A Dual to Lyapunov's Second Method for Linear Systems With Multiple Delays and Implementation Using SOS

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***Abstract*—We present a dual form of Lyapunov–Krasovskii functional which allows the problem of controller synthesis for multidelay systems to be formulated and solved in a convex manner. First, we give a generalized version of the dual stability condition formulated in terms of Lyapunov operators which are positive, self-adjoint, and preserve the structure of the state space. Second, we provide a class of such operators and express the stability conditions as positivity and negativity of quadratic Lyapunov–Krasovskii functional forms. Next, we adapt the Sum of Squares (SOS) methodology to express positivity and negativity of these forms as Linear Matrix Inequalities (LMIs), describing a new set of polynomial manipulation tools designed for this purpose. We apply the resulting LMIs to a battery of numerical examples and demonstrate that the stability conditions are not significantly conservative. Finally, we formulate a test for controller synthesis for systems with multiple delays, apply the test to a numerical example, and simulate the resulting closed-loop system.**

***Index Terms*—Controller synthesis, delay systems, LMIs.**

I. Introduction

Systems with delay have been well-studied for some time [1]–[3]. In recent years, however, there has been an increased emphasis on the use of optimization and SemiDefinite Programming (SDP) for stability analysis of linear and nonlinear time-delay systems. Although the computational question of the stability of a linear state-delayed system is believed to be NP-hard, several techniques have been developed that use LMI methods [4] to construct asymptotically exact algorithms. An asymptotically exact algorithm is a sequence of polynomial-time algorithms wherein each instance in the sequence provides sufficient conditions for stability, the computational complexity of the instances is increasing, the accuracy of the test is increasing, and the sequence converges to what appears to be a necessary and sufficient condition. Examples of such sequential algorithms include the piecewise-linear approach [2], the Wirtinger-based method of [5], and the SOS approach [6]. In addition, there are also frequency-domain approaches such as [7] and [8]. These asymptotic algorithms are sufficiently reliable so that for this paper, we may consider the problem of stability analysis of linear discrete-delay systems to be solved.

The purpose of this paper is to explore methods by which we may extend the success in the use of asymptotic algorithms for stability analysis of time-delay systems to the field of robust and optimal controller synthesis—an area that is relatively underdeveloped. Although there have been a number of results on controller synthesis for time-delay systems [9], none of these results has been able to resolve the fundamental bilinearity of the synthesis problem. Bilinearity here means that for a given feedback controller, the search for a Lyapunov functional is linear in the decision variables that define the functional and is relatively tractable. Furthermore, given a predefined Lyapunov functional, the search for a controller ensuring negativity of the time derivative of that functional is linear in the decision variables that define the feedback gains. However, if we are looking for both a controller and a Lyapunov functional that establishes the stability of that controller, then the resulting stability condition is nonlinear and nonconvex in the combined set of decision variables.

Without a convex formulation of the controller synthesis problem, we cannot search over the set of provably stabilizing controllers without significant conservatism, much less address the problems of robust and quadratic stability. To resolve this difficulty, some papers use iterative methods to alternately optimize the Lyapunov functional and then the controller as in [10] or [11] (via a "tuning parameter"). However, this iterative approach is not guaranteed to converge. Meanwhile, approaches based on frequency-domain methods, discrete approximation, or Smith predictors result in controllers that are not provably stable or are sensitive to variations in system parameters or in delay.

In this paper, we propose a dual Lyapunov-type stability criterion, wherein the decision variables do not parameterize a Lyapunov functional per se, but where the feasibility of this criterion *implies* the existence of such a functional. The advantage of such an approach for controller synthesis is that it allows for an invertible variable substitution, eliminating all bilinear terms in the criterion for controller synthesis.

Both our definition of duality (in the optimization sense) and our approach to controller synthesis are based on the LMI framework for controlling linear finite-dimensional state-space systems of the form $\dot{x} = Ax + Bu$. Specifically, if $u = 0$, the LMI condition for the existence of a quadratic Lyapunov function $V(x) = x^T Px$ is the existence of a $P > 0$ such that $A^T P + PA < 0$. The feasibility of this LMI implies that $V(x) = x^T Px > 0$ and $\dot{V}(x) = x^T(A^T P + PA)x < 0$. This

Manuscript received July 3, 2017; revised January 8, 2018; accepted April 5, 2018. Date of publication; date of current version. This work was supported by the National Science Foundation under Grant 1301660, Grant 1538374, and Grant 1739990. Recommended by Associate Editor W. Michiels.

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This paper has supplementary downloadable material available at http://ieeexplore.ieee.org, provided by the author.

Digital Object Identifier 10.1109/TAC.2018.2832470

LMI is in primal form because the decision variable P defines the Lyapunov function directly. However, when we add a controller $u = Kx$, we get $\dot{x} = (A + BK)x$ and the synthesis condition becomes $A^T P + PA + K^T B^T P + PBK < 0$, which is bilinear in decision variables P and K and hence intractable. Bilinearity can be eliminated, however, if we use the *implied* Lyapunov function $V(x) = x^T P^{-1} x$. Using this implied Lyapunov function the time derivative becomes $\dot{V}(x) = x^T(A^T P^{-1} + P^{-1}A)x = (P^{-1}x)^T(PA^T + AP)(P^{-1}x) = z^T(PA^T + AP)z$, where $z = P^{-1}x$. This implies that the stability of $\dot{x} = Ax$ is equivalent to the existence of $P > 0$ such that $AP + PA^T < 0$. If we now add a controller $u = Kx$, the controller synthesis condition becomes $(AP + BKP) + (AP + BKP)^T < 0$, which is still bilinear. However, if we consider the variable substitution $Z = KP$, then stabilizability is equivalent to the existence of a $P > 0$ and Z such that $(AP + BZ) + (AP + BZ)^T < 0$, which is an LMI. The stabilizing controller gains can then be reconstructed as $K = ZP^{-1}$. LMIs of this form were introduced in [12] and are the basis for a majority of LMI methods for controller synthesis (see the supplemental notes in [4, ch. 5] for a discussion). The first contribution of this paper, then, is an operator-valued equivalent of the dual Lyapunov inequality $P > 0$, $AP + PA^T < 0$ that implies the stability of a general class of infinite-dimensional systems. The second contribution of this paper is a computational framework for verifying this dual inequality using LMIs.

The standard approach to state-space representation of infinite-dimensional systems is to define the state as evolving on a Hilbert space Z and satisfying the derivative condition $\dot{x}(t) = \mathcal{A}x(t)$. The state is constrained to a subspace X of Z and the operator $\mathcal{A}$ is typically unbounded. It is known that if $\mathcal{A}$ generates a strongly continuous semigroup, then exponential stability of this system is equivalent to the existence of an operator $\mathcal{P}$ such that $\langle x, \mathcal{P}x\rangle \ge \|x\|^2$ and $\langle x, \mathcal{P}\mathcal{A}x\rangle + \langle \mathcal{P}\mathcal{A}x, x\rangle \le -\epsilon\|x\|^2$ [13]. In Section IV, we show that under mild additional conditions on $\mathcal{P}$, the dual version of this result also holds. Namely existence of an operator $\mathcal{P}$ such that $\langle x, \mathcal{P}x\rangle \ge \|x\|^2$ and $\langle x, \mathcal{A}\mathcal{P}x\rangle + \langle \mathcal{A}\mathcal{P}x, x\rangle \le -\epsilon\|x\|^2$ implies exponential stability of $\dot{x} = \mathcal{A}x$. Specifically, these additional conditions on $\mathcal{P}$ are that $\mathcal{P}$ be self-adjoint and preserve specified properties of the solution. This result applies to any well-posed infinite-dimensional system, and is not conservative if X is a closed subspace of Z.

Having formulated a general duality result, we next turn to the special case of systems with multiple delays and introduce a parametrization of a class of operators that are self-adjoint, preserve desired properties of the solution, and which are defined by the combination of multiplier and integral operators with constraints on the associated multipliers and kernels. This result allows us to represent the dual stability criterion in a manner similar to classical Lyapunov–Krasovskii stability conditions, but with an additional tridiagonal structure that may prove useful for solving these Lyapunov equations. Finally, we present an LMI/SOS method for enforcing positivity and negativity of the operators under the assumption that all multipliers and kernels are polynomial. Finally, we discuss how these results can be used to solve the controller synthesis problem and give a numerical example using the methods defined in [14] and [15].

Having stated the main contributions of this paper, we note that while we show how to enforce the operator inequalities using a slight generalization of existing SOS-based results, the duality results are presented in such a way as to encourage the reader to use other methods of enforcing these inequalities, methods including those contained in [5], or [16]. Indeed, we emphasize that Theorems 1 and 5 are formulated independent of whichever numerical method is used for enforcing the inequalities. In this way, our goal is to simply establish a new class of Lyapunov stability conditions that are well suited to the problem of controller synthesis, leaving the method of enforcement of these conditions to the reader.

Finally, we note that there have been a number of results on dual and adjoint systems [17]. Unfortunately, however, these dual systems are not delay-type systems and there is no clear relationship between the stability of these adjoint and dual systems and the stability of the original delayed system.

This paper is organized as follows. In Sections II and III, we develop a mathematical framework for expressing Lyapunov-based stability conditions as operator inequalities. In Section IV, we show that given additional constraints on the Lyapunov operator, satisfaction of the dual Lyapunov inequality $\langle x, \mathcal{A}\mathcal{P}x\rangle + \langle \mathcal{A}\mathcal{P}x, x\rangle \le -\epsilon\|x\|^2$ proves the stability of $\dot{x}(t) = \mathcal{A}x(t)$. In Sections VI and V, we define a restricted class of Lyapunov functionals and operators which are valid for the dual stability condition in both the single-delay and multiple-delay cases, applying these classes of operators in Sections VI-B and V-B to obtain dual stability conditions. These dual stability conditions are formulated as positivity and negativity of Lyapunov functionals. In Section VII, we show how SOS-based methods can be used to parameterize positive Lyapunov functionals and thereby enforce the inequality conditions in Sections VI-B and V-B, results which are summarized in Corollary 10. Finally, in Section VIII, we summarize our results with a set of LMI conditions for dual stability in both the single and multiple-delay cases. Section IX describes our MATLAB toolbox, available online, which facilitates construction and solution of the LMIs. Section X applies the results to a variety of stability problems and verifies that the dual stability test is not conservative. Finally, Section XI discusses the problem of full-state feedback controller synthesis and gives a numerical illustration in the case of a single delay.

A. Technical Summary of Results

Before proceeding, we give a brief summary of the main results of Section VI-B using as little mathematical formalism as possible in order to illustrate how these results differ from the classical Lyapunov–Krasovskii stability conditions. These results are stated for systems with a single delay in order to avoid much of the notation and mathematical progression needed for the multiple-delay case. That is, we consider the system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau).$$

1) Classical Lyapunov–Krasovskii Stability Conditions:

The standard necessary and sufficient conditions for stability in the single-delay case are the existence of a

$$V(\phi) = \int_{-\tau}^{0} \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix}^T \begin{bmatrix} M_{11} & \tau M_{12}(s) \\ \tau M_{21}(s) & \tau M_{22}(s) \end{bmatrix} \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix} ds$$
$$+ \tau \int_{-\tau}^{0}\int_{-\tau}^{0} \phi(s)^T N(s,\theta)\phi(\theta) d\theta ds$$

such that $V(\phi) \geq \|\phi(0)\|^2$ and

$$\dot{V}(\phi) =$$

$$\int_{-\tau}^{0} \begin{bmatrix} \phi(0) \\ \phi(-\tau) \\ \phi(s) \end{bmatrix}^T \begin{bmatrix} D_{11} + D_{11}^T & D_{12} & \tau D_{13}(s) \\ D_{12}^T & -M_{22}(-\tau) & \tau D_{23}(s) \\ \tau D_{13}(s)^T & \tau D_{23}(s)^T & -\tau \dot{M}_{22}(s) \end{bmatrix} \begin{bmatrix} \phi(0) \\ \phi(-\tau) \\ \phi(s) \end{bmatrix} ds$$

$$- \tau \int_{-\tau}^{0}\int_{-\tau}^{0} \phi(s)^T \left(\frac{d}{ds} N(s,\theta) + \frac{d}{d\theta} N(s,\theta) \right) \phi(\theta) d\theta ds$$

$$\leq -\epsilon \|\phi\|^2$$

$$D_{11} = M_{11}A_0 + M_{12}(0) + \frac{1}{2} M_{22}(0),$$

$$D_{12} = M_{11}A_1 - M_{12}(-\tau),$$

$$D_{23} = A_1^T M_{12}(s) - N(-\tau, s),$$

$$D_{13} = x A_0^T M_{12}(s) - \dot{M}_{12}(s) + N(0,s).$$

2) New Dual Lyapunov–Krasovskii Stability Conditions: As per Corollary 7, the single-delay system is stable if there exists a

$$V(\phi) = \int_{-\tau}^{0} \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix}^T \begin{bmatrix} \tau(R(0,0) + S(0)) & \tau R(0,s) \\ \tau R(s,0) & \tau S(s) \end{bmatrix} \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix} ds$$

$$+ \int_{-\tau}^{0} \int_{-\tau}^{0} \phi(s)^T R(s,\theta) \phi(\theta) d\theta ds \geq \left\| \begin{bmatrix} \phi(0) \\ \phi \end{bmatrix} \right\|^2$$

and

$$V_D(\phi)$$

$$= \int_{-\tau}^{0} \begin{bmatrix} \phi(0) \\ \phi(-\tau) \\ \phi(s) \end{bmatrix}^T \begin{bmatrix} S_{11} + S_{11}^T & S_{12} & \tau S_{13}(s) \\ S_{12}^T & S_{22} & 0_n \\ \tau S_{13}(s)^T & 0_n & \tau \dot{S}(s) \end{bmatrix} \begin{bmatrix} \phi(0) \\ \phi(-\tau) \\ \phi(s) \end{bmatrix} ds$$

$$+ \int_{-\tau}^{0} \int_{-\tau}^{0} \phi(s)^T \left(\frac{d}{ds} R(s,\theta) + \frac{d}{d\theta} R(s,\theta) \right) \phi(\theta) d\theta ds$$

$$\leq -\epsilon \left\| \begin{bmatrix} \phi(0) \\ \phi \end{bmatrix} \right\|$$

where

$$S_{11} := \tau A_0 (R(0,0) + S(0)) + \tau A_1 R(-\tau, 0) + \frac{1}{2} S(0)$$

$$S_{12} := \tau A_1 S(-\tau), \qquad S_{22} := -S(-\tau)$$

$$S_{13}(s) := A_0 R(0,s) + A_1 R(-\tau, s) + \dot{R}(s,0)^T.$$

Although this section only considers the single-delay case, one can see the two primary differences between the primal and dual stability conditions. First, as was the case for delay-free systems, the A_0, A_1 system matrices appear on the left as opposed to the right-hand side of the Lyapunov variables. This allows for controller synthesis via variable substitution as we will demonstrate in Section XI. The second difference is that in the dual stability conditions, the (2, 3) and (3, 2) blocks of the derivative condition are zero. This unexpected structure extends to the multiple-delay case, wherein *ALL* (i, j) blocks are zero for $i, j \neq 1, i \neq j$.

