm{tr}\, e^{-t\Delta_k}(x, x)dx \sim k_0 \log Y + c.$$

Here, the notation $A(Y) \sim B(Y)$ means that $A(Y) - B(Y) \to 0$ as $Y \to +\infty$, and k_0 and c are positive constants that depends on t. (In the case of 0-forms, $k_0 = \frac{h}{2\pi} \int_0^{+\infty} e^{-t(1+s^2)} ds$, where h is the number of cusps.)

In particular, for Y big enough, we have

$$\int_{M(Y)} \operatorname{tr} e^{-t\Delta_k}(x,x)dx \geq 2\mathrm{vol}(M).$$

When M has dimension $d = 3$, hyperbolic Dehn surgery constructs from M a closed hyperbolic manifold M' so that $M_{\leq Y}$ is almost isometrically embedded inside M' and $\mathrm{vol}(M')$ is close to $\mathrm{vol}(M)$. In particular, we may construct M' so that

$$\int_{M'} \operatorname{tr} e^{-t\Delta_k}(x,x)dx \geq \mathrm{vol}(M').$$

Now take $k = 1$. Starting from a sequence of finite volume, noncompact, hyperbolic manifolds that BS-converges toward $\mathbb{H}^d$, the construction above yields a sequence of closed hyperbolic manifolds (M_n) such that

$$\int_{M_n} \operatorname{tr} e^{-t\Delta_1}(x,x)dx \geq \mathrm{vol}(M_n).$$

On the other hand, the integral $\frac{1}{\mathrm{vol}(M_n)} \int_{M_n} \operatorname{tr} e^{-t\Delta_1^{(2)}}(x,x)dx$ is finite, bounded uniformly in n and approaches $\beta_1^{(2)}(\mathbb{H}^3) = 0$ as t tends to infinity. In particular, for t small enough, we may assume that

$$\int_{M_n} \operatorname{tr} e^{-t\Delta_1^{(2)}}(x,x)dx \leq \frac{1}{2}\mathrm{vol}(M_n).$$

And it follows that (7.13.1) cannot hold (when $d = 3$).

We now prove Theorem 7.13 when $d = 3$. Suppose that (M_n) is a sequence of finite volume[9] hyperbolic 3-manifolds that BS-converges to $\mathbb{H}^3$. In light of Proposition 7.6, we need to show that[10]

$$\lim_{n\to+\infty} \frac{b_1(M_n)}{\mathrm{vol}(M_n)} \leq \beta_1^{(2)}(\mathbb{H}^3). \tag{7.18.1}$$

Fix ϵ less than the Margulis constant. When M is a finite volume hyperbolic 3-manifold, we let M_T be the union of the ϵ-thick part of M and any components of the ϵ-thin part on which the injectivity radius is bounded below by $\epsilon/2$.

7.19. Lemma. *M_T is a closed, 3-dimensional submanifold whose boundary consists of tori or Klein bottles smoothly embedded in M, and the components of $M \setminus M_T$ are either solid tori or solid Klein bottles (i.e., disk bundles over*

[9]This argument even works for nonuniform lattices, while the estimates in the $d \geq 4$ case are just for uniform lattices.

[10]Note that the right-hand side below is 0, but we will not make use of that.

a circle) or are products $T^2 \times (0,\infty)$ *or* $K^2 \times (0,\infty)$. *Furthermore,* M_T *has "bounded geometry," in the sense of* [84, Def. 2.24].

As we will see below, "bounded geometry" requires that the boundary of M_T is not too distorted in M, which is why we take M_T instead of just the ϵ-thick part.

Proof. In dimension 3, the Margulis lemma implies that each component of the thin part $M \setminus M_T$ is the quotient of either a metric neighborhood of a geodesic in $\mathbb{H}^3$ or of a horoball; this implies that the boundary is smooth, and it gives the topological information above.[11] See also [17] for details.

