A Copula-Based Trend-Renewal Process Model for Analysis of Repairable Systems with Multi-type Failures

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Abstract: Reliability analysis of multi-component repairable systems with dependent component failures is challenging for two reasons. First, the failure mechanism of one component may depend on other components when considering component failure dependency. Second, imperfect repair actions can have accumulated effects on the repaired components and these accumulated effects are difficult to measure. In this paper, we propose a parametric statistical model to capture the failure dependency information with general component repair actions. We apply the maximum likelihood method to estimate the model parameters by utilizing the historical failure data. Statistical hypothesis tests are developed to determine the dependency structure of the component failures based on the proposed reliability model. The proposed methodology is demonstrated by a simulation study and case studies of a forklift vehicle system and a car body assembly process.

Keywords: Clayton copula, competing risks, failure dependency, lognormal, Weibull, Gaussian copula.

ACRONYMS

CDF Cumulative Distribution Function

AIC Akaike Information Criterion

NOTATIONS

T_i	Failure time of the i^{th} failure
Δ_i	Failure type of the i^{th} failure, $\Delta_i \in \{0,1,...,K\}$
τ	The predetermined ending time for failure observation
$N(t)$, $N_k(t)$	Numbers of system failures of all components and that of the k^{th} failure type occurring in the time interval (0, t]
$r_j(t)$	Most recently observed failure time of component j before time t
$a_k(t)$, $b_k(t)$	Age of component k in the original domain and that in the k^{th} transformed time domain, respectively
$\mathbf{W}_1$,	Random vector of the latent ages to failure of all components; $\mathbf{W}_1 = [W_{1,1},...,W_{1,K}]'$, where $W_{1,k}$ is the latent age to failure of the k^{th} component, and
$\mathbf{V}_1$	Random vector of the latent ages to failure of all components in the transformed time domain; $\mathbf{V}_1 = [V_{1,1},...,V_{1,K}]'$
$\Lambda_k(\cdot)$	The k^{th} trend function for failure type k
F, F_k	Joint and the k^{th} marginal distribution of renewal distribution
$\boldsymbol{\theta}$	Model parameters; $\boldsymbol{\theta} = \{\boldsymbol{\theta}_\zeta, \boldsymbol{\theta}_F\}$;
$\boldsymbol{\theta}_\zeta$, $\boldsymbol{\theta}_F$, $\boldsymbol{\theta}_{k,F}$	Model parameters for trend function, joint renewal distribution, and the k^{th} marginal distribution , respectively
$\boldsymbol{\beta}$, $\boldsymbol{\eta}$	Parameters for power law trend function
$\boldsymbol{\kappa}$, $\boldsymbol{\lambda}$	Shape and scale parameters of Weibull marginal distributions
$\boldsymbol{\Sigma}$	Correlation matrix in Gaussian copula
ρ	Association parameter in the Gumbal-Hougaard copula
$L(\boldsymbol{\theta})$	Likelihood function for unknown parameter vector $\boldsymbol{\theta}$
$\hat{\boldsymbol{\theta}}$	ML estimation of $\boldsymbol{\theta}$

D Likelihood ratio test statistic

L_n, L_f Likelihood value for the null model and that for the full model, respectively

1. Introduction

Modern engineering systems such as vehicles, aircrafts, and manufacturing systems generally consist of multiple subsystems/components that interact in a complex manner. As a result, the degradation or failure of one component may make the interacting components more prone to fail. Under most situations only the failed component, rather than the whole system, will be repaired. The repair actions include perfect, minimal, and situations in between, depending on the failure conditions. Reliability analysis of multi-component repairable systems plays a critical role for system safety and cost reduction. Ignoring the dependency of multi-component failures, however, will result in a biased reliability prediction.

Considering component failure dependency, there are two main challenges for reliability analysis of multi-component repairable systems. First, as the components compete to fail, only the potential failures with minimum failure time can be observed. Taking failure dependency into account, the repair actions of the failed component can change the failure mechanism of other components. Second, the accumulated effects due to imperfect repair make the failure mechanism of an individual component more complex. Under most situations, the repair actions can be distinctive for different components and for various repairs of each component. In addition, the effect of each repair action is difficult to quantify, and the repair actions may not be recorded.

The reliability analysis of a multi-component repairable system becomes more challenging for a single repairable system, when failure data is only collected from one realization of the multi-component repairable system. Although in this paper we focus on a single repairable system, the results can be extended to repairable systems with multiple realizations.

Traditional study on repairable systems mainly focuses on reliability models for systems with a single component under different repair actions. Kijima and Sumita [1] and Kijima [2] suggested two imperfect repair models by introducing the concept of virtual age of repairable systems. Lindqvist, Elvebakk and Heggland [3] proposed a trend-renewal Process (TRP) to generalize the inhomogeneous and modulated gamma process proposed by Berman [4], which deal with the imperfect repair conditions well. Other imperfect repair models for repairable systems with a single component include the modulated renewal process [5], the modulated power law process [6], the arithmetic reduction of age and arithmetic reduction of intensity models [7]. A comprehensive review on statistical methods of repairable systems is provided by Lindqvist [8].

For repairable systems under competing risks, most of the existing research assumes independency of component failure [9-11]. Thus, the reliability analysis of the entire system subjected to competing risks can be simplified by analyzing each component independently. The existing reliability models that consider failure dependency assume that when a failure of one component occurs, it will result in a possible shock to the other components with a certain probability [12-16]. Li and Pham [17] discussed a similar system with component failure dependency, and they assumed a binomial distribution of perfect and minimal repairs with certain probability. Langseth and Lindqvist [18] developed a model for systems consisting of multiple components associated with failures caused by multiple sources. Shaked and Shanthikumar [19] developed statistical models and investigated properties of repairable systems with dependent component failures. However, in their work, the parameters estimation approach was not given and the repair actions were not considered. Yang et al. [20] developed a statistical model to capture the failure dependency of multi-component repairable systems with the assumption that the failed component is replaced with one that is as good as new. Zhang and Yang [21] further developed optimal maintenance policies for multi-component repairable systems. However, in reality, there are many situations where the failed component is partially repaired rather than perfectly replaced, and the repair condition/effectiveness changes (e.g., becomes worse) along with the number of repairs.

Copulas become increasingly popular in modeling dependencies, due to their flexibility in capturing non-linear dependence and arbitrary marginal distributions. In the context of reliability, a comprehensive study of systems failure dependency by using copula was given in the book by Li and Xie [22]. In this paper, we propose a copula-based trend-renewal process model to analyze the multiple-component repairable systems under the dependent competing risks. The failed component is subject to general repair actions, including perfect and minimal repairs as well as situations in between. The repair conditions may change along with the number of repairs. The framework of this paper is as follows. After the introduction, Section 2 proposes a generalized parametric statistical model for reliability analysis of dependent competing-risk systems under the imperfect component repair assumption. Section 3 discusses the parameter estimation and statistical inference. Section 4 demonstrates the developed methodology through simulation studies, and Section 5 illustrates the developed methods by using two real-world applications. Finally, Section 6 gives a summary of the paper.

2. Data and model

2.1 Data notation

We consider a competing-risk system consisting of multiple (say K) components. The time scale is the time since installation. Upon each failure, only the failed component is repaired in terms of imperfect effectiveness between perfect and minimal. The successive failure events are recorded by $T_1, T_2, \ldots$, until a predetermined ending time τ. In addition, each event is labeled with a failure type $\Delta_i \in \{0, 1, \ldots, K\}$; where $\Delta_i = 0$ indicates there is no failure observed. We use pair (T_i, Δ_i) to represent failure information. An equivalent representation of the failure process is in terms of the marked point process ; where k denotes failure type and $N_k(t)$ denotes the cumulative number of failures for component k until time t. We use $N(t) = \sum_{k=1}^{K} N_k(t)$ to denote the total number of failures regardless of failure type until time t. We assume that two failures cannot occur simultaneously, which is a common assumption for repairable systems in the literature. In addition, we assume the repair action is immediate and the repair time is ignored.

2.2 Copula-based TRP

The TRP [3] is a statistical model to model the single-component repairable system under general system repair actions from perfect to minimal, in which both perfect and minimal repairs are included as two extreme cases. The basic idea of the TRP model is to apply a trend function $\Lambda(t)$ to transform the original failure times into a new time domain. The transformed failure times can be modeled by a renewal process that follows a renewal distribution F .