B. Notation

Shorthand notation used throughout this paper includes the Hilbert spaces $L_2^m[X] := L_2(X; \mathbb{R}^m)$ of square integrable functions from X to $\mathbb{R}^m$ and $W_2^m[X] := W^{1,2}(X;\mathbb{R}^m) = H^1(X;\mathbb{R}^m) = \{x : x, \dot{x} \in L_2^m[X]\}$. We use L_2^m and W_2^m when domains are clear from context. We also use the extensions $L_2^{n\times m}[X] := L_2(X;\mathbb{R}^{n\times m})$ and $W_2^{n\times m}[X] := W^{1,2}(X;\mathbb{R}^{n\times m})$ for matrix-valued functions. $\mathcal{C}[X] \supset W_2[X]$ denotes the continuous functions on X. $\mathbb{S}^n \subset \mathbb{R}^{n\times n}$ denotes the symmetric matrices. We say an operator $\mathcal{P} : Z \to Z$ is positive on a subset X of Hilbert space Z if $\langle x, \mathcal{P}x \rangle_Z \geq 0$ for all $x \in X$. $\mathcal{P}$ is coercive on X if $\langle x, \mathcal{P}x \rangle_Z \geq \epsilon \|x\|_Z^2$ for some $\epsilon > 0$ and for all $x \in X$. Given an operator $\mathcal{P} : Z \to Z$ and a set $X \subset Z$, we use the shorthand $\mathcal{P}(X)$ to denote the image of $\mathcal{P}$ on subset X. $I_n \in \mathbb{S}^n$ denotes the identity matrix. $0_{n\times m} \in \mathbb{R}^{n\times m}$ is the matrix of zeros with shorthand $0_n := 0_{n\times n}$. We will occasionally denote the intervals $T_i^j := [-\tau_i, -\tau_j]$ and $T_i^0 := [-\tau_i, 0]$. For a natural number $K \in \mathbb{N}$, we adopt the index shorthand notation, which denotes $[K] = \{1, \ldots, K\}$.

II. Standard Results on Lyapunov Stability of Linear Time-Delay Systems

In this paper, we consider the stability of linear discrete-delay systems of the form

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{K} A_i x(t - \tau_i) \quad \text{for all} \quad t \geq 0$$

$$x(t) = \phi(t) \qquad \text{for all} \quad t \in [-\tau_K, 0], \quad (1)$$

where $A_i \in \mathbb{R}^{n\times n}$, $\phi \in \mathcal{C}[-\tau_K, 0]$, $K \in \mathbb{N}$ and for convenience $\tau_1 < \tau_2 < \cdots < \tau_K$. We associate with any solution x and any time $t \geq 0$, the "state" of System (1), $x_t \in \mathcal{C}[-\tau_K, 0]$, where $x_t(s) = x(t+s)$. For linear discrete-delay systems of Form (1), the system has a unique solution for any $\phi \in \mathcal{C}[-\tau_K, 0]$ and global, local, asymptotic, and exponential stability are all equivalent.

Stability of (1) may be certified through the use of Lyapunov–Krasovskii functionals—an extension of Lyapunov theory to systems with infinite-dimensional state space. In particular, it is known [2] that System (1) is stable if and only if there exist functions M and N, continuous in their respective arguments everywhere except possibly at points $H := \{-\tau_1, \ldots, -\tau_{K-1}\}$, such that the quadratic Lyapunov–Krasovskii functional $V : \mathcal{C}[-\tau_k, 0] \to \mathbb{R}$

$$V(\phi) = \int_{-\tau_K}^{0} \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix}^T M(s) \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix} ds$$

$$+ \int_{-\tau_K}^{0} \int_{-\tau_K}^{0} \phi(s)^T N(s,\theta) \phi(\theta) \, ds \, d\theta \qquad (2)$$

satisfies $V(\phi) \geq \epsilon \|\phi(0)\|^2$ and the Lie (upper Dini) derivative of the functional is negative along any solution x of (1). That is

$$\dot{V}(x_t) = \lim_{h\to 0} \frac{V(x_{t+h}) - V(x_t)}{h} \leq -\epsilon \|x_t(0)\|^2$$

for all $t \geq 0$ and some $\epsilon > 0$.

For the dual stability conditions we propose in this paper, discontinuities in the unknown functions M and N pose challenges, which make this form of Lyapunov–Krasovskii functional poorly suited to controller synthesis. For this reason, we use an alternative formulation of the necessary Lyapunov–Krasovskii functional. Specifically, it has been shown in [19], Theorem 3, that exponential stability is also equivalent to the existence of a Lyapunov–Krasovskii functional of the form

$$\begin{aligned} V(\phi) = \tau_K \phi(0)^T P\phi(0) &+ \tau_K \sum_{i=1}^{K} \int_{-\tau_i}^{0} \phi(0)^T Q_i(s)\phi(s)ds \\ &+ \tau_K \sum_{i=1}^{K} \int_{-\tau_i}^{0} \phi(s)^T Q_i(s)^T \phi(0)ds \\ &+ \tau_K \sum_{i=1}^{K} \int_{-\tau_i}^{0} \phi(s)^T S_i(s)\phi(s) \\ &+ \sum_{i,j=1}^{K} \int_{-\tau_i}^{0} \int_{-\tau_j}^{0} \phi(s)^T R_{ij}(s,\theta)\phi(\theta)d\theta \geq \epsilon \left\|\phi(0)\right\|^2 \end{aligned} \tag{3}$$

where $\dot V(x_t) \leq -\epsilon \left\|x_t(0)\right\|^2$ for some $\epsilon > 0$ and the functions Q_i, S_i, and R_{ij} may be assumed continuous on their respective domains of definition.

III. Reformulating the Lyapunov Stability Conditions Using Positive Operators

In this section, we introduce the mathematical formalism, which will be used to express both the primal and dual stability conditions. We begin by reviewing the well-established semigroup framework—a generalization of the concept of differential equations. Sometimes known as the "flow map," a "strongly continuous semigroup" is an operator $S(t) : Z \to Z$ defined by the Hilbert space Z, which represents the evolution of the state of the system so that for any solution x, $x_{t+s} = S(s)x_t$. Associated with a semigroup on Z is an operator $\mathcal{A}$, called the "infinitesimal generator," which satisfies $\frac{d}{dt}S(t)\phi = \mathcal{A}S(t)\phi$ for any $\phi \in X$. The space $X \subset Z$ is often referred to as the domain of the generator $\mathcal{A}$, and is the space on which the generator is defined and need not be a closed subspace of Z. In this paper we will refer to X as the "state space."

For System (1), we define $Z_{m,n,K} := \{\mathbb{R}^m \times L_2^n[-\tau_1, 0] \times \cdots \times L_2^n[-\tau_K, 0]\}$ and for $\{x, \phi_1, \ldots, \phi_K\} \in Z_{m,n,K}$, we define the following shorthand notation:

$$\begin{bmatrix} x \\ \phi_i \end{bmatrix} := \{x, \phi_1, \ldots, \phi_K\}$$

which allows us to simplify expression of the inner product on $Z_{m,n,K}$, which we define to be

$$\left\langle \begin{bmatrix} y \\ \psi_i \end{bmatrix}, \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{m,n,K}} = \tau_K y^T x + \sum_{i=1}^{K} \int_{-\tau_i}^{0} \psi_i(s)^T \phi_i(s)ds.$$

When $m = n$, we simplify the notation using $Z_{n,K} := Z_{n,n,K}$. We may now conveniently write the state space for System (1) as follows:

$$X := \left\{ \begin{bmatrix} x \\ \phi_i \end{bmatrix} \in Z_{n,K} \ : \ \begin{matrix} \phi_i \in W_2^n[-\tau_i, 0] \text{ and} \\ \phi_i(0) = x \text{ for all } i \in [K] \end{matrix} \right\}.$$

Note that X is a subspace of $Z_{n,K}$, inherits the norm of $Z_{n,K}$, but is not closed in $Z_{n,K}$. We furthermore extend this notation to say

$$\begin{bmatrix} x \\ \phi_i \end{bmatrix}(s) = \begin{bmatrix} y \\ f(s,i) \end{bmatrix}$$

if $x = y$ and $\phi_i(s) = f(s, i)$ for $s \in [-\tau_i, 0]$ and $i \in [K]$. This also allows us to compactly represent the infinitesimal generator $\mathcal{A}$ of (1) as follows:

$$\mathcal{A} \begin{bmatrix} x \\ \phi_i \end{bmatrix}(s) := \begin{bmatrix} A_0 x + \sum_{i=1}^{K} A_i \phi_i(-\tau_i) \\ \dot\phi_i(s) \end{bmatrix}.$$

Using these definitions of $\mathcal{A}$, Z, and X, for matrix P and functions Q_i, S_i, and R_{ij}, we define an operator $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ of the "complete-quadratic" type as follows:

$$\left(\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} \begin{bmatrix} x \\ \phi_i \end{bmatrix}\right)(s) := \begin{bmatrix} Px + \sum_{i=1}^{K} \int_{-\tau_i}^{0} Q_i(s)\phi_i(s)ds \\ \tau_K Q_i(s)^T x + \tau_K S_i(s)\phi_i(s) + \sum_{j=1}^{K} \int_{-\tau_j}^{0} R_{ij}(s,\theta)\phi_j(\theta)\, d\theta \end{bmatrix}.$$

This notation will be used throughout this paper and allows us to associate P, Q_i, S_i, and R_{ij} with the corresponding complete-quadratic functional in (3) as follows:

$$V(\phi) = \left\langle \begin{bmatrix} \phi(0) \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} \begin{bmatrix} \phi(0) \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n,K}}.$$

That is, the Lyapunov functional is defined by the operator $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$, which is a variation of a classical combined multiplier and integral operator whose multipliers and kernel functions are defined by P, Q_i, S_i, R_{ij}.

The upper Dini derivative of the complete-quadratic functional can similarly be represented using complete quadratic operators as follows:

$$\begin{aligned} \dot V(\phi) &= \left\langle \begin{bmatrix} \phi(0) \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} \mathcal{A} \begin{bmatrix} \phi(0) \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n,K}} \\ &\quad + \left\langle \mathcal{A} \begin{bmatrix} \phi(0) \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} \begin{bmatrix} \phi(0) \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n,K}} \\ &= \left\langle \begin{bmatrix} \phi(0) \\ \vdots \\ \phi(-\tau_K) \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{D_1,V_i,\dot S_i,G_{ij}\}} \begin{bmatrix} \phi(0) \\ \vdots \\ \phi(-\tau_K) \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n(K+1),n,K}} \end{aligned}$$

where [20]

$$D_1 = \begin{bmatrix} \Delta_0 & \Delta_1 & \cdots & \Delta_K \\ \Delta_1^T & S_1(-\tau_1) & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ \Delta_K^T & 0 & 0 & S_K(-\tau_K) \end{bmatrix},$$

$$\Delta_0 = PA_0 + A_0^T P + \sum_{k=1}^{K} Q_k(0) + Q_k(0)^T + S_k(0),$$

$$\Delta_j = PA_j - Q_j(-\tau_j),$$

$$V_i(s) = \begin{bmatrix} \Pi_{0,i}(s)^T & \ldots & \Pi_{K,i}(s)^T \end{bmatrix}^T,$$

$$\Pi_{0j}(s) = A_0^T Q_j(s) + \frac{1}{\tau_K}\sum_{k=1}^{K} R_{jk}^T(s,0) - \dot{Q}_j(s),$$

$$\Pi_{ij}(s) = A_i^T Q_j(s) - \frac{1}{\tau_K} R_{ji}^T(s,-\tau_i),$$

$$G_{ij}(s,\theta) = -\frac{\partial}{\partial s} R_{ij}(s,\theta) - \frac{\partial}{\partial \theta} R_{ij}(s,\theta).$$

In this section, we have reformulated $\mathcal{A}^*\mathcal{P} + \mathcal{P}\mathcal{A} < 0$ as negativity of a multiplier/integral operator on a lifted space. The classical Lyapunov–Krasovskii stability condition, then, states that System (1) is stable if there exists an $\epsilon > 0$, matrix P, and functions Q_i, S_i, and R_{ij} such that $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} \geq \epsilon \hat{I}_1$ and $\mathcal{P}_{\{D_1,V_i,\dot{S}_i,G_{ij}\}} \leq -\epsilon \hat{I}_2$ for suitably defined $\hat{I}_1 = \operatorname{diag}(I_n, 0, \ldots)$ and $\hat{I}_2 = \operatorname{diag}(I_n, 0, \ldots)$.

IV. A Dual Stability Condition for Infinite-Dimensional Systems

Using the notation we have introduced in the preceding section, we compactly represent the dual stability condition that forms the main theoretical contribution of this paper. Note that the results of this section apply to infinite-dimensional systems in general and are not specific to systems with delay.

Theorem 1: Suppose that $\mathcal{A}$ generates a strongly continuous semigroup on Hilbert space Z with domain X. Furthermore, suppose there exists an $\epsilon > 0$ and a bounded, coercive linear operator $\mathcal{P} : X \to X$ with $\mathcal{P}(X) = X$ and which is self-adjoint with respect to the Z inner product and satisfies

$$\langle \mathcal{A}\mathcal{P}z, z\rangle_Z + \langle z, \mathcal{A}\mathcal{P}z\rangle_Z \leq -\epsilon \|z\|_Z^2$$

for all $z \in X$. Then a dynamical system which satisfies $\dot{x}(t) = \mathcal{A}x(t)$ generates an exponentially stable semigroup.

