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A simple procedure for construction of the orthonormal basis vectors of irreducible representations of $O(5)$ in the $O_T(3) \otimes O_{\mathcal{N}}(2)$ basis

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Abstract

A simple and effective algebraic isospin projection procedure for constructing orthonormal basis vectors of irreducible representations of $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ from those in the canonical $O(5) \supset SU_\Lambda(2) \otimes SU_I(2)$ basis is outlined. The expansion coefficients are components of null space vectors of the projection matrix with four nonzero elements in each row in general. Explicit formulae for evaluating $O_T(3)$-reduced matrix elements of $O(5)$ generators are derived.

Keywords: Isovector pairing interaction; Proton–neutron quasi-spin group; Dynamical symmetry

1. Introduction

It is well known that the proton–neutron quasi-spin group generated by an $O(5)$ algebra is very useful in dealing with nucleon pairing problems in a shell model framework [1–8]. Due to its importance in the nuclear spectroscopy, irreducible representations (irreps) of $O(5)$ have been studied in various ways. The most natural basis for irreps of $O(5)$ may be the branching multiplicity-free canonical one with $O(5) \supset O(4)$, where $O(4)$ is locally isomorphic to $SU_\Lambda(2) \otimes SU_I(2)$, of which the construction of the basis vectors was presented in [9–11]. The

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matrix representations of $O(5) \supset SU_\Lambda(2) \otimes SU_I(2)$ were provided in [9–12]. Since the isospin is approximately conserved in the charge-independent isovector pairing problem, it is more convenient to adopt the non-canonical $O(5) \supset O_T(3) \otimes O_\mathcal{N}(2)$ basis for this case, where $O_T(3)$ is the isospin group, and $O_\mathcal{N}(2) \sim U_\mathcal{N}(1)$ is related with the number of nucleons in the system. The main problem is the reduction $O(5) \downarrow O_T(3) \otimes O_\mathcal{N}(2)$ is no longer branching multiplicity-free in general. Basis vectors of $O(5)$ irreps in the $O(5) \supset O_T(3) \otimes O_\mathcal{N}(2)$ basis can be either expanded in terms of those in the $O(5) \supset SU_\Lambda(2) \otimes SU_I(2)$ or constructed by using tensor coupling methods directly, for which various attempts were made [9,13–18]. A recent survey on the subject with relevant references is provided in [19,20]. Though various procedures for the construction of basis vectors of $O(5)$ irreps in the $O(5) \supset O_T(3) \otimes O_\mathcal{N}(2)$ were provided in these works, only cases up to the branching multiplicity three were obtained explicitly in the past. Moreover, though there are closed expressions of the expansion coefficients (overlaps) [16] of the basis vectors of $O(5) \supset O_T(3) \otimes O_\mathcal{N}(2)$ in terms of those of $O(5) \supset SU_\Lambda(2) \otimes SU_I(2)$ for any irrep of $O(5)$, a triple sum is involved. Especially, the basis vectors of $O(5) \supset O_T(3) \otimes O_\mathcal{N}(2)$ obtained in all previous works [9,13–18] are non-orthogonal with respect to the branching multiplicity label, of which direct computation will be CPU time consuming.

Very recently, we have proposed a simple and effective angular momentum projection procedure to construct the non-canonical $O(5) \supset O(3)$ basis vectors from those in the $O(5) \supset O_1(3) \otimes U(1)$ basis for the symmetric irreps of $O(5)$ based on the group chain $U(5) \supset U(3) \otimes U(2)$ [21]. The same technique has also been used to construct basis vectors of $SU(3) \supset SO(3) \supset SO(2)$ from those of $U(3) \supset U(2) \supset U(1)$ for any irrep of $SU(3)$ [22]. It will be shown in this paper that the technique is also efficient for construction of orthonormal basis vectors of $O(5)$ irreps in the $O(5) \supset O_T(3) \otimes O_\mathcal{N}(2)$ basis from those of $O(5) \supset SU_\Lambda(2) \otimes SU_I(2)$.

In Sec. 2, the relevant canonical and non-canonical basis of $O(5)$ will be briefly reviewed. In Sec. 3, based on the results shown in Sec. 2, the basis vectors for irreps of $O(5)$ in the non-canonical $O(5) \supset O_T(3) \otimes O_\mathcal{N}(2)$ basis will be expanded in terms of those in the canonical $O(5) \supset SU_\Lambda(2) \otimes SU_I(2)$ basis, from which a four-term relation among the expansion coefficients are explicitly derived. In Sec. 4, explicit formulae for evaluating the $O_T(3)$-reduced matrix elements of $O(5)$ generators are derived. In Sec. 5, these formulae are used to evaluate the known eigenvalues of the pure isovector pairing Hamiltonian to check the validity of the results shown in Sec. 3 and 4.

2. $O(5)$ in the $SU_\Lambda(2) \otimes SU_I(2)$ and the $O_T(3) \otimes O_\mathcal{N}(2)$ basis

The generators of $O(5)$ can be expressed by linear combinations of a set of operators $\{E_{ij}\}$ $(1 \le i, j \le 4)$ satisfying

$$[E_{ij},\ E_{lk}] = \delta_{jl} E_{ik} - \delta_{ik} E_{lj}, \quad \left(E_{ij}\right)^\dagger = E_{ji}. \tag{1}$$

In the $SU_\Lambda(2) \otimes SU_I(2)$ basis, the generators of $O(5)$ may be expressed as

$$\begin{aligned}
&\nu_+ = E_{12}, \quad \nu_- = E_{21}, \quad \nu_0 = \tfrac{1}{2}(E_{11} - E_{22}),\\
&\tau_+ = E_{34}, \quad \tau_- = E_{43}, \quad \tau_0 = \tfrac{1}{2}(E_{33} - E_{44}),\\
&U_{\frac{1}{2}\frac{1}{2}} = \sqrt{\tfrac{1}{2}}(E_{14} + E_{32}), \quad U_{\frac{1}{2}-\frac{1}{2}} = \sqrt{\tfrac{1}{2}}(E_{42} - E_{13}),\\
&U_{-\frac{1}{2}\frac{1}{2}} = \sqrt{\tfrac{1}{2}}(E_{24} - E_{31}), \quad U_{-\frac{1}{2}-\frac{1}{2}} = -\sqrt{\tfrac{1}{2}}(E_{41} + E_{23}),
\end{aligned} \tag{2}$$

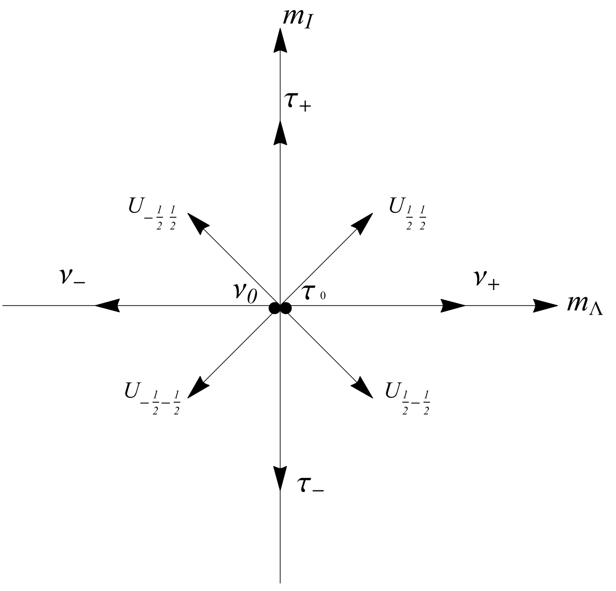

Fig. 1. Root diagram of $O(5)$ in the $SU_\Lambda(2) \otimes SU_I(2)$ basis, where m_Λ and m_I are the quantum number of ν_0 and that of τ_0, respectively, which is also the same diagram of $O(5)$ in the $O_T(3) \otimes O_{\mathcal{N}}(2)$ basis with the correspondence: $\nu_+ = A_1^\dagger$, $\nu_- = A_1$, $\tau_+ = A_{-1}^\dagger$, $\tau_- = A_{-1}$, $U_{\frac{1}{2}\frac{1}{2}} = A_0^\dagger$, $U_{-\frac{1}{2}-\frac{1}{2}} = -A_0$, $U_{-\frac{1}{2}\frac{1}{2}} = -\sqrt{\frac{1}{2}}T_-$, $U_{\frac{1}{2}-\frac{1}{2}} = -\sqrt{\frac{1}{2}}T_+$, $T_0 = \nu_0 - \tau_0$, and $\hat{\mathcal{N}} = \nu_0 + \tau_0$ shown in (2) and (10).

where $\{\nu_+, \nu_-, \nu_0\}$ and $\{\tau_+, \tau_-, \tau_0\}$ generate the subgroup $SU_\Lambda(2)$ and $SU_I(2)$, respectively, and the double tensor operators $\{U_{\mu\rho}\}$ satisfy the following Hermitian conjugation relation:

$$\left(U_{\mu\rho}\right)^\dagger = (-)^{\mu+\rho} U_{-\mu-\rho}, \tag{3}$$

which satisfy the following commutation relations:

$$\begin{aligned}
&[\nu_0,\ \nu_\pm] = \pm\nu_\pm, \quad [\nu_+,\ \nu_-] = 2\nu_0,\\
&[\tau_0,\ \tau_\pm] = \pm\tau_\pm, \quad [\tau_+,\ \tau_-] = 2\tau_0,\\
&[\nu_0,\ U_{\mu\rho}] = \mu U_{\mu\rho}, \quad [\tau_0,\ U_{\mu\rho}] = \rho U_{\mu\rho},\\
&[\nu_\pm,\ U_{\mu\rho}] = \sqrt{(\tfrac{1}{2} \mp \mu)(\tfrac{1}{2} \pm \mu + 1)}\,U_{\mu\pm 1\rho}, \quad [\tau_\pm,\ U_{\mu\rho}] = \sqrt{(\tfrac{1}{2} \mp \rho)(\tfrac{1}{2} \pm \rho + 1)}\,U_{\mu\rho\pm 1},\\
&[U_{\pm\frac{1}{2}\frac{1}{2}},\ U_{\pm\frac{1}{2}-\frac{1}{2}}] = \pm\nu_\pm,\ [U_{\frac{1}{2}\pm\frac{1}{2}},\ U_{-\frac{1}{2}\pm\frac{1}{2}}] = \pm\tau_\pm,\ [U_{\pm\frac{1}{2}\frac{1}{2}},\ U_{\mp\frac{1}{2}-\frac{1}{2}}] = -(\nu_0 \pm \tau_0).
\end{aligned} \tag{4}$$

The root diagram of $O(5)$ in the $SU_\Lambda(2) \otimes SU_I(2)$ basis is illustrated in Fig. 1.

An irrep of $O(5)$ may be denoted by (v_1, v_2) with $v_1 \geq v_2 \geq 0$, where v_1 and v_2 should be positive integers or positive half-integers simultaneously. Since $O(5) \downarrow O(4)$ is simply reducible and $O(4)$ is locally isomorphic to $SU_\Lambda(2) \otimes SU_I(2)$, the orthonormal basis vectors of $O(5) \supset SU_\Lambda(2) \otimes SU_I(2) \supset U_\Lambda(1) \otimes U_I(1)$ may be labeled as

$$\left| \begin{matrix} & (v_1, v_2) & \\ \Lambda = \frac{1}{2}(u_1 + u_2), & & I = \frac{1}{2}(u_1 - u_2) \\ m_\Lambda, & & m_I \end{matrix} \right\rangle, \tag{5}$$

where (u_1, u_2) labels possible irrep of $O(4)$ within the given irrep (v_1, v_2) of $O(5)$ restricted by $v_2 \le u_1 \le v_1$ and $-v_2 \le u_2 \le v_2$. Due to the fact that $O(5) \downarrow SU_\Lambda(2) \otimes SU_I(2)$ is simply reducible, $O(5) \supset SU_\Lambda(2) \otimes SU_I(2) \supset U_\Lambda(1) \otimes U_I(1)$ is called the canonical basis of $O(5)$.

