Homotopies for connected components of algebraic sets with application to computing critical sets

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Abstract. Given a polynomial system f, this article provides a general construction for homotopies that yield at least one point of each connected component on the set of solutions of $f = 0$. This algorithmic approach is then used to compute a superset of the isolated points in the image of an algebraic set which arises in many applications, such as computing critical sets used in the decomposition of real algebraic sets. An example is presented which demonstrates the efficiency of this approach.
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Introduction

For a polynomial system f with complex coefficients, the fundamental problem of algebraic geometry is to understand the set of solutions of the system $f = 0$, denoted $\mathcal{V}(f)$. *Numerical algebraic geometry* (see, e.g., [5,25] for a general overview) is based on using homotopy continuation methods for computing $\mathcal{V}(f)$. Geometrically, one can decompose $\mathcal{V}(f)$ into its irreducible components, which corresponds numerically to computing a numerical irreducible decomposition with each irreducible component represented by a witness set. The first step of

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computing a numerical irreducible decomposition is to compute witness point supersets with the algorithms [13,22,24] relying upon a sequence of homotopies. At each dimension where a solution component could exist, a generic linear space of complementary dimension is used to slice the solution set; the witness points are then the isolated points in the intersection of the solution component and the linear slice. Accordingly, a crucial property of the algorithms employed is that they must generate a finite set of points, say S, in the slice that includes all isolated points of the slice.

In this article, we change the focus from irreducible components to connected components. We present an approach that computes a finite set of points in $\mathcal{V}(f)$ containing at least one point on each connected component of $\mathcal{V}(f)$ using a single homotopy, built on a similar theoretical viewpoint as the nonconstructive approach presented in [19, Thm. 7]. This work is complementary to methods for computing a finite set of points in the set of real points in $\mathcal{V}(f)$, denoted $\mathcal{V}_{\mathbb{R}}(f)$, containing at least one point on each connected component of $\mathcal{V}_{\mathbb{R}}(f)$ [1,11,21,30].

Our approach is particularly relevant to *numerical elimination theory* [5, Chap. 16], which seeks to treat projections of algebraic sets in a similar fashion as general algebraic sets but without having on hand polynomials that vanish on the projection (and without computing such polynomials). This is a numerical alternative to symbolic elimination methods [29]. In particular, suppose that $f(x, y)$ is a polynomial system that is defined on a product of two projective spaces, and let $X = \pi(\mathcal{V}(f))$ where $\pi(x, y) = x$. We do not have a polynomial system that defines X, so we do all computations via points in its pre-image, $\pi^{-1}(X) \cap \mathcal{V}(f)$. In particular, if we wish to compute a finite set of points $S \subset \mathcal{V}(f)$ such that $\pi(S)$ includes all isolated points of X, it suffices if S contains a point on each connected component of $\mathcal{V}(f)$. Our new algorithm enables one to compute such a set S using a single homotopy; one does not need to separately consider each possible dimension of the fibers over the isolated points of X.

The viewpoint of computing based on connected components also has many other applications, particularly related to so-called critical point conditions. For example, the methods mentioned above in relation to real solutions, namely [1,11,21,30], compute critical points of $\mathcal{V}(f)$ with respect to the distance function (see also [9]). In [6,7], critical points of $\mathcal{V}(f)$ with respect to a linear projection are used to numerically decompose real algebraic sets. (We discuss this in more detail in § 3.) Other applications include computing witness point sets for irreducible components of rank-deficiency sets [2], isosingular sets [14], and deflation ideals [17].

To highlight the key point of this paper, consider computing rank-deficiency sets as in [2]. With this setup, one adds new variables related to the null space of the matrix. To make sure that all components of the rank-deficiency sets are computed, traditional approaches need to consider all possible dimensions of the null space. The point of this paper is to provide an algorithmic approach by which one only needs to consider the smallest possible null space dimension, thereby simplifying the computation.

The rest of the article is organized as follows. Section 1 derives an algorithmic approach that computes at least one point on every connected component of $\mathcal{V}(f)$ using one homotopy. This is discussed in relation to elimination theory in § 2, while § 3 focuses on computing critical sets of projections of real algebraic sets. An example illustrating this approach and its efficiency is presented in § 4.

1 Construction of homotopies

The starting point for constructing one homotopy that computes at least one point on each connected component of a solution set of polynomial equations is [19, Thm. 7]. Since this theorem is nonconstructive, we derive an algorithmic approach for performing this computation in Prop. 1 and sketch a proof. We refer to [25] for details regarding algebraic and analytic sets with [19, Appendix] providing a quick introduction to basic results regarding such sets.