Proof: Because $\mathcal{P}$ is coercive and bounded there exist $\gamma, \delta > 0$ such that $\langle x, \mathcal{P}x\rangle_Z \geq \gamma \|x\|_Z^2$ and $\|\mathcal{P}x\| \leq \delta \|x\|_Z$. By the Lax–Milgram theorem [21], $\mathcal{P}^{-1}$ exists and is bounded and $\mathcal{P}(X) = X$ implies $\mathcal{P}^{-1} : X \to X$. The inverse is self-adjoint since $\mathcal{P}$ is self-adjoint and hence $\langle \mathcal{P}^{-1}x, y\rangle_Z = \langle \mathcal{P}^{-1}x, \mathcal{P}\mathcal{P}^{-1}y\rangle_Z = \langle x, \mathcal{P}^{-1}y\rangle_Z$. Since $\sup_z \frac{\|\mathcal{P}z\|}{\|z\|} = \delta < \infty$, $\inf_y \frac{\|\mathcal{P}^{-1}y\|}{\|y\|} = \inf_x \frac{\|x\|}{\|\mathcal{P}x\|} = \frac{1}{\delta} > 0$ and hence $\langle y, \mathcal{P}^{-1}y\rangle_Z = \langle \mathcal{P}\mathcal{P}^{-1}y, \mathcal{P}^{-1}y\rangle_Z \geq \gamma \|\mathcal{P}^{-1}y\|_Z^2 \geq \frac{\gamma}{\delta^2}\|y\|_Z^2$. Hence, $\mathcal{P}^{-1}$ is coercive.

Define the Lyapunov functional $V(y) = \langle y, \mathcal{P}^{-1}y\rangle_Z \geq \frac{\gamma}{\delta^2}\|y\|_Z^2$, where positivity holds for any $y \in X$. If $y(t)$ satisfies $\dot{y}(t) = \mathcal{A}y(t)$, then V has time derivative

$$\begin{aligned} \frac{d}{dt}V(y(t)) &= \langle \dot{y}(t), \mathcal{P}^{-1}y(t)\rangle_Z + \langle y(t), \mathcal{P}^{-1}\dot{y}(t)\rangle_Z \\ &= \langle \mathcal{A}y(t), \mathcal{P}^{-1}y(t)\rangle_Z + \langle \mathcal{P}^{-1}y(t), \mathcal{A}y(t)\rangle_Z . \end{aligned}$$

Now, define $z(t) = \mathcal{P}^{-1}y(t) \in X$ for all $t \geq 0$. Then, $y(t) = \mathcal{P}z(t)$ and since $\mathcal{P}$ is bounded and $\mathcal{P}^{-1}$ is coercive

$$\begin{aligned} \dot{V}(y(t)) &= \langle \mathcal{A}y(t), \mathcal{P}^{-1}y(t)\rangle_Z + \langle \mathcal{P}^{-1}y(t), \mathcal{A}y(t)\rangle_Z \\ &= \langle \mathcal{A}\mathcal{P}z(t), z(t)\rangle_Z + \langle z(t), \mathcal{A}\mathcal{P}z(t)\rangle_Z \\ &\leq -\epsilon \|z(t)\|_Z^2 \leq -\frac{\epsilon}{\delta}\langle z(t), \mathcal{P}z(t)\rangle_Z \\ &= -\frac{\epsilon}{\delta}\langle y(t), \mathcal{P}^{-1}y(t)\rangle_Z \leq -\frac{\epsilon\gamma}{\delta^3}\|y(t)\|_Z^2 . \end{aligned}$$

Negativity of the derivative of the Lyapunov function implies exponential stability in the square norm of the state by, e.g., [13] or by the invariance principle. ■

The constraint $\mathcal{P}(X) = X$ ensures $\mathcal{P}^{-1} : X \to X$ and is satisfied if X is a closed subspace of Z or if X is itself a Hilbert space contained in Z and $\mathcal{P}$ is coercive on the space X with respect to the inner product in which X is closed. For the case of time-delay systems, X is not a closed subspace and we do not wish to constrain $\mathcal{P}$ to be coercive on X, since this space requires the Sobolev inner product in order to be closed. For these reasons, in Lemma 4, we will directly show that for our class of operators (to be defined) $\mathcal{P}(X) = X$.

In the following sections, we discuss how to parameterize operators which satisfy the conditions of Theorem 1, first in the case of multiple delays, and then for the special case of a single delay. We start with the constraints $\mathcal{P} = \mathcal{P}^*$ and $\mathcal{P} : X \to X$. Note that without additional restrictions on P, Q_i, S_i, R_{ij}, the operator $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ satisfies neither constraint.

Before moving to the next section, a natural question is whether the dual stability condition is significantly conservative. That is, does the stability of the system imply that the conditions of Theorem 1 are feasible. We refer to [14, Th. 5.1.3].

Theorem 2: Suppose that $\mathcal{A}$ is the infinitesimal generator of the C_0-semigroup $S(t)$ on the Hilbert space Z with domain $D(\mathcal{A})$. Then, $S(t)$ is exponentially stable if and only if there exists a positive, self-adjoint operator $\mathcal{P} \in \mathcal{L}(Z)$ such that

$$\langle \mathcal{P}\mathcal{A}z, z\rangle_Z + \langle z, \mathcal{P}\mathcal{A}z\rangle_Z = -\langle z, z\rangle_Z \quad \text{for all} \quad z \in D(\mathcal{A}).$$

Absent from the conditions of Theorem 2 is the restriction $\mathcal{P} : D(\mathcal{A}) \to D(\mathcal{A})$ and indeed the uniquely defined operator $\mathcal{P}$ from the proof of the theorem instead maps $D(\mathcal{A}) \to D(\mathcal{A}^*)$, with $D(\mathcal{A}^*)$ the domain defined by $\mathcal{A}^*$ and which has a structure significantly different than that of $D(\mathcal{A})$. Also absent from the conditions is the coercivity of $\mathcal{P}$. Several results show (e.g., [22, Th. 5.5]) that stability implies the existence of a coercive Lyapunov function (using a slightly weaker definition of coercivity). Finally, the image restriction $\mathcal{P}(X) = X$ is not satisfied by the operator in the proof of Theorem 2. However, if $\mathcal{P} : D(\mathcal{A}) \to D(\mathcal{A})$, in the following section we give conditions that guarantee $\mathcal{P}(X) = X$. In summary, however, we conclude that no definitive statement can be made regarding the necessity of Theorem 1.

V. Dual Conditions for Multiple-Delay Systems

In this section, we translate the results of Section IV into positivity and negativity of Lyapunov–Krasovskii-like functionals for systems with multiple delays. First, we give a class of operators $\mathcal{P}$, which satisfy the conditions of Theorem 1. Specifically, we give a parametrization of operators that are self-adjoint with respect to the Hilbert space $Z_{n,K}$, map $X \to X$ and satisfy $\mathcal{P}(X) = X$. Next, we show how the conditions of Theorem 1 can be applied to this class of operators to obtain stability conditions similar to the primal Lyapunov–Krasovskii conditions presented in Section II. Note that in Section VI, we will apply these results specifically to systems with a single delay and the exposition in that section is significantly reduced.

A. A Parametrization of Operators, P, Satisfying the Conditions of Theorem 1 on $Z_{n,K}$

In this section, we parameterize a class of operators which are self-adjoint and map $X \to X$, where

$$X := \left\{ \begin{bmatrix} x \\ \phi_i \end{bmatrix} \in Z_{n,K} \;:\; \begin{smallmatrix} \phi_i \in W_2^n[-\tau_i,0] \text{ and} \\ \phi_i(0)=x \text{ for all } i\in[K] \end{smallmatrix} \right\}.$$

The following lemma gives constraints on the matrix P and functions Q_i, S_i, and R_{ij} for which $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ is self-adjoint and maps $X \to X$.

Lemma 3: Suppose that $S_i \in W_2^{n\times n}[-\tau_i, 0]$, $R_{ij} \in W_2^{n\times n}[[-\tau_i, 0] \times [-\tau_j, 0]]$ and $S_i(s) = S_i(s)^T$, $R_{ij}(s,\theta) = R_{ji}(\theta, s)^T$, $P = \tau_K Q_i(0)^T + \tau_K S_i(0)$, and $Q_j(s) = R_{ij}(0, s)$ for all $i, j \in [K]$. Then, $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ is a bounded linear operator, maps $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} : X \to X$, and is self-adjoint with respect to the inner product defined on $Z_{n,K}$.

Proof: To simplify the presentation, let $\mathcal{P} := \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$. We first establish that $\mathcal{P} : X \to X$. If

$$\begin{bmatrix} x \\ \phi_i \end{bmatrix} \in X,$$

then $\phi_i \in \mathcal{C}[-\tau_i, 0]$ and $\phi_i(0) = x$. Now, if

$$\begin{bmatrix} y \\ \psi_i(s) \end{bmatrix} = \left(\mathcal{P}\begin{bmatrix} x \\ \phi_i \end{bmatrix}\right)(s),$$

then since $P = \tau_K Q_i(0)^T + \tau_K S_i(0)$ and $Q_j(s) = R_{ij}(0, s)$, we have that

$$\begin{aligned}\psi_i(0) = {}& \left(\tau_K Q_i(0)^T + \tau_K S_i(0)\right) x \\ &+ \sum_{j=1}^K \int_{-\tau_j}^0 R_{ij}(0,\theta)\phi_j(\theta)d\theta \\ = {}& Px + \sum_{j=1}^K \int_{-\tau_j}^0 Q_j(s)\phi_j(s)ds = y.\end{aligned}$$

Since $S_i \in W_2^{n\times n}[-\tau_i, 0]$ and $R_{ij} \in W_2^{n\times n}[[-\tau_i, 0] \times [-\tau_j, 0]]$, clearly $\psi_i \in W_2^n[-\tau_i, 0]$, and hence we have

$$\begin{bmatrix} y \\ \psi_i \end{bmatrix} \in X.$$

This proves that $\mathcal{P} : X \to X$. Furthermore, boundedness of the functions Q_i, S_i, and R_{ij} implies boundedness of the linear operator $\mathcal{P}$.

Now, to prove that $\mathcal{P}$ is self-adjoint with respect to the inner product $\langle \cdot,\cdot \rangle_{Z_{n,K}}$, we show $\langle y, \mathcal{P}x \rangle_{Z_{n,K}} = \langle \mathcal{P}y, x \rangle_{Z_{n,K}}$ for any $x, y \in Z_{n,K}$. Using the properties $S_i(s) = S_i(s)^T$ and $R_{ij}(s,\theta) = R_{ji}(\theta, s)^T$, we have the following:

$$\begin{aligned}&\left\langle \begin{bmatrix} y \\ \psi_i \end{bmatrix}, \mathcal{P}\begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n,K}} \\ &= \tau_K y^T \left(Px + \sum_{i=1}^K \int_{-\tau_i}^0 Q_i(\theta)\phi_i(\theta)d\theta \right) \\ &\quad + \sum_{i=1}^K \int_{-\tau_i}^0 \psi_i(s)^T \Bigg(\tau_K Q_i(s)^T x + \tau_K S_i(s)\phi_i(s) \\ &\qquad + \sum_{j=1}^K \int_{-\tau_j}^0 R_{ij}(s,\theta)\phi_j(\theta)d\theta \Bigg) \\ &= \tau_K \left(Py + \sum_{j=1}^K \int_{-\tau_j}^0 Q_i(s)\psi_j(s)ds \right)^T x \\ &\quad + \sum_{i=1}^K \int_{-\tau_i}^0 \Bigg(\tau_K Q_i(s)^T y + \tau_K S_i(s)^T \psi_i(s) \\ &\qquad + \sum_{j=1}^K \int_{-\tau_j}^0 R_{ji}(\theta,s)^T \psi_j(\theta)d\theta \Bigg)^T \phi_i(s)\,ds \\ &= \left\langle \mathcal{P}\begin{bmatrix} y \\ \psi_i \end{bmatrix}, \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n,K}}.\end{aligned}$$

■

Finally, we show that for this class of operators, if $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ is coercive with respect to the L_2-norm, then $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}(X) = X$.

Lemma 4: Suppose that there exist P, Q_i, S_i, and R_{ij} which satisfy the conditions of Lemma 3. If $\langle x, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}x \rangle_{Z_{n,K}} \geq \epsilon \|x\|^2_{Z_{n,K}}$ for all $x \in X$ and some $\epsilon > 0$, then $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}(X) = X$.