For a given irrep (v_1, v_2) of $O(5)$, the matrix representations of $O(5) \supset SU_\Lambda(2) \otimes SU_I(2)$ are well-known with the $SU_\Lambda(2) \otimes SU_I(2)$ reduced matrix elements given by [9,12]

$$\begin{aligned}
&\left\langle \begin{matrix} \Lambda - \frac{1}{2} \\ I + \frac{1}{2} \end{matrix} \middle\| U \middle\| \begin{matrix} \Lambda \\ I \end{matrix} \right\rangle \\
&= -\left[\frac{(v_1 - I + \Lambda + 1)(v_2 - I + \Lambda)(v_1 - \Lambda + I + 2)(v_2 - \Lambda + I + 1)}{2(2\Lambda)(2I + 2)} \right]^{\frac{1}{2}}, \\
&\left\langle \begin{matrix} \Lambda - \frac{1}{2} \\ I - \frac{1}{2} \end{matrix} \middle\| U \middle\| \begin{matrix} \Lambda \\ I \end{matrix} \right\rangle \\
&= \left[\frac{(v_1 + I + \Lambda + 2)(v_2 + I + \Lambda + 1)(v_1 - \Lambda - I + 1)(\Lambda + I - v_2)}{2(2\Lambda)(2I)} \right]^{\frac{1}{2}},
\end{aligned} \tag{6}$$

and

$$\left\langle \begin{matrix} \Lambda \\ I \end{matrix} \middle\| U \middle\| \begin{matrix} \Lambda' \\ I' \end{matrix} \right\rangle = \left[\frac{(2I' + 1)(2\Lambda' + 1)}{(2I + 1)(2\Lambda + 1)} \right]^{\frac{1}{2}} (-)^{I' - I + \Lambda' - \Lambda} \left\langle \begin{matrix} \Lambda' \\ I' \end{matrix} \middle\| U \middle\| \begin{matrix} \Lambda \\ I \end{matrix} \right\rangle. \tag{7}$$

According to the branching rule of $O(5) \downarrow O(4)$, the branching rule of $O(5) \downarrow SU_\Lambda(2) \otimes SU_I(2)$ can be expressed as

$$\begin{aligned}
&O(5) \quad \downarrow \qquad\qquad SU_\Lambda(2) \quad \otimes \quad SU_I(2) \\
&(v_1, v_2) \downarrow \bigoplus_{q=0,\ p=0}^{v_1 - v_2,\ 2v_2} \left(\Lambda = \tfrac{1}{2}(v_1 + v_2 - p - q),\ \ I = \tfrac{1}{2}(v_1 - v_2 + p - q) \right),
\end{aligned} \tag{8}$$

with which one gets the following sum rule:

$$\begin{aligned}
\mathrm{Dim}[O(5), (v_1, v_2)] &= \sum_{q=0}^{v_1 - v_2} \sum_{p=0}^{2v_2} (v_1 + v_2 - p - q + 1)(v_1 - v_2 + p - q + 1) \\
&= \tfrac{1}{6}(2v_1 + 3)(v_1 - v_2 + 1)(v_1 + v_2 + 2)(2v_2 + 1),
\end{aligned} \tag{9}$$

where $\mathrm{Dim}[O(5), (v_1, v_2)]$ is the dimension of the $O(5)$ irrep (v_1, v_2) with $v_1 \ge v_2 \ge 0$.

Alternatively, after a linear transformation, the generators of $O(5)$ in the $O(5) \supset O_T(3) \times O_{\mathcal{N}}(2)$ basis may be expressed as

$$\begin{aligned}
&A_1^\dagger = E_{12} = \nu_+, \quad A_{-1}^\dagger = E_{34} = \tau_+, \\
&A_1 = E_{21} = \nu_-, \quad A_{-1} = E_{43} = \tau_-, \\
&A_0^\dagger = \sqrt{\tfrac{1}{2}}(E_{14} + E_{32}) = U_{\frac{1}{2}\frac{1}{2}}, \quad A_0 = \sqrt{\tfrac{1}{2}}(E_{41} + E_{23}) = -U_{-\frac{1}{2}-\frac{1}{2}}, \\
&T_+ = E_{13} - E_{42} = -\sqrt{2}U_{\frac{1}{2}-\frac{1}{2}}, \quad T_- = E_{31} - E_{24} = -\sqrt{2}U_{-\frac{1}{2}\frac{1}{2}}, \\
&T_0 = \tfrac{1}{2}(E_{11} - E_{22} - E_{33} + E_{44}) = \nu_0 - \tau_0, \\
&\hat{\mathcal{N}} = \tfrac{1}{2}(E_{11} - E_{22} + E_{33} - E_{44}) = \nu_0 + \tau_0,
\end{aligned} \tag{10}$$

where $\{T_+, T_-, T_0\}$ generate the subgroup $O_T(3)$, and $\hat{\mathcal{N}}$ generates the $O_{\mathcal{N}}(2)$. $\hat{\mathcal{N}} = \frac{\hat{n}}{2} - \Omega$, where $\Omega = \sum_j (j + 1/2)$, in which the sum runs over all single-particle orbits considered, and $\hat{n}$ is the total number operator of valence nucleons, which is used in the isovector pairing model [1–8]. Additionally, $\{\nu_+ = A^\dagger_{+1},\ \nu_- = A_1,\ \nu_0 = \hat{n}_\pi/2 - \Omega/2\}$ and $\{\tau_+ = A^\dagger_{-1},\ \tau_- = A_{-1},\ \tau_0 = \hat{n}_\nu/2 - \Omega/2\}$, where $\hat{n}_\pi$ and $\hat{n}_\nu$ are valence neutron and proton number operator, respectively, generate the $SU_\Lambda(2) \otimes SU_I(2)$ related to the quasispin of protons and neutrons with $\Lambda = (\Omega - v_\pi)/2$ and $I = (\Omega - v_\nu)/2$, where v_π and v_ν are proton and neutron seniority number, respectively.

The Casimir (invariant) operator of $O(5)$ can be expressed as

$$\begin{aligned} C_2(O(5)) &= 2\boldsymbol{\nu}\cdot\boldsymbol{\nu} + 2\boldsymbol{\tau}\cdot\boldsymbol{\tau} + \sum_{\mu\rho}(-1)^{\mu+\rho}U_{\mu\rho}U_{-\mu-\rho} \\ &= \sum_{\mu}\left(A^\dagger_\mu A_\mu + A_\mu A^\dagger_\mu\right) + \mathbf{T}\cdot\mathbf{T} + \hat{\mathcal{N}}^2, \end{aligned} \tag{11}$$

where $\mathbf{l}\cdot\mathbf{l} = \frac{1}{2}(l_+l_- + l_-l_+) + l_0^2$. Eigenvalues of $C_2(O(5))$, $\boldsymbol{\nu}\cdot\boldsymbol{\nu}$, and $\boldsymbol{\tau}\cdot\boldsymbol{\tau}$ under (5) are given by

$$\begin{aligned} &\begin{pmatrix} C_2(O(5)) \\ \boldsymbol{\nu}\cdot\boldsymbol{\nu} \\ \boldsymbol{\tau}\cdot\boldsymbol{\tau} \end{pmatrix} \left| \begin{matrix} & (v_1, v_2) & \\ \Lambda = \frac{1}{2}(u_1+u_2), & & I = \frac{1}{2}(u_1-u_2) \\ m_\Lambda, & & m_I \end{matrix} \right\rangle = \\ &\begin{pmatrix} v_1(v_1+3) + v_2(v_2+1) \\ \Lambda(\Lambda+1) \\ I(I+1) \end{pmatrix} \left| \begin{matrix} & (v_1, v_2) & \\ \Lambda = \frac{1}{2}(u_1+u_2), & & I = \frac{1}{2}(u_1-u_2) \\ m_\Lambda, & & m_I \end{matrix} \right\rangle, \end{aligned} \tag{12}$$

where $u_1 = v_1 - q$ and $u_2 = v_2 - p$ with $p = 0, 1, \cdots, 2v_2$ and $q = 0, 1, \cdots, v_1 - v_2$.

3. The basis vectors of $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$

As can be observed from (10), the basis vector (5) is also an eigenstate of T_0 and $\hat{\mathcal{N}}$ with eigenvalues

$$M_T = m_\Lambda - m_I, \quad \mathcal{N} = m_\Lambda + m_I. \tag{13}$$

For a given irrep (v_1, v_2) of $O(5)$, all possible basis vectors of $O(5) \supset SU_\Lambda(2) \otimes SU_I(2) \supset U_\Lambda(1) \otimes U_I(1)$ shown in (5) restricted by the conditions (13) form a complete set for the fixed M_T and $\mathcal{N}$. Therefore, the basis vectors of $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ can be expanded in terms of them with the restriction on the quantum numbers $m_\Lambda = \frac{1}{2}(\mathcal{N} + M_T)$ and $m_I = \frac{1}{2}(\mathcal{N} - M_T)$. The possible basis vectors of $O(5) \supset SU_\Lambda(2) \otimes SU_I(2) \supset U_\Lambda(1) \otimes U_I(1)$ spanning the subspace with $m_\Lambda = \frac{1}{2}(\mathcal{N} + M_T)$ and $m_I = \frac{1}{2}(\mathcal{N} - M_T)$ can be illustrated in the weight projection diagram for the irrep $(3,\ 1)$ of $O(5)$ as an example shown in Fig. 2. In this example, the dimension of (3,1) irrep of $O(5)$ is $\text{Dim}[O(5), (3, 1)] = 81$, which involves $(\Lambda = 2, I = 1)$, $(\Lambda = 1, I = 2)$, $(\Lambda = 1, I = 1)$, $(\Lambda = 1, I = 0)$, $(\Lambda = 0, I = 1)$, $(\Lambda = \frac{3}{2}, I = \frac{3}{2})$, $(\Lambda = \frac{1}{2}, I = \frac{3}{2})$, $(\Lambda = \frac{3}{2}, I = \frac{1}{2})$, and $(\Lambda = \frac{1}{2}, I = \frac{1}{2})$ irreps of $SU_\Lambda(2) \otimes SU_I(2)$. In Fig. 2, the degeneracy equals to the number of possible (Λ, I) pairs determined from the branching rule of $O(5) \downarrow SU_\Lambda(2) \otimes SU_I(2)$ with the same m_Λ and m_I values, which thus equals to the number of $O(5) \supset SU_\Lambda(2) \otimes SU_I(2) \supset U_\Lambda(1) \otimes U_I(1)$ basis vectors involved in the projection with fixed M_T and $\mathcal{N}$. For example, as shown in Fig. 2, there should be 5 terms involved in the projection for $M_T = 0$ and $\mathcal{N} = 0$, while the number of terms involved for $m_T = 1$ and $\mathcal{N} = 3$ is 2.