In this paper, the TRP model is extended to systems consisting of multiple components that can fail dependently. We first develop a simplified reliability model when the failed component is repaired perfectly. Then this model is further extended to systems with imperfect component repair actions.

2.2.1 Partially perfect repair model for perfect component repair actions

The partially perfect repair model [20] assumes that the failed component is perfectly repaired whenever a failure occurs. Let $r_k(t)$ be the most recent failure time for component k . The age of component $k; k \in \{1,...,K\}$ at time t, denoted as $a_k(t)$, is defined as the cumulative running time since its last failure, i.e., $a_k(t) = t - r_k(t)$. Note that both $r_k(t)$ and $a_k(t)$ are defined as left-continuous functions. Thus, $r_k(t) = \lim_{x \to t^-} r_k(x)$ if a failure occurs at time t, and $r_k(t) = 0$ if no failure occurred by time t. Similarly, $a_k(t) = \lim_{x \to t^-} a_k(x)$.

In the partially perfect repair model [20], a component fails once its age reaches the corresponding life threshold, resulting in the entire system's failure. The life threshold for component *k*, called the latent age to failure of component k , is assumed to be a random variable. When considering the system from installation, $W_{1,k}$ is used to represent the first random latent age to failure of component k , and the random vector $\mathbf{W}_1 = [W_{1,1},...,W_{1,K}]'$ is modeled by a joint cumulative distribution function(CDF) F that can capture the component failure dependency. When the i^{th} system failure occurs at time point t_i, only the failed component is replaced and thus its age becomes zero, while all the other components' ages do not change. As a result, the $(i+1)^{\text{th}}$ latent age to failure of component k, denoted by $W_{i+1,k}$, should be larger than the age of component k at time point t_i, i.e., $W_{i+1,k} > a_k(t_i)$, $\forall\, k \in \{1,...,K\}, i = 1,2,...$. As a result, the random vector $\mathbf{W}_{i+1} = [W_{i+1,1},...,W_{i+1,K}]'$ follows a truncated distribution with a joint CDF F conditional on the vector of $[a_1(t_i),\ldots,a_K(t_i)]^T$.

2.2.2 A general reliability model for imperfect component repair actions

In the partially perfect repair model [20], the system failures are only determined by the last failure times of all the components because the repair actions are assumed to be perfect. When the repair action is imperfect, however, the component failures are affected by the effect of imperfect component repair accumulated from the all the repair history, which are coupled with the effect of failure dependency of other components in a complex manner.

To overcome this difficulty, we propose a multiple transformation procedure in this paper by applying the TRP model to transform the failure times of individual components to separate transformed time domains in which the effect of imperfect repair can be eliminated. Specifically, as shown in Figure 1, the failure times of component k (denoted by $T_{k,1}$, $T_{k,2}$, ...) are transformed into the k^{th} transformed time domain using a trend function $\Lambda_k(\cdot)$. Based on the properties of the TRP model, the transformed failure times of component k, $\Lambda_k(T_{k,1})$, $\Lambda_k(T_{k,2})$ ... follow a renewal process characterized by a renewal joint CDF F_k.

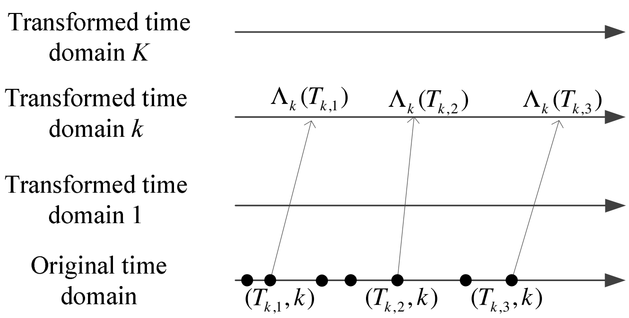

Figure 1. Illustration of the multiple transformation procedure based on different trend functions for different failure types

According to the properties of the renewal process, the accumulated effect of imperfect repair in the original time domain is eliminated in the transformed domain. Thus, the failure times in the transformed time domains can be modeled in a similar way to capture component failure dependency. We assume component k ($k=1,...,K$) has a latent age to failure $V_{i,k}$, in the k^{th} transformed time domain after the $(i-1)^{th}$ system failure and the corresponding repair action. When considering the first system failure from installation, $\mathbf{V}_1=[V_{1,1},...,V_{1,K}]'$ is modeled by a joint CDF F with a marginal distribution F_k, which denotes the renewal distribution of the k^{th} component in the transformed time domain. Let $a_k(t)$ and $r_k(t)$ be the component age and the most recent failure time for component k in the original time domain, respectively, whose definition are the same as in the previous section. Let $b_k(t)$ denote the age of component k in the k^{th} transformed time domain. When the i^{th} system failure occurs at time point t_i, the failed component can be treated as perfectly replaced in the corresponding transformed time domain according to the properties of renewal process. Hence, $b_k(t)=\Lambda_k(t)-\Lambda_k(r_k(t))$.

Because components may have a non-zero age right after a system failure and the corresponding repair, the $(i+1)^{\text{th}}$ latent age to failure of component k in the k^{th} transformed time domain, denoted by $V_{i+1,k}$, should be larger than $b_k(t_i)$, i.e., $V_{i+1,k} > b_k(t_i)$, $\forall\, k \in \{1,...,K\}$. As a result, the random vector $\mathbf{V}_{i+1} = [V_{i+1,1},...,V_{i+1,K}]'$ follows a truncated distribution with the joint CDF F conditional on the vector of $[b_1(t),\ldots,b_K(t_i)]^T$. It can be seen that the simplified model is a special case of the general model, when the trend function is the identity function, i.e., $\Lambda_k(t) = t$.

2.3 Parametric forms

The proposed copula-based TRP model is determined by the trend function and the joint CDF F. In this paper, the copula functions are used to build the joint CDF F based on the marginal distributions. Thus, the model parameters include those from the trend function, the marginal distribution, and the copula function.

2.3.1 Trend function

The power law relationship, which is generally used in the trend function of the TRP model, is also used in the multiple transformation procedure. In particular, the power law intensity function $\zeta_k(\cdot)$ for failure type k has the following form:

$$\zeta_k(t;\boldsymbol{\theta}_{k,\zeta}) = \frac{\beta_k}{\eta_k}\left(\frac{t}{\eta_k}\right)^{\beta_k - 1} \tag{1}$$

where $\boldsymbol{\theta}_{k,\zeta} = [\beta_k, \eta_k]'$ is the parameter vector of intensity function $\zeta_k(\cdot)$. We use $\boldsymbol{\theta}_\zeta = \left\{\boldsymbol{\theta}_{1,\zeta},...,\boldsymbol{\theta}_{K,\zeta}\right\}$ to denote the parameters in all trend functions.

2.3.2 Renewal distribution

In this paper, we choose the Weibull distribution as the marginal distribution of the joint CDF F . However, other distributions can also be applied in the model. The CDF of the Weibull marginal F_k is:

$$F_k(v_{i,k};\boldsymbol{\theta}_{k,F}) = 1 - \exp\left(-\left(\frac{v_{i,k}}{\lambda_k}\right)^{\kappa_k}\right); v_{i,k} \geq 0 \tag{2}$$

where $\kappa_k \in (0,\infty)$ and $\lambda_k \in (0,\infty)$ are called shape and scale parameter, respectively. $\boldsymbol{\theta}_{k,F} = [\kappa_k, \lambda_k]'$ is the parameter vector of the marginal distribution F_k .

Similar to the traditional TRP model for single-component systems, in our model the marginal expectations are restricted to one in order to reduce the degrees of freedom of the model, because if a trend function is multiplied by a constant then we can modify the corresponding marginal distribution accordingly by scaling the time. In practice, we add a constraint that $\lambda_k \cdot \Gamma(1+1/\kappa_k) = 1; k = 1,..., K$, where $\Gamma(\cdot)$ is the gamma function.

2.3.3 Copula function

Copula functions have been applied in reliability studies by many authors [23-25]. The copula functions can be classified into different families. In this paper, we consider two types of copula functions: the Clayton copula [26] and the Gaussian copula [27]. The Clayton copula is a typical one from the Archimedean family. It contains one association parameter ρ that relates to the dependency measurement Kendall's tau $\tau_{Kendall}$, by the relation $\tau_{Kendall} = \rho/(\rho+2)$ [26].