Proof: By Lemma 3, $\mathcal{P}$ is self-adjoint and maps $X \to X$. Since $\mathcal{P}$ is coercive, bounded, and self-adjoint, $\mathcal{P}^{-1}$ is coercive, bounded, and self-adjoint. To show $\mathcal{P}(X) = X$, we need only show that $\mathbf{y} = \mathcal{P}\mathbf{x} \in X$ implies that $\mathbf{x} \in X$. First, we show that if

$$\mathbf{y} = \begin{bmatrix} y \\ \psi_i(\theta) \end{bmatrix} \in X,$$

then

$$\mathbf{x} = \begin{bmatrix} x \\ \phi_i(\theta) \end{bmatrix} = \mathcal{P}^{-1}\mathbf{y}$$

satisfies $x = \phi_i(0)$. We proceed by contradiction. Suppose $x - \phi_i(0) \neq 0$ for some i. Then, we have

$$y = P(\phi_i(0) + x - \phi_i(0)) + \sum_{i=1}^K \int_{-\tau_i}^0 Q_i(s)\phi_i(s)ds.$$

Now, since $\mathbf{y} \in X$, $y = \psi_i(0)$, and hence,

$$y = P\phi_i(0)+\sum_{j=1}^{K}\int_{-\tau_j}^{0} R_{ij}(0,\theta)\phi_j(\theta)\, d\theta,$$

which implies $P(x-\phi_i(0))=0$. Now, $\langle \mathbf{x}, \mathcal{P}\mathbf{x}\rangle_{Z_{n,K}} \geq \epsilon \left\|\mathbf{x}\right\|^2_{Z_{n,K}}$ implies $P \geq \epsilon I$. Hence, $x - \phi(0) \neq 0$ implies $P(x-\phi(0)) \neq 0$, which is a contradiction. We conclude that $x = \phi_i(0)$. Next, we establish $\phi_i \in W_2^n$ for any i by showing $\left\|\dot{\phi}_i\right\|_{L_2} < \infty$. For this, we differentiate ψ_i to obtain

$$\begin{aligned}\dot{\psi}_i(s) &= \tau_K \dot{Q}_i(s)^T x+\tau_K \dot{S}_i(s)\phi_i(s)+\tau_K S_i(s)\dot{\phi}_i(s)\\ &\quad+\sum_{j=1}^{K}\int_{-\tau_j}^{0} \partial_s R_{ij}(s,\theta)\phi_j(\theta)\, d\theta,\end{aligned}$$

which we reverse to obtain

$$\begin{aligned}\tau_K S_i(s)\dot{\phi}_i(s) &= \dot{\psi}_i(s) - \tau_K \dot{Q}_i(s)^T x-\tau_K \dot{S}_i(s)\phi_i(s)\\ &\quad-\sum_{j=1}^{K}\int_{-\tau_j}^{0} \partial_s R_{ij}(s,\theta)\phi_j(\theta)\, d\theta,\end{aligned}$$

which is L_2 bounded since $\dot{\psi}_i, \phi_i, \dot{Q}_i \in L_2^n$, and $\dot{S}_i$ and $\partial_s R_{i,j}$ are continuous and thus bounded on $[-\tau_i, 0]$. Now, for $x=0$ and $\phi_j = 0$ for $j \neq i$, the constraint $\langle \mathbf{x}, \mathcal{P}\mathbf{x}\rangle_{Z_{n,K}} \geq \epsilon \left\|\mathbf{x}\right\|^2_{Z_{n,K}}$, implies that the operator on this subspace,

$$\tau_K S_i(s)\phi_i(s) + \int_{-\tau_i}^{0} R_{ii}(s,\theta)\phi_i(\theta)\, d\theta$$

is also coercive. Thus, since integral operators cannot be coercive for L_2-bounded kernels R_{ii}, we have that $S_i(s) \geq \eta I$ for some $\eta > 0$. Therefore, for each i, we conclude $\left\|\dot{\phi}_i\right\|_{L_2} \leq \frac{1}{\eta}\left\|S_i(s)\dot{\phi}_i(s)\right\|_{L_2} < \infty$. Hence, $\mathbf{x} \in X$. We conclude that $\mathcal{P}(X) = X$. ■

B. Duality Conditions for Multiple Delays

For the multiple-delay case, we apply the operator $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$, with P, Q_i, S_i, and R_{ij} satisfying the conditions of Lemma 4 to the dual stability condition in Theorem 1 and eliminate differential operators from the result. This section provides additional justification for the unique choice of state-space X and Hilbert space $Z_{m,n,K}$ used in this paper. Specifically, the elimination of differential operators and reformulation as negativity of a multiplier/integral operator on $Z_{n(K+1),n,K}$ would not be possible using the more classical state and inner product spaces, which allow for discontinuities in the state.

Theorem 5: Suppose that there exist P, Q_i, S_i, and R_{ij} satisfy the conditions of Lemma 3. If $\langle x, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} x\rangle_{Z_{n,K}} \geq \epsilon \left\|x\right\|^2$ for all $x \in Z_{n,K}$ and

$$\left\langle \begin{bmatrix}\begin{bmatrix} y_1\\ y_2\end{bmatrix}\\ \phi_i\end{bmatrix}, \mathcal{P}_{\{D_1,V_i,\dot{S}_i,G_{ij}\}} \begin{bmatrix}\begin{bmatrix} y_1\\ y_2\end{bmatrix}\\ \phi_i\end{bmatrix}\right\rangle_{Z_{n(K+1),n,K}} \leq -\epsilon \left\|\begin{bmatrix} y_1\\ \phi_i\end{bmatrix}\right\|^2_{Z_{n,K}}$$

for all $y_1 \in \mathbb{R}^n$ and

$$\begin{bmatrix}\begin{bmatrix} y_1\\ y_2\end{bmatrix}\\ \phi_i\end{bmatrix} \in Z_{n(K+1),n,K},$$

where

$$D_1 := \begin{bmatrix} C_0 + C_0^T & C_1 & \cdots & C_k\\ C_1^T & -S_1(-\tau_1) & 0 & 0\\ \vdots & 0 & \ddots & 0\\ C_{k^T} & 0 & 0 & -S_k(-\tau_K)\end{bmatrix}$$

$$C_0 := A_0 P + \sum_{i=1}^{K}\left(\tau_K A_i Q_i(-\tau_i)^T + \frac{1}{2}S_i(0)\right)$$

$$C_i := \tau_K A_i S_i(-\tau_i), \quad i \in [K]$$

$$V_i(s) := \begin{bmatrix} B_i(s)^T & 0 & \cdots & 0\end{bmatrix}^T, \quad i \in [K]$$

$$B_i(s) := A_0 Q_i(s)+\dot{Q}_i(s) + \sum_{j=1}^{K} A_j R_{ji}(-\tau_j, s), \;\; i \in [K]$$

$$G_{ij}(s,\theta) := \frac{\partial}{\partial s}R_{ij}(s,\theta) + \frac{\partial}{\partial \theta}R_{ji}(s,\theta)^T, \quad i,j \in [K]$$

then the system defined by (1) is exponentially stable.

Proof: Define the operators $\mathcal{A}$ and $\mathcal{P} = \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ as aforementioned. By Lemma 3, $\mathcal{P}$ is self-adjoint and maps $X \to X$. Since $\mathcal{P}$ is coercive by assumption, this implies by Theorem 1 and Lemma 4 that the system is exponentially stable if

$$\left\langle \mathcal{A}\mathcal{P}\begin{bmatrix} x\\ \phi_i\end{bmatrix}, \begin{bmatrix} x\\ \phi_i\end{bmatrix}\right\rangle_{Z_{n,K}} + \left\langle \begin{bmatrix} x\\ \phi_i\end{bmatrix}, \mathcal{A}\mathcal{P}\begin{bmatrix} x\\ \phi_i\end{bmatrix}\right\rangle_{Z_{n,K}} \leq -\epsilon \left\|\begin{bmatrix} x\\ \phi_i\end{bmatrix}\right\|^2_{Z_{n,K}}$$

for all

$$\begin{bmatrix} x\\ \phi_i\end{bmatrix} \in X.$$

We begin by constructing

$$\begin{bmatrix} y\\ \psi_i(s)\end{bmatrix} := \mathcal{A}\mathcal{P}\begin{bmatrix} x\\ \phi_i\end{bmatrix},$$

where

$$\begin{aligned} y &= A_0 P x + \sum_{i=1}^{K}\int_{-\tau_i}^{0} A_0 Q_i(s)\phi_i(s) ds\\ &\quad+\sum_{i=1}^{K} A_i\Bigg(\tau_K Q_i(-\tau_i)^T x + \tau_K S_i(-\tau_i)\phi_i(-\tau_i)\\ &\qquad\qquad+\sum_{j=1}^{K}\int_{-\tau_j}^{0} R_{ij}(-\tau_i,\theta)\phi_j(\theta) d\theta\Bigg)\end{aligned}$$

$$\begin{aligned}\psi_i(s) &= \tau_K \dot{Q}_i(s)^T x + \tau_K \dot{S}_i(s)\phi_i(s) + \tau_K S_i(s)\dot{\phi}_i(s)\\ &\quad+\sum_{j=1}^{K}\int_{-\tau_j}^{0} \frac{d}{ds}R_{ij}(s,\theta)\phi_j(\theta) d\theta.\end{aligned}$$

Now, divide the expression into terms as follows:

$$\left\langle \begin{bmatrix} x \\ \phi_i \end{bmatrix}, \mathcal{AP}\begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n,K}} := \tau_K x^T y + \sum_{i=1}^K \int_{-\tau_i}^0 \phi_i(s)^T \psi_i(s) ds.$$

Examining the first term and using $x = \phi_i(0)$, we have

$$\begin{aligned} x^T y = x^T A_0 P x &+ \sum_{i=1}^K \int_{-\tau_i}^0 x^T A_0 Q_i(s)\phi_i(s) ds \\ &+ \sum_{i=1}^K \tau_K x^T A_i Q_i(-\tau_i)^T x \\ &+ \sum_{i=1}^K \tau_K x^T A_i S_i(-\tau_i)\phi_i(-\tau_i) \\ &+ \sum_{i=1}^K \int_{-\tau_i}^0 \sum_{j=1}^K x^T A_j R_{ji}(-\tau_j, \theta)\phi_i(\theta) d\theta. \end{aligned}$$

Next, we examine the second term and use integration by parts to eliminate $\dot{\phi}$:

$$\begin{aligned} \sum_{i=1}^K \int_{-\tau_i}^0 \phi_i(s)^T \psi_i(s) ds &= \sum_{i=1}^K \tau_K \int_{-\tau_i}^0 \phi_i(s)^T \dot{Q}_i(s)^T x\, ds \\ &+ \sum_{i=1}^K \tau_K \int_{-\tau_i}^0 \phi_i(s)^T \dot{S}_i(s)\phi_i(s) ds \\ &+ \sum_{i=1}^K \tau_K \int_{-\tau_i}^0 \phi_i(s)^T S_i(s)\dot{\phi}_i(s) ds \\ &+ \sum_{i,j} \int_{-\tau_i}^0 \int_{-\tau_j}^0 \phi_i(s)^T \frac{\partial}{\partial s} R_{ij}(s,\theta)\phi_j(\theta)\, ds\, d\theta \\ = \sum_{i=1}^K \tau_K \int_{-\tau_i}^0 \phi_i(s)^T \dot{Q}_i(s)^T x\, ds & \\ &+ \frac{\tau_K}{2} \sum_{i=1}^K \int_{-\tau_i}^0 \phi_i(s)^T \dot{S}_i(s)\phi_i(s) ds \\ &+ \frac{\tau_K}{2} x^T \sum_{i=1}^K S_i(0) x \\ &- \frac{\tau_K}{2} \sum_{i=1}^K \phi_i(-\tau_i)^T S_i(-\tau_i)\phi_i(-\tau_i) \\ &+ \sum_{i,j} \int_{-\tau_i}^0 \int_{-\tau_j}^0 \phi_i(s)^T \frac{\partial}{\partial s} R_{ij}(s,\theta)\phi_j(\theta)\, ds\, d\theta. \end{aligned}$$

Combining both terms, we obtain

$$\begin{aligned} \left\langle \begin{bmatrix} x \\ \phi_i \end{bmatrix}, \mathcal{AP}\begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n,K}} &= \tau_K x^T y + \sum_{i=1}^K \int_{-\tau_i}^0 \phi_i(s)^T \psi_i(s) ds \\ = x^T \left(\tau_K A_0 P + \sum_{i=1}^K \tau_K^2 A_i Q_i(-\tau_i)^T + \frac{\tau_K}{2} \sum_{i=1}^K S_i(0) \right) x & \\ &+ \tau_K^2 \sum_{i=1}^K x^T A_i S_i(-\tau_i)\phi_i(-\tau_i) \\ &- \frac{\tau_K}{2} \sum_{i=1}^K \phi_i(-\tau_i)^T S_i(-\tau_i)\phi_i(-\tau_i) \\ &+ \tau_K \sum_{i=1}^K \int_{-\tau_i}^0 x^T \Bigg(A_0 Q_i(s) + \dot{Q}_i(s) \\ &\qquad + \sum_{j=1}^K A_j R_{ji}(-\tau_j, s) \Bigg) \phi_i(s) ds \\ &+ \frac{\tau_K}{2} \sum_{i=1}^K \int_{-\tau_i}^0 \phi_i(s)^T \dot{S}_i(s)\phi_i(s) ds \\ &+ \sum_{i,j} \int_{-\tau_i}^0 \int_{-\tau_j}^0 \phi_i(s)^T \frac{\partial}{\partial s} R_{ij}(s,\theta)\phi_j(\theta)\, ds\, d\theta. \end{aligned}$$

Combining the expression with its adjoint, we recover

$$\begin{aligned} &\left\langle \mathcal{AP}\begin{bmatrix} x \\ \phi_i \end{bmatrix}, \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n,K}} + \left\langle \begin{bmatrix} x \\ \phi_i \end{bmatrix}, \mathcal{AP}\begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n,K}} \\ &= \left\langle \begin{bmatrix} \begin{bmatrix} x \\ \phi_1(-\tau_1) \\ \vdots \\ \phi_k(-\tau_K) \end{bmatrix} \\ \phi_i \end{bmatrix}, \mathcal{D} \begin{bmatrix} \begin{bmatrix} x \\ \phi_1(-\tau_1) \\ \vdots \\ \phi_k(-\tau_K) \end{bmatrix} \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n(K+1),n,K}} \\ &\le -\epsilon \left\| \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\|_{Z_{n,K}}^2, \end{aligned}$$

where $\mathcal{D} := \mathcal{P}_{\{D_1, V_i, \dot{S}_i, G_{ij}\}}$. We conclude that all conditions of Theorem 1 are satisfied and hence System (1) is stable. ■

Theorem 5 provides stability conditions expressed as the positivity of $\mathcal{P}_{\{P, Q_i, S_i, R_{ij}\}}$ and negativity of the multiplier/integral operator $\mathcal{D} = \mathcal{P}_{\{D_1, V_i, \dot{S}_i, G_{ij}\}}$. Note that positivity is defined with respect to the inner product $Z_{m,n,K}$. In Section VII, we will show how to reformulate positivity on $Z_{m,n,K}$ as an equivalent positivity condition on the space $Z_{m,nK,1}$. Positive operators on $Z_{m,nK,1}$ are then parameterized using LMIs, as also described in Section VII. Before moving to the next section, we note that the derivative operator $\mathcal{D} = \mathcal{P}_{\{D_1, V_i, \dot{S}_i, G_{ij}\}}$ is sparse in the sense that no terms of the form $\phi(-\tau_i)^T \phi_j(-\tau_j)$ for $i \neq j$ or $\phi_i(-\tau_i)^T \phi_i(s)$ for any i appear in $\left\langle \phi, \mathcal{P}_{\{D_1, V_i, \dot{S}_i, G_{ij}\}} \phi \right\rangle$. This is extraordinary, as all such terms do appear in the similar formulation of the primal stability conditions (i.e., $\left\langle \phi, \mathcal{P}_{\{D_1, V_i, \dot{S}_i, G_{ij}\}} \phi \right\rangle$ from Section III). To emphasize this difference, we fully expand both versions of the form $\left\langle \phi, \mathcal{P}_{\{D_1, V_i, \dot{S}_i, G_{ij}\}} \phi \right\rangle$ to obtain the following.