Similar to the $SU(3)$ case [22], in constructing the basis vectors of $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ for the irrep (v_1, v_2) of O(5) with fixed $\mathcal{N}$, there is a freedom to choose a specific basis vector

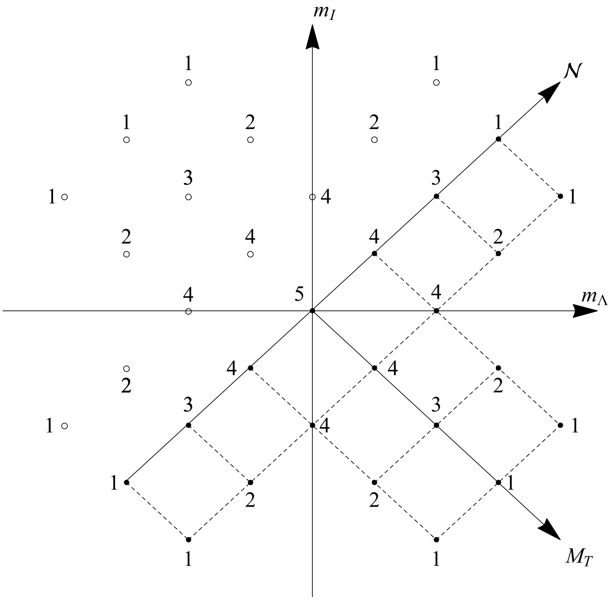

Fig. 2. The weight projection diagram for $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ for the irrep $(v_1, v_2) = (3, 1)$, where the solid dots and open circles are the corresponding $O(5)$ weights in the $O(5) \supset SU_\Lambda(2) \otimes SU_I(2) \supset U_\Lambda(1) \otimes U_I(1)$ basis labeled by the quantum numbers m_Λ and m_I with the corresponding degeneracy (the number near the dots or circles) clearly shown, in which only the weights denoted by the solid dots connected by the dashed lines are involved in the projection with fixed $M_T > 0$ for $M_T = 0, 1, \cdots, 3$ and $-3 \leq \mathcal{N} \leq 3$.

of $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ with isospin T and the quantum number of the third component of the isospin M_T. Practically, it is convenient to choose the highest or the lowest weight state of $O_T(3)$ with $M_T = T$ or $M_T = -T$. In this work, we choose the highest weight state of $O_T(3)$ with $M_T = T$ as a reference state with

$$\left| \begin{matrix} (v_1, v_2) \\ \zeta \; T = M_T, \mathcal{N} \end{matrix} \right\rangle, \tag{14}$$

where ζ is the multiplicity label needed in the reduction $(v_1, v_2) \downarrow (T, \mathcal{N})$ of $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$. Thus, (14) should satisfy

$$T_+ \left| \begin{matrix} (v_1, v_2) \\ \zeta \; T = M_T, \mathcal{N} \end{matrix} \right\rangle = 0. \tag{15}$$

Once the basis vector (14) for the highest weight state of $O_T(3)$ with $M_T = T$ is known, the basis vector of $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ for any M_T can be expressed in the standard way as

$$\left| \begin{matrix} (v_1, v_2) \\ \zeta \; T, M_T, \mathcal{N} \end{matrix} \right\rangle = \sqrt{\frac{(T + M_T)!}{(2T)!(T - M_T)!}} \, (T_-)^{T - M_T} \left| \begin{matrix} (v_1, v_2) \\ \zeta \; T, M_T = T, \mathcal{N} \end{matrix} \right\rangle, \tag{16}$$

where $T \geq 0$ should be satisfied.

In order to find all basis vectors of $O(5) \supset SU_\Lambda(2) \otimes SU_I(2)$ with fixed $M_T > 0$ and $\mathcal{N}$ in the irrep (v_1, v_2) of $O(5)$, one suffices to consider possible irreps (Λ, I) of $SU_\Lambda(2) \otimes SU_I(2)$ embedded in the canonical chain satisfying the condition (13) for this case. According to the restrictions $M_T = m_\Lambda - m_I$, $\mathcal{N} = m_\Lambda + m_I$, and the reduction rules shown in (8), we find that the following basis vectors are all possible within the $O(5)$ irrep (v_1, v_2) with $M_T \geq 0$ for fixed $\mathcal{N}$:

$$\left| \begin{array}{cc} & (v_1, v_2) \\ \Lambda, & I \\ \frac{1}{2}(\mathcal{N}+M_T), & \frac{1}{2}(\mathcal{N}-M_T) \end{array} \right\rangle \tag{17}$$

with the restrictions:

$$\tfrac{1}{2}|\mathcal{N}+M_T| \leq \Lambda \leq \tfrac{1}{2}(v_1+v_2), \quad \tfrac{1}{2}|\mathcal{N}-M_T| \leq I \leq \tfrac{1}{2}(v_1-v_2). \tag{18}$$

Hence, the basis vectors of $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ may be expanded in terms of (17) as

$$\left| \begin{array}{c} (v_1, v_2) \\ \zeta \; T = M_T, \mathcal{N} \end{array} \right\rangle = \sum_{q=0}^{v_1-v_2} \sum_{p=\mathrm{Max}[0,\; q-v_1+v_2+|\mathcal{N}-T|]}^{\mathrm{Min}[v_1+v_2-q-|\mathcal{N}+T|,\; 2v2]} c^{(\zeta)}_{p,q}$$
$$\times \left| \begin{array}{cc} & (v_1, v_2) \\ \Lambda = \frac{1}{2}(v_1+v_2-p-q), & I = \frac{1}{2}(v_1-v_2+p-q) \\ \frac{1}{2}(\mathcal{N}+T), & \frac{1}{2}(\mathcal{N}-T) \end{array} \right\rangle, \tag{19}$$

where the summations should also be restricted by the condition that $v_1 + v_2 - p - q - |\mathcal{N} + T|$ are even numbers, ζ is the multiplicity label needed in the reduction $(v_1, v_2) \downarrow (\mathcal{N}, T)$, and $\{c^{(\zeta)}_{pq} \equiv c^{(\zeta)}_{pq}((v_1, v_2), \mathcal{N}, T)\}$ are the expansion coefficients, which must satisfy

$$-\sqrt{\frac{1}{2}}T_+ \left| \begin{array}{c} (v_1, v_2) \\ \zeta \; T = M_T, \mathcal{N} \end{array} \right\rangle = U_{\frac{1}{2}-\frac{1}{2}} \left| \begin{array}{c} (v_1, v_2) \\ \zeta \; T = M_T, \mathcal{N} \end{array} \right\rangle = 0. \tag{20}$$

As shown in (19), the sum over possible values of p and q in the expansion is equivalent to expand the basis vector of $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ with fixed $\mathcal{N}$ and $M_T = T > 0$ in terms of the basis vectors of $O(5) \supset SU_\Lambda(2) \otimes SU_I(2) \supset U_\Lambda(1) \otimes U_I(1)$ corresponding to the degenerate weight states at fixed $\mathcal{N}$ and $M_T = T > 0$ as shown in Fig. 2. Since we choose the highest weight state of $O_T(3)$ with $M_T = T > 0$ as a reference state, only the degenerate weights on the lower right plane shown in Fig. 2 may be involved in the projection.

The action of $U_{\frac{1}{2}-\frac{1}{2}}$ onto the basis vector of $O(5) \supset SU_\Lambda(2) \otimes SU_I(2)$ shown in (17) useful for (20) can be summarized as follows:

$$U_{\frac{1}{2}-\frac{1}{2}} \left| \begin{array}{cc} & (v_1, v_2) \\ \Lambda, & I \\ \frac{1}{2}(\mathcal{N}+T), & \frac{1}{2}(\mathcal{N}-T) \end{array} \right\rangle =$$
$$\left\langle \begin{array}{cc} & (v_1, v_2) \\ \Lambda+\frac{1}{2}, & I+\frac{1}{2} \\ \frac{1}{2}(\mathcal{N}+T)+\frac{1}{2}, & \frac{1}{2}(\mathcal{N}-T)-\frac{1}{2} \end{array} \right| U_{\frac{1}{2}-\frac{1}{2}} \left| \begin{array}{cc} & (v_1, v_2) \\ \Lambda, & I \\ \frac{1}{2}(\mathcal{N}+T), & \frac{1}{2}(\mathcal{N}-T) \end{array} \right\rangle \left| \begin{array}{cc} & (v_1, v_2) \\ \Lambda+\frac{1}{2}, & I+\frac{1}{2} \\ \frac{1}{2}(\mathcal{N}+T)+\frac{1}{2}, & \frac{1}{2}(\mathcal{N}-T)-\frac{1}{2} \end{array} \right\rangle$$
$$+\left\langle \begin{array}{cc} & (v_1, v_2) \\ \Lambda+\frac{1}{2}, & I-\frac{1}{2} \\ \frac{1}{2}(\mathcal{N}+T)+\frac{1}{2}, & \frac{1}{2}(\mathcal{N}-T)-\frac{1}{2} \end{array} \right| U_{\frac{1}{2}-\frac{1}{2}} \left| \begin{array}{cc} & (v_1, v_2) \\ \Lambda, & I \\ \frac{1}{2}(\mathcal{N}+T), & \frac{1}{2}(\mathcal{N}-T) \end{array} \right\rangle \left| \begin{array}{cc} & (v_1, v_2) \\ \Lambda+\frac{1}{2}, & I-\frac{1}{2} \\ \frac{1}{2}(\mathcal{N}+T)+\frac{1}{2}, & \frac{1}{2}(\mathcal{N}-T)-\frac{1}{2} \end{array} \right\rangle$$
$$+\left\langle \begin{array}{cc} & (v_1, v_2) \\ \Lambda-\frac{1}{2}, & I+\frac{1}{2} \\ \frac{1}{2}(\mathcal{N}+T)+\frac{1}{2}, & \frac{1}{2}(\mathcal{N}-T)-\frac{1}{2} \end{array} \right| U_{\frac{1}{2}-\frac{1}{2}} \left| \begin{array}{cc} & (v_1, v_2) \\ \Lambda, & I \\ \frac{1}{2}(\mathcal{N}+T), & \frac{1}{2}(\mathcal{N}-T) \end{array} \right\rangle \left| \begin{array}{cc} & (v_1, v_2) \\ \Lambda-\frac{1}{2}, & I+\frac{1}{2} \\ \frac{1}{2}(\mathcal{N}+T)+\frac{1}{2}, & \frac{1}{2}(\mathcal{N}-T)-\frac{1}{2} \end{array} \right\rangle$$
$$+\left\langle \begin{array}{cc} & (v_1, v_2) \\ \Lambda-\frac{1}{2}, & I-\frac{1}{2} \\ \frac{1}{2}(\mathcal{N}+T)+\frac{1}{2}, & \frac{1}{2}(\mathcal{N}-T)-\frac{1}{2} \end{array} \right| U_{\frac{1}{2}-\frac{1}{2}} \left| \begin{array}{cc} & (v_1, v_2) \\ \Lambda, & I \\ \frac{1}{2}(\mathcal{N}+T), & \frac{1}{2}(\mathcal{N}-T) \end{array} \right\rangle \left| \begin{array}{cc} & (v_1, v_2) \\ \Lambda-\frac{1}{2}, & I-\frac{1}{2} \\ \frac{1}{2}(\mathcal{N}+T)+\frac{1}{2}, & \frac{1}{2}(\mathcal{N}-T)-\frac{1}{2} \end{array} \right\rangle. \tag{21}$$

By using (21) and (19), and the explicit matrix elements shown in (6) and (7), Eq. (20) can be written as

$$-\sqrt{\frac{1}{2}}T_+\left|\begin{matrix}(v_1,v_2)\\ \zeta\ T=M_T,\mathcal{N}\end{matrix}\right\rangle=$$
$$\sum_{q,p}\Big\{c^{(\zeta)}_{p,q+1}(-1)^{2\mathcal{N}-2q+2v_1-1}$$
$$\times\left[\frac{(1+q)(2v_1-q+2)(v_1+v_2-q+1)(v_1+v_2-p-q+T+\mathcal{N}+1)(v_1-v_2+T-\mathcal{N}+p-q+1)(v_1-v_2-q)}{8(v_1+v_2-p-q+1)(v_1+v_2-p-q)(v_1-v_2+p-q)(v_1-v_2+p-q+1)}\right]^{\frac{1}{2}}$$
$$+\,c^{(\zeta)}_{p+1,q}(-1)^{v_1+v_2+\mathcal{N}-p-q+T-1}$$
$$\times\left[\frac{(1+p)(2v_2-p)(v_1+v_2-p+1)(v_1+v_2+T+\mathcal{N}-p-q+1)(v_1-v_2+p+2)(v_1-v_2-T+\mathcal{N}+p-q+1)}{8(v_1+v_2-p-q+1)(v_1+v_2-p-q)(v_1-v_2+p-q+1)(v_1-v_2+p-q+2)}\right]^{\frac{1}{2}}$$
$$+\,c^{(\zeta)}_{p-1,q}(-1)^{v_1-v_2+\mathcal{N}+p-q-T-1}$$
$$\times\left[\frac{p(2v_2-p+1)(v_1+v_2-p+2)(v_1+v_2-T-\mathcal{N}-p-q+1)(v_1-v_2+p+1)(v_1-v_2+T-\mathcal{N}+p-q+1)}{8(v_1+v_2-p-q+2)(v_1+v_2-p-q+1)(v_1-v_2+p-q)(v_1-v_2+p-q+1)}\right]^{\frac{1}{2}}$$
$$-\,c^{(\zeta)}_{p,q-1}$$
$$\times\left[\frac{q(2v_1-q+3)(v_1+v_2-q+2)(v_1+v_2-T-\mathcal{N}-p-q+1)(v_1-v_2-T+\mathcal{N}+p-q+1)(v_1-v_2-q+1)}{8(v_1+v_2-p-q+2)(v_1+v_2-p-q+1)(v_1-v_2+p-q+1)(v_1-v_2+p-q+2)}\right]^{\frac{1}{2}}\Big\}$$
$$\times\left|\begin{matrix} & (v_1,v_2) & \\ \Lambda=\frac{1}{2}(v_1+v_2-p-q), & & I=\frac{1}{2}(v_1-v_2+p-q)\\ \frac{1}{2}(\mathcal{N}+T+1), & & \frac{1}{2}(\mathcal{N}-T-1)\end{matrix}\right\rangle=0, \tag{22}$$