When the Clayton copula is selected to construct the joint distribution, the dependency of the failure types in the transformed time domain is captured by the association parameter ρ. The range of the association parameter is $\rho \in [-1,0) \cup (0,\infty)$. The limiting case when $\rho \to 0$ represents the independent situation. In this paper, we define the Clayton copula as follows:

$$C_{Clayton}(u_1,\ldots,u_K) = \begin{cases} \left[\max\left(\sum_{k=1}^{K} u_k^{-\rho} - K + 1, 0 \right) \right]^{-1/\rho} & ; \rho \neq 0 \\ \prod_{k=1}^{K} u_k & ; \rho = 0 \end{cases} \tag{3}$$

where $u_k = F_k(v_{i,k}, \boldsymbol{\theta}_{k,F}); k = 1,\ldots,K$. Specifically, the multivariate Weibull cdf $F(\mathbf{v}_i; \boldsymbol{\theta}_F)$ can be obtained by substituting $u_k = 1 - \exp\left(-\left(v_{i,k} / \lambda_k \right)^{\kappa_k} \right)$ into (3), where $\boldsymbol{\theta}_F = \{\boldsymbol{\theta}_{1,F},\ldots,\boldsymbol{\theta}_{K,F},\rho\}$.

In comparison to Archimedean copulas that can only capture the overall dependency, Gaussian copula is able to capture full pairwise dependency of all marginals. Specifically, a Gaussian copula has the form:

$$C_{Gauss}(u_1,\ldots,u_K) = \Phi_\Sigma\left[(\Phi^{-1}(u_1),\ldots,\Phi^{-1}(u_K)) \right] \tag{4}$$

where Φ^{-1} is the inverse of the standard normal cdf, and Φ_Σ is the cdf of a multivariate normal distribution whose mean vector is $\mathbf{0}$ and covariance matrix equals to its correlation matrix. The Gaussian copula density function is given as follows [28]:

$$c_{Gauss}(u_1,\ldots,u_K) = \frac{1}{|\boldsymbol{\Sigma}|^{\frac{1}{2}}} \exp\left\{ -\frac{1}{2} \begin{pmatrix} \Phi^{-1}(u_1) \\ \cdots \\ \Phi^{-1}(u_K) \end{pmatrix}' (\boldsymbol{\Sigma}^{-1} - \mathbf{I}) \begin{pmatrix} \Phi^{-1}(u_1) \\ \cdots \\ \Phi^{-1}(u_K) \end{pmatrix} \right\} \tag{5}$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{\Sigma}$ is the correlation matrix. By using the chain rule, we obtain the probability density function (pdf) of the multivariate Weibull distribution using Gaussian copula, i.e.,

$$f_{Gauss}(\mathbf{v}_i;\boldsymbol{\theta}_F)=\frac{\partial^K C_{Gauss}}{\partial u_1\dots\partial u_K}\frac{du_1}{dv_{i,1}}\dots\frac{du_K}{dv_{i,K}}=c_{Gauss}(u_1,\dots u_K)\cdot f_1(v_{i,1})\dots f_K(v_{i,K}) \tag{6}$$

where $f_k(\cdot)$ is the marginal Weibull pdf for failure type k. Therefore, the joint CDF F can be obtained by integrating $f_{Gauss}(\mathbf{v}_i;\boldsymbol{\theta}_F)$, where $\boldsymbol{\theta}_F=\{\boldsymbol{\theta}_{1,F},...,\boldsymbol{\theta}_{K,F},\mathbf{\Sigma}\}$.

3. Parameter estimation and statistical inference

3.1 Construction of likelihood function

The maximum likelihood (ML) approach is used to estimate the model parameters, including those in the joint distribution and those in the trend functions. To implement the ML approach, the likelihood function is firstly calculated.

Let $\mathcal{F}_{t^-}=\left\{(T_1,\Delta_1),...,\left(T_{N(t^-)},\Delta_{N(t^-)}\right)\right\}$, which contains all the paired failure times and failure types until, but not include, time t. Thus, the whole dataset can be denoted by $\mathcal{F}_\tau=\mathcal{F}_{\tau^-}\bigcup\{(\tau,0)\}$. Given the failure data set $\mathcal{F}_\tau$, the likelihood function can be decomposed according to the conditional probability as follows:

$$L(\boldsymbol{\theta}\mid\mathcal{F}_\tau)=\prod_{i=1}^{N(\tau)+1}L_i \tag{7}$$

where L_i denotes the conditional likelihood function of failure i given all previous failures. The parameter set $\boldsymbol{\theta}=\left\{\boldsymbol{\theta}_\zeta,\boldsymbol{\theta}_F\right\}$ denotes all parameters in our model, where $\boldsymbol{\theta}_\zeta$ and $\boldsymbol{\theta}_F$ are parameters in trend functions and those in joint distributions, that are defined in Section 2.4. Specifically,

$$L_i = \begin{cases} \Pr(t_1, \delta_1) \text{ for } i=1 \\ \Pr(t_i, \delta_i \mid t_j, \delta_j; j=1,...,i-1) \text{ for } i=2,...,N(\tau) \\ \Pr(\tau, 0 \mid t_j, \delta_j; j=1,...,N(\tau)) \text{ for } i=N(\tau)+1 \end{cases}. \quad (8)$$

In equation (8), we interpret $\Pr(x)$ as $f(x)dx$, which is proportional to the density $f(x)$. For convenience of notation, we ignore dx in the likelihood functions.

Due to the cumulative effect of imperfect repair and component failure dependency, L_i in (8) is difficult to calculate in the original time domain as it depends on the entire failure history. To overcome this difficulty, we calculate the likelihood function in the transformed time domains, in which the cumulative effects of imperfect repair are eliminated so that L_i only depends on the most recent components' failures.

The following Proposition 1 shows the calculation of L_i in the transformed time domains. The detailed proof of Proposition 1 is available in Appendix Section 1.

Proposition 1: the conditional likelihood function of failure i given all previous failures, i.e., L_i, is given as follows

$$L_i = \frac{\left\{ -\frac{\partial S(v_{i1}, \cdots, v_{i,\delta_i}, \cdots, v_{i,K})}{\partial v_{i,\delta_i}} \Big|_{\mathbf{v}_i = [b_1(t_i), \cdots, b_K(t_i)]'} \right\} \lambda_{\delta_i}(t_i)}{S[b_1(t_{i-1}^+), \cdots, b_K(t_{i-1}^+)]}, \quad (9)$$

where $S(\cdot)$ denotes the survival function of $\mathbf{V}_1$; $\lambda_{\delta_i}(t_i)$ is the derivative of trend function that is used to transform the probability density from the original time domain to a transformed domain; and $b_k(t) = \Lambda_k(t) - \Lambda_k(r_k(t))$.

When $i = N(\tau)+1$ in (8), as there is no failure observed from $t_{N(\tau)}$ to the predetermined ending time τ, the conditional probability can be calculated as:

$$L_{N(\tau)+1} = \frac{S[b_1(\tau),\cdots,b_K(\tau)]}{S\{b_1[t_{N(\tau)}^{+}],\cdots,b_K[t_{N(\tau)}^{+}]\}}. \tag{10}$$

3.2 Maximization of likelihood function

Model parameters can be estimated by maximizing the likelihood function obtained in the previous section. In practice, however, several issues need to be addressed.

When the Gaussian copula is used, two constraints exist: (a) the covariance matrix needs to be positive definite; and (b) the covariance matrix is equal to its correlation matrix. Constraint (b) is satisfied by directly fixing the diagonal elements of the covariance matrix as one. To satisfy constraint (a), we apply the nearest correlation matrix method [29]. Specifically, at each iteration of the optimization process, the estimated correlation matrix is approximated by the nearest correlation matrix that is positive definite. In practice, the correlation matrix only needs to be approximated by the nearest correlation matrix at the first several iterations. After a number of iterations, the output correlation matrix will automatically become positive definite as the estimated correlation matrix converge to the real correlation matrix.