1) Dual Lyapunov–Krasovskii Form: Theorem 5 implies that system (1) is stable if there exists a

$$V(\phi) = \tau_K \phi(0)^T P\phi(0) + \tau_K \sum_{i=1}^{K} \int_{-\tau_i}^{0} \phi(0)^T Q_i(s)\phi(s)ds + \tau_K \sum_{i=1}^{K} \int_{-\tau_i}^{0} \phi(s)^T Q_i(s)^T \phi(0)ds + \tau_K \sum_{i=1}^{K} \int_{-\tau_i}^{0} \phi(s)^T S_i(s)\phi(s)ds + \sum_{i,j=1}^{K} \int_{-\tau_i}^{0}\int_{-\tau_j}^{0} \phi(s)^T R_{ij}(s,\theta)\phi(\theta)d\theta,$$

such that

$$V(\phi) \geq \epsilon \left\| \begin{bmatrix} \phi(0) \\ \phi_i \end{bmatrix} \right\|_{Z_{n,K}}^2$$

and

$$V_D(\phi) = \tau_K \phi(0)^T (C_0 + C_0^T)\phi(0) + 2\tau_K \sum_{i=1}^{K} \phi(0)^T C_i \phi_i(-\tau_i) - \tau_K \sum_{i=1}^{K} \phi_i(-\tau_i)^T S_i(-\tau_i)\phi_i(-\tau_i) + 2\tau_K \sum_{i=1}^{K} \int_{-\tau_i}^{0} \phi(0)^T B_i(s)\phi_i(s)ds + \tau_K \sum_{i=1}^{K} \int_{-\tau_i}^{0} \phi_i(s)^T \dot{S}_i(s)\phi_i(s)ds + \sum_{i,j=1}^{K} \int_{-\tau_i}^{0}\int_{-\tau_j}^{0} \phi_i(s)^T G_{ij}(s,\theta)\phi_i(\theta)\, ds\, d\theta \leq -\epsilon \left\| \begin{bmatrix} \phi(0) \\ \phi_i \end{bmatrix} \right\|_{Z_{n,K}}^2 .$$

2) Primal Lyapunov–Krasovskii Form: Now, compare with the associated primal classical Lyapunov–Krasovskii derivative condition [20] from Section III, which states that system (1) is stable if there exists a

$$V(\phi) = \phi(0)^T P\phi(0) + \sum_{i=1}^{K} \int_{-\tau_i}^{0} \phi(0)^T Q_i(s)\phi(s)ds + \sum_{i=1}^{K} \int_{-\tau_i}^{0} \phi(s)^T Q_i(s)^T \phi(0)ds + \sum_{i=1}^{K} \int_{-\tau_i}^{0} \phi(s)^T S_i(s)\phi(s)ds + \sum_{i,j=1}^{K} \int_{-\tau_i}^{0}\int_{-\tau_j}^{0} \phi(s)^T R_{ij}(s,\theta)\phi(\theta)d\theta$$

such that $V(\phi) \geq \epsilon \left\|\phi(0)\right\|^2$ and

$$\dot{V}(\phi) = \phi(0)^T \Delta_0 \phi(0) + \sum_{i=1}^{K} \phi_i(-\tau_i)^T S_i(-\tau_i)\phi_i(-\tau_i) + 2\sum_{i=1}^{K} \phi(0)^T \Delta_i \phi_i(-\tau_i) + 2\sum_{i=1}^{K} \int_{-\tau_i}^{0} \phi(0)^T \Pi_{0i}(s)\phi_i(s)ds + \sum_{i=1}^{K} \int_{-\tau_i}^{0} \phi_i(s)^T \dot{S}_i(s)\phi_i(s)ds + 2\sum_{i,j=1}^{K} \int_{-\tau_i}^{0} \phi_i(-\tau_i)^T \Pi_{ij}(s)\phi_j(s)ds - \sum_{i,j=1}^{K} \int_{-\tau_i}^{0}\int_{-\tau_j}^{0} \phi_i(s)^T G_{ij}(s,\theta)\phi_i(\theta)\, ds\, d\theta \leq -\epsilon \left\|\phi(0)\right\|^2 .$$

From this comparison, we see that the structure of the dual stability condition is very similar to the structure of the primal except for the fifth line of the derivative, which is absent from the dual. Roughly speaking, it is as if all the Π_{ij} terms in the primal form have been combined in Π_{0i}. This sparsity pattern yields a multiplier of the form

$$\begin{bmatrix} \cdot & \cdots \\ \vdots & \ddots \end{bmatrix}$$

consisting of a single row, single column, and diagonal. For an example of how to exploit such sparsity, the positivity of such a multiplier would be equivalent to positivity of the diagonal and positivity of the scalar $[\cdot] - \cdots \begin{bmatrix} \ddots \end{bmatrix}^{-1} \vdots$.

VI. Duality Conditions for Single-Delay Systems

In this section, we simplify the results of Section VIII-A for systems with a single delay. We find that in the case of single delay the parametrization of the operator $\mathcal{P}$ is direct (it does not rely on equality constraints to enforce the mapping conditions of Theorem 1), which allows us to arrive at the explicit forms described in Section I-A.

A. Parametrization of Operators, $\mathcal{P}$, Satisfying the Conditions of Theorem 1 on $Z_{n,1}$

First, we consider a class of operators that are self-adjoint with respect to Z and map $X \to X$. This is simplified in the case of a single-delay case partially due to the fact that $Z = Z_{n,1} = \mathbb{R}^n \times L_2^n$ equipped with the L_2^{2n} inner product and subspace $X := \{\{x,\phi\} \in \mathbb{R}^n \times W_2^n[-\tau, 0] \,:\, \phi(0) = x\}$. Specifically, given functions $S, R \in W_2^{n\times n}[-\tau, 0]$, in this section, we will define $\mathcal{P}$ as follows:

$$\left(\mathcal{P}\begin{bmatrix} x \\ \phi \end{bmatrix}\right)(s) := \begin{bmatrix} \tau(R(0,0)+S(0))x + \int_{-\tau}^{0} R(0,s)\phi(s)ds \\ \tau R(s,0)\phi(0) + \tau S(s)\phi(s) + \int_{-\tau}^{0} R(s,\theta)\phi(\theta)d\theta \end{bmatrix}. \tag{4}$$

Clearly, we have that $\mathcal{P}$ is a bounded linear operator and since S, R are continuous, it is trivial to show that $\mathcal{P} : X \to X$. Furthermore, $\mathcal{P}$ is self-adjoint with respect to the L_2^{2n} inner product, as indicated in the following lemma.

Lemma 6: Suppose $S \in W_2^{n\times n}[-\tau, 0]$, $R \in W_2^{n\times n}[[-\tau,0]\times[-\tau,0]]$, $R(s,\theta) = R(\theta,s)^T$, and $S(s) \in \mathbb{S}^n$. Then, the operator $\mathcal{P}$, as defined in (5), is self-adjoint with respect to the L_2^{2n} inner product. Furthermore, if there exists $\epsilon > 0$ such that $\langle x, \mathcal{P}x\rangle_{L_2^{2n}} \geq \epsilon \|x\|^2$ for all $x \in X$, then $\mathcal{P}(X) = X$.

Proof: The proof is a direct application of Lemma 3. First, we note that $\mathcal{P} = \mathcal{P}_{\{P,Q,S,R\}}$ where $P = \tau(R(0,0)+S(0))$ and $Q(s) = R(0,s)$. Noting that $P = \tau(R(0,0)+S(0)) = \tau Q(0)^T + \tau S(0)$, we see that $\mathcal{P}_{\{P,Q,S,R\}}$ satisfies the conditions of Lemma 3. ■

Note that the constraints $\mathcal{P} : X \to X$ and $\mathcal{P} = \mathcal{P}^*$ significantly reduce the number of free variables. In the single-delay case, we make this explicit by replacing P and Q with $P = \tau(R(0,0)+S(0))$ and $Q(s) = R(0,s)$.

Having introduced a parametrization of $\mathcal{P}$ and established properties of this operator, we now apply this structured operator to Theorem 1 to obtain Lyapunov-like conditions on S and R for which stability holds.

B. Dual Stability Conditions: Single Delay

In this section, we specialize the results of Theorem 5 to single-delay systems. First, recall that the dynamics of the single-delay system are represented by the infinitesimal generator $\mathcal{A}$ defined as follows:

$$\left(\mathcal{A}\begin{bmatrix} x \\ \phi \end{bmatrix}\right)(s) = \begin{bmatrix} A_0 x + A_1\phi(-\tau) \\ \frac{d}{ds}\phi(s) \end{bmatrix}.$$

Then, we have the following.

Corollary 7: Suppose S and R satisfy the conditions of and Lemma 6 and there exists $\epsilon > 0$ such that

$$\langle x, \mathcal{P}_{\{P,Q,S,R\}}x\rangle_{L_2^{2n}} \geq \epsilon \|x\|^2_{L_2^{2n}}$$

for all $x \in \mathbb{R}^n \times L_2^n[-\tau, 0]$ where $P = \tau(R(0,0)+S(0))$ and $Q(s) = R(0,s)$. Furthermore, suppose

$$\left\langle \begin{bmatrix} x \\ y \\ \phi \end{bmatrix}, \mathcal{D}\begin{bmatrix} x \\ y \\ \phi \end{bmatrix}\right\rangle_{L_2^{3n}} \leq -\epsilon \left\|\begin{bmatrix} x \\ \phi \end{bmatrix}\right\|^2_{L_2^{2n}}$$

for all

$$\begin{bmatrix} x \\ y \\ \phi \end{bmatrix} \in \mathbb{R}^n \times \mathbb{R}^n \times L_2^n[-\tau, 0]$$

where $\mathcal{D} = \mathcal{P}_{\{D_1, V, \dot{S}, G\}}$ and

$$D_1 := \begin{bmatrix} C_0 + C_0^T & C_1 \\ C_1^T & -S(-\tau) \end{bmatrix}, \quad V(s) = \begin{bmatrix} B(s) \\ 0 \end{bmatrix}$$

$$C_0 := \tau A_0(R(0,0)+S(0)) + \tau A_1 R(-\tau,0) + \frac{1}{2}S(0)$$

$$C_1 := \tau A_1 S(-\tau)$$

$$B(s) := A_0 R(0,s) + A_1 R(-\tau,s) + \dot{R}(s,0)^T$$

$$G(s,\theta) := \frac{d}{ds}R(s,\theta) + \frac{d}{d\theta}R(s,\theta).$$

Then, the system defined by (1) in the case $K = 1$ with $\tau_1 = \tau$ is exponentially stable.

Proof: The proof is a direct application of Lemma 6 and Theorem 5. ■

Note that expanding the term

$$\left\langle \begin{bmatrix} \phi(0) \\ \phi(-\tau) \\ \phi \end{bmatrix}, \mathcal{D}\begin{bmatrix} \phi(0) \\ \phi(-\tau) \\ \phi \end{bmatrix}\right\rangle_{L_2^{3n}}$$

from Corollary 7 yields the new dual stability conditions previously described in Section I-A.

VII. Using LMIs to Solve Linear Operator Inequalities (LOIs) on $Z_{m,n,K}$

In previous sections, we have formulated dual stability conditions, with decision variables parameterized by the matrix P and functions Q_i, S_i, and R_{ij}. The dual stability conditions were reformulated as the positivity of

$$\langle x, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}x\rangle_{Z_{n,K}} \geq \epsilon \|x\|^2_{Z_{n,K}}$$

for all $x \in Z_{n,K}$ and the negativity of

$$\left\langle \begin{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{D_1,V_i,\dot{S}_i,G_{ij}\}}\begin{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \phi_i \end{bmatrix}\right\rangle_{Z_{n(K+1),n,K}} \leq -\epsilon \left\|\begin{bmatrix} y_1 \\ \phi_i \end{bmatrix}\right\|^2_{Z_{n,K}}$$

for $y_1 \in \mathbb{R}^n$ and

$$\begin{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \phi_i \end{bmatrix} \in Z_{n(K+1,n,K)}$$

where $D_1, V_i, \dot{S}_i$, and G_{ij} are as defined in Theorem 5. Operator feasibility conditions of this form are termed linear operator inequalities and, in this section, we will show how LMIs can be used to solve LOIs under the presumption that the functions Q_i, S_i, and R_{ij} are polynomial (which implies $D_1, V_i, \dot{S}_i$, and G_{ij} are polynomial). Specifically, the variables in this case become the coefficients of the polynomials Q_i, S_i, and R_{ij} and the goal of the section is to find LMI constraints on P and these polynomial coefficients, which ensure that

$$\langle x, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}x\rangle_{Z_{m,n,K}} \geq 0.$$

Our approach to solving LOIs on $Z_{m,n,K}$ is to construct an equivalent feasibility condition using operators on $Z_{m,nK,1} = \mathbb{R}^m \times L_2^{nK}[-\tau_K, 0]$. This is accomplished in two steps. First, in Section VII-A, we construct polynomials $\hat{Q}$, $\hat{S}$, and $\hat{R}$

such that $\mathcal{P}_{\{P,\hat{Q},\hat{S},\hat{R}\}}$ is coercive on $Z_{m,nK,1}$ if and only if $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ is coercive on $Z_{m,n,K}$. Second, in Section VII-B, we impose LMI constraints on P and the coefficients of these polynomials $\hat{Q}$, $\hat{S}$, and $\hat{R}$, constraints which are denoted $\{P,\hat{Q},\hat{S},\hat{R}\} \in \Xi_{d,m,nK}$ and which ensure that $\mathcal{P}_{\{P,\hat{Q},\hat{S},\hat{R}\}}$ is coercive on $Z_{m,nK,1}$.

Both steps are combined into a single summarizing statement in Corollary 10.

