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ORIGINAL RESEARCH

Global dynamics of stochastic predator–prey model with mutual interference and prey defense

Ranjit Kumar Upadhyay[1] · **Rana D. Parshad**[2] · **Kwadwo Antwi-Fordjour**[3] · **Emmanuel Quansah**[4] · **Sarita Kumari**[1]

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Abstract

Predator–prey interactions with stochastic forcing have been extensively investigated in the literature. However there are not many investigations of such models, that include prey defense. The goal of the current manuscript is to investigate a stochastic predator–prey model with mutual interference, and various Holling type functional responses, where the prey is able to release toxins as defense against a predator. This can also be generalized to include group or herd defense, toxin production and mimicry. We establish local and global existence for the stochastic model, and perform various numerical simulations to support our theoretical results. Our key result is that we have globally existing solutions independent of the magnitude of the toxin release parameter, or the predation rates. We also show that large enough noise intensity in solely the prey, can lead to extinction in the noisy model, for both species, whilst there is persistence in the deterministic model.

Keywords Stochastic predator–prey model · Global existence · Extinction · Multiplicative noise · Millstein scheme

Mathematics Subject Classification Primary 60H10 · 65C30; Secondary 92D25 · 92D40

✉ Ranjit Kumar Upadhyay
ranjit.chaos@gmail.com

Rana D. Parshad
rparshad@iastate.edu

[1] Department of Applied Mathematics, Indian Institute of Technology (Indian School of Mines), Dhanbad, Jharkhand 826004, India

[2] Department of Mathematics, Iowa State University, Ames, IA 50011, USA

[3] Department of Mathematics and Computer Science, Samford University, Birmingham, AL 35229, USA

[4] PacificSource Health Plans, Springfield, OR 97477, USA

1 Introduction

In recent years, applied mathematicians and ecologists have been interested in investigating the dynamical properties of biological models. Nonlinear factors such as functional response and mutual interference are among several crucial factors affecting biological models. Predator–prey models with varied functional responses and mutual interference types have been of great interest lately. Mutual interference [1,2] is defined as the behavioral interactions among feeding organisms, that reduce the time that each individual spends obtaining food, or the amount of food each individual consumes. It occurs most commonly where the amount of food is scarce, or the population of feeding organisms is large [3]. Wang [4,5] studied the existence, permanence and global asymptotic stability for a predator–prey system with mutual interference. Wang and his group [6,7] also obtained some sufficient conditions, for the permanence and global attractivity of positive periodic solution for a Volterra model, and delayed predator–prey model with mutual interference. Periodic solutions, permanence and global attractivity of a delayed impulsive predator–prey system has also been studied in [8]. Recently, Chen [9] investigated the permanence of the corresponding discrete periodic system model

$$\begin{aligned}\frac{dx}{dt} &= x(r_1(t) - b_1(t)x) - c(t)xy^m/(k+x),\\ \frac{dy}{dt} &= y(-r_2(t) - b_2(t)x) + kc(t)xy^{m-1}/(k+x),\end{aligned}$$

where x (prey) and y (predator) are population densities and all parameter are positive. Lin and Chen [10] have studied the almost periodic solution for system with Beddington-DeAngelis functional response. Wang et al. [11] have investigated the existence and global attractivity of positive periodic solution to the above system with Holling type III functional response.

All biological processes are fundamentally affected by enviromental white noise. In stochastic modeling, the environmental noise is conceived as stochastic fluctuations. May [12] analyzed a biological system under stochastic fluctuations, considering white noise forcing for the population. He observed that when the population deviates more from equilibrium point, the system shows irregular behavior (i.e., instability). Thus, after one investigates a deterministic system, extending the results to the stochastic case becomes an important research endeavor. Upadhyay et al. [13] investigated the influence of environmental noise on a fairly realistic ecological model with generalist top predator and showed the importance of the noise amplitude, the trophic level susceptibility of populations to environmental noise. Liu and Wang [14–16] studied the population dynamics of a Lotka Volterra cooperative model, two prey one predator system, with random perturbation and persistence-extinction dynamics in a stochastic single species model, with regime switching in a polluted environment. Rudnicki [17] studied the long time behavior of a stochastic prey–predator model. Ji et al. [18] analysed a stochastic predator–prey model, with modified Leslie-Gower and Holling type II schemes. Upadhyay and Agrawal [19] studied the effect of mutual interference on the dynamics of a predator–prey system, with gestation delay. Du [20] considered a stochastic predator–prey model with mutual interference and studied the existence, extinction and its global asymptotic stability. Recently Du et al. [30,32] studied the dynamical behavior for a stochastic predator–prey model with Hassell-Varley type functional response, as well as investigated a neural network approach.

In the works outlined above, the classical setting of predator–prey models were considered. That is the predator feeds on a prey species according to some functional response, and dies out due to a natural death rate. The loss in the prey population is due to predation, and often times the prey population is logistically controlled. However, in most cases the prey species is able to fight back, due to inducible defenses that it possesses. These might include group or herd defense, toxin production, mimicry etc. [3]. Thus understanding the dynamics of systems with prey defense, in a noisy environment is an interesting scientific question, but not well studied in the literature. The goal of the current manuscript is to investigate a stochastic predator–prey model with mutual interference, and various Holling type functional responses, where the prey is able to release toxins, as a form of defense against the predator. Our primary contributions in the manuscript are

1. We formulate and propose a two species predator–prey model, with a general form of mutual interference and toxin release effect. The model is subjected to stochastic forcing.
2. We show global existence of solutions to the model via Theorem 6.1. This result is derived independent of the relation between the predation rate w_1 and the toxin release parameter θ. That is, we *do not* require the restriction $\theta > w_1$, for global existence. We provide biological justification to this end, see remark 1.
3. We investigate extinction criteria in the model, finding that the stochastic forcing can be responsible for extinction of both species, under certain conditions. This is shown via Theorem 5.1.
4. We carry out extensive numerical simulations to support our analytical results.

2 Model system

In this paper, we consider the following predator–prey model with mutual interferences

$$\begin{aligned}\frac{dx_1}{dt} &= a_1x_1 - b_1x_1^2 - w_0\left(\frac{x_1}{x_1+d}\right)^{m_1} x_2^{m_2},\\ \frac{dx_2}{dt} &= -a_2x_2 - \theta f(x_1)x_2 + w_1\left(\frac{x_1}{x_1+d}\right)^{m_1} x_2^{m_2},\end{aligned}$$

There are few assumptions we place on the toxin release function f, and the mutual interference paramaters m_i.

- $0 < m_i \leq 1, i = 1, 2$, as per literature on mutual interference [1,2].
- f depends only on prey density, that is $f = f(x_1)$.
- f is identically 0, at 0 prey density. So $f(0) = 0$.
- $f \in C^0(\Re_+)$, and grows sublinearly. That is, $|f(x_1)| \leq Cx_1$, where C is a pure constant.

Table 1 List of parameters used in the model

Symbols	Meaning
x_1	Prey population
x_2	Predator population
$f(x_1)$	Toxin release function
a_1	Per capita rate of self-reproduction for the prey
a_2	Intrinsic death rate of the predator population in the absence of the only food x_1
w_0	Maximum rate of per capita removal of prey
w_1	Measure efficiency of biomass conversion from prey to predator
b_1	Effect of the density of one species on the rate of growth rate of other
d	Value of population density at which per capita removal rate is half of x_1
θ	Rate of toxin released by prey
m_i	Mutual interference parameter that model the intraspecific competition among predators when hunting for preys

All parameters considered are positive constants

- f is saturating in x_1, that is $\frac{df}{dx_1}$ decreases, (or is at least nonincreasing) as x_1 increases.
- Thus we can assume standard Holling type responses for the toxin release function f.