which leads to the following four-term relation to determine the expansion coefficients $\{c^{(\zeta)}_{p,q}\}$:

$$c^{(\zeta)}_{p,q+1}(-1)^{2\mathcal{N}-2q+2v_1}$$
$$\times\left[\frac{(1+q)(2v_1-q+2)(v_1+v_2-q+1)(v_1+v_2-p-q+T+\mathcal{N}+1)(v_1-v_2+T-\mathcal{N}+p-q+1)(v_1-v_2-q)}{(v_1+v_2-p-q)(v_1-v_2+p-q)}\right]^{\frac{1}{2}}$$
$$+\,c^{(\zeta)}_{p+1,q}(-1)^{v_1+v_2+\mathcal{N}-p-q+T}$$
$$\times\left[\frac{(1+p)(2v_2-p)(v_1+v_2-p+1)(v_1+v_2+T+\mathcal{N}-p-q+1)(v_1-v_2+p+2)(v_1-v_2-T+\mathcal{N}+p-q+1)}{(v_1+v_2-p-q)(v_1-v_2+p-q+2)}\right]^{\frac{1}{2}}$$
$$+\,c^{(\zeta)}_{p-1,q}(-1)^{v_1-v_2+\mathcal{N}+p-q-T}$$
$$\times\left[\frac{p(2v_2-p+1)(v_1+v_2-p+2)(v_1+v_2-T-\mathcal{N}-p-q+1)(v_1-v_2+p+1)(v_1-v_2+T-\mathcal{N}+p-q+1)}{(v_1+v_2-p-q+2)(v_1-v_2+p-q)}\right]^{\frac{1}{2}}$$
$$+\,c^{(\zeta)}_{p,q-1}$$
$$\times\left[\frac{q(2v_1-q+3)(v_1+v_2-q+2)(v_1+v_2-T-\mathcal{N}-p-q+1)(v_1-v_2-T+\mathcal{N}+p-q+1)(v_1-v_2-q+1)}{(v_1+v_2-p-q+2)(v_1-v_2+p-q+2)}\right]^{\frac{1}{2}}=0. \tag{23}$$

Similar to the projection procedure for $O(5)\supset O(3)$ shown in [21], one can construct a matrix equation of (23) with

$$\mathbf{P}((v_1,v_2),\mathcal{N},T)\mathbf{c}^{(\zeta)}=\Lambda\mathbf{c}^{(\zeta)}, \tag{24}$$

where $\mathbf{c}^{(\zeta)}\equiv\mathbf{c}^{(\zeta)}((v_1,v_2),\mathcal{N},T)$, of which the transpose is arranged as $\left(\mathbf{c}^{(\zeta)}\right)^{\mathrm{T}}=(c^{(\zeta)}_{0,0},c^{(\zeta)}_{1,0},c^{(\zeta)}_{2,0},\cdots,c^{(\zeta)}_{0,1},c^{(\zeta)}_{1,1},\cdots)$. Possible nonzero components of $\mathbf{c}^{(\zeta)}$ for some specific cases are shown in

Table 1
Allowed (p, q) combinations in the basis vectors (19) of $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ for some specific (v_1, v_2) cases with fixed $\mathcal{N}$ and T expanded in terms of those of $O(5) \supset SU_\Lambda(2) \otimes SU_I(2)$ with the corresponding multiplicity Multi$((v_1, v_2), \mathcal{N}, T)$, where $d(\mathcal{N}, T)$ is the total number of terms needed in the expansion, in which only some specific $(\mathcal{N}, T)$ combinations with $\mathcal{N} > 0$ are shown.

(v_1, v_2)	$\mathcal{N}, T$	(p, q)	$d(\mathcal{N}, T)$	Multi$((v_1, v_2), \mathcal{N}, T)$
(6, 0)	0, 0	(0, 0), (0, 2), (0, 4), (0, 6)	4	1
	1, 1	(0, 0), (0, 2), (0, 4)	3	1
$(\frac{11}{2}, \frac{1}{2})$	$\frac{1}{2}, \frac{1}{2}$	(1, 0), (0, 1), (1, 2), (0, 3), (1, 4), (0, 5)	6	1
	$\frac{3}{2}, \frac{1}{2}$	(0, 0), (1, 1), (0, 2), (1, 3), (0, 4)	5	1
	$\frac{3}{2}, \frac{3}{2}$	(1, 0), (0, 1), (1, 2), (0, 3)	4	1
(5, 1)	0, 0	(0, 0), (2, 0), (1, 1), (0, 2), (2, 2), (1, 3), (0, 4), (2, 4)	8	1
	0, 1	(1, 0), (0, 1), (2, 1), (1, 2), (0, 3), (2, 3), (1, 4)	7	1
	0, 2	(0, 0), (2, 0), (1, 1), (0, 2), (2, 2), (1, 3)	6	2
	0, 3	(1, 0), (0, 1), (2, 1), (1, 2)	4	1
	1, 1	(0, 0), (2, 0), (1, 1), (0, 2), (2, 2), (1, 3), (0, 4)	7	2
	1, 2	(1, 0), (0, 1), (2, 1), (1, 2), (0, 3)	5	1
(4, 2)	0, 0	(0, 0), (2, 0), (4, 0), (1, 1), (3, 1), (0, 2), (2, 2), (4, 2)	8	1
	0, 2	(0, 0), (2, 0), (4, 0), (1, 1), (3, 1), (2, 2)	6	3
	0, 3	(1, 0), (3, 0), (2, 1)	3	2
	1, 1	(0, 0), (2, 0), (4, 1), (1, 1), (3, 1), (0, 2), (2, 2)	7	2
	1, 2	(1, 0), (3, 0), (0, 1), (2, 1), (1, 2)	5	2
(3, 3)	0, 0	(0, 0), (2, 0), (4, 0), (6, 0)	4	1
	0, 1	(1, 0), (3, 0), (5, 0)	3	1
	0, 2	(2, 0), (4, 0)	2	1
	0, 3	(3, 0)	1	1
	1, 1	(0, 0), (2, 0), (4, 0)	3	1
	1, 2	(1, 0), (3, 0)	2	1
	2, 2	(2, 0), (4, 0)	2	1
	2, 3	(1, 0)	1	1

Table 1. Entries of the isospin projection matrix $\mathbf{P}((v_1, v_2), \mathcal{N}, T)$ can easily be read out from Eq. (23). The components of eigenvector $\mathbf{c}^{(\zeta)}$ corresponding to $\Lambda = 0$ provide the expansion coefficients $\{c^{(\zeta)}_{p,q}\}$ shown in (19). Once the matrix $\mathbf{P}((v_1, v_2), \mathcal{N}, T)$ is constructed, it can be verified that the number of $\Lambda = 0$ solutions of Eq. (24) equals exactly to the number of rows of $\mathbf{P}((v_1, v_2), \mathcal{N}, T)$ with all entries zero. Actually, the eigenvectors $\mathbf{c}^{(\zeta)}((v_1, v_2), T, \mathcal{N})$ belong to the null space of $\mathbf{P}((v_1, v_2), \mathcal{N}, T)$. Since there are many ways to find null space vectors of a matrix, to find solutions of Eq. (24) with $\Lambda = 0$ becomes practically easy. Furthermore, $\left(\mathbf{c}^{(\zeta')}\right)^{\mathrm{T}} \cdot \mathbf{c}^{(\zeta)} \neq 0$ when the multiplicity is greater than 1 mainly because the projection matrix $\mathbf{P}((v_1, v_2), \mathcal{N}, T)$ is nonsymmetric. Therefore, the $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ basis vectors (19) constructed from the expansion coefficients obtained according to (23) are also non-orthogonal with respect to the multiplicity label ζ in general. The Gram–Schmidt process may be adopted in order to construct orthonormalized basis vectors of $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$. Nevertheless, in the Wolfram Mathematica after version 10, the built-in function NullSpace of a matrix with non-integer entries generates orthonormalized null space vectors automatically, with which the Gram–Schmidt orthogonalization can be avoided. In the following, we use $\tilde{\mathbf{c}}^{(\zeta)}$ to denote the orthonormalized null space vectors of $N[\mathbf{P}((v_1, v_2), \mathcal{N}, T)]$ with respect to the multiplicity label ζ obtained from the Wolfram Mathematica (version 10.3) numerically, where $N[\mathbf{P}]$ means to take $\mathbf{P}$ with numerical valued entries with a default precision.

It is known that CPU time cost and memory space needed for a computer to solve the null space problem (24) depend mainly on the number of terms $d(\mathcal{N}, T)$ needed in the expansion (19), which equals to the number of columns of $\mathbf{P}((v_1, v_2), \mathcal{N}, T)$. Generally, it would take CPU time on the order of $\mathrm{O}(d^3)$ with a unit inversely proportional to the CPU frequency and memory space on the order of $\mathrm{O}(d^2)$ bytes. When v_1 and v_2 are integers, for example, we observe from Eq. (19) that the maximal number of terms occurs in $T = \mathcal{N} = 0$ case. In such extreme case, the upper bound of the number of terms involved in the expansion can be estimated by

$$d(\mathcal{N}=0, T=0) \le \sum_{q=0}^{v_1-v_2} \sum_{p=\mathrm{Max}[0,\ q-v_1+v_2]}^{\mathrm{Min}[v_1+v_2-q,\ 2v_2]} 1 = (1+v_1-v_2)(2v_2+1), \tag{25}$$

which shows that $\mathrm{Max}[d(\mathcal{N}, T)] \le d(\mathcal{N}=0, T=0)$ increases with v_1 linearly and with v_2 quadratically.

When $v_2 = 0$, only $p = 0$ is allowed. There are only two terms involved in (23) for this case with

$$c^{(\zeta)}_{0,q+1}(-1)^{2\mathcal{N}-2q+2v_1}\left[\frac{(1+q)(2v_1-q+2)(v_1-q+T+\mathcal{N}+1)(v_1+T-\mathcal{N}-q+1)}{(v_1-q)}\right]^{\frac{1}{2}} +$$
$$c^{(\zeta)}_{0,q-1}\left[\frac{q(2v_1-q+3)(v_1-T-\mathcal{N}-q+1)(v_1-T+\mathcal{N}-q+1)}{(v_1-q+2)}\right]^{\frac{1}{2}} = 0. \tag{26}$$

For the special case considered in [9] for the symmetric irrep of $O(5)$ with the parameters

$$v_1 = 2J_m,\ \Lambda = I = J_m - \mu,\ \mathcal{N} = 2J_m - 2b - a,\ \ T = a,$$

where $\mu = q/2$, the $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ basis vector with $M_T = T = a$ and $\mathcal{N} = 2J_m - 2b - a$ is expanded in terms of the $O(5) \supset SU_\Lambda(2) \otimes SU_I(2) \supset U_\Lambda(1) \otimes U_I(1)$ basis vectors as

$$\left|\begin{array}{c}(2J_m, 0)\\ T = M_T, \mathcal{N}\end{array}\right\rangle = \sum_{\mu=0}^{b} c_{0,\mu}\left|\begin{array}{c}(2J_m, 0)\\ J_m-\mu,\ \ J_m-\mu\\ J_m-b,\ \ J_m-b-a\end{array}\right\rangle, \tag{27}$$

where the multiplicity label ζ is omitted because the reduction $O(5) \downarrow O_T(3) \otimes O_{\mathcal{N}}(2)$ for symmetric irreps of $O(5)$ in this case is multiplicity-free, according to (19), in which the expansion coefficients

$$c_{0,\mu} = c_{0,0}(-1)^\mu\left[\frac{(2J_m+1)!(2J_m-1)!!b!(2J_m-a-b)!}{(4J_m+1)!!(2J_m+1)!!(a+b)!(2J_m-b)!}\right]^{\frac{1}{2}} \times$$
$$\left[\frac{(2\mu-1)!!(4J_m+1-2\mu)!!(a+b-\mu)!(2J_m-b-\mu)!(2J_m+1-2\mu)}{\mu!(b-\mu)!(2J_m+1-\mu)!(2J_m-a-b-\mu)!}\right]^{\frac{1}{2}} \tag{28}$$

derived from (26) are, up to a normalization constant, equivalent to the expansion coefficients derived in [9] for this case.