A. Equivalence Between $Z_{m,n,K}$ and $Z_{m,nK,1}$

In this section, we address the positivity of $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ on $Z_{m,n,K}$ by constructing a linear map from the matrix P and coefficients of Q_i, S_i, and R_{ij} to the coefficients of new polynomial variables $\hat{Q}$, $\hat{S}$, and $\hat{R}$, where the coercivity of $\mathcal{P}_{\{P,\hat{Q},\hat{S},\hat{R}\}}$ on $Z_{m,nK,1}$ is equivalent to the coercivity of $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ on $Z_{m,n,K}$.

Given matrix P and polynomials Q_i, S_i, and R_{ij}, define the linear map $\mathcal{L}_1$ by

$$\{\hat{P},\hat{Q},\hat{S},\hat{R}\} := \mathcal{L}_1(P,Q_i,S_i,R_{ij}) \tag{5}$$

if $a_i = \frac{\tau_i}{\tau_K}$, $\hat{P} = P$ and

$$\hat{Q}(s) := \left[\sqrt{a_1}Q_1(a_1 s) \quad \cdots \quad \sqrt{a_K}Q_K(a_K s)\right]$$

$$\hat{S}(s) := \begin{bmatrix} S_1(a_1 s) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & S_K(a_K s) \end{bmatrix}$$

$$\hat{R}(s,\theta) :=$$

$$\begin{bmatrix} \sqrt{a_1 a_1}R_{11}(sa_1,\theta a_1) & \cdots & \sqrt{a_1 a_K}R_{1K}(sa_1,\theta a_K) \\ \vdots & \cdots & \vdots \\ \sqrt{a_K a_1}R_{K1}(sa_K,\theta a_1) & \cdots & \sqrt{a_K a_K}R_{KK}(sa_K,\theta a_K) \end{bmatrix}.$$

Then, we have the following result.

Lemma 8: Let $\{\hat{P},\hat{Q},\hat{S},\hat{R}\} := \mathcal{L}_1(P,Q_i,S_i,R_{ij})$. Then

$$\left\langle \begin{bmatrix} x \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{m,n,K}} \ge \alpha \left\| \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\|_{Z_{m,n,K}}$$

for all $\begin{bmatrix} x \\ \phi_i \end{bmatrix} \in Z_{m,n,K}$ if and only if

$$\left\langle \begin{bmatrix} x \\ \hat{\phi} \end{bmatrix}, \mathcal{P}_{\{\hat{P},\hat{Q},\hat{R},\hat{S}\}} \begin{bmatrix} x \\ \hat{\phi} \end{bmatrix} \right\rangle_{Z_{m,nK,1}} \ge \alpha \left\| \begin{bmatrix} \hat{x} \\ \hat{\phi} \end{bmatrix} \right\|_{Z_{m,nK,1}}$$

for all $\begin{bmatrix} \hat{x} \\ \hat{\phi} \end{bmatrix} \in Z_{m,nK,1}$.

Proof: The proof is straightforward. For necessity, let

$$\hat{\phi} = \begin{bmatrix} \sqrt{a_i}\phi_1(sa_1) \\ \vdots \\ \sqrt{a_K}\phi_K(sa_K) \end{bmatrix}.$$

Then,

$$\begin{bmatrix} x \\ \hat{\phi} \end{bmatrix} \in Z_{m,nK,1}$$

and define the change of variables $s_i' = \frac{\tau_K}{\tau_i} s_i = \frac{1}{a_i} s_i$. Then, $s_i = \frac{\tau_i}{\tau_K} s_i' = a_i s_i'$ and $ds_i = a_i ds_i'$ and

$$\begin{aligned} \left\| \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\|_{Z_{m,n,K}} &= \tau_K x^T x + \sum_{i=1}^{K} \int_{-\tau_i}^{0} \|\phi_i(s_i)\|^2 ds_i \\ &= \tau_K x^T x + \sum_{i=1}^{K} \int_{-\tau_K}^{0} \|\sqrt{a_i}\phi_i(s_i' a_i)\|^2 ds_i' \\ &= \tau_K x^T x + \int_{-\tau_K}^{0} \left\|\hat{\phi}(s)\right\|^2 ds = \left\| \begin{bmatrix} x \\ \hat{\phi} \end{bmatrix} \right\|_{Z_{m,nK,1}}. \end{aligned}$$

Now, using a similar change of integration variables, we have the following:

$$\begin{aligned} &\left\langle \begin{bmatrix} x \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{m,n,K}} \\ &= \tau_K x^T P x + 2\tau_K \int_{-\tau_K}^{0} \sum_{i=1}^{K} x^T \sqrt{a_i} Q_i(sa_i)\hat{\phi}_i(s) ds \\ &\quad + \tau_K \int_{-\tau_K}^{0} \sum_{i=1}^{K} \hat{\phi}_i(s)^T S_i(sa_i)\hat{\phi}_i(s) ds \\ &\quad + \int_{-\tau_K}^{0}\int_{-\tau_K}^{0} \sum_{i,j=1}^{K} \hat{\phi}_i(s)^T \sqrt{a_i a_j} R_{ij}(sa_i,\theta a_j)\hat{\phi}_j(\theta) d\theta ds \\ &= \int_{-\tau_K}^{0} \begin{bmatrix} x \\ \hat{\phi}(s) \end{bmatrix}^T \begin{bmatrix} P & \tau_K \hat{Q}(s) \\ \tau_K \hat{Q}(s)^T & \tau_K \hat{S}(s) \end{bmatrix} \begin{bmatrix} x \\ \hat{\phi}(s) \end{bmatrix} ds \\ &\quad + \int_{-\tau_K}^{0}\int_{-\tau_K}^{0} \hat{\phi}(s)^T \hat{R}(s,\theta)\hat{\phi}(\theta) d\theta ds \\ &= \left\langle \begin{bmatrix} x \\ \hat{\phi} \end{bmatrix}, \mathcal{P}_{\{P,\hat{Q},\hat{R},\hat{S}\}} \begin{bmatrix} x \\ \hat{\phi} \end{bmatrix} \right\rangle_{Z_{m,nK,1}} \\ &\ge \alpha \left\| \begin{bmatrix} x \\ \hat{\phi} \end{bmatrix} \right\|_{Z_{m,nK,1}} = \alpha \left\| \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\|_{Z_{m,n,K}}. \end{aligned}$$

For the sufficiency, we reverse the steps using

$$\phi_i(s) = \frac{1}{\sqrt{a_i}} \hat{\phi}_i\left(\frac{s}{a_i}\right). \qquad \blacksquare$$

Note that if Q_i, S_i, and R_{ij} are polynomials whose coefficients are variables in the optimization problem, then the constraint $\{\hat{P},\hat{Q},\hat{S},\hat{R}\} = \mathcal{L}_1(P,Q_i,S_i,R_{ij})$ defines a linear equality constraint between the coefficients of Q_i, S_i, and R_{ij} and the coefficients of the polynomials that define $\hat{Q}$, $\hat{S}$, and $\hat{R}$. In the following section, we will discuss how to enforce the positivity of operators on $Z_{m,nK,1}$.

B. LMI Conditions for the Positivity of Multiplier and Integral Operators on $Z_{m,nK,1}$

In this section, we define LMI-based conditions for the positivity of operators $\mathcal{P}_{\{P,Q,R,S\}}$ on $Z_{m,nK,1}$ where Q, S, and R are continuous on $[-\tau_K, 0]$.

Our approach to positivity is based on the observation that a positive operator will always have a square root. If we assume that this square root is also of the form $\mathcal{P}_{\{P,Q,R,S\}}$ with functions Q, S, and R polynomial of bounded degree, then the results of this section give necessary and sufficient conditions. Note that although this assumption is restrictive, it is unclear whether it implies conservatism. For example, while not all positive polynomials are sum-of-squares, any positive polynomial can be approximated arbitrarily well in the sup norm on a bounded domain by a polynomial with a polynomial "root." Specifically, the following theorem assumes a square root of the form

$$\left(\mathcal{P}^{\frac{1}{2}}\begin{bmatrix} x \\ \phi \end{bmatrix}\right)(s) := N_1\sqrt{g(s)}x + N_2\sqrt{g(s)}Y_1(s)\phi(s) + \int_{-\tau_K}^{0} N_3\sqrt{g(s)}Y_2(s,\theta)\phi(\theta)d\theta$$

where the matrices N_i are unknown, the matrix-valued functions, Y_i are chosen apriori, and g is either $g(s) = 1$ or $g(s) = -s(s+\tau_K)$ (meaning $g(s)$ is nonnegative on the interval $[-\tau_K, 0]$).

Theorem 9: For any functions $Y_1 : [-\tau_K, 0] \to \mathbb{R}^{m_1 \times n}$ and $Y_2 : [-\tau_K, 0] \times [-\tau_K, 0] \to \mathbb{R}^{m_2 \times n}$, square integrable on $[-\tau_K, 0]$ with $g(s) \geq 0$ for $s \in [-\tau_K, 0]$, suppose that

$$P = M_{11} \cdot \frac{1}{\tau_K}\int_{-\tau_K}^{0} g(s)ds$$

$$Q(s) = \frac{1}{\tau_K}\left(g(s)M_{12}Y_1(s) + \int_{-\tau_K}^{0} g(\eta)M_{13}Y_2(\eta, s)d\eta\right)$$

$$S(s) = \frac{1}{\tau_K}g(s)Y_1(s)^T M_{22}Y_1(s)$$

$$R(s,\theta) = g(s)Y_1(s)^T M_{23}Y_2(s,\theta) + g(\theta)Y_2(\theta,s)^T M_{32}Y_1(\theta) + \int_{-\tau_K}^{0} g(\eta)Y_2(\eta,s)^T M_{33}Y_2(\eta,\theta)d\eta$$

where $M_{11} \in \mathbb{R}^{m\times m}$, $M_{22} \in \mathbb{R}^{m_1\times m_1}$, $M_{33} \in \mathbb{R}^{m_2\times m_2}$, and

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \geq 0.$$

Then, $\langle x, \mathcal{P}_{\{P,Q,R,S\}}x\rangle_{Z_{m,n,1}} \geq 0$ for all $x \in Z_{m,n,1}$.

Proof: Since $M \geq 0$, there exists a matrix $N = [\,N_1 \quad N_2 \quad N_3\,]$ such that $M = N^T N$ where $N_1 \in \mathbb{R}^{m+m_1+m_2\times m}$, $N_2 \in \mathbb{R}^{m+m_1+m_2\times m_2}$, and $N_3 \in \mathbb{R}^{m+m_1+m_2\times m_2}$. Using the definition of $\mathcal{P}^{\frac{1}{2}}$ introduced previously, it is straightforward to show that

$$\langle x, \mathcal{P}_{\{P,Q,R,S\}}x\rangle_{Z_{m,n,1}} = \left\langle \mathcal{P}^{\frac{1}{2}}x, \mathcal{P}^{\frac{1}{2}}x\right\rangle_{L_2^{m+m_1+m_2}} \geq 0. \quad \blacksquare$$

Theorem 9 gives a linear parametrization of a cone of positive operators using positive semidefinite matrices. Inclusion of g is inspired by the Positivstellensatz approach to local positivity of polynomials, as can be found in, e.g., [23]–[25]. For example, under mild conditions, Putinar's P-Satz states that a polynomial $p(x)$ is positive for all $x \in \{x : g(x) \geq 0\}$ if and only if it can be represented as $p(x) = s_1(x) + g(x)s_2(x)$ for some sum-of-squares polynomials s_1, s_2. In this way, Theorem 9 can be seen as an operator-valued version of this classical result. Note, however, in our case g is a function of the variable of integration and not the state and so the analogy is somewhat specious. Furthermore, for this paper, we restrict ourselves to *linear* maps of the state space. A partial discussion of parametrization of positive nonlinear operators for the stability of nonlinear time-delay systems can be found in [26] and [27].

Note that there are few constraints on the matrix-valued functions Y_1 and Y_2, functions whose elements are a basis for the multiplier and kernel functions found in $\mathcal{P}^{\frac{1}{2}}$. In this paper, these are chosen as $Y_1(s) = Z_d(s) \otimes I_n$ and $Y_2(s,\theta) = Z_d(s,\theta) \otimes I_n$, where Z_d is the vector of monomials of degree d or less in variables s and s, θ, respectively. Likewise, as mentioned, g is chosen as both $g(s) = 1$ and $g(s) = -s(s+\tau_K)$, with the resulting P, Q, R, S being the sum of the results of applying Theorem 9 to each case. To simplify notation, throughout the remainder of this paper, we will use the notation $\{P, Q, S, R\} \in \Xi_{d,m,n}$ to denote the LMI constraints on the coefficients of the polynomials P, Q, R, S implied by the conditions of Theorem 9 using both $g_i(s) = 1$ and $g_i = -s(s+\tau_K)$ as follows:

$$\Xi_{d,m,n} := \left\{ \{P,Q,R,S\} : \begin{array}{c} \{P,Q,S,R\} = \{P_1,Q_1,S_1,R_1\} + \{P_2,Q_2,S_2,R_2\}, \\ \text{where } \{P_1,Q_1,S_1,R_1\} \text{ and } \{P_2,Q_2,S_2,R_2\} \text{ satisfy} \\ \text{Theorem 9 with } g=1 \text{ and } g=-s(s+\tau_K), \text{ respectively.} \end{array} \right\}$$

C. Summary of Conditions for Positivity on $Z_{m,n,K}$

The following corollary summarizes the main result of this section.

Corollary 10: Suppose there exist $d \in \mathbb{N}$, constant $\epsilon > 0$, matrix $P \in \mathbb{R}^{m\times m}$, polynomials Q_i, S_i, and R_{ij} for $i, j \in [K]$ such that

$$\mathcal{L}_1(P, Q_i, S_i, R_{ij}) \in \Xi_{d,m,nK}.$$

Then, $\langle x, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}x\rangle_{Z_{m,n,K}} \geq 0$ for all $x \in Z_{m,n,K}$.