All parameters in the above model are positive constants.
The above model together with $x(0) = x_0 > 0$, $y(0) = y_0 > 0$, will be referred to as model (DM). The variables and parameters used in the model are defined in Table 1:

The deterministic system assumes that the parameters in the model are all deterministic, irrespective of the environmental fluctuations. Hence they have some limitations in mathematical modeling of ecological systems, as they are unable to fit data well, and thus, to predict the future dynamics of the system accurately. May [12] has pointed out that because of environmental noise, the birth rate, carrying capacity, competition coefficient and other parameter involved in the system *always* exhibit random fluctuations. Thus we construct a stochastic model by introducing a noise term, into the growth terms of both the predator and prey populations. Essentially the growth rate can be written as an average growth rate plus an error term. In general, by the well-known central limit theorem, the error term follows a normal distribution; thus for short correlation time, we can approximate the error term by a white noise, $\sigma_i(t)\frac{dB_i}{dt}$, where $\sigma_i^2(t)$ is a continuous bounded function on $\Re_+$ representing the intensity of the noise at time t and $\frac{dB_i}{dt}$ is a standard white noise. Then corresponding to the deterministic system (DM), we obtain the following stochastic system (SM).

$$\mathrm{d}x_1 = \left[a_1x_1 - b_1x_1^2 - w_0\left(\frac{x_1}{x_1+d}\right)^{m_1} x_2^{m_2}\right]\mathrm{d}t + \sigma_1(t)x_1\mathrm{d}B_1(t), \tag{1a}$$

$$\mathrm{d}x_2 = \left[-a_2x_2 - \theta f(x_1)x_2 + w_1\left(\frac{x_1}{x_1+d}\right)^{m_1} x_2^{m_2}\right]\mathrm{d}t + \sigma_2(t)x_2\mathrm{d}B_2(t), \tag{1b}$$

where $\sigma_i(t) > 0\,\forall i$ and $B_1(t), B_2(t)$ are statistically independent Brownian motions at time, t. We show that the system admits a unique positive asymptotically global solution, starting from a positive initial value, with different functional responses. The solutions are interpreted in the Ito sense [21,23]. Also note that in the ensuing analysis $C, C_i, 1 \leq i \leq 5$, are generic pure constants, and can change in value from line to line, and also within the same line if so required.

3 Existence and uniqueness of global positive solution

In this section, we will present some results on existence and uniqueness of a global positive solution of the system (SM). Our approach to study the properties of (SM) is a combination of some stochastic calculus and martingale methods. If $f(t)$ is a continuous bounded function on $\Re_+$, define

$$f^u = \sup_{t \in \Re_+} f(t),$$
$$f^l = \inf_{t \in \Re_+} f(t).$$

Additionally, assume there exist positive constants $\sigma_1^l, \sigma_1^u, \sigma_2^l, \sigma_2^u$, such that

$$\sigma_1^l < \sigma_1(t) < \sigma_1^u, \forall t \in \Re_+$$
$$\sigma_2^l < \sigma_2(t) < \sigma_2^u, \forall t \in \Re_+$$

We first present some lemmas that give us almost sure eventual boundedness of the state variables x_1, x_2.

Lemma 3.1 *Assume* $(x_1(t), x_2(t))$ *are solutions to system (SM) then* $x_1 \leq K_1, x_2 \leq K_2$*, a.s.*

Proof We provide a few details for x_1, the result for x_2 follows similarly. Via positivity of solutions and the stochastic comparison principle, see [26] and the references within, it follows that

$$x_1(t) \leq \Phi(t),\ t \in [0, \tau),\ a.s. \tag{2}$$

where $\Phi(t)$ is the solution to the following SDE,

$$\mathrm{d}\Phi = \left[a_1\Phi - b_1\Phi^2\right] + \sigma_1(t)\Phi \mathrm{d}B_1(t) \tag{3}$$

Since global existence for (3) is standard [21,23], clearly $\tau = \infty$, a.s. Also we have via standard results [23] that if $a < \frac{(\sigma_1)^2}{2}$ then $\lim_{t\to\infty} \Phi(t) \to 0,\ a.s.$. And if $a > \frac{(\sigma_1)^2}{2}$, then one has persistence in mean, that is $\lim_{t\to\infty} \frac{1}{t}\int_0^t \Phi(s)ds = \frac{a_1 - \frac{(\sigma_1)^2}{2}}{b_1},\ a.s.$

Thus one easily infers

$$\lim_{t\to\infty} \mathbf{E}[x_1] \leq \lim_{t\to\infty} \mathbf{E}[\Phi] = \frac{a_1 - \frac{(\sigma_1)^2}{2}}{b_1}, \tag{4}$$

and so setting $K_1 = \frac{a_1 - \frac{(\sigma_1)^2}{2}}{b_1}$, we obtain

$$\lim_{t\to\infty} x_1 \leq K_1,\ a.s. \tag{5}$$

The almost sure eventual bound for x_2 follows similarly. □

Theorem 3.2 *There is a unique positive local solution* $(x_1(t), x_2(t))$ *for* $t \in [0, \tau_e)$ *to system (SM) almost surely (a.s.) for the initial* $x_1(0) > 0$, $x_2(0) > 0$ *where* τ_e *is the explosion time.*

Proof Let us assume $x_1(t) = e^{u(t)}$ and $x_2(t) = e^{v(t)}$. Therefore, $u(t) = \ln(x_1(t))$ and $v(t) = \ln(x_2(t))$. Then by using Itô's formula [21] we obtain

$$\begin{aligned} du(t) &= \frac{1}{x_1}\left\{x_1\left(a_1 - b_1x_1 - w_0\frac{x_1^{m_1-1}}{(x_1+d)^{m_1}}x_2^{m_2}\right)dt + \sigma_1 x_1 dB_1(t)\right\} \\ &\quad + \frac{1}{2}(\sigma_1 x_1)^2\left(\frac{-1}{x_1^2}\right)dt, \\ &= \left(a_1 - b_1x_1 - w_0\frac{x_1^{m_1-1}}{(x_1+d)^{m_1}}x_2^{m_2} - \frac{1}{2}\sigma_1^2\right)dt + \sigma_1 dB_1(t), \\ &= \left(a_1 - b_1e^{u(t)} - w_0\frac{e^{(m_1-1)u(t)}}{\left(e^{u(t)}+d\right)^{m_1}}e^{m_2v(t)} - \frac{1}{2}\sigma_1^2\right)dt + \sigma_1 dB_1(t). \end{aligned}$$

Similarly,

$$\begin{aligned} dv(t) &= \frac{1}{x_2}\left\{x_2\left(-a_2 - \theta f(x_1) + w_1\left(\frac{x_1}{x_1+d}\right)^{m_1} x_2^{m_2-1}\right)dt + \sigma_2 x_2 dB_2(t)\right\} \\ &\quad + \frac{1}{2}(\sigma_2 x_2)^2\left(\frac{-1}{x_2^2}\right)dt, \\ &= \left(-a_2 - \theta f(x_1) + w_1\left(\frac{x_1}{x_1+d}\right)^{m_1} x_2^{m_2-1} - \frac{1}{2}\sigma_2^2\right)dt + \sigma_2 dB_2(t), \\ &= \left(-a_2 - \theta f(e^{u(t)}) + w_1\left(\frac{e^{u(t)}}{e^{u(t)}+d}\right)^{m_1} e^{(m_2-1)v(t)} - \frac{1}{2}\sigma_2^2\right)dt + \sigma_2 dB_2(t), \end{aligned}$$

with $u(0) = \ln x_1(0)$, $v(0) = \ln x_2(0)$.

Therefore the system (SM) becomes

$$du(t) = \left(a_1 - b_1 e^{u(t)} - w_0 \frac{e^{(m_1-1)u(t)}}{\left(e^{u(t)}+d\right)^{m_1}} e^{m_2 v(t)} - \frac{1}{2}\sigma_1^2\right) dt + \sigma_1 dB_1(t),$$

$$dv(t) = \left(-a_2 - \theta f(e^{u(t)}) + w_1 \left(\frac{e^{u(t)}}{e^{u(t)}+d}\right)^{m_1} e^{(m_2-1)v(t)} - \frac{1}{2}\sigma_2^2\right) dt + \sigma_2 dB_2(t),$$

along with $u(0) = \ln x_1(0)$, $v(0) = \ln x_2(0)$, at $t = 0$.