The first order derivative of the survival function in (9) needs to be evaluated many times during the process of optimizing the likelihood function. To speed up the parameter estimation process, we evaluate the first order derivative of survival function of Gaussian copula as follows:

$$-\frac{\partial S(v_{i,1},...,v_{i,k},...,v_{i,K})}{\partial v_{i,k}} = f_k(v_{i,k}) S_{Normal}(\gamma_1,...,\gamma_j,...,\gamma_K); j \neq k \tag{11}$$

where $f_k(v_{i,k})$ is the pdf of random variable $v_{i,k}$. In this paper, we consider $f_i(v_{i,k})$ as Weibull marginal distribution. However, equation (11) still holds for other marginal distributions. Here, $S_{Normal}(\cdot)$ is the survival function of a $K-1$ dimensional multivariate normal, and $\gamma_j = \Phi^{-1}(u_j)$, where u_j denotes the cumulative density of the j^{th} marginal. The proof of (11) is given in Appendix Section 2.

Under a large-sample assumption, the ML estimate $\hat{\boldsymbol{\theta}}$ is asymptotically normally distributed based on ML theory [30]. Thus, the asymptotic covariance $\hat{\boldsymbol{\Sigma}}_{\hat{\boldsymbol{\theta}}}$ for $\hat{\boldsymbol{\theta}}$ can be calculated from the observed Fisher information matrix $\mathbf{I}(\hat{\boldsymbol{\theta}})$, i.e., $\hat{\boldsymbol{\Sigma}}_{\hat{\boldsymbol{\theta}}} = \mathbf{I}(\hat{\boldsymbol{\theta}})^{-1}$, where $\mathbf{I}(\hat{\boldsymbol{\theta}})$ is the negative of the Hessian matrix $\mathbf{H}(\boldsymbol{\theta})$ evaluated at $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}$, i.e., $\mathbf{I}(\hat{\boldsymbol{\theta}}) = -\left.\frac{\partial^2 \log(L(\boldsymbol{\theta}))}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^T}\right|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}}$.

3.3 Statistical hypothesis test

Dependency information of different component failures is important for the maintenance and design of complex systems. In addition, distinguishing subsets of components which do not fail independently can simplify the reliability model and efficiently reduce the parameter dimension.

In this section, we propose hypothesis testing procedures to determine the dependency among different component failures based on the proposed model. The likelihood ratio test statistic is calculated as follows,

$$D = -2 \cdot \ln\left(\frac{\sup\{L_n\}}{\sup\{L_f\}}\right) \tag{12}$$

where L_n indicates the likelihood values for the null model; and L_f indicates the likelihood values for the full model that includes both the null and the alternative models.

The likelihood ratio test statistic in (12) follows a χ^2 distribution with the degrees of freedom ω; where ω is the difference between the number of parameters in the full model and that in the null model. The null and alternative hypotheses, in practice, depend on the specific copula function that is used to construct the reliability model. In this paper, hypothesis tests for statistical model via Clayton copula and that via Gaussian copula are developed in Sections 3.3.1 and 3.3.2, respectively.

3.3.1 Hypothesis test for Clayton copula

When the Clayton copula is selected to construct the reliability model, the overall dependency among all failure types can be tested using the following hypothesis test.

H_0: all failure types are independent vs H_1: not all failure types are independent. (13)

Hypothesis test (13) can be tested based on the likelihood ratio test statistic that is defined in (12). In (12), $\sup\{L_f\}$ can be obtained by maximizing (8), while $\sup\{L_n\}$ can be obtained by maximizing (8) with a constraint that the association parameter ρ is fixed as zero.

3.3.2 Hypothesis test for Gaussian copula

The pairwise dependency among all failure types can be tested, when the Gaussian copula is selected to construct the reliability model. The following test is proposed:

H_0: failure types i, j are independent vs H_1: failures types i, j are dependent . (14)

Similar to hypothesis test (13), hypothesis test (14) can be tested based on the likelihood ratio test statistic that is defined in (12). Under this situation, $\sup\{L_f\}$ can be obtained by maximizing (8), while $\sup\{L_n\}$ can be obtained by maximizing (8) with a constraint that the correlation for failure types i, j in the Gaussian copula is fixed as zero.

If multiple pairwise dependency of failure types are tested simultaneously, the Bonferroni correction [31] can be used to counteract the problem of multiple comparisons. Specifically, each individual hypothesis is tested at a significance level of α / m , where α is the desired significance level and m is the number of hypothesis tests.

4. Simulation study

A comprehensive simulation study is conducted to verify the developed model. We consider five scenarios. Scenarios 1-4 are used to examine the effect of different copula functions, different degree of dependency, different marginal distributions, and different trend functions, respectively; while Scenario 5 is used to verify the parameter estimation method. To keep the setting simple, we consider a two-component system for Scenario 1-4 and a three-component system for Scenario 5. For each scenario, the failure data are simulated based on the proposed reliability model.

4.1 Parameters setting

1) Scenario 1: examine the effect of form of copula.

We use the Weibull marginal distribution and the power law trend function with increasing trend. We consider two copula functions with moderate dependency: Gaussian copula and Clayton copula. The parameters of the copula functions are chosen such that the copula functions have the same overall dependency. When the Gaussian and Clayton copulas are chosen, the parameters are listed in Table I and Table II, respectively.

Table I. Parameter setting in simulation Scenario 1 (Gaussian copula)

Component	Trend function		Joint distribution (Gaussian copula + Weibull marginal)			
	β	η	κ (shape)	λ (scale)	Mean	Correlation matrix
1	1.200	1.000	2.000	1.128	0	1.000 0.500
2	1.200	1.000	2.000	1.128	0	0.500 1.000

Table II. Parameter setting in simulation Scenario 1 (Clayton copula)

Component	Trend function		Joint distribution (Clayton copula + Weibull marginal)		
	β	η	κ (shape)	λ (scale)	Association parameter
1	1.200	1.000	2.000	1.128	1.000
2	1.200	1.000	2.000	1.128	

2) Scenario 2: examine the effect of dependency in copula.

We use the Weibull marginal distribution, power law trend function with increasing trend, and the Gaussian copula. By choosing different values of the copula, we consider three situations: component failure independency, moderate failure dependency, and strong failure dependency. For the moderate dependency case, the simulation parameters setting are the same as listed in Table I. For independency and the strong dependency cases, we set the values of the correlation coefficients to be 0 and 0.9, respectively, while all other parameters are the same as those in Table 1.

3) Scenario 3: Examine the effect of marginal distribution:

We use the power law trend function with increasing trend and Gaussian copula. We consider two marginal distributions: the Weibull and the lognormal distribution. For the Weibull case, the parameters are the same as listed in Table I. The parameters for the lognormal distribution case are listed in Table III.

Table III. Parameter setting in simulation Scenario 3 (lognormal marginal)

Component	Trend function		Joint distribution (Gaussian copula + lognormal marginal)				
	$\boldsymbol{\beta}$	$\boldsymbol{\eta}$	$\boldsymbol{\mu}$	$\boldsymbol{\sigma}$	Mean	Correlation matrix	
1	1.200	1.000	-0.125	0.500	0	1.000	0.500
2	1.200	1.000	-0.125	0.500	0	0.500	1.000

4) Scenario 4: examine the effect of trend function:

We use the Weibull marginal distribution and Gaussian copula function. We consider three situations of the power law trend function: increasing trend, constant, or decreasing trend. For increasing trend function case, the parameters are the same as those in Table I. For constant and decreasing trend functions, we set $\boldsymbol{\beta} = [1.0, 1.0]'$ and $\boldsymbol{\beta} = [0.8, 0.8]'$, respectively, while all the other parameters are the same as those in Table 1.

5) Scenario 5: validate the parameter estimation method:

We use the Weibull marginal distribution, Gaussian copula function, and the power law trend function with increasing trend. A three-component system is considered, and the parameters are given in Table IV.

Table IV. Parameter setting in simulation Scenario 5

Component	Trend function		Joint distribution (Gaussian copula + Weibull marginal)					
	$\boldsymbol{\beta}$	$\boldsymbol{\eta}$	$\boldsymbol{\kappa}$ (shape)	$\boldsymbol{\lambda}$ (scale)	Mean	Correlation matrix		
1	1.200	1.000	2.000	1.128	0	1.000	0.100	0.400
2	1.200	1.000	2.000	1.128	0	0.100	1.000	0.800
3	1.200	1.000	2.000	1.128	0	0.400	0.800	1.000

4.2 Parameter estimation

In the simulation study, we vary the value of stopping time τ to obtain different values of the expected number of events. We think it is more informative to show the number of events, instead of the value of τ. We consider four different numbers of events for each scenario, i.e., 100, 200, 500 and 1000 respectively.