In [84, Def. 2.24], "bounded geometry" means the following. First, the injectivity radius of M_T should be bounded below, which is true by definition. Second, the geodesic flow starting from the inward normal vector field on ∂M_T should give a collar neighborhood of the boundary with radius bounded below; this follows since the injectivity radius of M_T is bounded below and since the components of the preimage of $M \setminus M_T$ in $\mathbb{H}^3$ are convex. (This again is a 3 dimensional phenomenon and is false in higher dimension.) Finally, the derivatives of the metric tensor and its inverse should be bounded, both in exponential coordinates and the "boundary normal coordinates" on the collar of ∂M_T above. In exponential coordinates, the bounds come from differentiating the metric tensor on $\mathbb{H}^3$, while in boundary normal coordinates, one uses that the second fundamental form of $\partial M_T \subset M$ has bounded derivatives, as it is the quotient of a horosphere or of a metric neighborhood of a geodesic with radius bounded below by $\epsilon/4$. □

For all four topological types of components of $M \setminus M_T$, the first cohomology of the boundary surjects, so using Mayer–Vietoris sequence we see that

$$b_1(M) \leq b_1(M_T).$$

Therefore, to prove (7.18.1) it suffices to estimate the Betti numbers of $(M_n)_T$.

Let Δ be the Laplacian operator on differential 1-forms on M, and let $e^{-t\Delta}(x,x)$ be the corresponding heat kernel. We also let Δ_1^T be the Laplacian operator on differential 1-forms on M_T with absolute boundary conditions, and

[11] Isometries of $\mathbb{H}^3$ that translate along an axis c are compositions of pure translations and 2-dimensional rotations in the orthogonal direction. So for a given $r > 0$, a loxodromic isometry of $\mathbb{H}^3$ with geodesic axis c acts with the same translation distance on every point of the boundary $\partial N_r(c)$ of the r-neighborhood around c. This is not true in higher dimensions, since the rotational part of an isometry can be more complicated, and in fact the components of $M \setminus M_T$ that are (nonmetric) neighborhoods of closed geodesics may not have smooth boundary.

denote by $e^{-t\Delta_1^T}(x,x)$ its integral kernel. It follows from [50, Th. 6.1] that

$$b_1(M_T) = \lim_{t\to\infty} \operatorname{Tr} e^{-t\Delta_1^T} = \lim_{t\to\infty} \int_{M_{\geq\varepsilon}} tr\, e^{-t\Delta_1^T}(x,x)\, dx; \tag{7.19.1}$$

note that since $\operatorname{Tr} e^{-t\Delta_1} = \sum_i e^{-t\lambda_i}$, where λ_i are the eigenvalues of Δ_1^T, the expression above is monotone decreasing in t, so the limit exists.

Recall that $\beta_1^{(2)}(\mathbb{H}^3) = \lim_{t\to\infty} \operatorname{tr} e^{-t\Delta_1^{(2)}}(\tilde{x},\tilde{x})$, where $e^{-t\Delta_1^{(2)}}(\tilde{x},\tilde{x})$ is the L^2-heat kernel of $\mathbb{H}^3$. In light of (7.19.1), it suffices to fix $t>0$ and show that

$$\limsup_{n\to\infty} \frac{1}{\operatorname{vol}(M_n)} \int_{(M_n)_T} \operatorname{tr} e^{-t\Delta_1^T}(x,x)\, dx \leq \operatorname{tr} e^{-t\Delta_1^{(2)}}(\tilde{x},\tilde{x}) \tag{7.19.2}$$

for some (arbitrary) $\tilde{x}\in\mathbb{H}^3$. Then taking $t\to\infty$ proves (7.18.1).

Fix some large $R \gg 1 > \epsilon$, and consider the subset $(M_n)_{\geq R} \subset (M_n)_T$. Then the boundary of $(M_n)_T$ is uniformly far from $(M_n)_{\geq R}$, and by a theorem of Lück and Schick [84, Th. 2.26] we have that for all $x\in(M_n)_{\geq R}$,

$$||e^{-t\Delta_1^T}(x,x) - e^{-t\Delta_1}(x,x)|| \leq C(t,R),$$

where $C(t,R)\to 0$ as $R\to\infty$. (Note: although their statement assumes that M_n has bounded geometry, which in this case means the global injectivity radius $\operatorname{InjRad} M_n > 0$, it suffices in their proof to assume a lower injectivity radius bound on $(M_n)_{\geq R}$, which is automatic.) So, by Lemma 7.4, for all $x\in(M_n)_{\geq R}$,