Suppose that $\mathcal{E}$ is a complex algebraic vector bundle on an n-dimensional irreducible and reduced complex projective set X. Denote the bundle projection from $\mathcal{E}$ to X by $\pi_{\mathcal{E}}$. A section s of $\mathcal{E}$ is a complex algebraic map $s : X \to \mathcal{E}$ such that $\pi_{\mathcal{E}} \circ s$ is the identity; i.e., for all $x \in X$, $(\pi_{\mathcal{E}} \circ s)(x) = \pi_{\mathcal{E}}(s(x)) = x$.

There is a nonempty Zariski open set $\mathcal{U} \subset X$ over which $\mathcal{E}$ has a trivialization. Using such a trivialization, an algebraic section of $\mathcal{E}$ becomes a system of $\mathrm{rank}(\mathcal{E})$ algebraic functions. In fact, *all* polynomial systems arise in this way and results about special homotopies which track different numbers of paths, e.g., [16,20,26], are based on this interpretation (see also [25, Appendix A]).

Let us specialize this to a concrete situation.

Example 1. Suppose that $X \subset \prod_{j=1}^{r} \mathbb{P}^{n_j}$ is an irreducible and reduced n-dimensional algebraic subset of a product of projective spaces. For example, X could be an irreducible component of a system of multihomogeneous polynomials in the variables

$$z_{1,0}, \ldots, z_{1,n_1}, \ldots, z_{r,0}, \ldots, z_{r,n_r},$$

where $[z_{j,0}, \ldots, z_{j,n_j}]$ are the homogeneous coordinates on the j^{th} projective space, $\mathbb{P}^{n_j}$. Each homogeneous coordinate $z_{j,k}$ has a natural interpretation as a section of the hyperplane section bundle, denoted $\mathcal{L}_{\mathbb{P}^{n_j}}(1)$. The d^{th} power of the hyperplane section bundle is denoted by $\mathcal{L}_{\mathbb{P}^{n_j}}(d)$. A multihomogeneous polynomial defined on $\prod_{j=1}^{r} \mathbb{P}^{n_j}$ with multidegree $(d_1, \ldots, d_r)$ is naturally interpreted as a section of the line bundle

$$\mathcal{L}_{\prod_{j=1}^{r} \mathbb{P}^{n_j}}(d_1, \ldots, d_r) := \otimes_{j=1}^{r} \pi_j^* \mathcal{L}_{\mathbb{P}^{n_j}}(d_j),$$

where $\pi_k : \prod_{j=1}^{r} \mathbb{P}^{n_j} \to \mathbb{P}^{n_k}$ is the product projection onto the k^{th} factor. A system of n multihomogeneous polynomials

$$f := \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \tag{1}$$

where f_i has multidegree $(d_{i,1}, \ldots, d_{i,n_i})$ is interpreted as a section of

$$\mathcal{E} := \bigoplus_{i=1}^{n} \mathcal{L}_{\prod_{j=1}^{r} \mathbb{P}^{n_j}}(d_{i,1}, \ldots, d_{i,r}).$$

The solution set of $f = 0$ is simply the set of zeroes of the section f.

The n^{th} Chern class of $\mathcal{E}$ [8,10], which lies in the $2n^{\text{th}}$ integer cohomology group $H^{2n}(X, \mathbb{Z})$, is denoted by $c_n(\mathcal{E})$. Let $d := c_n(\mathcal{E})[X] \in \mathbb{Z}$, i.e., d denotes the evaluation of $c_n(\mathcal{E})$ on X.

Example 2. Continuing from Example 1, let $c := \sum_{j=1}^{r} n_j - n$ be the codimension of X. Using multi-index notation for $\alpha = (\alpha_1, \ldots, \alpha_r)$ where each $\alpha_i \geq 0$ and $|\alpha| = \sum_{i=1}^{r} \alpha_i$, we can represent X in homology by

$$\sum_{|\alpha|=c} e_\alpha \mathcal{H}^\alpha$$

where $\mathcal{H}_i := \pi_i^{-1}(H_i)$ with hyperplane $H_i \subset \mathbb{P}^{n_i}$ and $\mathcal{H}^\alpha = \mathcal{H}_1^{\alpha_1} \cdots \mathcal{H}_r^{\alpha_r}$. Moreover, $d := c_n(\mathcal{E})[X]$ is simply the multihomogeneous Bézout number of the system of multihomogeneous polynomials restricted to X, i.e., the coefficient of $\prod_{j=1}^{r} z_j^{n_j}$ in the expression

$$\left(\sum_{|\alpha|=c} e_\alpha z^\alpha \right) \cdot \prod_{i=1}^{n} \left(\sum_{j=1}^{r} d_{i,j} z_j \right).$$

In particular, d is simply the number of zeroes of a general section of $\mathcal{E}$ restricted to X.