To evaluate the performance of the parameter estimation method, we calculate both the MSEs of estimators and the coverage probabilities for the 95% confidence intervals based on 1000 replicates under each parameter setting. Figures 4-8 plot the MSEs (left) and coverage probabilities (right). From Figures 4-8, we can see that when the sample size is large enough, the MSEs are approaching zero, and the coverage probabilities of 95% confidence intervals for the unknown parameters are approaching 95%. Thus, the estimators of the parameters perform well.

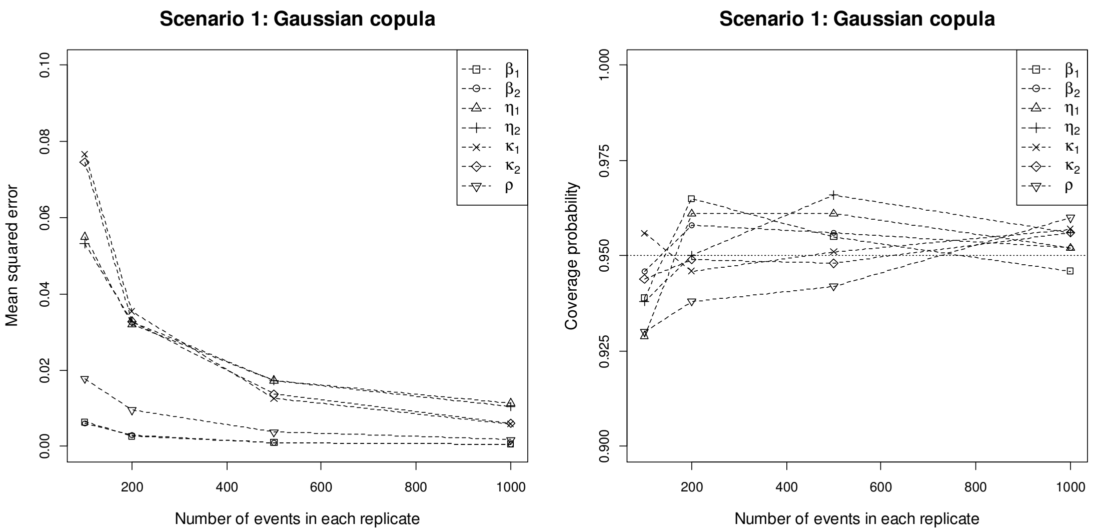

Figure 2. Simulation results for scenario 1 with Gaussian copula.