The possible $\mathcal{N}$ and T values for an arbitrary irrep (v_1, v_2) of $O(5)$ were obtained by several techniques previously [6–8]. Specifically, for a given irrep (v_1, v_2) of $O(5)$, the allowed values T are given by the following isospin couplings

$$T_u \otimes v_2\ \downarrow\ T \le v_1,$$
$$T_u = u, u-2, u-4, \cdots,$$
$$u = 0, 1, 2, \cdots, u_{\max} \le v_1, \tag{29}$$

Table 2
$O(5) \downarrow O_T(3) \otimes O_{\mathcal{N}}(2)$ for (v_1, v_2) with $v_1 = 6 - v/2$ and $v_2 = t$ for $(v = 0, t = 0)$, $(v = 1, t = 1/2)$, $(v = 2, t = 1)$, $(v = 4, t = 2)$, and $(v = 6, t = 3)$. For given $\mathcal{N}$, the multiplicity of T, if greater than 1, is shown by the superscript of T.

(v_1, v_2)	$[\mathcal{N}, T]$	Dimension$[(v_1, v_2)]$
$(6, 0)$	$[\pm 6, 0]$ $[\pm 5, 1]$ $[\pm 4, 0]$ $[\pm 4, 2]$ $[\pm 3, 1]$ $[\pm 3, 3]$ $[\pm 2, 0]$ $[\pm 2, 2]$ $[\pm 2, 4]$ $[\pm 1, 1]$ $[\pm 1, 3]$ $[\pm 1, 5]$ $[0, 0]$ $[0, 2]$ $[0, 4]$ $[0, 6]$	140
$(\frac{11}{2}, \frac{1}{2})$	$[\pm\frac{11}{2}, \frac{1}{2}]$ $[\pm\frac{9}{2}, \frac{1}{2}]$ $[\pm\frac{9}{2}, \frac{3}{2}]$ $[\pm\frac{7}{2}, \frac{1}{2}]$ $[\pm\frac{7}{2}, \frac{3}{2}]$ $[\pm\frac{7}{2}, \frac{5}{2}]$ $[\pm\frac{5}{2}, \frac{1}{2}]$ $[\pm\frac{5}{2}, \frac{3}{2}]$ $[\pm\frac{5}{2}, \frac{5}{2}]$ $[\pm\frac{5}{2}, \frac{7}{2}]$ $[\pm\frac{3}{2}, \frac{1}{2}]$ $[\pm\frac{3}{2}, \frac{3}{2}]$ $[\pm\frac{3}{2}, \frac{5}{2}]$ $[\pm\frac{3}{2}, \frac{7}{2}]$ $[\pm\frac{3}{2}, \frac{9}{2}]$ $[\pm\frac{1}{2}, \frac{1}{2}]$ $[\pm\frac{1}{2}, \frac{3}{2}]$ $[\pm\frac{1}{2}, \frac{5}{2}]$ $[\pm\frac{1}{2}, \frac{7}{2}]$ $[\pm\frac{1}{2}, \frac{9}{2}]$ $[\pm\frac{1}{2}, \frac{11}{2}]$	224
$(5, 1)$	$[\pm 5, 1]$ $[\pm 4, 0]$ $[\pm 4, 1]$ $[\pm 4, 2]$ $[\pm 3, 1^2]$ $[\pm 3, 2]$ $[\pm 3, 3]$ $[\pm 2, 0]$ $[\pm 2, 1]$ $[\pm 2, 2^2]$ $[\pm 2, 3]$ $[\pm 2, 4]$ $[\pm 1, 1^2]$ $[\pm 1, 2]$ $[\pm 1, 3^2]$ $[\pm 1, 4]$ $[\pm 1, 5]$ $[0, 0]$ $[0, 1]$ $[0, 2^2]$ $[0, 3]$ $[0, 4^2]$ $[0, 5]$	260
$(4, 2)$	$[\pm 4, 2]$ $[\pm 3, 1]$ $[\pm 3, 2]$ $[\pm 3, 3]$ $[\pm 2, 0]$ $[\pm 2, 1]$ $[\pm 2, 2^2]$ $[\pm 2, 3]$ $[\pm 2, 4]$ $[\pm 1, 1^2]$ $[\pm 1, 2^2]$ $[\pm 1, 3^2]$ $[\pm 1, 4]$ $[0, 0]$ $[0, 1]$ $[0, 2^3]$ $[0, 3^2]$ $[0, 4]$	220
$(3, 3)$	$[\pm 3, 3]$ $[\pm 2, 2]$ $[\pm 2, 3]$ $[\pm 1, 1]$ $[\pm 1, 2]$ $[\pm 1, 3]$ $[0, 0]$ $[0, 1]$ $[0, 2]$ $[0, 3]$	84

with which the corresponding $\mathcal{N} = \pm|v_1 - u|$. If the possible couplings of $T_u \otimes v_2$ lead to a specific T more than once, the specific T occurs at most Min$[v_1 - T + 1, v_1 - v_2 + 1]$ times [13]. Table 2 shows some examples of the reduction obtained in this way. Usually, the above branching rules should be checked by the dimension formula of $O(5)$ and that of $O_T(3)$ to determine the multiplicity of T in some cases. However, for given $\mathcal{N}$ and T, the number of solutions, Multi$((v_1, v_2), \mathcal{N}, T)$, of Eq. (24) with $\zeta = 1,\ 2, \cdots$, Multi$((v_1, v_2), \mathcal{N}, T)$ equals exactly to the multiplicity in the reduction $O(5) \downarrow O_T(3) \otimes O_{\mathcal{N}}(2)$ for the $O(5)$ irrep (v_1, v_2). Multi$((v_1, v_2), \mathcal{N}, T)$ of some examples determined by Eq. (24) is also shown in the last column of Table 1. Therefore, in the new isospin projection, it is not necessary to know the branching rule of $O(5) \downarrow O_T(3) \otimes O_{\mathcal{N}}(2)$ beforehand. For a given $\mathcal{N}$ and T, the solutions of the projection matrix $\mathbf{P}$ shown in (24) not only provide the expansion coefficients $\mathbf{c}^{(\zeta)}((v_1, v_2), T, \mathcal{N})$, but also determine the branching multiplicity of T, which is just the number of null space vectors obtained according to (24). Moreover, when no nontrivial null space vector of $\mathbf{P}((v_1, v_2), \mathcal{N}, T)$ exists, the only solution for this case is $\mathbf{c}((v_1, v_2), T, \mathcal{N}) = 0$, which occurs, for example, when $\mathcal{N} = 1$ and $T = 0$ for the $O(5)$ irrep $(4, 2)$. Therefore, a state with $\mathcal{N} = 1$ and $T = 0$ for the $O(5)$ irrep $(4, 2)$ does not exist as shown in Table 1.

In solving the four-term relation (23), there is always a freedom in choosing the global phase. In our calculation, we always set $c^{(\zeta)}_{0,0} > 0$, while the relative phase is completely determined by the eigen-equation (24). The multiplicity label ζ will be omitted if Multi$((v_1, v_2), \mathcal{N}, T) = 1$. One can verify that the multiplicity Multi$((v_1, v_2), \mathcal{N}, T)$ in the reduction $O(5) \downarrow O_T(3) \otimes O_{\mathcal{N}}(2)$ for the irrep $(v_1, v_2) \downarrow (\mathcal{N}, T)$ determined by Eq. (24) is indeed consistent with the branching rule calculated according to the rules provided in (29). The advantage of the projection (24) lies in the fact that the null space vectors of the projection matrix $\mathbf{P}$ can now be obtained easily, e.g., by using the built-in function NullSpace$[N[\mathbf{P}]]$ in Wolfram Mathematica, from which the null space vectors $\{\tilde{\mathbf{c}}^{(\zeta)}\}$ have already been orthonormalized with respect to ζ. Therefore, the matrix projection (24) is more suitable to be used in numerical calculations, which is useful, for example, in the isovector pairing problems [8].

In the following, we provide the $\mathbf{P}((v_1, v_2), \mathcal{N}, T)$ matrix and the corresponding expansion coefficients $\{c^{(\zeta)}_{p,q}\}$ for the branching multiplicity four case with the smallest v_1, v_2, $|\mathcal{N}|$, and

Table 3
The orthonormalized expansion coefficients $\tilde{c}^{(\zeta)}_{p,q}((v_1,v_2),\mathcal{N},T)$ of (19) for $(v_1,v_2)=(6,3)$, $\mathcal{N}=0$, and $T=3$, which is a branching multiplicity four case with $\zeta=1,2,3$, and 4.

$\tilde{c}^{(\zeta)}_{p,q}$				
$\tilde{c}^{(1)}_{0,0}=0.049645$,	$\tilde{c}^{(1)}_{2,0}=0.470707$,	$\tilde{c}^{(1)}_{4,0}=-0.633030$,	$\tilde{c}^{(1)}_{6,0}=-0.268207$,	$\tilde{c}^{(1)}_{1,1}=0.244431$,
$\tilde{c}^{(1)}_{3,1}=-0.112461$,	$\tilde{c}^{(1)}_{5,1}=-0.436399$,	$\tilde{c}^{(1)}_{2,2}=-0.010915$,	$\tilde{c}^{(1)}_{4,2}=-0.177676$,	$\tilde{c}^{(1)}_{3,3}=-0.093713$
$\tilde{c}^{(2)}_{0,0}=0.280671$,	$\tilde{c}^{(2)}_{2,0}=-0.044311$,	$\tilde{c}^{(2)}_{4,0}=0.44212$,	$\tilde{c}^{(2)}_{6,0}=-0.735426$,	$\tilde{c}^{(2)}_{1,1}=0.129513$,
$\tilde{c}^{(2)}_{3,1}=0.275610$,	$\tilde{c}^{(2)}_{5,1}=-0.18844$,	$\tilde{c}^{(2)}_{2,2}=0.220414$,	$\tilde{c}^{(2)}_{4,2}=-0.069694$,	$\tilde{c}^{(2)}_{3,3}=0.035434$
$\tilde{c}^{(3)}_{0,0}=0.743249$,	$\tilde{c}^{(3)}_{2,0}=0.057792$,	$\tilde{c}^{(3)}_{4,0}=-0.058706$,	$\tilde{c}^{(3)}_{6,0}=0.349662$,	$\tilde{c}^{(3)}_{1,1}=0.424036$,
$\tilde{c}^{(3)}_{3,1}=-0.000633$,	$\tilde{c}^{(3)}_{5,1}=0.159727$,	$\tilde{c}^{(3)}_{2,2}=0.236929$,	$\tilde{c}^{(3)}_{4,2}=0.106101$,	$\tilde{c}^{(3)}_{3,3}=0.214051$
$\tilde{c}^{(4)}_{0,0}=0.342470$,	$\tilde{c}^{(4)}_{2,0}=-0.623821$,	$\tilde{c}^{(4)}_{4,0}=-0.127977$,	$\tilde{c}^{(4)}_{6,0}=-0.126084$,	$\tilde{c}^{(4)}_{1,1}=-0.105715$,
$\tilde{c}^{(4)}_{3,1}=-0.520861$,	$\tilde{c}^{(4)}_{5,1}=-0.126637$,	$\tilde{c}^{(4)}_{2,2}=-0.184859$,	$\tilde{c}^{(4)}_{4,2}=-0.302202$,	$\tilde{c}^{(4)}_{3,3}=-0.193074$