We denote the above system along with the initial conditions as (SM). Note that the coefficients of (SM) satisfy the local Lipschitz condition, then for given initial values $u(0) > 0$, $v(0) > 0$ there is a unique maximal local solution $u(t)$, $v(t)$ on $[0, \tau_e)$, where τ_e is the explosion time of the solution. By Itô's formula [21], $x_1(t) = e^{u(t)}$, $x_2(t) = e^{v(t)}$ is the positive local solution to (SM) with initial value $x_1(0) > 0$, $x_2(0) > 0$. ☐

Theorem 3.3 *For any given initial value* $(x_1(0), x_2(0)) \in \Re_+^2$ *there is a unique solution* $(x_1(t), x_2(t))$ *on* $t \geq 0$ *to (SM), and this solution will remain in* $\Re_+^2$ *with probability 1, where* $\Re_+^2 = \left\{(x, y) \in \Re^2 \,|x, y > 0\right\}$.

Proof For convenience, let

$$F(x_1, x_2) = x_1 \left(a_1 - b_1 x_1 - w_0 \frac{x_1^{m_1-1}}{(x_1+d)^{m_1}} x_2^{m_2}\right),$$

$$G(x_1, x_2) = x_2 \left(-a_2 - \theta f(x_1) + w_1 \left(\frac{x_1}{x_1+d}\right)^{m_1} x_2^{m_2-1}\right).$$

Let $n_0 > 0$, be large enough, such that for $x_1(0)$ and $x_2(0)$ given, they lie in the interval $\left[\frac{1}{n_0}, n_0\right]$. For each integer $n > n_0$, define the stopping times [15]:

$$\tau_n = \inf\left\{t \in [0, \tau_e] : x_1(t) \notin \left(\frac{1}{n}, n\right) \quad or\ x_2(t) \notin \left(\frac{1}{n}, n\right)\right\}.$$

Clearly, τ_n is increasing as $n \to \infty$. Let $\tau_\infty = \lim_{n\to\infty} \tau_n$, where $\tau_\infty \leq \tau_e$. We only need to show that $\tau_\infty = \infty$. If this statement is false, there exist constants $T < \infty$ and $\varepsilon \in (0, 1)$ such that $P\{\tau_\infty \leq T\} > \varepsilon$. Consequently, there exists an integer $n_1 \geq n_0$ such that

$$P\{\tau_n \leq T\} > \varepsilon,\ n \geq n_1. \tag{6}$$

Define a C^2 function $V : \Re_+^2 \to \Re_+$ by

$$V(x_1, x_2) = \left(\sqrt{x_1} - 1 - 0.5 \ln x_1\right) + \left(\sqrt{x_2} - 1 - 0.5 \ln x_2\right).$$

If $(x_1, x_2) \in \Re^2_+$, then Itô's formula [21] yields

$$\begin{aligned} dV(x_1,x_2) &= \frac{dV}{dx_1}dx_1 + \frac{1}{2}\frac{d^2V}{dx_1^2}(dx_1)^2 + \frac{dV}{dx_2}dx_2 + \frac{1}{2}\frac{d^2V}{dx_2^2}(dx_2)^2, \\ &= \left(\frac{1}{2\sqrt{x_1}} - \frac{0.5}{x_1}\right)\big(F(x_1,x_2)dt + \sigma_1 x_1 dB_1\big) + \frac{1}{2}(\sigma_1^2 x_1^2) \\ &\quad \left(\frac{-1}{4}x_1^{-3/2} + \frac{0.5}{x_1^2}\right)dt \\ &\quad + \left(\frac{1}{2\sqrt{x_2}} - \frac{0.5}{x_2}\right)(G(x_1,x_2)dt + \sigma_2 x_2 dB_2) + \frac{1}{2}(\sigma_2^2 x_2^2) \\ &\quad \left(\frac{-1}{4}x_2^{-3/2} + \frac{0.5}{x_2^2}\right)dt, \qquad (7) \\ &= 0.5\left(x_1^{-0.5} - x_1^{-1}\right)F(x_1,x_2)dt + \frac{1}{8}\sigma_1^2\left(-x_1^{0.5} + 2\right)dt \\ &\quad + 0.5\sigma_1(x_1^{0.5} - 1)dB_1 \\ &\quad + 0.5\left(x_2^{-0.5} - x_2^{-1}\right)G(x_1,x_2)dt + \frac{1}{8}\sigma_2^2\left(-x_2^{0.5} + 2\right)dt \\ &\quad + 0.5\sigma_2(x_2^{0.5} - 1)dB_2. \end{aligned}$$

Now,

$$\begin{aligned} \left(x_1^{-0.5} - x_1^{-1}\right)F(x_1,x_2) &= \left(x_1^{-0.5} - x_1^{-1}\right)\left(a_1x_1 - b_1x_1^2 - \frac{w_0 x_1^{m_1}}{(x_1+d)^{m_1}}x_2^{m_2}\right), \\ &= a_1x_1^{0.5} - b_1x_1^{1.5} - w_0\frac{x_1^{m_1-0.5}}{(x_1+d)^{m_1}}x_2^{m_2} - a_1 + b_1x_1 \\ &\quad + w_0\frac{x_1^{m_1-1}}{(x_1+d)^{m_1}}x_2^{m_2}, \\ &\le K_1, a.s. \end{aligned}$$

This follows via Lemma 3.1, and here we assume sufficient time has passed so the bound is achieved (the time here may depend on the IC, that is large IC may require a larger time for the absorption). Similarly, we have

$$\begin{aligned} (x_2^{-0.5} - x_2^{-1})G(x_1,x_2) &= (x_2^{-0.5} - x_2^{-1})\left(-a_2x_2 - \theta f(x_1)x_2 + w_1\left(\frac{x_1}{x_1+d}\right)^{m_1}x_2^{m_2}\right), \\ &= -a_2x_2^{0.5} - \theta f(x_1)x_2 + w_1\left(\frac{x_1}{x_1+d}\right)^{m_1}x_2^{m_2-0.5} + a_2 + \theta f(x_1) \\ &\quad - w_1\left(\frac{x_1}{x_1+d}\right)^{m_1}x_2^{m_2-1}, \\ &\le K_2, a.s. \end{aligned}$$

This follows via Lemma 3.1. Here K_1 and K_2 are constants as in Lemma 3.1. The terms $\frac{1}{8}\sigma_1^2\left(-x_1^{0.5} + 2\right)dt$ and $\frac{1}{8}\sigma_2^2\left(-x_2^{0.5} + 2\right)dt$ are used as follows, we bring the

$\frac{1}{8}\sigma_1^2\left(-x_1^{0.5}\right)dt$ and $\frac{1}{8}\sigma_2^2\left(-x_2^{0.5}\right)dt$ terms to the left and use their positivity, and integrate both sides of the above Eq. (7) from 0 to $\tau_n \wedge T$ and then take expectation on both sides to yield

$$\begin{aligned}
&E\,V\left(x_1\left(\tau_n \wedge T\right), x_2\left(\tau_n \wedge T\right)\right) + \frac{1}{8}\mathbb{E}\left[\int_0^{\tau_n \wedge T} \sigma_1^2(t) x_1^{0.5} dt\right] \\
&\quad + \frac{1}{8}\mathbb{E}\left[\int_0^{\tau_n \wedge T} \sigma_2^2(t) x_2^{0.5} dt\right] \\
&\leq V\left(x_1(0), x_2(0)\right) + \left(K_1 + K_2 + K_3 + K_4\right) E\left(\tau_n \wedge T\right) \\
&\quad + 0.5\sigma_1\mathbb{E}\left[\int_0^{\tau_n \wedge T} x_1^{0.5} dB_1\right] + 0.5\sigma_2\mathbb{E}\left[\int_0^{\tau_n \wedge T} x_1^{0.5} dB_2\right].
\end{aligned} \tag{8}$$

Here K_3, K_4 are the bounds on the $\sigma_1^2(t)$, $\sigma_2^2(t)$ respectively. The control of the Itô's integral terms is handled via the embedding of $L^2 \hookrightarrow L^1$, the Ito isometry [23], and a standard application of Holder and Young's inequality,

$$\begin{aligned}
&0.5\sigma_1\mathbb{E}\left[\int_0^{\tau_n \wedge T} x_1^{0.5} dB_1\right] \\
&\leq C\mathbb{E}\left[\left(\int_0^{\tau_n \wedge T} x_1^{0.5} dB_1\right)^2\right] \\
&= C\mathbb{E}\left[\int_0^{\tau_n \wedge T} x_1 ds\right] \\
&\leq \frac{b_1}{2}\mathbb{E}\left[\int_0^{\tau_n \wedge T} x_1^{1.5} ds\right] + K_5(\tau_n \wedge T).
\end{aligned}$$