A vector space V of global sections of $\mathcal{E}$ is said to *span* $\mathcal{E}$ if, given any point $e \in \mathcal{E}$, there is a section $\sigma \in V$ of $\mathcal{E}$ with $\sigma(\pi_{\mathcal{E}}(e)) = e$. We assume that the rank of $\mathcal{E}$ is $n = \dim X$. If V spans $\mathcal{E}$, then Bertini's Theorem asserts that there is a Zariski dense open set $U \subset V$ with the property that, for all $\sigma \in U$, σ has d nonsingular isolated zeroes contained in the smooth points of X, i.e., the graph of σ meets the graph of the identically zero section of $\mathcal{E}$ transversely in d points in the set of smooth points of X.

Let $|V| := (V \setminus \{0\})/\mathbb{C}^*$ be the space of lines through the origin of V. Given a complex analytic vector bundle $\mathcal{E}$ spanned by a vector space of complex analytic sections V, the total space $\mathcal{Z} \subset X \times |V|$ of solution sets of $s \in V$ is

$$\mathcal{Z} := \{(x, s) \in X \times |V| \ : \ s(x) = 0\}. \tag{2}$$

For simplicity, let $p : \mathcal{Z} \to X$ and $q : \mathcal{Z} \to |V|$ denote the maps induced by the product projections $X \times |V| \to X$ and $X \times |V| \to |V|$, respectively.

Since V spans $\mathcal{E}$, the evaluation map

$$X \times V \to \mathcal{E}$$

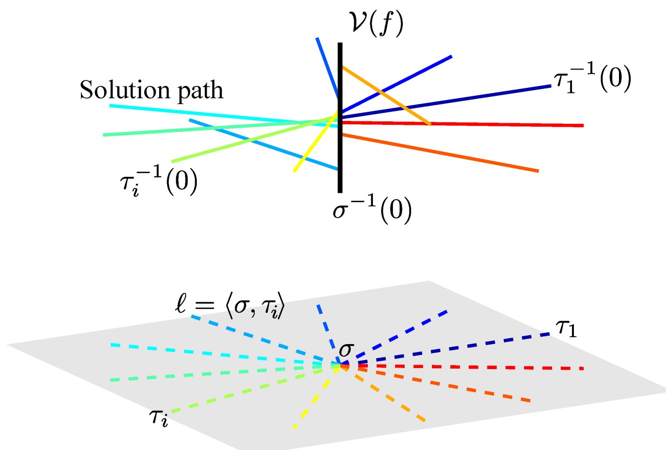

Fig. 1. Illustration of the terminology of the paper. The upper space is in terms of the variables of the problem, with solid lines representing solutions paths, starting at the finite nonsingular zeros of some τ_i, and ending at some zero of σ. We show here many τ_i systems, which all are deformed into σ. At the bottom, the patch represents the vector space V and the lines ℓ interpolate from some τ_i to σ.

is surjective so that the kernel is a vector bundle of rank $\dim V - \operatorname{rank}(\mathcal{E})$. Let $\mathcal{K}$ denote the dual of this kernel and $\mathbb{P}(\mathcal{K})$ denote $(\mathcal{K}^* \setminus X)/\mathbb{C}^*$, the space of lines through the vector space fibers of the bundle projection of $\mathcal{K}^* \to X$. The standard convention of denoting $(\mathcal{K}^* \setminus X)/\mathbb{C}^*$ by $\mathbb{P}(\mathcal{K})$ and not $\mathbb{P}(\mathcal{K}^*)$ is convenient in many calculations.

The space $\mathbb{P}(\mathcal{K})$ is easily identified with $\mathcal{Z}$ and the map p is identified with the map $\mathbb{P}(\mathcal{K}) \to X$ induced by the bundle projection. From this identification, we know that $\mathcal{Z}$ is irreducible.