T as a non-trivial example. Using the branching rules (29), one can verify that the branching multiplicity four case occurs at least when $(v_1,v_2)=(6,3)$ for $\mathcal{N}=0$ and $T=3$. According to (23), the corresponding $\mathbf{P}$ matrix is 10 dimensional with

$$\mathbf{P}((6,3),0,3)$$
$$=\begin{pmatrix} 1 & \frac{\sqrt{3}}{2} & 0 & 0 & -\sqrt{\frac{7}{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{10}{7}} & \sqrt{\frac{10}{7}} & 0 & 0 & -\sqrt{\frac{125}{42}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}} & 1 & 0 & 0 & -\sqrt{\frac{7}{2}} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & \sqrt{\frac{27}{14}} & \frac{2}{\sqrt{3}} & 0 & -\sqrt{\frac{39}{7}} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & \frac{2}{\sqrt{3}} & \sqrt{\frac{27}{14}} & 0 & -\sqrt{\frac{39}{7}} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{\frac{325}{2}} & 0 & \sqrt{\frac{56}{25}} & \sqrt{\frac{56}{25}} & -\frac{12}{5} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (30)$$

Since there are four rows with all entries zero in (30), the multiplicity of $T=3$ for $\mathcal{N}=0$ is Multi$((6,3),0,3)=4$, which is consistent with the multiplicity provided by (29). The normalized expansion coefficients $\mathbf{c}^{(\zeta)}((v_1,v_2),T,\mathcal{N})$ corresponding to $\Lambda=0$ shown in (24) are provided in Table 3.

4. Matrix representations of $O(5)\supset O_T(3)\otimes O_{\mathcal{N}}(2)$

Once the orthonormalized expansion coefficients $\tilde{\mathbf{c}}^{(\zeta)}$ are obtained according to the isospin projection shown in the previous section, one can easily calculate matrix elements of $O(5)$ generators $\{A^\dagger_\mu,\ A_\mu,\ T_\mu,\mathcal{N}\}$ ($\mu=-1,\ 0,\ 1$) given in (10) in the $O_T(3)\otimes O_{\mathcal{N}}(2)$ basis. Since matrix elements of $\{T_\mu,\ \hat{\mathcal{N}}\}$ are well-known, which only depend on T or $\mathcal{N}$, and are irrelevant to the irrep of $O(5)$ and the multiplicity label ζ, only formulae of matrix elements of $A^\dagger_\mu$ and A_μ in the $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ basis will be provided.

In the $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ basis, the pair creation operators $\mathcal{A}^+_\mu$ with $\{\mathcal{A}^+_{+1} = -A^\dagger_{+1},\ \mathcal{A}^+_0 = A^\dagger_0,\ \mathcal{A}^+_{-1} = A^\dagger_{-1}\}$ and the pair annihilation operators $\mathcal{A}_\mu$ with $\{\mathcal{A}_{+1} = A_{-1},\ \mathcal{A}_0 = -A_0,\ \mathcal{A}_{-1} = -A_{+1}\}$ are $T = 1$ irreducible tensor operators of $O_T(3)$ satisfying the following conjugation relation [23]:

$$\mathcal{A}_\mu = (-1)^{1-\mu}\left(\mathcal{A}^+_{-\mu}\right)^\dagger . \tag{31}$$

These $T = 1$ irreducible tensor operators shift $\mathcal{N}$ by one unit, while $A^\dagger_1 = \nu_+$, $A^\dagger_0 = U_{\frac{1}{2}\frac{1}{2}}$, and $A^\dagger_{-1} = \tau_+$ in the $O(5) \supset SU_\Lambda(2) \otimes SU_I(2)$ basis shown in (2). Using the Wigner–Eckart theorem for matrix elements of $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$, we have

$$\begin{aligned}
&\left\langle \begin{matrix}(v_1, v_2)\\ \zeta'\ T'\ M'_T, \mathcal{N}'\end{matrix} \middle| \mathcal{A}^+_\mu \middle| \begin{matrix}(v_1, v_2)\\ \zeta\ T\ M_T, \mathcal{N}\end{matrix} \right\rangle \\
&= \delta_{\mathcal{N}',\mathcal{N}+1} \langle T M_T, 1\mu | T'\ M'_T \rangle \left\langle \begin{matrix}(v_1, v_2)\\ \zeta'\ T', \mathcal{N}+1\end{matrix} \middle\| \mathcal{A}^+ \middle\| \begin{matrix}(v_1, v_2)\\ \zeta\ T, \mathcal{N}\end{matrix} \right\rangle,
\end{aligned} \tag{32}$$

where $\langle T M_T, 1\mu | T'\ M'_T \rangle$ is the CG coefficient of $O_T(3)$, and $\left\langle \begin{matrix}(v_1, v_2)\\ \zeta'\ T', \mathcal{N}'\end{matrix} \middle\| \mathcal{A}^+ \middle\| \begin{matrix}(v_1, v_2)\\ \zeta\ T, \mathcal{N}\end{matrix} \right\rangle$ is the $O_T(3)$-reduced matrix element. In the calculation, we ensure that T' is always involved in the $O_T(3)$ coupling $T \otimes 1$, and $M'_T = M_T + \mu$ is always satisfied. By using (19) and the expressions of $A^\dagger_\mu$ in terms of the generators of $O(5)$ in the $SU_\Lambda(2) \otimes SU_I(2)$ basis shown in (2), the left-hand-side of (32) can be expressed in terms of expansion coefficients $\tilde{\mathbf{c}}^{(\zeta)}$ and the matrix elements of $O(5)$ generators in the $SU_\Lambda(2) \otimes SU_I(2)$ basis. In the following, we list nonzero $O_T(3)$-reduced matrix elements of $\mathcal{A}^\dagger$ derived in this way:

$$\begin{aligned}
&\left\langle \begin{matrix}(v_1, v_2)\\ \zeta'\ T+1, \mathcal{N}+1\end{matrix} \middle\| \mathcal{A}^+ \middle\| \begin{matrix}(v_1, v_2)\\ \zeta\ T, \mathcal{N}\end{matrix} \right\rangle = -\tfrac{1}{2} \textstyle\sum_{q,p} \tilde{c}^{(\zeta')}_{p,q}(\mathcal{N}+1, T+1)\tilde{c}^{(\zeta)}_{p,q}(\mathcal{N}, T) \times \\
&\sqrt{(v_1 + v_2 - p - q - \mathcal{N} - T)(v_1 + v_2 - p - q + \mathcal{N} + T + 2)},
\end{aligned} \tag{33}$$

$$\begin{aligned}
&\left\langle \begin{matrix}(v_1, v_2)\\ \zeta'\ T, \mathcal{N}+1\end{matrix} \middle\| \mathcal{A}^+ \middle\| \begin{matrix}(v_1, v_2)\\ \zeta\ T, \mathcal{N}\end{matrix} \right\rangle = \sqrt{\frac{T+1}{8T}} \sum_{q,p} \tilde{c}^{(\zeta)}_{q,t}(\mathcal{N}, T) \Big(\tilde{c}^{(\zeta')}_{p,q-1}(\mathcal{N}+1, T) \\
&\times (-1)^{2\mathcal{N}-2q+2v_1+1} \\
&\times \left[\frac{q(2v_1-q+3)(v_1-v_2+p-q+\mathcal{N}-T+2)(v_1+v_2-p-q+T+\mathcal{N}+2)(v_1-v_2-q+1)(v_1+v_2-q+2)}{(v_1-v_2+p-q+1)(v_1-v_2+p-q+2)(v_1+v_2-p-q+1)(v_1+v_2-p-q+2)} \right]^{\frac{1}{2}} \\
&+ \tilde{c}^{(\zeta')}_{p-1,q}(\mathcal{N}+1, T)(-1)^{v_1+v_2+\mathcal{N}-p-q+T} \\
&\times \left[\frac{p(2v_2-p+2)(v_1-v_2+p-q-\mathcal{N}+T)(v_1+v_2-p-q+T+\mathcal{N}+2)(v_1-v_2+p+1)(v_1+v_2-p+2)}{(v_1-v_2+p-q+1)(v_1-v_2+p-q)(v_1+v_2-p-q+1)(v_1+v_2-p-q+2)} \right]^{\frac{1}{2}} \\
&+ \tilde{c}^{(\zeta')}_{p+1,q}(\mathcal{N}+1, T)(-1)^{v_1-v_2+\mathcal{N}+p-q-T} \\
&\times \left[\frac{(p+1)(2v_2-p)(v_1-v_2+\mathcal{N}-T+p-q+2)(v_1+v_2-T-\mathcal{N}-p-q)(v_1-v_2+p+2)(v_1+v_2-p+1)}{(v_1-v_2+p-q+1)(v_1-v_2+p-q+2)(v_1+v_2-p-q)(v_1+v_2-p-q+1)} \right]^{\frac{1}{2}}
\end{aligned}$$

$$+\tilde{c}^{(\zeta')}_{p,q+1}(\mathcal{N}+1,T)$$
$$\times\left[\frac{(q+1)(2v_1-q+2)(v_1-v_2-T+\mathcal{N}+p-q)(v_1+v_2-T-\mathcal{N}-p-q)(v_1-v_2-q)(v_1+v_2-q+1)}{(v_1-v_2+p-q)(v_1-v_2+p-q+1)(v_1+v_2-p-q)(v_1+v_2-p-q+1)}\right]^{\frac{1}{2}}\Bigg) \quad (34)$$

for $T \geq \frac{1}{2}$, and

$$\left\langle \begin{matrix}(v_1,v_2)\\ \zeta'\ T-1,\mathcal{N}+1\end{matrix} \middle\| \mathcal{A}^+ \middle\| \begin{matrix}(v_1,v_2)\\ \zeta\ T,\mathcal{N}\end{matrix}\right\rangle = \frac{1}{2}\sqrt{\frac{2T+1}{2T-1}}\sum_{q,p}\tilde{c}^{(\zeta')}_{p,q}(\mathcal{N}+1,T-1)\tilde{c}^{(\zeta)}_{p,q}(\mathcal{N},T)\times \sqrt{(v_1-v_2+p-q-\mathcal{N}+T)(v_1-v_2+p-q+\mathcal{N}-T+2)} \quad (35)$$

for $T \geq 1$.

By using (33)–(35), non-zero reduced matrix elements of $\mathcal{A}$ can be obtained by the conjugation relation:

$$\left\langle \begin{matrix}(v_1,v_2)\\ \zeta'\ T',\mathcal{N}'\end{matrix} \middle\| \mathcal{A} \middle\| \begin{matrix}(v_1,v_2)\\ \zeta\ T,\mathcal{N}\end{matrix}\right\rangle = (-1)^{T'-T+1}\sqrt{\frac{2T+1}{2T'+1}}\left\langle \begin{matrix}(v_1,v_2)\\ \zeta\ T,\mathcal{N}\end{matrix} \middle\| \mathcal{A}^+ \middle\| \begin{matrix}(v_1,v_2)\\ \zeta'\ T',\mathcal{N}'\end{matrix}\right\rangle. \quad (36)$$

Thus, the matrix representations of $O(5) \supset O_T(3) \supset O_{\mathcal{N}}(2)$ are obtained completely.