$$||e^{-t\Delta_1^T}(x,x) - e^{-t\Delta_1^{(2)}}(\tilde{x},\tilde{x})|| \leq C(t,R) + C(t)R^{-3} = C'(t,R),$$

where again $C'(t,R)\to 0$ as $R\to\infty$. Hence, for all n, we have the average value

$$\frac{1}{\operatorname{vol}(M_n)_{\geq R}} \int_{(M_n)_{\geq R}} \operatorname{tr} e^{-t\Delta_1^T}(x,x)\, dx \leq \operatorname{tr} e^{-t\Delta_1^{(2)}}(\tilde{x},\tilde{x}) + C'(t,R). \tag{7.19.3}$$

Next, if $x\in D_n = (M_n)_T \setminus (M_n)_{\geq R}$, we have by [84, Th. 2.35] that

$$||e^{-t\Delta_1^T}(x,x)|| \leq C(t),$$

since by Lemma 7.19 the manifold with boundary $(M_n)_T$ has bounded geometry, in the sense of [84, Def. 2.24]. So, we also have the average value

$$\frac{1}{\operatorname{vol} D_n} \int_{D_n} \operatorname{tr} e^{-t\Delta_1^T}(x,x)\, dx \leq C(t). \tag{7.19.4}$$

Combining (7.19.3) and (7.19.4), we obtain that for all n,

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{\mathrm{vol}(M_n)_T}\int_{(M_n)_T}&\operatorname{tr} e^{-t\Delta_1^T}(x,x)\,dx\\ &\le \frac{\mathrm{vol}\,(M_n)_{\ge R}}{\mathrm{vol}\,M_n}\left(\operatorname{tr} e^{-t\Delta_1^{(2)}}(\tilde{x},\tilde{x})+C'(t,R)\right)+\frac{\mathrm{vol}\,D_n}{\mathrm{vol}M_n}C(t)\\ &\le \operatorname{tr} e^{-t\Delta_1^{(2)}}(\tilde{x},\tilde{x})+C'(t,R)+\frac{\mathrm{vol}\,D_n}{\mathrm{vol}M_n}C(t).\end{aligned}$$

For a fixed R, by first letting $n\to\infty$, we deduce from BS-convergence that the last term vanishes in the limit. Finally, by sending $R\to\infty$, the term $C'(t,R)$ disappears. □

8. **Growth of torsion**

In this last section we consider only those $X=G/K$ for which all $\beta_k^{(2)}(X)$ vanish. It is then natural to consider the secondary invariant given by the L^2-torsion. We first review its definition and then consider the corresponding approximation problems. We continue with the notation of the preceeding sections. In particular, we let Γ be a cocompact torsion-free subgroup of G and let $M=\Gamma\backslash X$.

8.1. L^2- *and analytic torsion.* We will work in the setting of [24]: here we will be as brief as possible concerning definitions, etc., and refer to that paper for all details.

Given a finite-dimensional representation ρ of $G_{\mathbb{C}}$ on a vector space E one can construct a canonical G-equivariant Hermitian bundle E_ρ on X with fiber E. The space of square-integrable k-forms with coefficients in E_ρ is then endowed with a Laplacian $\Delta_k^{(2)}$ and associated heat kernels $e^{-t\Delta_k^{(2)}(\rho)}$ which are bounded operators given by convolution with a G-equivariant kernel $e^{-t\Delta_k^{(2)}(\rho)}(x,y)$ (a section of a bundle over $X\times X$). The trace $\operatorname{tr} e^{-t\Delta_k^{(2)}(\rho)}(x,x)$ does not depend on $x\in X$. Let $\Gamma(s)$ denote the Euler Gamma-function; the determinant $\det\Delta_k^{(2)}$ is then defined by

$$\begin{aligned}\log\det\Delta_k^{(2)}=\frac{d}{ds}\Big|_{s=0}&\left(\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\operatorname{tr} e^{-t\Delta_k^{(2)}}(x,x)dt\right)\\ &+\int_1^{+\infty} t^{-1}\operatorname{tr} e^{-t\Delta_k^{(2)}}(x,x)dt\end{aligned}$$