The first term in the above can be absorbed into the left hand side with the same term, with coefficient b_1. The Ito integral involving dB_2 is handled in the same way, and there is a bound generated via a constant K_6. This finally yields

$$E\,V\left(x_1\left(\tau_n \wedge T\right), x_2\left(\tau_n \wedge T\right)\right) \tag{9}$$

$$\leq V\left(x_1(0), x_2(0)\right) + \left(K_1 + K_2 + K_3 + K_4 + K_5 + K_6\right) E\left(\tau_n \wedge T\right) \tag{10}$$

Now, we set $\Omega_n = \{\tau_n \leq T\}$. So, by Eq. (6), we have $P\left(\Omega_n\right) \geq \varepsilon$. Note that for each $w \in \Omega_n$, there is some i such that $x_i\left(\tau_n, w\right)$ equals to n or $\frac{1}{n}$ for $i = 1, 2$. Hence $V\left(x_1\left(\tau_n \wedge T\right), x_2\left(\tau_n \wedge T\right)\right)$ is no less than

$$\min\left\{\sqrt{n} - 1 - 0.5\ln n\ ,\ \sqrt{1/n} - 1 - 0.5\ln 1/n\right\}.$$

So by Eq. (8), we have

$$\begin{aligned}
&V\left(x_1(0), x_2(0)\right) + \left(K_1 + K_2 + K_3 + K_4 + K_5 + K_6\right) E\left(\tau_n \wedge T\right) \\
&\geq E\left[1_{\Omega_n(w)} V\left(x_1\left(\tau_n\right), x_2\left(\tau_n\right)\right)\right], \\
&> \varepsilon \min\left\{\sqrt{n} - 1 - 0.5\ln n\ ,\ \sqrt{{}^1\!/_n} - 1 - 0.5\ln {}^1\!/_n\right\},
\end{aligned}$$

where 1_{Ω_n} is the indicator function of Ω_n. Letting $n \to \infty$, leads to a contradiction

$$\infty > V\left(x_1(0), x_2(0)\right) + (K_1 + K_2 + K_3 + K_4 + K_5 + K_6)\,T = \infty.$$

This completes the proof. □

4 Persistence

In this section, persistent in time average of the model (1a)–(1b) is intended to be proved under certain conditions on the parameters.

Definition 4.1 The model (1a)–(1b) is said to be persistent in time average if for all $i = 1, 2$

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t x_i(s)ds > 0, \quad a.s.$$

The lemma below from Xia et al. [22] is used to prove persistent in time average for (1a)–(1b).

Lemma 4.2 *Let $f \in C[[0,\infty)\times\Omega,(0,\infty)]$, $F(t) \in ((0,\infty)\times\Omega,\mathbb{R})$. If there exist positive constants λ_0 and λ such that*

$$\log f(t) \geq \lambda t - \lambda_0 \int_0^t f(s)ds + F(t), \qquad t \geq 0,\ a.s.,$$

and

$$\lim_{t\to\infty} \frac{F(t)}{t} = 0,\ a.s.,$$

then

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t f(s)ds \geq \frac{\lambda}{\lambda_0}, \quad a.s.$$

Proof This proof is shown in Lemma 17 of [22].

Theorem 4.3 *Let us assume $a_1 - k - 0.5\sigma_1^2(s) > 0$, where k is a positive constant, that is an upper bound for $\frac{w_0}{d}\mathbb{E}[x_2]$, then the solution $(x_1(t), x_2(t))$ of the system (1a)–(1b) with any given initial value $(x_1(0), x_2(0)) \in \Re_+^2$ is persistent in time average.*

Proof Using Itô's formula on (1a), we obtain

$$d[\ln x_1(t)] = \left[a_1 - b_1 x_1(t) - w_0 \frac{x_1^{m_1-1}(t)}{(x_1(t)+d)^{m_1}} x_2^{m_2}(t) - \frac{1}{2}\sigma_1^2(t)\right] dt + \sigma_1(t)dB_1(t). \tag{11}$$

Integrating both sides of Eq. (11) from 0 to t yields,

$$\frac{\ln\left({x_1(t)}/{x_1(0)}\right)}{t} = \frac{\int_0^t \left(a_1 - 0.5\sigma_1^2(s)\right) ds}{t} - \frac{b_1 \int_0^t x_1(s)ds}{t} + \frac{\int_0^t \sigma_1(s) dB_1(s)}{t} - \frac{\int_0^t w_0 \frac{x_1^{m_1-1}(s)}{(x_1(s)+d)^{m_1}} x_2^{m_2}(s) ds}{t}. \tag{12}$$

But noting via positivity, Theorem 3.3 and the embedding of $L^1 \hookrightarrow L^{m_2}$ we have

$$w_0 \frac{x_1^{m_1-1}(s)}{(x_1(s)+d)^{m_1}} x_2^{m_2}(s) \leq \frac{w_0}{d} x_2^{m_2}(s) \leq k,\ a.s$$

for some positive constant k, then

$$-\frac{\int_0^t w_0 \frac{x_1^{m_1-1}(s)}{(x_1(s)+d)^{m_1}} x_2^{m_2}(s) ds}{t} \geq -\frac{\int_0^t k ds}{t},\ a.s$$

Thus

$$\frac{\ln\left({x_1(t)}/{x_1(0)}\right)}{t} \geq \frac{\int_0^t \left(a_1 - 0.5\sigma_1^2(s)\right) ds}{t} - \frac{b_1 \int_0^t x_1(s)ds}{t} + \frac{\int_0^t \sigma_1(s) dB_1(s)}{t} - \frac{\int_0^t k ds}{t} \tag{13}$$

$$= \frac{\int_0^t \left(a_1 - k - 0.5\sigma_1^2(s)\right) ds}{t} - \frac{b_1 \int_0^t x_1(s)ds}{t} + \frac{\int_0^t \sigma_1(s) dB_1(s)}{t}. \tag{14}$$

Moreover, $M_i = \int_0^t \sigma_i(s) dB_i(s)$ is a local martingale whose quadratic variation is given by

$$\langle M_i, M_i \rangle_t = \int_0^t \sigma_i^2(s) ds \leq ||\sigma||_\infty^2 t.$$

The above bound allows us to apply the strong law of large numbers for Martingales [21] to obtain

$$\lim_{t\to\infty} \frac{M_i(t)}{t} = 0,\ a.s. \tag{15}$$

Applying Lemma 4.2 and condition $a_1 - k - 0.5\sigma_1^2(s) > 0$, we obtain the following result

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t x_1(s) ds \geq \frac{1}{b_1} \liminf_{t\to\infty} \frac{1}{t} \int_0^t \left(a_1 - k - 0.5\sigma_1^2(s)\right)(s) ds > 0, \quad a.s.$$

Now assume that x_1 persists. We can use standard comparison on the predator equation to obtain

$$\begin{aligned} dx_2 &= \left(-a_2x_2 - \theta f(x_1)x_2 + w_1\left(\frac{x_1}{x_1+d}\right)^{m_1} x_2^{m_2}\right)dt + \sigma_2(t)x_2\mathrm{d}B_2(t),\\ &\geq \left(-a_2x_2 - \theta G x_2 + w_1\left(\frac{\epsilon}{G+d}\right)^{m_1} x_2^{m_2}\right)dt + \sigma_2(t)x_2\mathrm{d}B_2(t),\\ &= \left(-C_1x_2 + C_2x_2^{m_2}\right)dt + \sigma_2(t)x_2\mathrm{d}B_2(t), \text{ where } C_1, C_2 > 0. \end{aligned} \tag{16}$$

This follows via the persistence of x_1, the boundedness of x_1 via Theorem 3.3, and the assumptions on the toxic release function. We now consider a subsolution x_2^* that solves

$$dx_2^* = (-C_1x_2^* + C_2(x_2^*)^{m_2})dt + \sigma_2^l x_2^*\mathrm{d}B_2(t),$$

Our goal is to derive conditions under which x_2^* will persist, as then x_2 will persist by simple comparison. Via standard methods for SDE see [23], the above equation is easily solved in closed form. We have

$$\begin{aligned} x_2^*(t) &= e^{\left(\sigma_2^l B_2(t) - \frac{(\sigma_2^l)^2}{2}t\right)}\\ &\quad\times\Bigg[\left(e^{-C_1(1-m_2)t}(x_2^*(0))^{1-m_2}\right) + e^{-C_1(1-m_2)t}\\ &\quad\int_0^t e^{C_1(1-m_2)s}\left(e^{-(1-m_2)\sigma_2^l B_2(t) + (1-m_2)(\sigma_2^l)^2 s)}\right)ds\Bigg]^{\frac{1}{1-m_2}} \end{aligned} \tag{17}$$

Under conditions that $x_2^*(t)$ given by the above will persist, we have by simple comparison that x_2 will persist. This completes the proof. □

5 Extinction criteria

The following theorem shows that under certain sufficient conditions the environmental noise may drive the predator and prey populations to extinction, whilst the equilibrium populations are stable *in the absence of the noise*.