Let $\mathcal{E}$ denote a rank n algebraic vector bundle on a reduced and irreducible projective algebraic set spanned by a vector space V of algebraic sections of $\mathcal{E}$. Suppose that $\sigma \in V$ and $\tau \in V$ have distinct images in $|V|$ and let $\ell := \langle \sigma, \tau \rangle \subset |V|$ denote the unique projective line, i.e., linear $\mathbb{P}^1$, through the images of σ and τ in $|V|$. Letting λ and μ be homogeneous coordinates on ℓ, i.e, spanning sections of $\mathcal{L}_\ell(1)$, we have the section

$$H(x, \lambda, \mu) := \lambda\sigma + \mu\tau \qquad (3)$$

of $q^*_{q^{-1}(\ell)}\mathcal{L}_\ell(1) \otimes p^*\mathcal{E}$. Choosing a trivialization of $\mathcal{E}$ over a Zariski open dense set U and a trivialization of $\mathcal{L}_\ell(1)$ over a Zariski open dense set of ℓ, e.g., the set where $\mu \neq 0$, H is naturally interpreted as a homotopy. See Figure 1 for an illustration.

With this general setup, we are now ready to state a specialization of the nonconstructive result [19, Thm. 7]. The key difference is that this specialization immediately yields a constructive algorithm for computing a finite set of points containing at least one point on each connected component of $\sigma^{-1}(0)$.

Proposition 1. *Let $\mathcal{E}$ denote a rank n algebraic vector bundle over an irreducible and reduced n-dimensional projective algebraic set X. Let V be a vector space of sections of $\mathcal{E}$ that spans $\mathcal{E}$. Assume that $d := c_n(\mathcal{E})[X] > 0$ and $\tau \in V$ which has d nonsingular zeroes all contained in the smooth points of X. Let $\sigma \in V$ be a nonzero section of $\mathcal{E}$, which is not a multiple of τ. Let $\ell = \langle \sigma, \tau \rangle$ and H as in (3). Then, there is a nonempty Zariski open set $\mathcal{Q} \subset \ell$ such that*

1. *the map $q_{\mathcal{Z}_\mathcal{Q}}$ of $\mathcal{Z}_\mathcal{Q} := \left\{ H^{-1}(0) \cap (X \times \mathcal{Q}) \right\}$ to ℓ is d-to-one; and*
2. *the finite set $\overline{\mathcal{Z}_\mathcal{Q}} \cap \sigma^{-1}(0)$ contains at least one point of every connected subset of $\sigma^{-1}(0)$.*

Proof. Let $\mathcal{Z}$ as in (2). The projection map $q : \mathcal{Z} \to |V|$ may be Stein factorized [25, Thm. A.4.8] as $q = s \circ r$ where $r : \mathcal{Z} \to Y$ is an algebraic map with connected fibers onto an algebraic set Y and $s : Y \to |V|$ is an algebraic map with finite fibers. The surjectivity of q implies that s is surjective and $\dim Y = \dim |V|$. Since $\mathcal{Z}$ is irreducible, Y is irreducible.

It suffices to show that given any $y \in Y$, there is a complex open neighborhood U of y with $s(U)$ an open neighborhood of $s(y)$. A line $\ell \subset |V|$ is defined by $\dim |V| - 1$ linear equations. Thus, $s^{-1}(\ell)$ has all components of dimension at least 1. The result follows from [25, Thm. A.4.17].

Remark 1. If X is a codimension c irreducible component of multiplicity one of the solution set of a polynomial system $f_1, \ldots, f_c$ in the total space, we can choose our homotopy so that the paths over $(0, 1]$ are in the set where $df_1 \wedge \cdots \wedge df_c$ is non-zero.

2 Isolated points of images

With the theoretical foundation presented in § 1, this section focuses on computing a finite set of points containing at least one point on each connected component in the image of an algebraic set which, in particular, provides a finite superset of the isolated points in the image. Without loss of generality, it suffices to consider projections of algebraic sets which corresponds algebraically with computing solutions of an elimination ideal.

Lemma 1. *Let V be a closed algebraic subset of a complex quasiprojective algebraic set X. Let $\pi : X \to Y$ denote a proper algebraic map from X to a complex quasiprojective algebraic set Y. If S is a finite set of points in V that contains a point on each connected component of V, then $\pi(S)$ is a finite set of points in $\pi(V)$ which contains a point on each connected component of $\pi(V)$. In particular, $\pi(S)$ is a finite superset of the zero-dimensional components of $\pi(V)$.*

Proof. The image of a connected set under a proper algebraic map is connected.