(see [85, Def. 3.128] for a justification) and the L^2-torsion $t_X^{(2)}(\rho)$ is defined by

$$t_X^{(2)}(\rho)=\frac{1}{2}\sum_{k\ge 0}(-1)^k k\log\det\Delta_k^{(2)}. \tag{8.1.1}$$

The bundle E_ρ descends to a bundle V on M, with Laplacians Δ_k and heat kernels $e^{-t\Delta_k}$; similar to the L^2-case one can define determinants of the Δ_k and analytic torsion $T_M(\rho)$. We raise the following question/conjecture.

8.2. Conjecture. *Let (M_n) be a* uniformly discrete *sequence of compact X-manifolds that* BS*-converges to X. Then we have*

$$\frac{\log T_{M_n}(\rho)}{\mathrm{vol}(M_n)} \to t_X^{(2)}(\rho).$$

We note that $t_X^{(2)}(\rho)$ is nonzero if and only if $\delta(G) = 1$, i.e., if G is one of the groups $\mathrm{SL}_2(\mathbb{C})$, $\mathrm{SL}_3(\mathbb{R}), \mathrm{SO}_{n,m}$, nm odd. In principle one can compute an explicit value of $t_X^{(2)}$ for all G and ρ; see [24, §5]. When $G = \mathrm{SO}_{2p+1,1}$, the space X is the real hyperbolic space $\mathbb{H}^{2p+1}$. For trivial ρ [85, Th. 3.152], we have

$$t_{\mathbb{H}^3}^{(2)} = -\frac{1}{6\pi},\ t_{\mathbb{H}^5}^{(2)} = \frac{31}{45\pi^2}, \dots .$$

8.3. *Strongly acyclic coefficients.* The representation ρ is said to be *strongly acyclic* if there is a constant η such that for every cocompact $\Gamma \subset G$ and for every k, the spectrum of the Laplace operator Δ_k on $\Gamma\backslash X$ is contained in $[\eta, +\infty[$. (In particular, this implies that $H^*(M; V) = 0$.) When ρ is strongly acyclic, Conjecture 8.2 was proven for normal chains in [24, Th. 4.5]. The proof of loc. cit. adapts immediately to the setting of Benjamini–Schramm convergence, simply by replacing the main lemma there by Lemma 7.4. Thus we obtain

8.4. Theorem. *Assume that $\rho : G \to \mathrm{GL}(E)$ is strongly acyclic. Let (M_n) be a uniformly discrete sequence of compact X-manifolds that* BS*-converges toward X. Then we have*

$$\frac{\log(T_{M_n}(\rho))}{\mathrm{vol}(M_n)} \to t_X^{(2)}(\rho).$$

8.5. *Example.* Given any orientable compact hyperbolic 3-manifold $M = \Gamma\backslash\mathbb{H}^3$, we can consider the discrete faithful $\mathrm{SL}_2(\mathbb{C})$-representation $\alpha_{\mathrm{can}} : \Gamma \hookrightarrow \mathrm{SL}_2(\mathbb{C})$. It is strongly acyclic. (See Example (3) of [24, §5.9.3] with $(p, q) = (1, 0)$.) In particular, the corresponding twisted chain complex

$$C_*(\widetilde{M}) \otimes_{\mathbb{Z}[\Gamma]} \mathbb{C}^2$$

is acyclic and it follows that the corresponding Reidemeister torsion $\tau(M, \alpha_{\mathrm{can}}) \in \mathbb{R}^*$ is defined. According to the Cheeger–Müller theorem extended to unimodular representation by Müller [89], we have $T_M(\rho) = |\tau(M, \alpha_{\mathrm{can}})|$, and Theorem 1.13 follows from Theorem 8.4.