Theorem 5.1 *Assume that we have Trace* $(\mathbf{J}) < 0$ *and det* $(\mathbf{J}) > 0$*, where* $\mathbf{J}$ *is the linearized variational matrix of system (DM) around the interior equilibrium* $E^*(x_1^*, x_2^*)$*. Further assume we choose* σ_1, a_1 *s.t* $a_1 < \frac{1}{2t}\int_0^t \left(\sigma_1^2(s)\right)ds$*. Then the equilibrium populations to* (DM) *will be locally asymptotically stable, whilst the populations* $x_1(t)$ *and* $x_2(t)$*, that are solutions to (SM), will be driven to extinction exponentially, with probability one.*

Proof The linearized variational matrix of system (SM) around the interior equilibrium $E^*(x_1^*, x_2^*)$ is

$$\mathbf{J} = \begin{bmatrix} a_1 - 2b_1x_1^* - \frac{dm_1w_0{x^*}_2^{m_2}{x^*}_1^{m_1}}{x_1^*(x_1^*+d)^{m_1+1}} & -\frac{m_2w_0x_2^{*m_2-1}{x^*}_1^{m_1}}{(x_1^*+d)^{m_1}} \\ -\theta\frac{\mathrm{d}f(x_1^*)}{\mathrm{d}x_1^*}x_2^* + m_1dw_1x_2^{*m_2}\left(\frac{x_1^{*m_1-1}}{(x_1^*+d)^{m_1+1}}\right) & -\theta f(x_1^*) - a_2 + m_2w_1x_2^{*m_2-1}\left(\frac{x_1^*}{x_1^*+d}\right)^{m_1} \end{bmatrix}$$
$$= \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix},$$

with

$$Trace\,(\mathbf{J}) = a_1 - a_2 - 2b_1x_1^* - \theta f(x_1^*) + m_2w_1{x^*}_2^{m_2-1}\left(\frac{x_1^*}{x_1^*+d}\right)^{m_1} - dm_1w_0{x^*}_2^{m_2}\left(\frac{x_1^{*m_1}}{x_1^*(x_1^*+d)^{m_1+1}}\right),$$

and

$$det\,(\mathbf{J}) = \left(a_1 - 2b_1x_1^* - \frac{dm_1w_0{x^*}_2^{m_2}{x^*}_1^{m_1}}{x_1^*(x_1^*+d)^{m_1+1}}\right)\left(-\theta f(x_1^*) - a_2 + m_2w_1x_2^{*m_2-1}\left(\frac{x_1^*}{x_1^*+d}\right)^{m_1}\right) - \left(-\frac{m_2w_0x_2^{*m_2-1}{x^*}_1^{m_1}}{(x_1^*+d)^{m_1}}\right)\left(-\theta\frac{\mathrm{d}f(x_1^*)}{\mathrm{d}x_1^*}x_2^* + m_1dw_1x_2^{*m_2}\left(\frac{x_1^{*m_1-1}}{(x_1^*+d)^{m_1+1}}\right)\right).$$

Thus the local stability of the interior equilibrium (x_1^*, x_2^*) requires that we have $Trace\,(\mathbf{J}) < 0$ and $det\,(\mathbf{J}) > 0$. Assume, that the above criteria are met, then (x_1^*, x_2^*) are locally asymptotically stable in the absence of noise. Now define $V(x) = c_1x_1 + c_2x_2$, then by It$\hat{o}$'s formula, we have (11).

Now, by the Eqs. (11), (15), and by using the condition $a_1 < \frac{1}{2t}\int_0^t \left(\sigma_1^2(s)\right) ds$, we obtain

$$\lim_{t\to\infty} \sup \frac{\ln(x_1(t))}{t} \le \frac{\int_0^t \left(a_1 - 0.5\sigma_1^2(s)\right) ds}{t} < 0, \quad a.s$$

We show extinction for x_2, using a simple approach. Under the criteria posed in Theorem 5.1, x_1 goes to extinction, a.s, thus the equation for x_2 reduces to

$$\mathrm{d}x_2 = -a_2x_2\mathrm{d}t + \sigma_2(t)x_2\mathrm{d}B_2(t),$$

This follows via the assumption on the toxic release function f. Standard comparison now yields

$$\mathrm{d}x_2 \le -a_2x_2\mathrm{d}t + \sigma_2^u x_2\mathrm{d}B_2(t),$$

Note that the solution to

$$\mathrm{d}x_2^* = -a_2x_2^*\mathrm{d}t + \sigma_2^u x_2^*\mathrm{d}B_2(t),$$

goes to extinction trivially by standard theory [23], if $-a_2 < \frac{(\sigma_2^u)^2}{2}$. But this follows as $a_2 > 0$. Thus x_2^* goes to extinction, a.s, and then so does x_2 by simple comparison.

This proves the theorem. □

It is important to note that the extinction result depends solely on the noise intensity in the prey species $\sigma_1(t)$. It is independent of the magnitude of noise intensity in the predator species $\sigma_2(t)$, or the magnitude of toxin release θ.

6 Global existence

Theorem 6.1 *Let $x_1(t), x_2(t)$ be a solution to (SM) with positive initial conditions $(x_1(0), x_2(0))$. If $a_1 < \frac{1}{2t}\int_0^t \left(\sigma_1^2(s)\right) ds$, then there exist constants C_4 and C_5 depending only on the parameters and initial conditions, such that*

$$E\left[x_1(t)\right] \leq C_4,\ \ E\left[x_2(t)\right] \leq C_5.$$

Proof We first multiply the equation for x_1 in (1a) by w_1, and the equation for x_2 in (1b) by w_0. Then using Itô's lemma [21] we obtain

$$\begin{aligned} & d[w_1 \ln x_1(t)] \\ & \quad = \left[w_1 a_1 - w_1 b_1 x_1(t) - w_1 w_0 \frac{x_1^{m_1-1}(t)}{(x_1(t)+d)^{m_1}} x_2^{m_2}(t) - \frac{1}{2} w_1 \sigma_1^2(t) \right] dt \\ & \qquad + w_1 \sigma_1(t) dB_1(t). \end{aligned} \tag{18}$$

$$\begin{aligned} & d\left[w_0 \ln x_2(t)\right] \\ & \quad = \left[-w_0 a_2 - w_0 \theta f\left(x_1(t)\right) + w_1 \left(\frac{x_1(t)}{x_1(t)+d}\right)^{m_1} x_2^{(m_2-1)}(t) - \frac{1}{2} w_0 \sigma_2^2(t) \right] dt \\ & \qquad + w_0 \sigma_2 dB_2(t). \end{aligned} \tag{19}$$

We now integrate (18), (19) in time from $[0, T]$ and take expectations to yield

$$\begin{aligned} & w_1 \mathbb{E}[\ln x_1(T)] + w_1 b_1 \int_0^T \mathbb{E}[x_1(t)] dt + \frac{1}{2} w_1 \int_0^T \sigma_1^2(t) dt \\ & = a_1 w_1 T - w_1 w_0 \int_0^T \mathbb{E}[\frac{x_1^{m_1-1}(t)}{(x_1(t)+d)^{m_1}} x_2^{m_2}] dt + w_1 \mathbb{E}[\ln x_1(0)] \end{aligned} \tag{20}$$

and

$$\begin{aligned} & w_0 \mathbb{E}[\ln x_2(T)] + w_0 b_1 \int_0^T \mathbb{E}[\theta f\left(x_1(t)\right)] dt + \frac{1}{2} w_0 \int_0^T \sigma_2^2(t) dt + w_0 a_2 T \\ & = w_1 w_0 \int_0^T \mathbb{E}[\frac{x_1^{m_1-1}(t)}{(x_1(t)+d)^{m_1}} x_2^{m_2}] dt + w_0 \mathbb{E}[\ln x_2(0)]. \end{aligned} \tag{21}$$