Consider the concrete case where f is a polynomial system defined on $\mathbb{C}^N \times \mathbb{P}^M$. Let $\mathcal{V}(f) \subset \mathbb{C}^N \times \mathbb{P}^M$ and $Z(f) \subset \mathbb{P}^N \times \mathbb{P}^M$ be the closure of $\mathcal{V}(f)$ under the natural embedding of $\mathbb{C}^N$ into $\mathbb{P}^N$. The approach of Prop. 1 provides one homotopy which can be used to compute a point on each connected component of $Z(f)$. However, it may happen that a point computed on each connected component of $\mathcal{V}(f)$ is at "infinity." One special case is the following for isolated points in the projection of $\mathcal{V}(f)$ onto $\mathbb{C}^N$.

Corollary 1. *Let f be a polynomial system defined on $\mathbb{C}^N \times \mathbb{P}^M$ and π denote the projection $\mathbb{C}^N \times \mathbb{P}^M \to \mathbb{C}^N$. By considering the natural inclusion of $\mathbb{C}^N$ into $\mathbb{P}^N$, let $Z(f)$ be the closure of $\mathcal{V}(f)$ in $\mathbb{P}^N \times \mathbb{P}^M$. Let S be a finite set of points in $Z(f)$ which contains a point on each connected component of $Z(f)$ and $S_{\mathbb{C}} = S \cap (\mathbb{C}^N \times \mathbb{P}^M)$. Then, $\pi(S_{\mathbb{C}})$ is a finite set of points in $\pi(\mathcal{V}(f))$ which contains the isolated points in $\pi(\mathcal{V}(f))$.*

Proof. Suppose that $x \in \pi(\mathcal{V}(f)) \subset \mathbb{C}^N$ is isolated. Let $y \in \mathbb{P}^M$ such that $(x, y) \in \mathcal{V}(f)$. By abuse of notation, we have $(x, y) \in Z(f)$ so that there is a connected component, say C, of $Z(f)$ which contains (x, y). Since x is isolated in $\pi(\mathcal{V}(f))$, we must have $C \subset \{x\} \times \mathbb{P}^M$. The statement follows from the fact that C is thus naturally contained in $\mathbb{C}^N \times \mathbb{P}^M$.

Example 3. To illustrate, consider the polynomial system

$$F(x) = \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix} = \begin{bmatrix} x_1^2 + x_2^2 + x_3^2 + x_4^2 \\ x_1^3 + x_2^3 + x_3^3 + x_4^2 \end{bmatrix}$$

defined on $\mathbb{C}^4$. The set $\mathcal{V}(F) \subset \mathbb{C}^4$ is an irreducible surface of degree six containing one real point, namely the origin, which is an isolated singularity. Since the total derivatives dF_1 and dF_2 are linearly dependent at a singular point, we can consider the following system defined on $\mathbb{C}^4 \times \mathbb{P}^1$:

$$G(x, v) = \begin{bmatrix} F(x) \\ v_0 \cdot dF_1(x) + v_1 \cdot dF_2(x) \end{bmatrix}.$$

Since G consists of 6 polynomials defined on a 5 dimensional space, we reduce to a square system via randomization[4] which, for example, yields:

$$f(x, v) := \begin{bmatrix} x_1^2 + x_2^2 + x_3^2 + x_4^2 \\ x_1^3 + x_2^3 + x_3^3 + x_4^2 \\ v_0(x_1 + x_4) + v_1(3x_1^2 + x_4) \\ v_0(x_2 + x_4) + v_1(3x_2^2 + x_4) \\ v_0(x_3 + x_4) + v_1(3x_3^2 + x_4) \end{bmatrix}.$$

[4] In usual practice, "randomization" means replacing a set of polynomials with some number of random linear combinations of the polynomials. When the appropriate number of combinations is used, then in a Zariski-open subset of the Cartesian space of coefficients of the linear combinations, the solution set of interest is preserved. See, for example, [25, §13.5]. Here, for simplicity of illustration, we take very simple linear combinations involving small integers. These happen to suffice, but in general one would use a random number generator and possibly hundreds of digits to better approximate the probability-one chance of success that is implied in a continuum model of the coefficient space.