8.6. *Torsion homology.* In this (largely speculative) section we suppose that ρ is trivial. According to the Cheeger–Müller theorem [35], [89], the analytic torsion T_M decomposes as a product of

$$\prod_{k=0}^{\dim X} (-1)^{k+1} |H_k(M, \mathbb{Z})_{\text{tors}}|$$

by a so-called *regulator*; see [24, eq. (2.2.4)]. This relates Conjecture 8.2 to the following question: Let M_n be a sequence of compact X-manifolds that BS-converges to X. Do we have

$$\frac{\log |H_k(M_n, \mathbb{Z})_{\text{tors}}|}{\text{vol}(M_n)} \to \begin{cases} |t_X^{(2)}| & \text{if } k = \frac{\dim X - 1}{2}, \\ 0 & \text{otherwise} \end{cases}$$

for every $k \leq \dim(X)$?

To avoid discussing the growth of $H_k(M_n)$ for $k \neq (\dim(X) - 1)/2$, here we will restrict to the case $X = \mathbb{H}^3$ so that $H_k(M_n)$ are torsion-free if $k \neq \frac{\dim X - 1}{2} = 1$ and $t_X^{(2)} = -(6\pi)^{-1}$. In this setting there are extensive computations by Şengun [43] for covers of a fixed manifold that suggest the answer to the question above is negative, indicating that the contribution of the regulator to the limit in Conjecture 8.2 should be nonzero in general. However the same computations suggest that this is not the case when considering only congruence covers of an arithmetic manifold. See [21] for a detailed discussion on regulators and the differences between congruence and noncongruence covers.

The following result of Brock–Dunfield [31] finally shows that Conjecture 8.2 cannot hold for general (nonuniformly discrete) sequences.

8.7. Theorem. *There exists a sequence of hyperbolic* integer *homology* 3*-spheres that* BS*-converges toward the hyperbolic* 3*-space.*

According to the Cheeger-Müller theorem, if M is a homology sphere then $T_M = 1$. Thus the theorem above provides us with a sequence M_n that converges to $\mathbb{H}^3$ in the Benjamini–Schramm sense but such that the conclusion of the conjecture is violated in an extreme way.

8.8. *Knot exteriors.* Given a hyperbolic knot, Dunfield, Friedl and Jackson [51] have introduced an invariant $\mathcal{T}_K(t) \in \mathbb{C}[t^{\pm 1}]$ that is defined as the normalized twisted Alexander polynomial of K corresponding to the discrete and faithful $\text{SL}_2(\mathbb{C})$-representation of the knot group. It follows from [77, Th. 4] that the following holds. Let M_n be the n-th cyclic ramified cover of $\mathbb{S}^3$ along K. Then for n large enough, M_n is hyperbolic and

$$\lim_{n \to +\infty} \frac{1}{n} \log |\tau(M_n, \alpha_{\text{can}})| = -\log m(\mathcal{T}_K), \tag{8.8.1}$$

where m is the exponential Mahler measure. On the other hand, Friedl and Jackson [57] produce computations that suggest that $\log m(\mathcal{T}_K)$ correlates strongly with $\mathrm{vol}(K)$: as $\mathrm{vol}(K)$ tends to infinity, the ratio $\log m(\mathcal{T}_K)/\mathrm{vol}(K)$ seems to tend to a constant ≈ 0.29.

Let $\overline{M}_n$ be the hyperbolic orbifold with underlying space $\mathbb{S}^3$ and n-th cyclic singularity along K. Then M_n is a regular n-sheeted cover of $\overline{M}_n$. Now recalling that $\overline{M}_n$ BS-converges toward $\mathbb{S}^3 - K$ (and, in particular, that $\mathrm{vol}(\overline{M}_n) \to \mathrm{vol}(K)$) as n tends to infinity and that $11/12\pi \approx 0.29$, in view of Theorem 1.13 and equation (8.8.1) it is natural to ask the following question (compare [97]):

Question. Let (K_n) be a sequence of hyperbolic knots in $\mathbb{S}^3$ such that $\mathrm{vol}(K_n) \to +\infty$. Can it happen that the sequence of finite volume hyperbolic manifolds $\mathbb{S}^3 - K_n$ BS-converge toward $\mathbb{H}^3$?

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