5. Applications to the pairing model for nuclei

In the spherical shell model, we consider n valence nucleons with $J=0$ and $T=1$ pairing interactions in p single-particle orbits. In general, the spherical shell model mean-field plus the isovector pairing interaction Hamiltonian may be written as [8]

$$\hat{H} = \sum_j \epsilon_j n_j - G_\pi A^\dagger_{+1}A_{+1} - G_{\pi\nu}A^\dagger_0 A_0 - G_\nu A^\dagger_{-1}A_{-1}, \quad (37)$$

where ϵ_j is the single particle energy of the j-orbit, $G_\pi > 0$, $G_\nu > 0$, and $G_{\pi\nu} > 0$ are proton–proton (pp), neutron–neutron (nn), and neutron–proton (np) pairing interaction strength, respectively, $n_j = \sum_{mm_t} a^\dagger_{jm,m_t} a_{jm,m_t}$ is the valence nucleon number operator in the j-orbit, in which $a^\dagger_{jm,m_t}$ (a_{jm,m_t}) is the creation (annihilation) operator for a valence nucleon in the state with angular momentum j, angular momentum projection m, and isospin projection m_t with $m_t = 1/2,\ -1/2$. When $G_\pi = G_\nu = G_{\pi\nu} = G$, the isospin is a good quantum number. In this isospin conserved case, the Hamiltonian (37) is exactly solvable [23,29]. Since neutron and proton single-particle energy of the j-orbit are the same, it is expected that $G_\pi = G_\nu = G$ may be approximately satisfied, while, in general, $G_{\pi\nu} \neq G$, for which the Bethe ansatz method used in [23,29] will no longer be useful. In such a case, the Hamiltonian (37) may be diagonalized in the $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ basis [24–27]. For the sake of simplicity, in the following, we consider the degenerate case with $\epsilon_j = \epsilon\ \forall\ j$, with which the first term of (37) becomes a constant for fixed number of nucleons n, and is neglected. Thus, the Hamiltonian can be expressed as

$$\hat{H}_P = -G\mathcal{A}^+ \cdot \mathcal{A}, \quad (38)$$

where $G_\pi = G_\nu = G_{\pi\nu} = G$ is assumed.

According to (11), the Hamiltonian (38) is $O_T(3)$ invariant, and can be expressed as

$$\hat{H}_{O_T(3)} = \hat{H}_P = -G\mathcal{A}^+ \cdot \mathcal{A} = -\frac{1}{2}G\left(C_2(O(5)) - \hat{\mathcal{N}}(\hat{\mathcal{N}}-3) - \mathbf{T}\cdot\mathbf{T}\right), \quad (39)$$

which is diagonal under the $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ basis with

$$\hat{H}_{O_T(3)} \left| \begin{matrix} (v_1, v_2) \\ \zeta \; T, M_T, \mathcal{N} \end{matrix} \right\rangle = -\frac{1}{2} G \left(v_1(v_1+3) + v_2(v_2+1) - \mathcal{N}(\mathcal{N}-3) - T(T+1) \right) \left| \begin{matrix} (v_1, v_2) \\ \zeta \; T, M_T, \mathcal{N} \end{matrix} \right\rangle . \quad (40)$$

In this case, the labels of the $O(5)$ irrep (v_1, v_2) are related with the seniority number of nucleons v and the reduced isospin t with $v_1 = \Omega - v/2$ and $v_2 = t$, where v and t indicate that there are v nucleons coupled to the isospin t, which are free from $J = 0$ and $T = 1$ pairs.

In order to check the validity of the results shown in previous sections, the matrix elements of $\mathcal{A}^+ \cdot \mathcal{A}$ for some specific $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ states will be calculated by using the results shown in Sec. 3 and 4 directly, which can be expressed as

$$\left\langle \begin{matrix} (v_1, v_2) \\ \zeta \; T, M_T, \mathcal{N} \end{matrix} \right| \mathcal{A}^+ \cdot \mathcal{A} \left| \begin{matrix} (v_1, v_2) \\ \zeta \; T, M_T, \mathcal{N} \end{matrix} \right\rangle = \left\langle \begin{matrix} (v_1, v_2) \\ \zeta \; T, \mathcal{N} \end{matrix} \right\| \mathcal{A}^+ \cdot \mathcal{A} \left\| \begin{matrix} (v_1, v_2) \\ \zeta \; T, \mathcal{N} \end{matrix} \right\rangle = \sum_{\zeta' T'} \left| \left\langle \begin{matrix} (v_1, v_2) \\ \zeta \; T, \mathcal{N} \end{matrix} \right\| \mathcal{A}^+ \left\| \begin{matrix} (v_1, v_2) \\ \zeta' \; T', \mathcal{N}-1 \end{matrix} \right\rangle \right|^2 \quad (41)$$

by using the Racah–Wigner calculus, in which the relation (36) is used. Since there is the analytical result shown in (40) for this case, it can be used to check the results shown in the previous sections via Eq. (41). In the following, the matrix element of $\mathcal{A}^+ \cdot \mathcal{A}$ for $(v_1, v_2) = (6, 0)$, $(5, 1)$, and $(4, 2)$ with $\mathcal{N} = T = 0$ will be calculated by using (41) with the method shown in the previous sections as examples.

For $(v_1, v_2) = (6, 0)$ and $\mathcal{N} = T = 0$, the corresponding projection matrix $\mathbf{P}$ shown in (24) is 4 dimensional with

$$\mathbf{P}((6,0), 0, 0) = \begin{pmatrix} -\sqrt{\frac{3}{2}} & -\sqrt{\frac{39}{10}} & 0 & 0 \\ 0 & -3\sqrt{\frac{2}{5}} & -\sqrt{\frac{22}{3}} & 0 \\ 0 & 0 & -\frac{5}{\sqrt{6}} & -3\sqrt{\frac{3}{2}} \\ 0 & 0 & 0 & 0 \end{pmatrix} . \quad (42)$$

In order to calculate the matrix element of $\mathcal{A}^+ \cdot \mathcal{A}$ for this case, one also needs the expansion coefficients $c_{p,q}$ for $\mathcal{N} = -1$ and $T = 1$ according to (41), for which the projection matrix $\mathbf{P}$ is 3 dimensional with

$$\mathbf{P}((6,0), -1, 1) = \begin{pmatrix} -1 & -\sqrt{\frac{26}{5}} & 0 \\ 0 & -\frac{3}{\sqrt{5}} & -\sqrt{11} \\ 0 & 0 & 0 \end{pmatrix} . \quad (43)$$

For $(v_1, v_2) = (5, 1)$ and $\mathcal{N} = T = 0$, the corresponding projection matrix $\mathbf{P}$ is 8 dimensional with

$$\mathbf{P}((5,1),0,0)=\begin{pmatrix} \sqrt{\frac{3}{10}} & \sqrt{\frac{3}{10}} & -\sqrt{\frac{42}{25}} & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{\frac{6}{5}} & 0 & \sqrt{\frac{21}{50}} & -\sqrt{\frac{33}{10}} & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{\frac{6}{5}} & \sqrt{\frac{21}{50}} & 0 & -\sqrt{\frac{33}{10}} & 0 & 0 & 0 \\ 0 & 0 & -3\sqrt{\frac{11}{50}} & \sqrt{\frac{7}{10}} & \sqrt{\frac{7}{10}} & -\sqrt{\frac{25}{6}} & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{\frac{5}{2}} & 0 & \sqrt{\frac{7}{6}} & -\sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{\frac{5}{2}} & \sqrt{\frac{7}{6}} & 0 & -\sqrt{6} \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{2} & \sqrt{\frac{7}{2}} & \sqrt{\frac{7}{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{44}$$

For $(v_1, v_2) = (5, 1)$ and $\mathcal{N} = -1$ and $T = 1$, the corresponding projection matrix $\mathbf{P}$ is 7 dimensional with

$$\mathbf{P}((5,1),-1,1)=\begin{pmatrix} \sqrt{\frac{2}{5}} & \sqrt{\frac{1}{5}} & -2\sqrt{\frac{14}{25}} & 0 & 0 & 0 & 0 \\ -\sqrt{\frac{3}{5}} & 0 & \frac{\sqrt{21}}{10} & -3\sqrt{\frac{11}{20}} & 0 & 0 & 0 \\ 0 & -\sqrt{\frac{2}{5}} & \sqrt{\frac{14}{25}} & 0 & -\sqrt{\frac{22}{5}} & 0 & 0 \\ 0 & 0 & -3\frac{\sqrt{11}}{10} & \sqrt{\frac{21}{20}} & \sqrt{\frac{7}{20}} & -\frac{5}{2} & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{\frac{5}{4}} & \sqrt{\frac{7}{4}} & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{45}$$

For $(v_1, v_2) = (4, 2)$ and $\mathcal{N} = T = 0$, the corresponding projection matrix $\mathbf{P}$ is 8 dimensional with

$$\mathbf{P}((4,2),0,0)$$
$$=\begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{9}{10}} & 0 & -\sqrt{\frac{7}{6}} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{9}{10}} & \sqrt{\frac{2}{3}} & 0 & -\sqrt{\frac{7}{6}} & 0 & 0 & 0 \\ -\sqrt{\frac{5}{6}} & 0 & 0 & \sqrt{\frac{14}{15}} & 0 & -3\sqrt{\frac{3}{10}} & 0 & 0 \\ 0 & -\sqrt{\frac{7}{10}} & 0 & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{2}} & 0 & -\sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & -\sqrt{\frac{5}{6}} & 0 & \sqrt{\frac{14}{15}} & 0 & 0 & -3\sqrt{\frac{3}{10}} \\ 0 & 0 & 0 & -\sqrt{\frac{9}{10}} & 0 & \sqrt{\frac{14}{5}} & \sqrt{\frac{5}{2}} & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{\frac{9}{10}} & 0 & \sqrt{\frac{5}{2}} & \sqrt{\frac{14}{5}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{46}$$

For $(v_1, v_2) = (4, 2)$ and $\mathcal{N} = -1$ and $T = 1$, the corresponding projection matrix $\mathbf{P}$ is 7 dimensional with

$$\mathbf{P}((4,2),-1,1)=\begin{pmatrix} 1 & \sqrt{\frac{9}{20}} & 0 & -\sqrt{\frac{7}{4}} & 0 & 0 & 0 \\ 0 & \sqrt{\frac{6}{5}} & \frac{2}{3} & 0 & -\sqrt{\frac{14}{9}} & 0 & 0 \\ 0 & -\sqrt{\frac{7}{20}} & 0 & \frac{3}{2} & \frac{\sqrt{3}}{2} & -\frac{3}{2} & 0 \\ 0 & 0 & -\frac{\sqrt{5}}{3} & 0 & \sqrt{\frac{56}{45}} & 0 & -\sqrt{\frac{18}{5}} \\ 0 & 0 & 0 & 0 & -\sqrt{\frac{9}{20}} & \sqrt{\frac{15}{4}} & \sqrt{\frac{7}{5}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{47}$$

One can check that the multiplicity of T in an $O(5)$ irrep equals exactly to the number of rows with all entries zero in $\mathbf{P}$ for fixed $\mathcal{N}$. The null space of $\mathbf{P}((v_1,v_2),\mathcal{N},T)$ provides the corresponding orthonormalized expansion coefficients $\tilde{c}^{(\zeta)}_{p,q}((v_1,v_2),\mathcal{N},T)$ of (19). The orthonormalized expansion coefficients $\tilde{c}^{(\zeta)}_{p,q}((v_1,v_2),\mathcal{N},T)$ for $(v_1,v_2)=(6,0)$, $(5,1)$, and $(4,2)$ with $\mathcal{N}=0$, $T=0$ and $\mathcal{N}=-1$, $T=1$ needed in the evaluation of the matrix elements of $\mathcal{A}^+\cdot\mathcal{A}$ according to (41) for these cases are shown in Table 4. In addition, as shown in [21,22,28], there is an arbitrary $SO(\mathrm{Multi}((v_1,v_2),\mathcal{N},T))$ rotational transformation with respect to the multiplicity labels $\zeta=1,2,\cdots,\mathrm{Multi}((v_1,v_2),\mathcal{N},T)$. When $\mathrm{Multi}((v_1,v_2),\mathcal{N},T)=2$ for example, let $|\zeta=1\rangle=\left|\begin{matrix}(v_1,v_2)\\ \zeta=1\ T,M_T,\mathcal{N}\end{matrix}\right\rangle$ and $|\zeta=2\rangle=\left|\begin{matrix}(v_1,v_2)\\ \zeta=2\ T,M_T,\mathcal{N}\end{matrix}\right\rangle$ be orthonormalized basis vectors of $O(5)\supset O_T(3)\otimes O_{\mathcal{N}}(2)$. New basis vectors $\{|\bar{\zeta}\rangle\}$ after an arbitrary $SO(2)$ rotation with respect to the multiplicity labels with

$$\begin{aligned}|\bar{\zeta}=1\rangle&=\cos\theta|\zeta=1\rangle-\sin\theta|\zeta=2\rangle,\\ |\bar{\zeta}=2\rangle&=\sin\theta|\zeta=1\rangle+\cos\theta|\zeta=2\rangle\end{aligned} \tag{48}$$

are also an orthonormalized basis vectors of $O(5)\supset O_T(3)\otimes O_{\mathcal{N}}(2)$ of the same irrep of $O(5)$ with T and $\mathcal{N}$ unchanged, where $0\le\theta\le 2\pi$. As a result, matrix elements of $\mathcal{A}^+$ and $\mathcal{A}$ in the $O(5)\supset O_T(3)\otimes O_{\mathcal{N}}(2)$ basis may be numerically different when they are derived by using different methods for non-multiplicity-free cases.