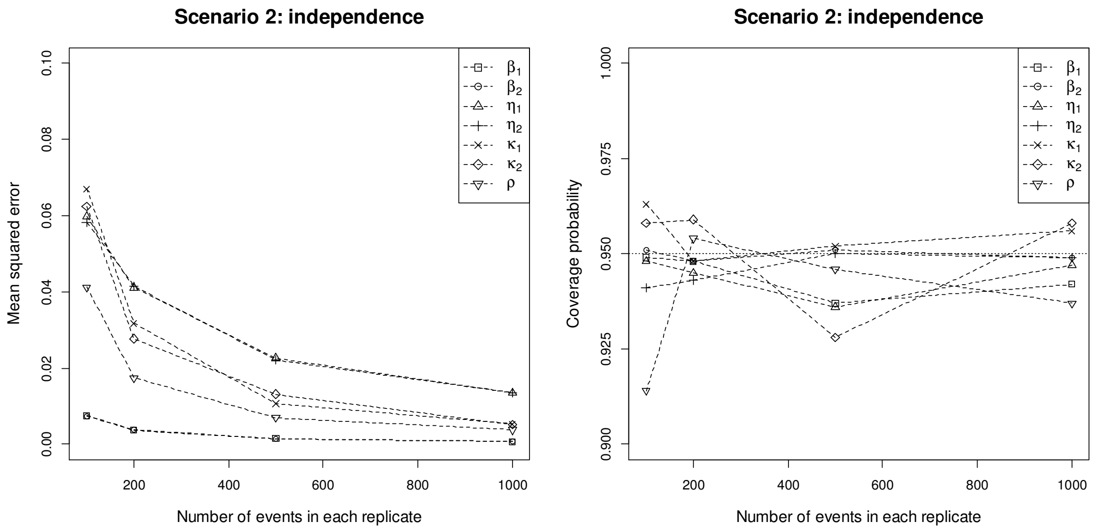

Figure 3. Simulation results for scenario 2 with independent failures

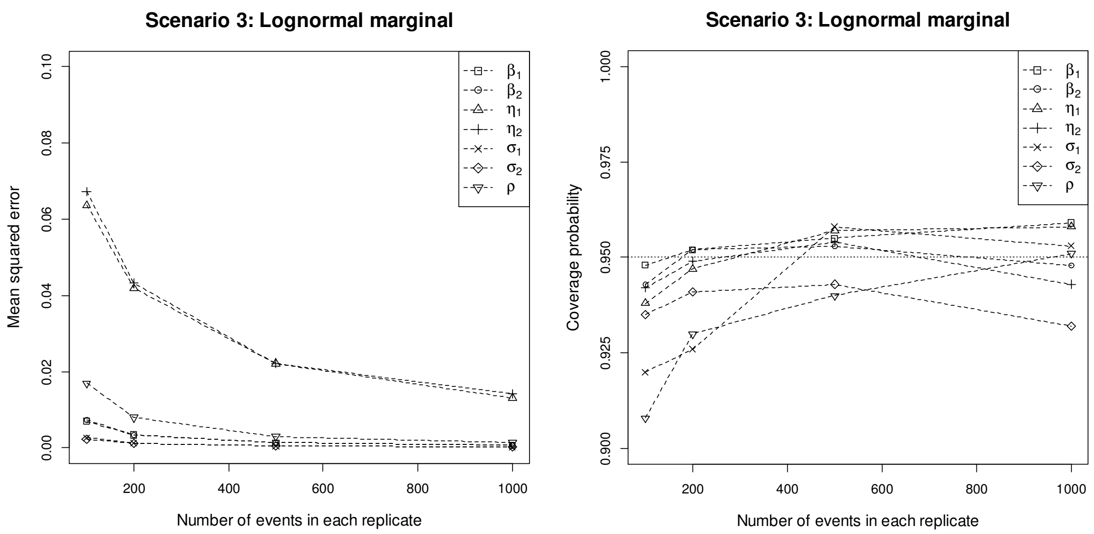

Figure 4. Simulation results for scenario 3 with lognormal marginal

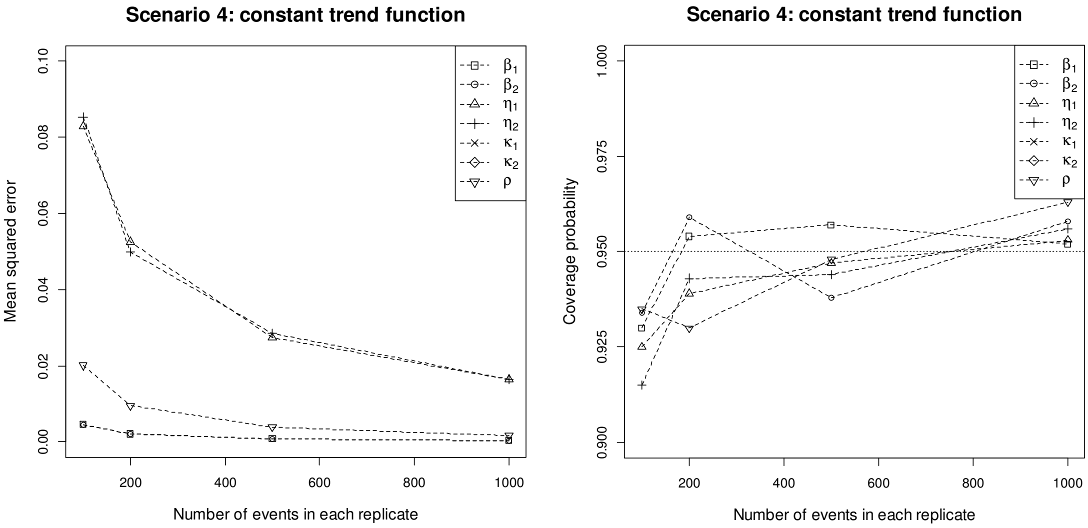

Figure 5. Simulation results for scenario 4 with constant trend function

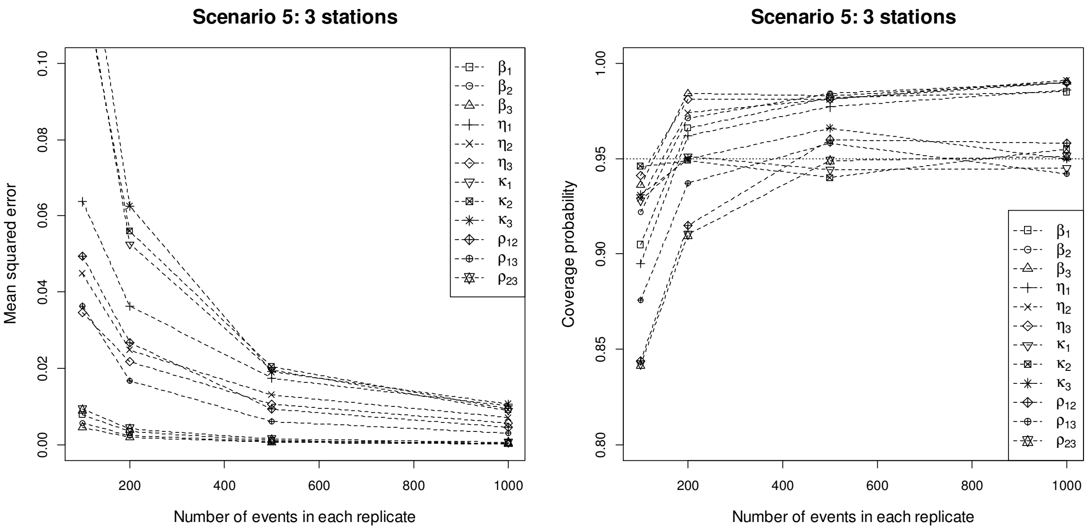

Figure 6. Simulation results for scenario 5 with 3 stations

5 Application

5.1 Application for Example 1

The developed methodology is applied to two real world applications. The first application comes from a car body assembly machine of an automotive assembly production line in the United States. The machine is repaired when a failure occurs, and the repair action is determined by the failure condition. However, the repair actions are not recorded. Figure 7 shows the cumulative number of failures from two subsystems of the machine that work simultaneously during the assembly process. The time is rescaled and the subsystems are denoted by subsystem A and subsystem B to protect proprietary information.

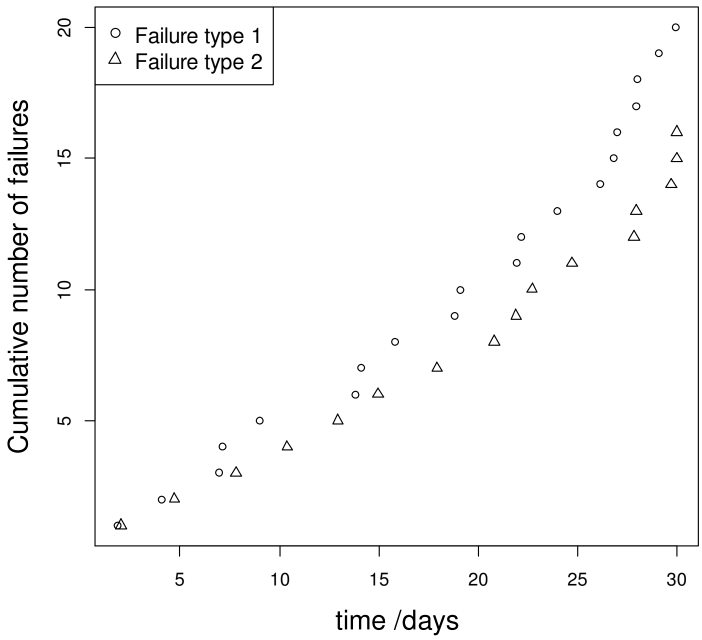

Figure 7. Failure data from stations A and B

When applying the proposed model, the overall likelihood is obtained by substituting (9) and (10) into (8). The parameters are estimated by maximizing the likelihood function. When the Clayton copula is applied, the estimated parameters and the standard errors are listed in the following Table V.

Table V. Parameter estimates and standard errors (values in the bracket) when choosing the Clayton copula

failure type	Trend function		Joint distribution		
	$\hat{\beta}$	$\hat{\eta}$	$\hat{\kappa}$	$\hat{\lambda}$	Association parameter ρ
1	1.38(0.28)	169.79(1.61)	1.06(0.20)	1.02(0.68)	0.33(0.04)
2	1.24(0.33)	110.95(1.90)	1.20(0.34)	1.06(0.23)	

When applying the Gaussian copula to obtain the joint distribution, the estimated parameters and the corresponding standard errors are listed in Table VI.

Table VI. Parameter estimates and standard errors (values in the bracket) when choosing the Gaussian copula

failure type	Trend function		Joint distribution			
	$\hat{\beta}$	$\hat{\eta}$	$\hat{\kappa}$	$\hat{\lambda}$	$\hat{\Sigma}$	
1	1.52(0.24)	3.27(1.61)	0.89(0.22)	0.95(0.13)	1.00	0.47(0.20)
2	1.65(0.31)	4.45(2.01)	0.90(0.26)	0.95(0.15)	-	1.00

The negative log-likelihood values for the models are 49.909 and 47.255 for the model via the Clayton copula and that via the Gaussian copula, respectively. As both models have the same number of model parameters, the model via the Gaussian copula fits the data much better so that we select the reliability model via the Gaussian copula. Thus, hypothesis test (14) is applied to test the overall failure dependency. The log-likelihood value is -47.255 for the full model and -50.561 for the null model. As the p-value equals 0.01, the reliability model indicates that different failure types are dependent.

We apply a graphical tool that is traditionally used in reliability literature [32] to show the goodness of fit for the proposal model. Specifically, we compare the cumulative number of observed events to the estimated expected number of events. The following Figure 9 shows the estimated expected number of events for the reliability model via the Clayton copula and that via the Gaussian copula, respectively.

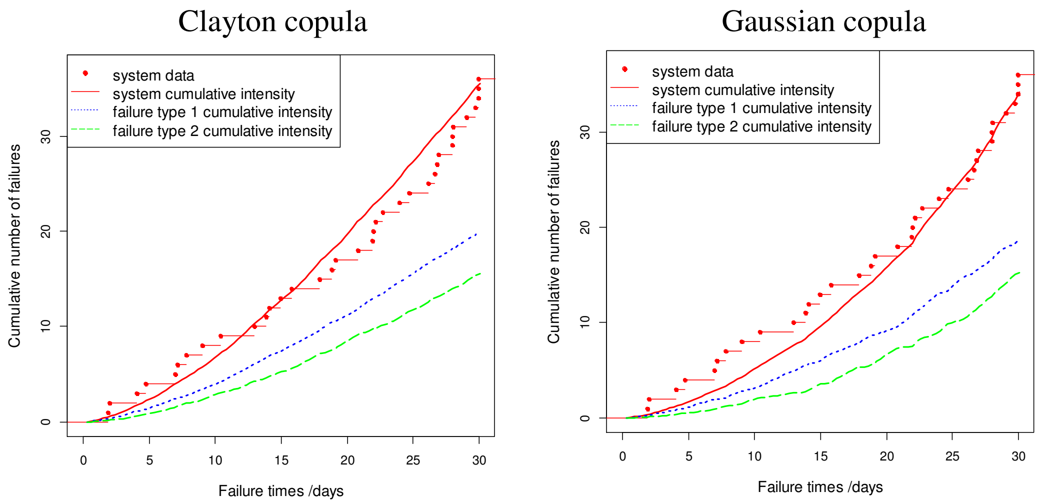

Figure 8. Estimated mean cumulative intensity function of failures compared to the real failure data, conditional on event history.

We compared the propose model to three simpler models: 1) perfect repair with dependent component failures; 2) imperfect repair with independent component failures; and 3) perfect repair with independent component failures. Specifically, we calculate both the negative log-likelihood and the Akaike Information Criterion (AIC) values through $AIC = -2\ln\hat{L} + 2w$; where $\hat{L}$ is the estimated likelihood value; and w is the number of free parameters in a model. In general, the smaller negative log-likelihood or the AIC values, the better the model.

The following Table VII shows the comparison results. As the proposed model has the smallest negative log-likelihood and the AIC values, it fits the data best.