Consider the linear product [26] system:

$$g(x,v) := \begin{bmatrix} x_1^2 + x_2^2 + x_3^2 + x_4^4 \\ x_1^3 + x_2^3 + x_3^3 + x_4^2 \\ (v_0+v_1)(x_1 - 4x_4 - 1)(x_1 - 2) \\ (v_0 - v_1)(x_2 + 2x_4 - 1)(x_2 - 3) \\ (v_0 + 2v_1)(x_3 - 3x_4 - 1)(x_3 - 4) \end{bmatrix}$$

together with the homotopy

$$H((x,v),[\lambda,\mu]) = \lambda f(x,v) + \mu g(x,v).$$

The symbols λ and μ are spanning sections from (3); in this context, they are scalar values interpolating between f and g, and the homotopy "path" variables. With this setup, $g^{-1}(0)$ has exactly $d = 72$ nonsingular isolated solutions which can be computed easily. Further, 72 is the coefficient of a^4b in the polynomial $(2a)(3a)(2a+b)^3$, one way to compute the 2-homogeneous root count [20].

We used `Bertini` [4] to track the 72 paths along a real arc contained in the line $\langle\sigma,\tau\rangle$ in which 30 paths diverge to infinity and 42 paths end at finite points. Of the latter, 20 endpoints are nonsingular isolated solutions which are extraneous in that they arose from the randomization and not actually in $\mathcal{V}(G)$. The other 22 paths converged to points in $\{0\}\times\mathbb{P}^1$: 18 of which ended with $v = [0,1] \in \mathbb{P}^1$ while the other 4 break into 2 groups of 2 with v of the form $[1,\alpha]$ and $[1, \mathrm{conj}(\alpha)]$ where $\alpha \approx -0.351 + 0.504\cdot\sqrt{-1}$. In particular, even though $\{0\}\times\mathbb{P}^1$ is a positive-dimensional solution component of $\mathcal{V}(f)$ and also of $\mathcal{V}(G)$, we always obtain at least one point on this component showing that the origin is the only point in $\mathcal{V}(F)$ which is singular with respect to F.

3 Computing critical points of projections

An application of Corollary 1 is to compute the critical points of an irreducible curve $X \subset \mathbb{C}^N$ with respect to a nonconstant linear projection $\pi : X \to \mathbb{C}$. In particular, assume that $f = \{f_1, \ldots, f_{N-1}\}$ is a polynomial system on $\mathbb{C}^N$ such that X is an irreducible component of $\mathcal{V}(f)$ which has multiplicity one with respect to f. A critical point of π with respect to X is a point $x \in X$ such that either

- x is a smooth point and $d\pi$ is zero on the tangent space of X at x; or
- x is a singular point of X.

In terms of rank-deficiency sets, the set of critical points is the set of points on X such that

$$\operatorname{rank}\begin{bmatrix} d\pi \\ df_1 \\ \vdots \\ df_{N-1} \end{bmatrix} \leq N-1. \tag{4}$$

With this setup, there are finitely many critical points. In [7], which includes an implementation of the curve decomposition algorithm of [18], a finite superset of the critical points are needed to compute a cellular decomposition of the real points of X. In fact, the points that are not critical points simply make the cellular decomposition finer which can be merged away in a post-processing step. Hence, one needs to compute at least one point in each connected component in $X \times \mathbb{P}^{N-1}$ intersected with the solution set in $\mathbb{C}^N \times \mathbb{P}^{N-1}$ of

$$\begin{bmatrix} f_1 \\ \vdots \\ f_{N-1} \\ \begin{bmatrix} d\pi \\ df_1 \\ \vdots \\ df_{N-1} \end{bmatrix} \cdot \xi \end{bmatrix} = 0.$$

The advantage here is that we obtain a finite superset of the critical points using one homotopy regardless of the possibly different dimensions of the corresponding null spaces, i.e., there is no need to cascade down the possible null space dimensions.

The setup above naturally extends to computing witness point supersets for the critical set of dimension $k-1$ of an irreducible component of dimension k, e.g., critical curves of a surface.

4 Example

Consider the 12-bar spherical linkage from [27,28]. This device can be viewed as 20 rigid rods meeting in spherical joints at 9 points, or since a loop of three such rods forms a rigid triangle, as 12 rigid links meeting in rotational hinges with the axes of rotation all intersecting at a central point. The arrangement is most clearly seen in Figure 2(c). The irreducible decomposition of the variety in $\mathbb{C}^{18}$ for the polynomial system F defined below for this linkage was first computed in [11] and summarized in Table 1. Here, we consider computing a superset of the critical points of the the curve C which is the union of the eight one-dimensional irreducible components having degree 36 with respect to the projection π defined below in (5). We will compare approaches computed using `Bertini` [4].