We can now add up (20) and (21), use the condition that $a_1 < \frac{1}{2t}\int_0^t \left(\sigma_1^2(s)\right) ds$ to obtain,

$$w_1\mathbb{E}[\ln x_1(T)] + w_0\mathbb{E}[\ln x_2(T)] \leq w_1\mathbb{E}[\ln x_1(0)] + w_0\mathbb{E}[\ln x_2(0)]. \tag{22}$$

Thus we obtain

$$\mathbb{E}[\ln x_1(T)] \leq C_3,\ \mathbb{E}[\ln x_2(T)] \leq C_3,\ \forall T. \tag{23}$$

Here C_3 depends only on initial conditions and parameters in the problem. Thus, trivially we can ascertain that,

$$\mathbb{E}[x_1(T)] \leq C_4,\ \mathbb{E}[x_2(T)] \leq C_5,\ \forall T. \tag{24}$$

We can take the limit in above as $T \to \infty$, and the bound still holds, as the constants C_4 and C_5, is independent of T. Thus $\mathbb{E}[x_1(t)]$, $\mathbb{E}[x_2(t)]$ cannot blow-up in finite time, because if without loss of generality say x_1 did, we would have

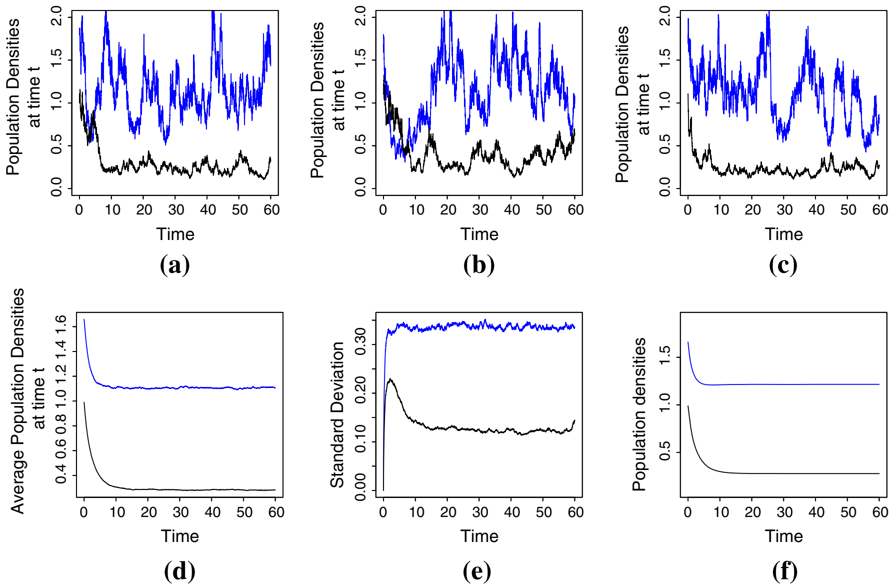
$$\lim_{t\to T^*<\infty} \mathbb{E}[\ln x_1(t)] = \mathbb{E}[\lim_{t\to T^*} \ln(x_1(t))] = \mathbb{E}[\ln(\lim_{t\to T^*} x_1(t))] = \infty,$$

but this is a contradiction via (23). And thus we have global existence.

Remark 1 Note that the above result holds independent of the relation between θ and w_1. That is we do not need that $\theta > w_1$, for global existence. If we enforce $\theta > w_1$ then one has very strong damping in predator population—this leads decline of the predators, and thus global existence easily follows via simple comparison method. Also such a restriction might not be interesting biologically, because it will probably always lead to predator extinction, and prey going to carrying capacity. Thus will not be a feasible biological model for ecosystems where one observes persistence. Also, $\theta < w_1$, is indicative of the fact that the predator gains more from his hunting efforts, than he his harmed by the prey release—which provides a net positive feedback to the predator population. If we restricted $\theta > w_1$, then in a sense we are saying that there is very strong prey defense mechanism, and in a sense the particular prey is not huntable or optimal for the predator to hunt. In this setting the predator prey formulism might not make sense from a optimal foraging theory as well [31], which says that predators want the most “bang” for the “buck”. □

7 Numerical simulation results

In this section, we will use the semi-implicit Milstein method described in [24] to give some numerical finding to system (SM). The semi-implicit Milstein method is used to approximate the $It\hat{o}'s$ stochastic ordinary differential equation given in system (SM), with a time-step size of $\Delta t = 0.001$. This reduces system (SM) into a difference equation given as



(a) (b) (c)

(d) (e) (f)

Fig. 1 Solution of system (SM) with Holling type I functional response with $m_1 > m_2$ where (blue line) represents the prey behavior and (black line) represents the predator behavior. **a**–**c** are 3 sample runs of $x_1(t)$ and $x_2(t)$. Hence **a** is when $k = 1$, **b** is when $k = 2$ and **c** is when $k = 3$. **d**, **e** are the mean and standard deviation of $x_1(t)$ and $x_2(t)$ and **f** being the behavior of population size of $x_1(t)$ and $x_2(t)$ where $\sigma_1 = \sigma_2 = 0.0$ and $\theta = 0.25$. Parameter set given in Table 2. (Color figure online)

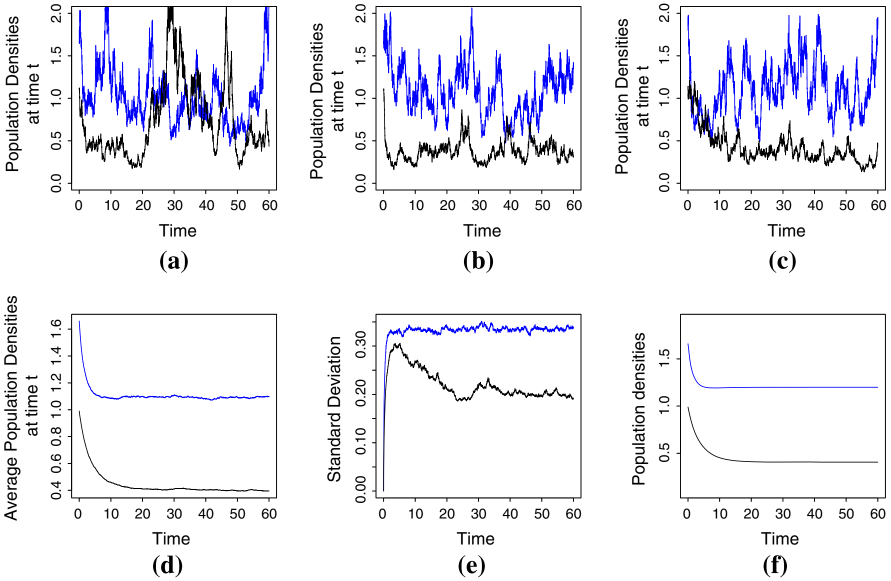
$$\begin{aligned}\mathbf{x}_1^{n+1} - \mathbf{x}_1^n = {} & \left[a_1\mathbf{x}_1^{n+1} - b_1(\mathbf{x}_1^{n+1})^2 - w_0\left(\frac{\mathbf{x}_1^{n+1}}{\mathbf{x}_1^{n+1}+d}\right)^{m_1}(\mathbf{x}_2^{n+1})^{m_2}\right]\Delta t \\ & + \sigma_1\mathbf{x}_1^n\Delta B_1 + \frac{\sigma_1^2}{2}\mathbf{x}_1^n\Big((\Delta B_1)^2 - \Delta t\Big),\end{aligned} \tag{25}$$

$$\begin{aligned}\mathbf{x}_2^{n+1} - \mathbf{x}_2^n = {} & \left[-a_2\mathbf{x}_2^{n+1} - \theta f(\mathbf{x}_1^{n+1})\mathbf{x}_2^{n+1} + w_1\left(\frac{\mathbf{x}_1^{n+1}}{\mathbf{x}_1^{n+1}+d}\right)^{m_1}(\mathbf{x}_2^{n})^{m_2}\right]\Delta t \\ & + \sigma_1\mathbf{x}_2^n\Delta B_2 + \frac{\sigma_1^2}{2}\mathbf{x}_2^n\Big((\Delta B_2)^2 - \Delta t\Big),\end{aligned} \tag{26}$$

where $\mathbf{x}_1^n = x_1(t_n)$ and $\mathbf{x}_2^n = x_2(t_n)$ are the discretization of the solutions at discrete time t_n. Also $\Delta B_1 = z_1\sqrt{\Delta t}$ and $\Delta B_2 = z_2\sqrt{\Delta t}$, where $z_1, z_2 \sim \mathcal{N}(0, 1)$ are independent normally distributed random numbers with mean 0 and standard deviation 1. These random numbers are generated using Box–Muller–Wiener algorithm [25].