Table VII. Comparison of the proposed model to other three simple models

	Negative Log-likelihood	AIC	Degrees of freedom
Proposed model: imperfect repair with dependent failures	47.255	108.511	7
Model 1: imperfect repair with independent failures	50.561	113.122	6
Model 2: perfect repair with dependent failures	51.932	113.864	5
Model 3: perfect repair with independent failures	54.307	116.613	4

5.2 Application for Example 2

We also apply the developed method to a forklift vehicle system, in which two major subsystems are a transportation and drive subsystem and a lift mechanism subsystem. Figure 9 illustrates the cumulative number of failures from the two different subsystems of a forklift vehicle used in a manufacturing plant for about three years. The failure types are denoted by failure types 1 and 2 to protect proprietary information.

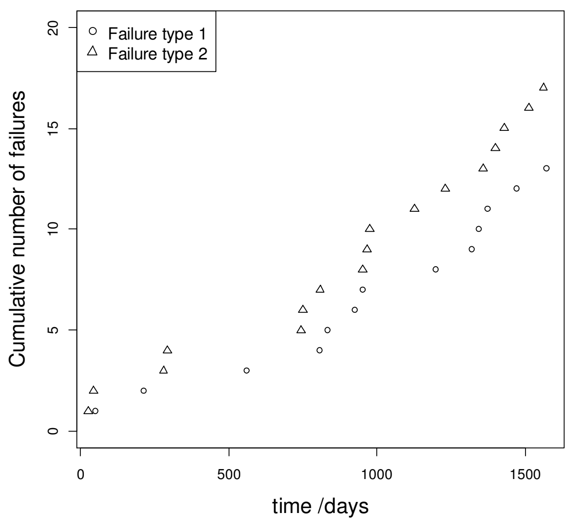

Figure 9. Failure data from failure type 1 and failure type 2

We apply the developed methods to the lifting fork vehicle system. The following Tables VIII and IX list the estimated model parameters and the standard errors, when applying the Clayton copula and the Gaussian copula to obtain the joint distribution, respectively.

Table VIII. Parameter estimates and standard errors (values in the bracket) when choosing the Clayton

failure type	Trend function		Joint distribution		
	$\hat{\beta}$	$\hat{\eta}$	$\hat{\kappa}$	$\hat{\lambda}$	Association parameter ρ
1	1.38(0.26)	169.79(77.44)	0.84(0.23)	0.91(0.16)	2.41(1.88)
2	1.24(0.33)	110.95(80.73)	0.63(0.17)	0.70(0.23)	

Table IX. Parameter estimates and standard errors (values in the bracket) when choosing the Gaussian copula

failure type	Trend function		Joint distribution			
	$\hat{\beta}$	$\hat{\eta}$	$\hat{\kappa}$	$\hat{\lambda}$	$\hat{\Sigma}$	
1	1.32(0.30)	155.11(84.05)	1.01(0.29)	1.00(0.12)	1.00	0.56(0.18)
2	1.19(0.31)	96.43(70.31)	0.75(0.18)	0.84(0.17)	-	1.00

The negative log-likelihood values for the models are 165.967 and 165.752 for the model via the Clayton copula and that via the Gaussian copula, respectively. As the negative log-likelihood values are close, both models have the similar ability to fit the data. Hypothesis tests are applied in the case study to examine the failure dependency structure of the stations. When applying hypothesis test (13) to test the failure dependency, the negative log likelihood value is 165.967 for the full model and is 168.017 for the null model, and the p-value equals 0.043. When applying hypothesis test

(14) to test the failure dependency, the log-likelihood values is -165.752 for the full model and is -168.016 for the null model, and the p-value equals 0.033. Thus, the different failure types are dependent. However, the estimation and standard deviation of $\hat{\boldsymbol{\beta}}$ in both Tables VIII and IX show that the parameters $\hat{\boldsymbol{\beta}}$ in the trend function are not statistical different from 1, indicating perfect component repairs in the system.

We apply the propose model as well as other three simpler models to fit the data. The following Table X shows the comparison results.

Table X**.** Comparison of the proposed model to other three simple models

	Negative Log-likelihood	AIC	Degrees of freedom
Proposed model: imperfect repair with dependent failures	165.752	345.504	7
Model 1: imperfect repair with independent failures	168.017	348.033	6
Model 2: perfect repair with dependent failures	166.943	343.882	5
Model 3: perfect repair with independent failures	168.598	345.196	4

For Table X it can be seen that the models assuming component failure dependency (proposed model and Model 2) fit the data better than those assuming component failure independency (Model 1 and Model 3), while the models assuming perfect repair (Model 2 and Model 3) fit the data better than those assuming imperfect repair (proposed model and Model 1). The results are consistent with what obtained previously. Thus we conclude that the two subsystems are subject to perfect repair with dependent failures.

6. Conclusions

In this paper, we propose a parametric reliability model for dependent competing-risk systems. This model can handle two challenges in general multi-component repairable systems, which are to deal with imperfect component repair from perfect to minimal, and to capture the dependency among different failure types. We extend the TRP model for single-component systems to competing-risk systems by transforming original failure times into new time domains for each component respectively. Then, the dependency of different component failures is captured by a joint distribution established from marginal in the transformed time domains. The model parameters are estimated using the ML method. The dependency is further examined by the suggested hypothesis tests. Finally, a case study from an engineering head assembly system consisting of three failure types is conducted to verify the model.

The proposed model can be useful in maintenance planning. Based on the proposed model, one can predict the system reliability given the failure history. To briefly discuss the idea, suppose one observed the failure history up to time π, which is denoted as $\mathcal{F}_{\pi}$. One can compute the reliability of the system at a future time $t > \pi$. Specifically, the probability can be computed as $R(t \mid \mathcal{F}_{\pi}) = S[b(1,t^{-}),\ldots,b(K,t^{-})] / S[b(1,t_{\pi}^{+}),\ldots,b(K,t_{\pi}^{+})]$, which is the probability that no failure occurs before t given the history. Applying this reliability information and the model predictive ability for the maintenance planning can be an interesting topic for further research. In some applications, window observations may occur for the event history data (e.g., Hong, Li and Osborn [33]). It would be interesting to consider the proposed model under the window-observed recurrent event data. In this paper, we use a parametric method. In future research, it would also be interesting to build nonparametric models. The estimation of nonparametric models, however, can be challenging as identifiability problems may arise.

Appendix

1. Proof of Proposition 1

Note that $V_{i,k}$ is defined as the latent age to failure of component k after the $(i-1)^{th}$ failures in the k^{th} transformed time domain. Let $W_{i,k}$ be the corresponding random variable for the age to failure of component k after the $(i-1)^{th}$ failure in the original time domain. Suppose that the i^{th} system failure occurs at time point t_i in the original time domain.

Because $r_k(t_i)$ is left continuous, $r_k(t_i) = r_k(t_{i-1}^+)$. Thus,

$$V_{i,k} = \Lambda_k[W_{i,k} + r_k(t_i)] - \Lambda_k[r_k(t_i)] = \Lambda_k[W_{i,k} + r_k(t_{i-1}^+)] - \Lambda_k[r_k(t_{i-1}^+)]$$

As $b_k(t_i) = \Lambda_k(t_i) - \Lambda_k[r_k(t_i)]$, and $b_k(t_i)$ is also left continuous. Thus,

$$b_k(t_{i-1}^+) = \Lambda_k[a_k(t_{i-1}^+) + r_k(t_{i-1}^+)] - \Lambda_k[r_k(t_{i-1}^+)] = \Lambda_k(t_{i-1}) - \Lambda_k[r_k(t_i)].$$

Note that

$$\begin{aligned} W_{i,k} > a_k(t_i) &\Leftrightarrow W_{i,k} + r_k(t_i) > a_k(t_i) + r_k(t_i) \\ &\Leftrightarrow \Lambda_k[W_{i,k} + r_k(t_i)] > \Lambda_k[a_k(t_i) + r_k(t_i)] \\ &\Leftrightarrow \Lambda_k[W_{\xi_{i,k},k} + r_k(t_i)] - \Lambda_k[r_k(t_i)] > \Lambda_k[a_k(t_i) + r_k(t_i)] - \Lambda_k[r_k(t_i)] \\ &\Leftrightarrow V_{i,k} > b_k(t_i). \end{aligned} \tag{A1}$$