The ground link for the linkage is specified by fixing three points, namely $P_0 = (0,0,0)$, $P_7 = (-1,1,-1)$, and $P_8 = (-1,-1,-1)$. The three coordinates of the other six points, $P_1, \ldots, P_6$, are the 18 variables of polynomial system

Table 1. Decomposition of 12-bar spherical linkage system.

dimension	degree	# components
3	8	2
2	4	2
	8	14
	12	12
	16	1
	20	4
	24	1
1	4	6
	6	2

$F : \mathbb{C}^{18} \to \mathbb{C}^{17}$. The 17 polynomials in F are the following quadratics:

$$\begin{aligned} G_{ij} &= \|P_i - P_j\|^2 - 4, \\ &\quad (i,j) \in \{(1,2),(3,4),(5,6),(1,5),(2,6),(3,7),(4,8),(1,3),(2,4),(5,7),(6,8)\}; \\ H_k &= \|P_k\|^2 - 3, \\ &\quad k \in \{1,2,3,4,5,6\}. \end{aligned}$$

Denoting the coordinates of P_i as P_{i1}, P_{i2}, P_{i3}, we choose[5] a projection map $\pi : \mathbb{C}^{18} \to \mathbb{C}$ defined by

$$\begin{aligned} \pi(P) = {} & \tfrac{3}{5}P_{11} + \tfrac{13}{17}P_{12} - \tfrac{5}{16}P_{13} + \tfrac{26}{27}P_{21} - \tfrac{1}{10}P_{22} + \tfrac{1}{6}P_{23} + \tfrac{3}{5}P_{31} + \tfrac{7}{17}P_{32} + \tfrac{3}{10}P_{33} + \\ & \tfrac{1}{4}P_{41} - \tfrac{4}{5}P_{42} + \tfrac{1}{3}P_{43} + \tfrac{18}{25}P_{51} + \tfrac{14}{29}P_{52} - \tfrac{12}{13}P_{53} - \tfrac{17}{30}P_{61} - \tfrac{5}{17}P_{62} + \tfrac{13}{20}P_{63} \quad (5) \end{aligned}$$

and consider the following system defined on $C \times \mathbb{P}^{17} \subset \mathbb{C}^{18} \times \mathbb{P}^{17}$:

$$f(P,\xi) = \begin{bmatrix} F(P) \\ \begin{bmatrix} d\pi \\ dF(P) \end{bmatrix} \cdot \xi \end{bmatrix}.$$

Since each irreducible component in C has multiplicity one with respect to F, the irreducible components of $\mathcal{V}(f) \cap (C \times \mathbb{P}^{17})$ must be of the form $\{x\} \times L$ for some point $x \in C$ and linear space $L \subset \mathbb{P}^{17}$. We aim to compute all such points x.

With traditional methods, one would need to consider various dimensions of the corresponding null spaces L. The advantage is that one obtains additional information, namely witness point supersets for the irreducible components. The first approach is to consider each possible dimension of $\mathbb{P}^{17}$ independently. Since the zero-dimensional case is equivalent in terms of the setup and number of paths to the new approach discussed below, we will just quickly summarize what would be needed to perform this full computation. In particular, for each $0 \le i \le 16$, starting with a witness set for $C \times \mathbb{P}^{17}$, the corresponding start system, after

[5] As before, we choose simple rational coefficients for simplicity of presentation.

possible randomization, would require tracking $36 \cdot (17 - i)$, totaling 5508, paths related to moving linear slices and the same number of paths to compute witness point supersets.

Rather than treat each dimension independently, another option is to cascade down through the dimensions, e.g., using the regenerative extension [15]. The implementation in `Bertini`, starting with a witness set for $C \times \mathbb{P}^{17}$, requires tracking 6276 paths for solving as well as tracking 3216 paths related to moving linear slices. Using 64, 2.3 GHz processors, this computation took 618 seconds.

Instead of using a method designed for computing witness point supersets, our new approach uses one homotopy to compute a point on each connected component. This is all that is needed for the current application via Corollary 1. Since $d\pi$ is constant and dF is a matrix with linear entries, we take our start system to be

$$g(P, \xi) = \begin{bmatrix} F(P) \\ \xi_0 \\ \ell_1(P) \cdot \xi_1 \\ \vdots \\ \ell_{17}(P) \cdot \xi_{17} \end{bmatrix}$$

restricted to $C \times \mathbb{P}^{17}$ where each ℓ_i is a random linear polynomial. In particular, $\mathcal{V}(g) \cap (C \times \mathbb{P}^{17})$ consists of $d = 36 \cdot \binom{17}{1} = 612$ points, each of which is nonsingular with respect to g. The 612 solutions can be computed from a witness set for C by tracking 612 paths related to moving linear slices. Then, a point on each connected component of $\mathcal{V}(f) \cap (C \times \mathbb{P}^{17})$ is computed via Corollary 1 by tracking 612 paths. This computation in total, using the same parallel setup as above, took 20 seconds.