Remark 2 The semi-implicit Milsten scheme used to advance the difference equation in time necessitates solving a nonlinear system of equations at each time-step. This is achieved through a Newton–Raphson method.

We next investigate the asymptotic stability of system (SM) by studying the behavior of θ on system (SM) as shown on Figs. 1, 2 and 3. Numerical experiments are conducted for fixed values of $b_1, w_0, w_1, d, a_1, a_2, m_1, m_2$ and varying values of θ and $f(x_1)$ as



(a) **(b)** **(c)**

(d) **(e)** **(f)**

Fig. 2 Solution of system (SM) with Holling type II functional response with $m_1 > m_2$ where (blue line) represents the prey behavior and (black line) represents the predator behavior. **a**–**c** are 3 sample runs of $x_1(t)$ and $x_2(t)$. Hence **a** is when $k = 1$, **b** is when $k = 2$ and **c** is when $k = 3$. **d**, **e** are the average and standard deviation of $x_1(t)$ and $x_2(t)$ for 3.0×10^3 runs where $\sigma_1 = \sigma_2 = 0.3$ and **f** being the behavior of population size of $x_1(t)$ and $x_2(t)$ where $\sigma_1 = \sigma_2 = 0.0$ and $\theta = 0.85$. Parameter set given in Table 2. (Color figure online)

given in Table 2 with fixed intensity of the Gaussian noise with $\sigma = 0$ (Deterministic case) and $\sigma > 0$. In Figs. 4, 5 and 6, we then investigate numerically conditions under which extinction occurs using Theorem 5.1. Thus for each set of values of the parameters, a set of numerical approximations is conducted.

To understand the stochastic behavior of system (SM) with solutions $x_1(t)$ and $x_2(t)$, $M = 3.0 \times 10^3$ Monte Carlo simulations are conducted and the behavior of both the sample means and sample standard deviations are given. Both the sample mean and sample deviation of the sample population are calculated at each time of the total M runs. We first denote $x_1^k(t_j)$ as the approximate solution of $x_1(t_j)$ at run k. Thus for M Monte Carlo runs, there exist M possible solutions of x_1 at time t_j given as $x_1^k(t_j)$. Hence the average population density at time t_j is taken over all the possible M solution of $x_i(t_j)$. This definition also applies for $x_2^k(t_j)$. Hence

$$\hat{x}_1(t_j) = \frac{1}{M} \sum_{k=1}^{M} x_1^k(t_j), \tag{27a}$$

$$\hat{x}_2(t_j) = \frac{1}{M} \sum_{k=1}^{M} x_2^k(t_j), \tag{27b}$$

where as $j = 0, \ldots, N$ with $N = \lfloor \frac{60}{\Delta t} \rfloor$ and M is the number of Monte Carlo runs.

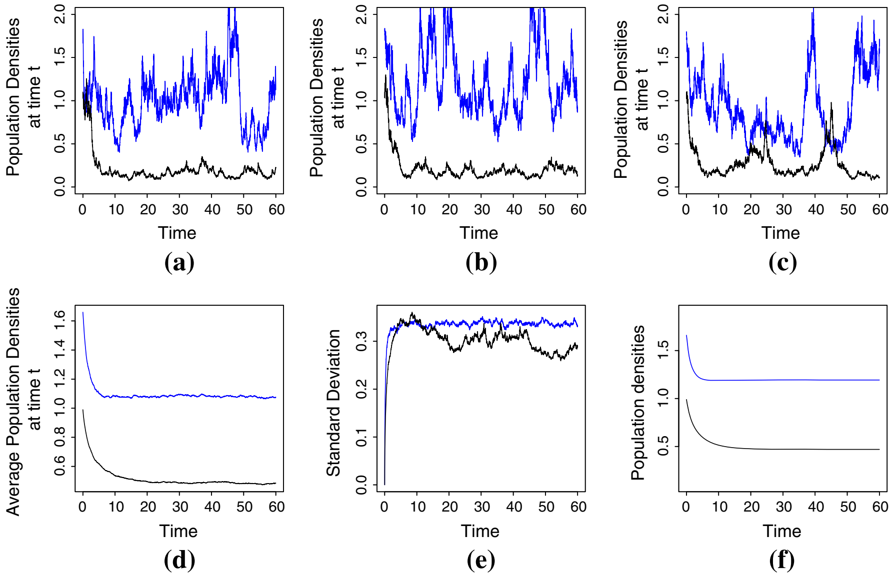


Fig. 3 Solution of system (SM) with Holling type III functional response with $m_1 > m_2$ where (blue line) represents the prey behavior and (black line) represents the predator behavior. **a**–**c** are 3 sample runs of $x_1(t)$ and $x_2(t)$. Hence **a** is when $k = 1$, **b** is when $k = 2$ and **c** is when $k = 3$. **d**, **e** are the average and standard deviation of $x_1(t)$ and $x_2(t)$ for 3.0×10^3 runs where $\sigma_1 = \sigma_2 = 0.3$ and **f** being the behavior of population size of $x_1(t)$ and $x_2(t)$ where $\sigma_1 = \sigma_2 = 0.0$ and $\theta = 2.0$. Parameter set given in Table 2. (Color figure online)

Table 2 Lists of parameters used in the simulations of Figs. 1, 2 and 3

$a_1 = 0.7$	$b_1 = 0.5$	$w_0 = 0.3$	$w_1 = 0.3$
$m_1 = 0.5$	$m_2 = 0.2$	$a_2 = 0.1$	$d = 4.0$

Similarly the sample standard deviation for each $x_1^k(t_j)$ and $x_2^k(t_j)$, measures how much the individual solution at times t_j in each run are dispersed or scattered around individual average population densities.

Figures 1, 2 and 3 shows the effect of varying θ and $f(x_1)$ on system (SM). Clearly as seen in the prey density's population standard deviation, there is a fairly high standard deviation which signifies that the individual times of $x_1^k(t_j)$ for each k run spreads out over a large range of values. The predator population density also has a fairly low standard deviation as compared to the standard deviation associated with the density population of the prey. Low standard deviations indicates that the individual times of each $x_2^k(t_j)$ of the predator population are very close to the average population density $\hat{x}_2(t_j)$, which means the predator behavior in time is fairly the same within each run. This therefore implies that if the deterministic system solution is globally stable then the corresponding stochastic system will preserve this property when the noise σ_i are sufficiently small. Hence the stability of system (SM) may depends on the noise intensity σ_i.