Proof: Define $\{\hat{P}, \hat{Q}, \hat{S}, \hat{R}\} = \mathcal{L}_1(P, Q_i, S_i, R_{ij})$. $\{\hat{P}, \hat{Q}, \hat{S}, \hat{R}\} \in \Xi_{d,m,nK}$, by Theorem 9, $\left\langle x, \mathcal{P}_{\{\hat{P},\hat{Q},\hat{S},\hat{R}\}}x\right\rangle_{Z_{m,nK,1}} \geq 0$ for all $x \in Z_{m,nK,1}$. Next, since $\{\hat{P}, \hat{Q}, \hat{S}, \hat{R}\} = \mathcal{L}_1(P, Q_i, S_i, R_{ij})$, by Lemma 8, $\langle x, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}x\rangle_{Z_{m,n,K}} \geq 0$ for all $x \in Z_{m,n,K}$. $\blacksquare$

To simplify presentation, the main results of the following section will reference Corollary 10 instead of the individual lemma and theorem statements, which it combines.

VIII. LMI Formulation of the Dual Stability Test

In this section, we apply the positivity conditions developed in Section VII to the operators parameterized in Section V-B, yielding a computational method for verification of the dual stability conditions of Theorem 5 and Corollary 7.

A. LMI Test for Dual Stability With Multiple Delays

We first consider the case of systems with multiple delays. The variables in the LMI are the matrix P and the coefficients of the polynomial functions Q_i S_i, and R_{ij}. The polynomial constraints $\in \Xi_{d,n,nK}$ and $\in \Xi_{d,n(K+1),nK}$ represent LMI constraints on the coefficients of the polynomials as per Theorem 9.

Theorem 11: Suppose there exist $d \in \mathbb{N}$, constant $\epsilon > 0$, matrix $P \in \mathbb{R}^{n\times n}$, polynomials $S_i, Q_i \in W_2^{n\times n}[T_i^0]$ and $R_{ij} \in W_2^{n\times n}\left[T_i^0 \times T_j^0\right]$ for $i, j \in [K]$ such that

$$\mathcal{L}_1(P - \epsilon I_n, Q_i, S_i - \epsilon I_n, R_{ij}) \in \Xi_{d,n,nK}$$

$$\mathcal{L}_1(D_1 + \epsilon \hat{I}, V_i, \dot{S}_i + \epsilon I_n, G_{ij}) \in \Xi_{d,n(K+1),nK}$$

where $\hat{I} = \operatorname{diag}(I_n, 0_{nK})$, $\mathcal{L}_1$ is as defined in (6), and where P_1, V_1, G_{ij} are as defined in Theorem 5.

Furthermore, suppose

$$P = \tau_K Q_i(0)^T + \tau_K S_i(0) \quad \text{for } i \in [K]$$

$$S_i(s) = S_i(s)^T, \qquad R_{ij}(s,\theta) = R_{ji}(\theta, s)^T \quad \text{for } i, j \in [K]$$

$$Q_j(s) = R_{ij}(0, s) \quad \text{for } i, j \in [K].$$

Then, the system defined by (1) is exponentially stable.

Proof: Clearly, $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ satisfies the conditions of Lemma 3. By Corollary 10, we have

$$\begin{aligned}&\left\langle x, \mathcal{P}_{\{P-\epsilon I_n, Q_i, S_i - \epsilon I_n, R_{ij}\}} x\right\rangle_{Z_{n,K}}\\ &\quad = \left\langle x, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} x\right\rangle_{Z_{n,K}} - \epsilon \left\|x\right\|^2_{Z_{n,K}} \geq 0\end{aligned}$$

for all $x \in Z_{n,K}$. Similarly, we have

$$\begin{aligned}&\left\langle \begin{bmatrix}\begin{bmatrix} y_1 \\ y_2\end{bmatrix} \\ \phi_i\end{bmatrix}, \mathcal{P}_{\{D_1+\epsilon\hat{I}, V_i, \dot{S}_i + \epsilon I_n, G_{ij}\}} \begin{bmatrix}\begin{bmatrix} y_1 \\ y_2\end{bmatrix} \\ \phi_i\end{bmatrix}\right\rangle_{Z_{n(K+1),n,K}}\\ &= \left\langle \begin{bmatrix}\begin{bmatrix} y_1 \\ y_2\end{bmatrix} \\ \phi_i\end{bmatrix}, \mathcal{P}_{\{D_1, V_i, \dot{S}_i, G_{ij}\}} \begin{bmatrix}\begin{bmatrix} y_1 \\ y_2\end{bmatrix} \\ \phi_i\end{bmatrix}\right\rangle_{Z_{n(K+1),n,K}}\\ &\quad + \epsilon \left\| \begin{bmatrix} y_1 \\ \phi_i \end{bmatrix}\right\|^2_{Z_{n,K}} \leq 0.\end{aligned}$$

Hence, Theorem 5 establishes the exponential stability of (1). ■

B. LMI for Dual Stability of Single-Delay Systems

We now state an LMI representation of the dual stability condition for a single delay ($\tau_1 = \tau_K = \tau$). This is a simplified version of Theorem 11, where we have eliminated the variables P and Q.

Theorem 12: Suppose there exist $d \in \mathbb{N}$, constant $\epsilon > 0$, polynomials $S \in W_2^{n\times n}[-\tau, 0]$ and $R \in W_2^{n\times n}[[-\tau, 0] \times [-\tau, 0]]$, with $R(s,\theta) = R(\theta, s)^T$ and $S(s) \in \mathbb{S}^n$ such that

$$\begin{aligned}&\{\tau(R(0,0) + S(0)) - \epsilon I_n, R(0,\cdot), S - \epsilon I_n, R\} \in \Xi_{d,2n,1}\\ &\qquad -\left\{D_1 + \epsilon I_n, V, \dot{S} + \epsilon I_n, G\right\} \in \Xi_{d,2n,n}\end{aligned}$$

where D_1, V, and G are as defined in Corollary 7.

Then, the system defined by (1) in the case $K = 1$ with $\tau_1 = \tau$ is exponentially stable.

Proof: The proof follows from Theorem 11 by defining $P = \tau(R(0,0) + S(0))$ and $Q(s) = R(0,s)$ and noting that when $K = 1$

$$\{P, Q, S, R\} = \mathcal{L}_1(P, Q, S, R).$$ ■

IX. MATLAB Toolbox Implementation

To assist with the application of these results, we have created a library of functions for verifying the stability conditions described in this paper. These libraries make use of modified versions of the SOSTOOLS [28] and MULTIPOLY toolboxes coupled with either SeDuMi [29] or Mosek. A complete package can be downloaded from [30] or [31] and all scripts and functions are well documented and commented. Key examples of functions included are as follows.

1) `sosjointpos_mat_ker_ndelay_PQRS_vZ.m`
   a) Declares a $[P, Q_i, R_{ij}, S_i]$ that defines an operator, which is positive on $Z_{m,n,K}$.
2) `sosmateq.m`
   a) Declare a matrix-valued equality constraint.
3) `solver_ndelay_dual_joint_nd_RL2.m`
   a) A script that combines the functions listed previously to test the stability of a user-defined problem.

These functions are implemented within the pvar framework of SOSTOOLS and are available on Code Ocean.

Pseudocode: The following is a pseudocode implementation of the conditions of Theorem 11.

```
(a) [P,Q,R,S]=sosjointpos_mat_ker_ndelay
    _PQRS
(b) [D,E,G,H]=F(P,Q,R,S)
(c) [L,M,N,O]=sosjointpos_mat_ker_ndelay
    _PQRS
(d) sosmateq(D+L); sosmateq(E+M)
(e) sosmateq(G+N); sosmateq(H+O)
```

Here, we use the function `F` to represent the derivative construction defined in Theorem 11. This is not an actual function in the toolbox. The derivative construction can be found in `solver_ndelay_dual_joint_nd_RL2`, however.

X. Numerical Validation

In the preceding sections, we proposed a sufficient condition for stability. However, as discussed, this condition is not necessary and there are several potential sources of conservatism, including the constraint $\mathcal{P}(X) = X$ and the assumption of an SOS representation of the positive operator. In this section, we apply the dual stability condition to a battery of numerical examples in order to determine whether this potential conservatism is significant.

In each case, a table is given that lists the maximum provably stable value of a specified parameter for each degree d. This maximum value is found using bisection on the parameter. In each case d is increased until the maximum parameter value converges to several decimal places. The true maximum is also provided as either the "limit" or "analytic" value, depending on whether this limiting value is known analytically or is a best estimate based on simulation. The computation time is also listed in CPU seconds on an Intel i7-5960X 3.0-GHz processor. This time corresponds to the interior-point (IPM) iteration in SeDuMi and does not account for preprocessing, postprocessing, or for the time spent on polynomial manipulations formulating the SDP using SOSTOOLS. Such polynomial manipulations can significantly exceed SDP computation time for small problems.

b) Example A: First, we consider a simple example that is known to be stable for $\tau \leq \frac{\pi}{2}$:

$$\dot{x}(t) = -x(t-\tau)$$

d	1	2	3	analytic
$\tau_{\max}$	1.558	1.5707	1.5707	1.5707
CPU sec	0.309	0.516	0.776	

c) Example B: Next, we consider a well-studied two-dimensional (2-D), single-delay system:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & .1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t-\tau)$$

d	1	2	3	limit
$\tau_{\max}$	1.693	1.7176	1.71785	1.71785
$\tau_{\min}$	0.10018	0.100174	0.100174	0.100174
CPU sec	0.478	0.879	2.48	

d) Example C: We consider a scalar, two-delay system:

$$\dot{x}(t) = ax(t) + bx(t-1) + cx(t-2).$$

In this case, we fix $a = -2$ and $c = -1$ and search for the maximum b, which is 3 [32]–[34]:

d	1	2	3	analytic
$b_{\max}$	0.829	2.999	2.999	3
CPU sec	0.603	1.50	3.89	

e) Example D: We consider a 2-D, two-delay system where $\tau_1 = \tau_2/2$ and search for the maximum stable τ_2:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & .1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} x(t-\tau/2) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t-\tau)$$

d	1	2	3	limit
$\tau_{\max}$	1.354	1.3722	1.3722	1.3722
CPU sec	1.75	7.51	27.2	

f) Example E: Next, we consider a 4-D, one-delay static output feedback system which, in [35], was found to be challenging for SOS-based methods. This example considers the static feedback system

$$\dot{x}(t) = (A - BKC)x(t) + BKCx(t-\tau)$$

where

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10 & 10 & 0 & 0 \\ 5 & -15 & 0 & -.25 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T.$$

TABLE I
COMPUTATION TIME (IN CPU SEC) INDEXED BY THE NUMBER OF STATES (n) AND THE NUMBER OF DELAYS (K)

$K \downarrow n \rightarrow$	1	2	3	5	10
1	.366	.094	.158	.686	12.8
2	.112	.295	1.260	10.83	61.05
3	.177	1.311	6.86	96.85	5223
5	.895	13.05	124.7	2014	80950
10	13.09	59.5	5077	80231	NA

In this case, we take $K = 1$. It has been reported that it requires polynomials of degree 10 even in the primal case to prove stability of $h = 3$. However, using the dual stability condition, we find a stability proof for degree $d = 4$, perhaps due to the use of the new parametrization of positive operators. The computation times for increasing degrees are listed in the following table:

d	1	2	3
CPU sec	2.23	7.45	21.6
Stability?	no	yes	yes

g) Example F: In this example, we consider a generalized n-D system with K delays and examine the computational scalability of the stability test. Our system has the form

$$\dot{x}(t) = -\sum_{i=1}^{K} \frac{x(t-i/K)}{K}.$$

For this example, we only search for polynomials of degree 2 and leave off the second kernel function. All results indexed in Table I list IPM computation time in seconds and all establish the stability of the system. The table is jointly indexed by the number of states and the number of delays.

These numerical examples indicate little, if any conservatism in the LMI implementation of the dual stability conditions, and moreover, the method is accurate for relatively low degree. Example E shows that computational complexity is a function of nK and that the results scale well to high-dimensional systems and large numbers of delay. Specifically, current desktop computers with 128-GB RAM can solve problems where $\cong nK \leq 50$. This scaling can be improved if the delay channel is low dimensional through the use of the differential-difference framework [19]. In the following section, we introduce a controller synthesis condition. Note that adding the controller to the optimization problem does *not* significantly change the computational complexity of the problem.

XI. LMI CONTROLLABILITY TEST

Establishment of dual stability conditions is the first step in developing full-state feedback controller synthesis conditions. Obtaining the stabilizing controller requires two more steps. Specifically, consider the system $\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(t)$, where $u(t) \in \mathbb{R}^m$. First, we define the controllability test.