Of the 612 paths, 492 diverge to infinity while 120 have finite endpoints. Of the 120 finite endpoints of the form (P, ξ), 78 are real (i.e., have $P \in \mathbb{R}^{18}$) with 22 distinct real points P since some points appear with multiplicity while others have a null space with dimension greater than one so that the same P can appear with several different null directions ξ. In detail, the breakdown of the 22 real points is as follows:

- 14 real points are the endpoint of one path each. These points are smooth points of C with $\operatorname{rank} dF = 17$. Each lie on one of the degree 4 irreducible components of C and is an equilateral spherical four-bar linkage of the type illustrated in Figure 2(a).
- 6 real points are the endpoint of 10 paths each. Each of these points has $\operatorname{rank} dF = 12$ with $\operatorname{rank} \begin{bmatrix} d\pi \\ dF \end{bmatrix} = 13$ and arise where an irreducible component of degree 4 in C intersects another irreducible component of $\mathcal{V}(F)$. The corresponding 12-bar linkage appears as in Figure 2(b).
- 2 real points are the endpoint of 2 paths each. Each of these points P has $\operatorname{rank} dF = 16$ and $\operatorname{rank} \begin{bmatrix} d\pi \\ dF \end{bmatrix} = 17$ so that the corresponding null vector $\xi \in \mathbb{P}^{17}$ is unique. Hence, the points (P, ξ) have multiplicity 2 with respect

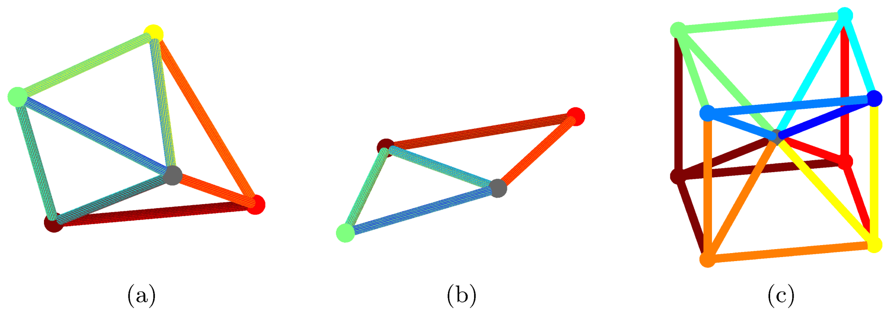

Fig. 2. Solutions to the 12 bar spherical linkage obtained from the critical point computation: (a) an equilateral spherical four-bar configuration, corresponding to a nonsingular critical point on a degree four irreducible component; (b) a degenerate configuration, coming from the intersection of such a component with a higher-dimensional irreducible component; (c) a rigid configuration arising from the intersection of the irreducible curves of degree six.

to f. These points correspond to a rigid arrangement as shown in Figure 2(c), one the mirror image of the other.

To clarify the accounting, note that $14 \cdot 1 + 6 \cdot 10 + 2 \cdot 2 = 78$.

5 Conclusion

We have described an algorithmic approach for constructing one homotopy that yields a finite superset of solutions to a polynomial system containing at least one point on each connected component of the solution set. This idea naturally leads to homotopies for solving elimination problems, such as computing critical points of projections as well as other rank-constraint problems. This method allows one to compute such points directly without having to cascade through all the possible dimensions of the auxiliary variables. This can provide considerable computational savings, as we have demonstrated on an example arising in kinematics, where the endpoints of a single homotopy include all the critical points on a curve even though the associated null spaces at these points have various dimensions.

We note that our approach has application to numerical elimination theory but in that case leaves an open problem concerning sorting isolated from nonisolated points. In the classical setting, when one finds a superset of the isolated solutions, one can sift out the set of isolated solutions from a superset by using, for example, either the global homotopy membership test [23] or the numerical local dimension test [3]. In the elimination setting, a modified version of the homotopy membership test as developed in [12] can sort out which points are isolated under projection, but there is no local dimension test in this setting as yet.

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