In Fig. 4, we give a numerical evidence to support Theorem 5.1. This reveals that if the condition $a_1 - 0.5\sigma_1^2 < 0$ holds then $x_1(t)$ and $x_2(t)$ by system (SM) will become

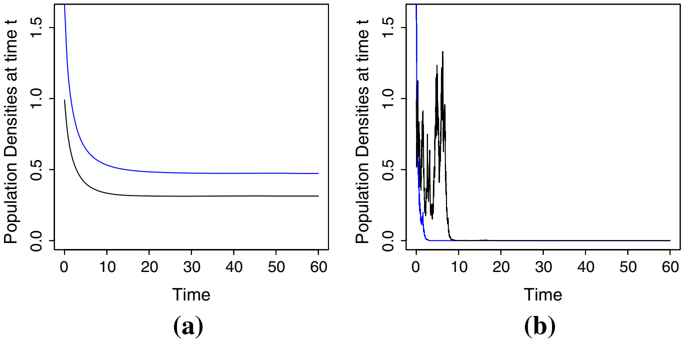


Fig. 4 Numerical behavior of system (DM) versus (SM) is shown. Here we are numerically illustrating the results of Theorem 5.1 using Holling type II functional response with $\sigma_1 = \sigma_2 = 0.895$. Parameter set given in Table 3

Table 3 List of parameters used in the simulations of Figs. 4, 5 and 6

$a_1 = 0.4$	$b_1 = 0.5$	$w_0 = 0.3$	$\theta = 0.31$	$w_1 = 0.3$
$m_1 = 0.5$	$m_2 = 0.2$	$a_2 = 0.1$	$d = 4.0$	

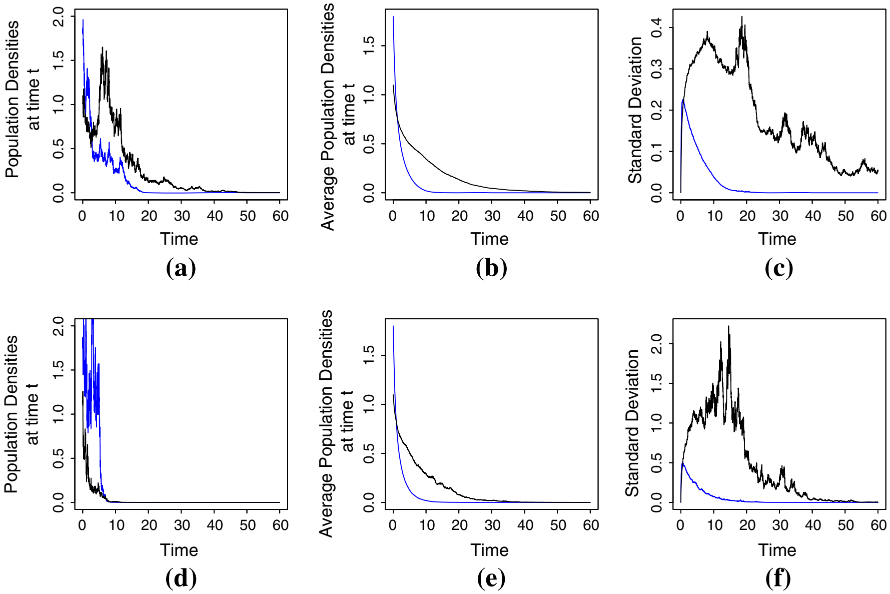


Fig. 5 Numerical behavior of system (SM) with Holling type I functional response with $\sigma_i = 0.3$ shown in (**a**–**c**) and $\sigma_i = 0.7$ shown in (**d**–**f**), where (blue line) represents the prey behavior and (black line) represents the predator behavior. **a**, **d** are sample runs of $x_1(t)$ and $x_2(t)$ when $k = 1$. Parameter set given in Table 3. (Color figure online)

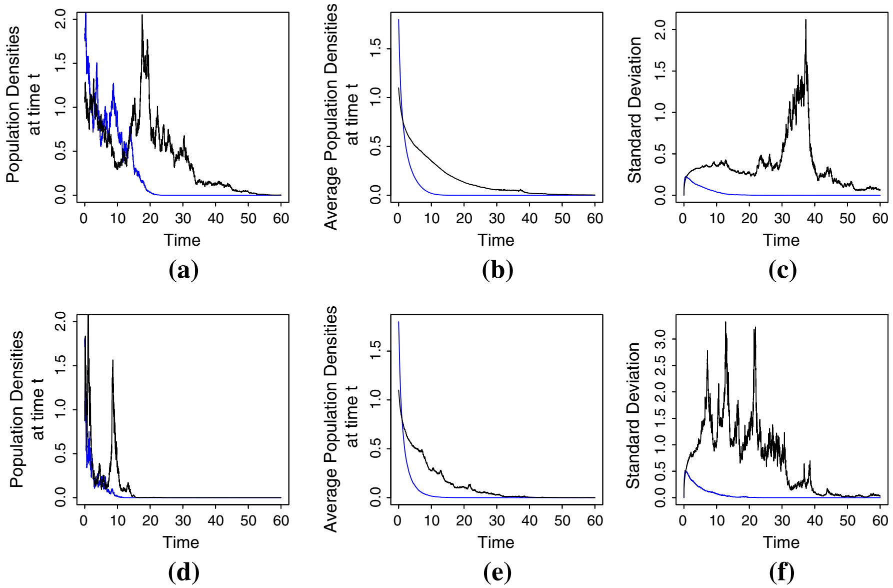


Fig. 6 Numerical behavior of system (SM) with Holling type II functional response with $\sigma_i = 0.3$ shown in (**a**–**c**) and $\sigma_i = 0.7$ shown in (**d**–**f**), where (blue line) represents the prey behavior and (black line) represents the predator behavior. **a**, **d** are sample runs of $x_1(t)$ and $x_2(t)$ when $k = 1$. Parameter Parameter set given in Table 3. (Color figure online)

extinct exponentially with probability 1. We therefore again investigated system (SM) choosing $f(x)$ as Holling type I and Holling type II functional response and varying the intensity of the noise $\sigma_i > 0$ respectively. This is done to show how the noise can affect system (SM) even when extinct is possible.

Figure 5 shows the effect of the noise intensity σ_i on system (SM) with $f(x_1)$ chosen as Holling type I functional response. Clearly from Fig. 5, increasing σ_i from 0.3 to 0.7 does cause an increase in the variability in the average predator population. This behavior is due to the increasing growth rate of the standard deviation in time as σ_i increases. Therefore as shown in Fig. 5, increasing the noise intensity in Theorem 5.1 for system (SM) does have a significant effect on the average predator population behavior. But the effect of σ_i on the average prey population is relatively low even as σ_i increases. A similar behavior for both the prey and predator population is observed even either $f(x_1)$ chosen as Holling type II functional response as shown in Fig. 6. Therefore even though noise does affect both the prey and predator populations, its effect is fairly high on the predator population even as it goes into extinction.

8 Conclusions and discussions

In this paper, we have studied a two dimensional predator–prey system with mutual interferences m_1 and m_2 , with different Holling's type functional responses in predator equation. Sufficient conditions for the local existence, persistence, extinction and global existence of positive solutions are obtained in stochastic environment, which is fundamental to many problems in population dynamics. Note that our global existence result is derived independent of the relation between the predation rate w_1 and the toxin release parameter θ. That is, we *do not* require the restriction $\theta > w_1$, for global existence. We have added white Gaussian noise in the growth rate parameter of the prey population and death rate parameter of predator populations and observe that the stochastic system preserves the property of the deterministic system when the noise is sufficiently small. However for large enough noise the stochastic system can be driven to extinction, whilst the deterministic system persists, via Theorem 5.1. This result depends solely on the noise intensity of the prey species, and is independent of the noise intensity of the predator species. Also note, the global existence result given via Theorem 6.1 does not depend on the toxin release parameter θ. This warrants some discussion. In the vent of no toxin release that is $\theta = 0$, the model reduces to stochastic predator–prey model, with interference, as studied by many authors. Here global existence also follows, from the form of the functional responses, in that x_1, x_2 cannot explode in finite time. Thus the toxin release parameter provides further damping to the system, by decreasing the predator population, and so actually "helps" in global existence. The effect of toxin release as well as pollution effects, is well studied in literature and the interested reader is refereed to [27,28]. It would be a very interesting problem to consider negative values of θ as modeled, where this could be interpreted as additional predation. Depending on the form of the function $f(x_1)$, one may or may not expect global existence [29].

Numerically, we have simulated both the model systems for different values of mutual interference parameters and same values of noise strength i.e., $\sigma_1 = \sigma_2 = 0.2$, and present the results in the form of time series. Sufficient conditions for extinction for both the population are obtained. Note, the stochastic system has a unique local positive solution $(x_1(t), x_2(t))$ for $t \in [0, \tau_e)$ via Theorem 3.2, as well as a globally existing solution via Theorem 6.1.

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