Similarly,

$$W_{i,k} > a_k(t_{i-1}^+) \Leftrightarrow V_{i,k} > b_k(t_{i-1}^+). \tag{A2}$$

In addition,

$$
\begin{aligned}
W_{i,k} = a_k(t_i) &\Leftrightarrow a_k(t_i) \le W_{i,k} < a_k(t_i) + dt \\
&\Leftrightarrow a_k(t_i) + r_k(t_i) \le W_{i,k} + r_k(t_i) < a_k(t_i) + r_k(t_i) + dt \\
&\Leftrightarrow \Lambda_k[a_k(t_i) + r_k(t_i)] \le \Lambda_k[W_{i,k} + r_k(t_i)] < \Lambda_k[a_k(t_i) + r_k(t_i) + dt] \\
&\Leftrightarrow \Lambda_k[a_k(t_i) + r_k(t_i)] - \Lambda_k[r_k(t_i)] \le \Lambda_k[W_{i,k} + r_k(t_i)] - \Lambda_k[r_k(t_i)] \\
&\quad < \Lambda_k[a_k(t_i) + r_k(t_i) + dt] - \Lambda_k[r_k(t_i)] \\
&\Leftrightarrow b_k(t_i) \le V_{i,k} < \Lambda_k[a_k(t_i) + r_k(t_i)] + \lambda_k[a_k(t_i) + r_k(t_i)]dt - \Lambda_k[r_k(t_i)] \\
&\Leftrightarrow b_k(t_i) \le V_{i,k} < b_k(t_i) + \lambda_k(t_i)dt.
\end{aligned} \tag{A3}
$$

From (8), we have

$$
\begin{aligned}
L_i &= \Pr(t_i, \delta_i \mid t_j, \delta_j; j = 1, \cdots, i-1) \qquad i = 1, \cdots, N(\tau) \\
&= \Pr[W_{i,\delta_i} = a_{\delta_i}(t_i), W_{i,l} > a_l(t_i); l \ne \delta_i \mid W_{i,k} > a_k(t_{i-1}^+); k = 1, \cdots, K] \\
&= \frac{\Pr[W_{i,\delta_i} = a_{\delta_i}(t_i), W_{i,l} > a_l(t_i); l \ne \delta_i]}{\Pr[W_{i,k} > a_k(t_{i-1}^+); k = 1, \cdots, K]}.
\end{aligned} \tag{A4}
$$

Substituting (A1), (A2), and (A3) into (A4), we obtain

$$
\begin{aligned}
L_i &= \frac{\Pr[b_{\delta_i}(t_i) \le V_{i\delta_i} < b_{\delta_i}(t_i) + \lambda_{\delta_i}(t_i)dt, V_{i,l} > b_l(t_i); l \ne \delta_i]}{\Pr[V_{i,k} > b_k(t_{i-1}^+); k = 1, \cdots, K]} \\
&= \frac{\left\{ -\dfrac{\partial S(v_{i1}, \cdots, v_{i,\delta_i}, \cdots, v_{i,K})}{\partial v_{i,\delta_i}} \Big|_{\mathbf{v}_i = [b_1(t_i), \cdots, b_K(t_i)]'} \right\} \lambda_{\delta_i}(t_i)}{S[b_1(t_{i-1}^+), \cdots, b_K(t_{i-1}^+)]},
\end{aligned}
$$

where $\mathbf{v}_i = (v_{i,1}, \cdots, v_{i,K})'$; and $S(\cdot)$ denotes the survival function of $\mathbf{V}_1$.

When $i = N(\tau) + 1$, the conditional probability can be calculated as:

$$
\begin{aligned}
L_{N(\tau)+1} &= \Pr[\tau, 0 \mid t_j, \delta_j; j = 1, \cdots, N(\tau)] \\
&= \Pr\left\{ W_{N(\tau),k} > a_k(\tau); k = 1, \cdots, K \mid W_{N(\tau),k} > a_k[t_{N(\tau)}^+]; k = 1, \cdots, K \right\}.
\end{aligned} \tag{A5}
$$

Substituting (A1) and (A2) into (A5), we obtain

$$L_{N(\tau)+1} = \frac{\Pr[V_{N(\tau),k} > b_k(\tau); k = 1, \cdots, K]}{\Pr\{V_{N(\tau),k} > b_k[t^+_{N(\tau)}]; k = 1, \cdots, K\}}$$
$$= \frac{S[b_1(\tau), \cdots, b_K(\tau)]}{S\{b_1[t^+_{N(\tau)}], \cdots, b_K[t^+_{N(\tau)}]\}}.$$

2. Proof of Equation (11)

We use γ_j to denote $\Phi^{-1}(u_j)$, i.e., $u_j = \Phi(\gamma_j)$. Based on (4), the Gaussian copula function can be written as $C_{Gauss}(u_{1,}, ..., u_K) = \Phi_\Sigma(\gamma_1, ..., \gamma_K)$. Thus, the pdf of Gaussian copula becomes:

$$f_{Gauss}(v_1, ..., v_K; \mathbf{\theta}_F) = \frac{\partial^K C_{Gauss}}{\partial \gamma_1 ... \partial \gamma_K} \left(\frac{d\gamma_1}{dv_1} ... \frac{d\gamma_K}{dv_K} \right) = \phi(\gamma_1, ..., \gamma_K) \left(\frac{d\gamma_1}{dv_1} ... \frac{d\gamma_K}{dv_K} \right),$$

where $\phi(\cdot)$ denotes the pdf of multivariate normal distribution Φ_Σ. In particular, we use $\mathbf{\Sigma}_{1,1}$ and $\mathbf{\Sigma}_{1,2}$ to denote the covariance of $[\gamma_1, ..., \gamma_j, ..., \gamma_K]', j \neq i$ and the covariance between $[\gamma_1, ..., \gamma_j, ..., \gamma_K]' j \neq i$ and γ_i, respectively. Here $[\gamma_1, ..., \gamma_j, ..., \gamma_K]' j \neq i$ is the vector without γ_i. By using the result in Eaton [34], the pdf of multivariate normal distribution can be calculated by conditional probability, i.e., $\phi(\gamma_1, ..., \gamma_K) = g(\gamma_i) \cdot h(\gamma_1, ..., \gamma_j, ..., \gamma_K); j \neq i$, where $g(\cdot)$ denotes the standard normal pdf, and $h(\cdot)$ denotes a $K-1$ dimensional multivariate normal with a mean vector of $\Sigma_{1,2} \cdot \gamma_i$ and a covariance vector of $\mathbf{\Sigma}_{1,1} - \mathbf{\Sigma}_{1,2} \cdot \mathbf{\Sigma}^T_{1,2}$. Thus, the first order partial derivative of the survival function becomes:

$$
\begin{aligned}
&-\frac{\partial S(v_1,...,v_i,...,v_K)}{\partial v_i} \\
&=\int_{v_K}^{\infty}...\int_{v_j}^{\infty}...\int_{v_1}^{\infty}\{f_{Gauss}(v_1,...,v_K;\boldsymbol{\theta}_F)\}\,dv_1...dv_j...dv_K;\, j\neq i \\
&=\int_{v_K}^{\infty}...\int_{v_j}^{\infty}...\int_{v_1}^{\infty}\left\{g(\gamma_i)h(\gamma_1,...,\gamma_j,...,\gamma_K)\left(\frac{d\gamma_1}{dv_1}....\frac{d\gamma_K}{dv_K}\right)\right\}dv_1...dv_j...dv_K;\, j\neq i \\
&=\left\{g(\gamma_i)\left(\frac{du_i}{d\gamma_i}\right)^{-1}f_i(v_i)\right\}\left\{\int_{v_K}^{\infty}...\int_{v_j}^{\infty}...\int_{v_1}^{\infty}h(\gamma_1,...,\gamma_j,...,\gamma_K)\,d\gamma_1...d\gamma_j...d\gamma_K;\, j\neq i\right\} \\
&=\left\{g(\gamma_i)\left(g(\gamma_i)\right)^{-1}f_i(v_i)\right\}S_{Normal}(\gamma_1,...,\gamma_j,...,\gamma_K);\, j\neq i \\
&=f_i(v_i)S_{Normal}(\gamma_1,...,\gamma_j,...,\gamma_K);\, j\neq i
\end{aligned}
$$

where $f_i(\cdot)$ denotes the i^{th} marginal distribution in the Gaussian copula; $S_{Normal}(\gamma_1,...,\gamma_j,...,\gamma_K);\, j\neq i$, is the survival function of a multivariate normal distribution whose pdf is $h(\gamma_1,...,\gamma_j,...,\gamma_K);\, j\neq i$.

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