Using the expansion coefficients shown in Table 4 and Eq. (35), we have

$$\left\langle\begin{matrix}(6,0)\\0,0\end{matrix}\right\|\mathcal{A}^+\left\|\begin{matrix}(6,0)\\1,-1\end{matrix}\right\rangle=5.19615. \tag{49}$$

$$\left\langle\begin{matrix}(5,1)\\0,0\end{matrix}\right\|\mathcal{A}^+\left\|\begin{matrix}(5,1)\\\zeta=1,\ 1,-1\end{matrix}\right\rangle=3.32942,\quad \left\langle\begin{matrix}(5,1)\\0,0\end{matrix}\right\|\mathcal{A}^+\left\|\begin{matrix}(5,1)\\\zeta=2,\ 1,-1\end{matrix}\right\rangle=3.14880. \tag{50}$$

$$\left\langle\begin{matrix}(4,2)\\0,0\end{matrix}\right\|\mathcal{A}^+\left\|\begin{matrix}(4,2)\\\zeta=1,\ 1,-1\end{matrix}\right\rangle=4.06054,\quad \left\langle\begin{matrix}(4,2)\\0,0\end{matrix}\right\|\mathcal{A}^+\left\|\begin{matrix}(4,2)\\\zeta=2,\ 1,-1\end{matrix}\right\rangle=-0.715557. \tag{51}$$

Substituting these values into Eq. (41), one can check that each result of Eq. (41) is exactly the same as the corresponding one shown by (40), which validates the isospin projection shown in the previous sections.

Moreover, besides the $O_T(3)$ isospin dynamical symmetry limit case shown above, there is the well known $SU_\Lambda(2)\otimes SU_I(2)$ quasispin dynamical symmetry limit for any value of G_π

Table 4
The orthonormalized expansion coefficients $\tilde{c}^{(\zeta)}_{p,q}((v_1, v_2), \mathcal{N}, T)$ of (19) for $(v_1, v_2) = (6, 0)$, $(5, 1)$, and $(4, 2)$ with $\mathcal{N} = 0$, $T = 0$ and $\mathcal{N} = -1$, $T = 1$.

(v_1, v_2)	$\zeta, \mathcal{N}, T$	$\tilde{c}_{p,q}$
(6, 0)	1, 0, 0	$\tilde{c}_{0,0} = 0.782852$, $\tilde{c}_{0,2} = -0.485504$, $\tilde{c}_{0,4} = 0.340168$, $\tilde{c}_{0,6} = -0.188982$
	1, −1, 1	$\tilde{c}_{0,0} = 0.90396$, $\tilde{c}_{0,2} = -0.396412$, $\tilde{c}_{0,4} = 0.160357$
(5, 1)	1, 0, 0	$\tilde{c}_{0,0} = 0.574456$, $\tilde{c}_{2,0} = -0.574456$, $\tilde{c}_{1,1} = 0$, $\tilde{c}_{0,2} = -0.34641$, $\tilde{c}_{2,2} = 0.34641$, $\tilde{c}_{1,3} = 0$, $\tilde{c}_{0,4} = 0.223607$, $\tilde{c}_{2,4} = -0.223607$
	1, −1, 1	$\tilde{c}^{(\zeta=1)}_{0,0} = 0.0797736$, $\tilde{c}^{(\zeta=1)}_{2,0} = 0.893054$, $\tilde{c}^{(\zeta=1)}_{1,1} = 0.300561$, $\tilde{c}^{(\zeta=1)}_{0,2} = 0.0341335$, $\tilde{c}^{(\zeta=1)}_{2,2} = -0.273573$, $\tilde{c}^{(\zeta=1)}_{1,3} = -0.170371$, $\tilde{c}^{(\zeta=1)}_{2,4} = 0.0268284$
	2, −1, 1	$\tilde{c}^{(\zeta=2)}_{0,0} = 0.858363$, $\tilde{c}^{(\zeta=2)}_{2,0} = -0.150337$, $\tilde{c}^{(\zeta=2)}_{1,1} = 0.317803$, $\tilde{c}^{(\zeta=2)}_{0,2} = -0.233385$, $\tilde{c}^{(\zeta=2)}_{2,2} = 0.177481$, $\tilde{c}^{(\zeta=2)}_{1,3} = -0.180144$, $\tilde{c}^{(\zeta=2)}_{2,4} = -0.145579$
(4, 2)	1, 0, 0	$\tilde{c}_{0,0} = 0.519615$, $\tilde{c}_{2,0} = -0.447214$, $\tilde{c}_{4,0} = 0.519615$, $\tilde{c}_{1,1} = 0$, $\tilde{c}_{3,1} = 0$, $\tilde{c}_{0,2} = -0.288675$, $\tilde{c}_{2,2} = 0.305505$, $\tilde{c}_{4,2} = -0.288675$
	1, −1, 1	$\tilde{c}^{(\zeta=1)}_{0,0} = 0.373586$, $\tilde{c}^{(\zeta=1)}_{2,0} = -0.52682$, $\tilde{c}^{(\zeta=1)}_{4,0} = 0.663391$, $\tilde{c}^{(\zeta=1)}_{1,1} = 0.0152581$, $\tilde{c}^{(\zeta=1)}_{3,1} = -0.108114$, $\tilde{c}^{(\zeta=1)}_{2,2} = 0.160619$, $\tilde{c}^{(\zeta=1)}_{4,2} = -0.32417$
	2, −1, 1	$\tilde{c}^{(\zeta=2)}_{0,0} = 0.34123$, $\tilde{c}^{(\zeta=2)}_{2,0} = -0.337929$, $\tilde{c}^{(\zeta=2)}_{4,0} = -0.592498$, $\tilde{c}^{(\zeta=2)}_{1,1} = 0.0865847$, $\tilde{c}^{(\zeta=2)}_{3,1} = -0.61351$, $\tilde{c}^{(\zeta=2)}_{2,2} = -0.134344$, $\tilde{c}^{(\zeta=2)}_{4,2} = -0.127955$

and G_ν when $G_{\pi\nu} = 0$, where Λ and I are the quasi-spin of the proton and neutron pairing, respectively. In this case, the pairing interaction part of (37)

$$\hat{H}_{SU_\Lambda(2)\otimes SU_I(2)} = -G_\pi A^\dagger_{+1} A_{+1} - G_\nu A^\dagger_{-1} A_{-1} \tag{52}$$

is diagonal under the $O(5) \supset SU_\Lambda(2) \otimes SU_I(2)$ basis with

$$\hat{H}_{SU_\Lambda(2)\otimes SU_I(2)} \left| \begin{matrix} (v_1, v_2) \\ \Lambda, \; I \\ m_\Lambda, m_I \end{matrix} \right\rangle$$

$$= \left(-G_\pi \left(\Lambda(\Lambda+1) - m_\Lambda(m_\Lambda+1)\right) - G_\nu \left(I(I+1) - m_I(m_I+1)\right)\right) \left| \begin{matrix} (v_1, v_2) \\ \Lambda, \; I \\ m_\Lambda, m_I \end{matrix} \right\rangle, \tag{53}$$

where $\Lambda = (\Omega - v_\pi)/2$ and $I = (\Omega - v_\nu)/2$, in which v_π (v_ν) is the proton (neutron) seniority number, $m_\Lambda = n_\pi/2 - \Omega/2$, and $m_I = n_\nu/2 - \Omega/2$, in which n_π (n_ν) is the number of valence protons (neutrons), which shows, though the interpretation of (v_1, v_2) in terms of v and t is no longer appropriate in this case due to the fact that the isospin symmetry is broken, (52) is still block diagonal with respect to the O(5) irreps labeled by (v_1, v_2).

For other values of the pairing interaction strengths, the pairing interaction part of (37) can only be diagonalized under any basis of $O(5)$, of which the eigenstates may be expanded in terms of either the basis vectors of $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ or those of $O(5) \supset SU_\Lambda(2) \otimes SU_I(2)$. The parameter rectangle of the pure isovector pairing Hamiltonian is illustrated in Fig. 3, which shows the pure isovector pairing Hamiltonian may be diagonalized in the $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ basis, except the $G_{\pi\nu} = 0$ case indicated by the left leg of the rectangle with the $SU_\Lambda(2) \otimes SU_I(2)$ quasispin dynamical symmetry.

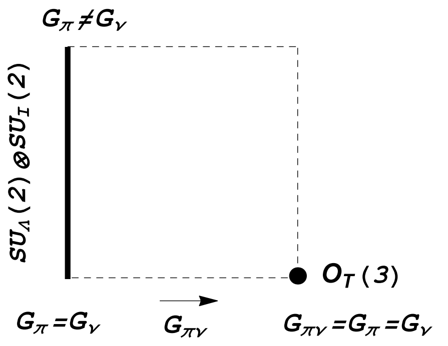

Fig. 3. The parameter rectangle of the isovector pairing Hamiltonian, where the left leg marked by the solid line represents the Hamiltonian with arbitrary values of G_π and G_ν and $G_{\pi\nu} = 0$ corresponding to the $SU_\Lambda(2) \otimes SU_I(2)$ quasispin dynamical symmetry, and the vertex marked by the solid dot represents the Hamiltonian with $G_\pi = G_\nu = G_{\pi\nu}$ corresponding to the $O_T(3)$ isospin dynamical symmetry. The Hamiltonian for other values of the parameters shown by the other area of the rectangle may be diagonalized in either the $O(5) \supset SU_\Lambda(2) \otimes SU_I(2)$ or the $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ basis.

6. Summary

A simple and effective algebraic isospin projection procedure for constructing basis vectors of irreducible representations of the non-canonical $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ basis from those in the canonical $O(5) \supset SU_\Lambda(2) \otimes SU_I(2)$ basis is proposed. We show that the expansion coefficients can be obtained as components of the null-space vectors of the projection matrix, of which, similar to the $SU(3) \supset SO(3)$ projection shown in [22], there are only four nonzero elements in each row in general. There are currently available well-optimized algorithms for computing the null-space vectors of a matrix, for example, the Wolfram Mathematica, from which the null space vectors obtained are orthonormalized. Hence, to evaluate the expansion coefficients of the orthonormal basis vectors of $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ in terms of the basis of the canonical chain becomes straightforward. The advantage of this work lies in the fact that the basis vectors of $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ thus obtained are orthnormalized with respect to the $O(5) \downarrow O_T(3) \otimes O_{\mathcal{N}}(2)$ branching multiplicity label ζ for any irrep of $O(5)$. Explicit formulae for evaluating $O_T(3)$-reduced matrix elements of $O(5)$ generators are derived, of which the validity is checked in the evaluation of some matrix elements of the pure isovector pairing Hamiltonian. For the non-degenerate case of (37), one needs to diagonalize it in the $\bigotimes_{i=1}^{p} O_i(5)$ subspace when there are p non-degenerate orbits, which will be a part of our future work.

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