Theorem 13: Suppose there exist $d \in \mathbb{N}$, constant $\epsilon > 0$, matrix $P \in \mathbb{R}^{n\times n}$, polynomials $S_i, Q_i \in W_2^{n\times n}[T_i^0]$, $R_{ij} \in W_2^{n\times n}\left[T_i^0 \times T_j^0\right]$ for $i, j \in [K]$, matrices $W_i \in \mathbb{R}^{m\times n}$, and polynomials $Y_i \in W_2^{m\times n}$ for $i \in [K]$ such that

$$\mathcal{L}_1(P - \epsilon I_n, Q_i, S_i - \epsilon I_n, R_{ij}) \in \Xi_{d,n,nK}$$

$$-\mathcal{L}_1(D_1 + W + \epsilon\hat{I}, V_i + BY_i, \dot{S}_i + \epsilon I_n, G_{ij}) \in \Xi_{d,n(K+1),nK}$$

where $\hat{I}$, D_1, V_i, and G_{ij} are as defined in Theorem 5, $\mathcal{L}_1$ is as defined in (6), and

$$W = \begin{bmatrix} BW_0 + W_0^T B^T & BW_1 & \dots & BW_K \\ W_1^T B^T & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ W_K B^T & 0 & \dots & 0 \end{bmatrix}.$$

Furthermore, suppose P, Q_i, S_i, and R_{ij} satisfy the conditions of Lemma 3. Then, the system $\dot{x}(t) = A_0 x(t) = \sum_i A_i x(t-\tau_i) + Bu(t)$ is exponentially stabilizable and $u(t) = \mathcal{Z}\mathcal{P}^{-1}x(t)$ is an exponentially stabilizing controller where

$$\left(\mathcal{Z}\begin{bmatrix} x \\ \phi_i \end{bmatrix}\right)(s) := W_0 x + \sum_{i=1}^K W_i \phi_i(-\tau_i) + \sum_{i=1}^K \int_{-\tau_i}^0 Y_i(s)\phi_i(s) ds.$$

Proof: If $u(t) = \mathcal{Z}\mathcal{P}^{-1}x(t)$, then $\dot{x}(t) = \left(\mathcal{A} + \mathcal{B}\mathcal{Z}\mathcal{P}^{-1}\right) x(t)$ where

$$(\mathcal{B}u)(s) = \begin{bmatrix} Bu(t) \\ 0 \end{bmatrix}.$$

Hence, as in Theorem 5, the closed-loop system is stable if

$$\left\langle \left(\mathcal{A} + \mathcal{B}\mathcal{Z}\mathcal{P}^{-1}\right)\mathcal{P}\begin{bmatrix} x \\ \phi_i \end{bmatrix}, \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n,K}} + \left\langle \begin{bmatrix} x \\ \phi_i \end{bmatrix}, \left(\mathcal{A} + \mathcal{B}\mathcal{Z}\mathcal{P}^{-1}\right)\mathcal{P}\begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n,K}}$$

$$= \left\langle \begin{bmatrix} \begin{bmatrix} x \\ \phi_1(-\tau_1) \\ \vdots \\ \phi_k(-\tau_K) \end{bmatrix} \\ \phi_i \end{bmatrix}, \mathcal{D} + \mathcal{D}_Z \begin{bmatrix} \begin{bmatrix} x \\ \phi_1(-\tau_1) \\ \vdots \\ \phi_k(-\tau_K) \end{bmatrix} \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n(K+1),n,K}} \leq -\epsilon \left\| \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\|^2_{Z_{n,K}} \quad \forall \begin{bmatrix} x \\ \phi_i \end{bmatrix} \in X$$

where

$$\mathcal{D}_Z := \mathcal{P}_{\{W, BY_i, 0, 0\}} \quad \text{and} \quad \mathcal{D} := \mathcal{P}_{\{D_1, V_i, \dot{S}_i, G_{ij}\}}.$$

Now, from Corollary 10, we have

$$\mathcal{P}_{\{D_1 + W + \epsilon\hat{I}, V_i + BY_i, \dot{S}_i + \epsilon I_n, G_{ij}\}} \leq 0$$

and hence

$$\left\langle \begin{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{D_1 + W + \epsilon\hat{I}, V_i + BY_i, \dot{S}_i + \epsilon I_n, G_{ij}\}} \begin{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \phi_i \end{bmatrix} \right\rangle$$

$$= \left\langle \begin{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \phi_i \end{bmatrix}, (\mathcal{D} + \mathcal{D}_Z) \begin{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \phi_i \end{bmatrix} \right\rangle + \epsilon \left\| \begin{bmatrix} y_1 \\ \phi_i \end{bmatrix} \right\|^2_{Z_{n,K}}$$

$$\leq 0.$$

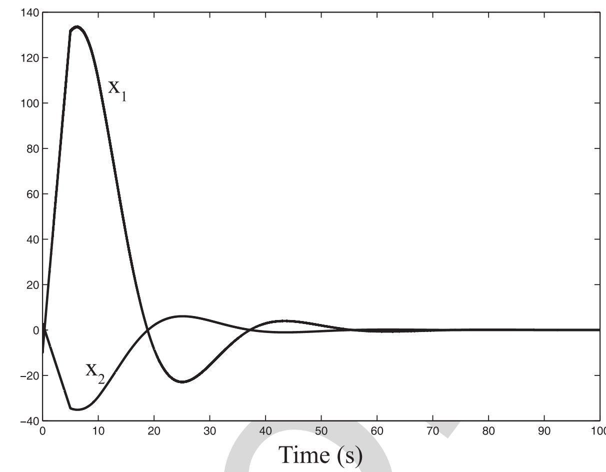

Fig. 1. MATLAB DDE23 simulation of System (6) and Controller (8) and delay $\tau = 5$ s.

Therefore, by Theorem 5, the closed-loop system is exponentially stable. ■

The second step in controller synthesis is the construction of the stabilizing controller $u(t) = \mathcal{Z}\mathcal{P}^{-1}_{\{P,Q_i,S_i,R_{ij}\}}$, which requires inversion of the operator $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$—a topic which is addressed in the sequel to this paper [36]. We illustrate these results in the single-delay case using the well-studied system

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -2 & -.5 \\ 0 & -1 \end{bmatrix} x(t-\tau) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t). \tag{6}$$

For $\tau = 5$ using simple polynomials of degree 2, we obtained the following exponentially stabilizing controller:

$$u(t) = \begin{bmatrix} -3601 \\ -944 \end{bmatrix}^T x(t) + \begin{bmatrix} -.00891 \\ .872 \end{bmatrix}^T x(t-\tau) + \int_{-5}^0 \begin{bmatrix} 52.1 + 6.98s + .00839s^2 - .0710s^3 \\ 12.7 + 1.50s - .0407s^2 - .0190s^3 \end{bmatrix}^T x(t+s) ds. \tag{7}$$

Simulations for fixed initial conditions were performed and can be seen in Fig. 1.

XII. Conclusion

We have proposed a new form of dual Lyapunov stability condition that allows the convexification of the controller synthesis problem for delayed and other infinite-dimensional systems. This duality principle requires a Lyapunov operator that is positive, invertible, and self-adjoint and preserves the structure of the state space. We have proposed such a class of operators and used them to create stability conditions that can be expressed as positivity and negativity of quadratic Lyapunov functions. These dual stability conditions have a tridiagonal structure, which is distinct from standard Lyapunov–Krasovskii forms and may be exploited to increase performance when studying systems with large numbers of delays. The dual stability condition is presented in a format that can be adapted to many existing computational methods for Lyapunov stability analysis. We have applied the sum-of-squares approach to enforce the positivity of the quadratic forms and tested the stability condition in both the single- and multiple-delay cases. Numerical testing on several examples indicates the method is not likely to be conservative. The contribution of this paper is not in the efficiency of the stability test, however, as these are likely less efficient when compared to, e.g., previous SOS results due to the structural constraints imposed upon the operator. Rather, the contribution is in the convexification of the synthesis problem, which opens the door for dynamic output-feedback H_∞ synthesis for infinite-dimensional systems. This potential is demonstrated in the numerical example of controller synthesis for a single-delay system.

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A Dual to Lyapunov's Second Method for Linear Systems With Multiple Delays and Implementation Using SOS

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***Abstract*—We present a dual form of Lyapunov–Krasovskii functional which allows the problem of controller synthesis for multidelay systems to be formulated and solved in a convex manner. First, we give a generalized version of the dual stability condition formulated in terms of Lyapunov operators which are positive, self-adjoint, and preserve the structure of the state space. Second, we provide a class of such operators and express the stability conditions as positivity and negativity of quadratic Lyapunov–Krasovskii functional forms. Next, we adapt the Sum of Squares (SOS) methodology to express positivity and negativity of these forms as Linear Matrix Inequalities (LMIs), describing a new set of polynomial manipulation tools designed for this purpose. We apply the resulting LMIs to a battery of numerical examples and demonstrate that the stability conditions are not significantly conservative. Finally, we formulate a test for controller synthesis for systems with multiple delays, apply the test to a numerical example, and simulate the resulting closed-loop system.**

***Index Terms*—Controller synthesis, delay systems, LMIs.**

I. Introduction

Systems with delay have been well-studied for some time [1]–[3]. In recent years, however, there has been an increased emphasis on the use of optimization and SemiDefinite Programming (SDP) for stability analysis of linear and nonlinear time-delay systems. Although the computational question of the stability of a linear state-delayed system is believed to be NP-hard, several techniques have been developed that use LMI methods [4] to construct asymptotically exact algorithms. An asymptotically exact algorithm is a sequence of polynomial-time algorithms wherein each instance in the sequence provides sufficient conditions for stability, the computational complexity of the instances is increasing, the accuracy of the test is increasing, and the sequence converges to what appears to be a necessary and sufficient condition. Examples of such sequential algorithms include the piecewise-linear approach [2], the Wirtinger-based method of [5], and the SOS approach [6]. In addition, there are also frequency-domain approaches such as [7] and [8]. These asymptotic algorithms are sufficiently reliable so that for this paper, we may consider the problem of stability analysis of linear discrete-delay systems to be solved.

The purpose of this paper is to explore methods by which we may extend the success in the use of asymptotic algorithms for stability analysis of time-delay systems to the field of robust and optimal controller synthesis—an area that is relatively underdeveloped. Although there have been a number of results on controller synthesis for time-delay systems [9], none of these results has been able to resolve the fundamental bilinearity of the synthesis problem. Bilinearity here means that for a given feedback controller, the search for a Lyapunov functional is linear in the decision variables that define the functional and is relatively tractable. Furthermore, given a predefined Lyapunov functional, the search for a controller ensuring negativity of the time derivative of that functional is linear in the decision variables that define the feedback gains. However, if we are looking for both a controller and a Lyapunov functional that establishes the stability of that controller, then the resulting stability condition is nonlinear and nonconvex in the combined set of decision variables.

Without a convex formulation of the controller synthesis problem, we cannot search over the set of provably stabilizing controllers without significant conservatism, much less address the problems of robust and quadratic stability. To resolve this difficulty, some papers use iterative methods to alternately optimize the Lyapunov functional and then the controller as in [10] or [11] (via a "tuning parameter"). However, this iterative approach is not guaranteed to converge. Meanwhile, approaches based on frequency-domain methods, discrete approximation, or Smith predictors result in controllers that are not provably stable or are sensitive to variations in system parameters or in delay.

In this paper, we propose a dual Lyapunov-type stability criterion, wherein the decision variables do not parameterize a Lyapunov functional per se, but where the feasibility of this criterion *implies* the existence of such a functional. The advantage of such an approach for controller synthesis is that it allows for an invertible variable substitution, eliminating all bilinear terms in the criterion for controller synthesis.

Both our definition of duality (in the optimization sense) and our approach to controller synthesis are based on the LMI framework for controlling linear finite-dimensional state-space systems of the form $\dot{x} = Ax + Bu$. Specifically, if $u = 0$, the LMI condition for the existence of a quadratic Lyapunov function $V(x) = x^T Px$ is the existence of a $P > 0$ such that $A^T P + PA < 0$. The feasibility of this LMI implies that $V(x) = x^T Px > 0$ and $\dot{V}(x) = x^T(A^T P + PA)x < 0$. This

Manuscript received July 3, 2017; revised January 8, 2018; accepted April 5, 2018. Date of publication; date of current version. This work was supported by the National Science Foundation under Grant 1301660, Grant 1538374, and Grant 1739990. Recommended by Associate Editor W. Michiels.

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This paper has supplementary downloadable material available at http://ieeexplore.ieee.org, provided by the author.

Digital Object Identifier 10.1109/TAC.2018.2832470

LMI is in primal form because the decision variable P defines the Lyapunov function directly. However, when we add a controller $u = Kx$, we get $\dot{x} = (A + BK)x$ and the synthesis condition becomes $A^T P + PA + K^T B^T P + PBK < 0$, which is bilinear in decision variables P and K and hence intractable. Bilinearity can be eliminated, however, if we use the *implied* Lyapunov function $V(x) = x^T P^{-1} x$. Using this implied Lyapunov function the time derivative becomes $\dot{V}(x) = x^T(A^T P^{-1} + P^{-1}A)x = (P^{-1}x)^T(PA^T + AP)(P^{-1}x) = z^T(PA^T + AP)z$, where $z = P^{-1}x$. This implies that the stability of $\dot{x} = Ax$ is equivalent to the existence of $P > 0$ such that $AP + PA^T < 0$. If we now add a controller $u = Kx$, the controller synthesis condition becomes $(AP + BKP) + (AP + BKP)^T < 0$, which is still bilinear. However, if we consider the variable substitution $Z = KP$, then stabilizability is equivalent to the existence of a $P > 0$ and Z such that $(AP + BZ) + (AP + BZ)^T < 0$, which is an LMI. The stabilizing controller gains can then be reconstructed as $K = ZP^{-1}$. LMIs of this form were introduced in [12] and are the basis for a majority of LMI methods for controller synthesis (see the supplemental notes in [4, ch. 5] for a discussion). The first contribution of this paper, then, is an operator-valued equivalent of the dual Lyapunov inequality $P > 0$, $AP + PA^T < 0$ that implies the stability of a general class of infinite-dimensional systems. The second contribution of this paper is a computational framework for verifying this dual inequality using LMIs.

The standard approach to state-space representation of infinite-dimensional systems is to define the state as evolving on a Hilbert space Z and satisfying the derivative condition $\dot{x}(t) = \mathcal{A}x(t)$. The state is constrained to a subspace X of Z and the operator $\mathcal{A}$ is typically unbounded. It is known that if $\mathcal{A}$ generates a strongly continuous semigroup, then exponential stability of this system is equivalent to the existence of an operator $\mathcal{P}$ such that $\langle x, \mathcal{P}x\rangle \geq \|x\|^2$ and $\langle x, \mathcal{P}\mathcal{A}x\rangle + \langle \mathcal{P}\mathcal{A}x, x\rangle \leq -\epsilon\|x\|^2$ [13]. In Section IV, we show that under mild additional conditions on $\mathcal{P}$, the dual version of this result also holds. Namely existence of an operator $\mathcal{P}$ such that $\langle x, \mathcal{P}x\rangle \geq \|x\|^2$ and $\langle x, \mathcal{A}\mathcal{P}x\rangle + \langle \mathcal{A}\mathcal{P}x, x\rangle \leq -\epsilon\|x\|^2$ implies exponential stability of $\dot{x} = \mathcal{A}x$. Specifically, these additional conditions on $\mathcal{P}$ are that $\mathcal{P}$ be self-adjoint and preserve specified properties of the solution. This result applies to any well-posed infinite-dimensional system, and is not conservative if X is a closed subspace of Z.

Having formulated a general duality result, we next turn to the special case of systems with multiple delays and introduce a parametrization of a class of operators that are self-adjoint, preserve desired properties of the solution, and which are defined by the combination of multiplier and integral operators with constraints on the associated multipliers and kernels. This result allows us to represent the dual stability criterion in a manner similar to classical Lyapunov–Krasovskii stability conditions, but with an additional tridiagonal structure that may prove useful for solving these Lyapunov equations. Finally, we present an LMI/SOS method for enforcing positivity and negativity of the operators under the assumption that all multipliers and kernels are polynomial. Finally, we discuss how these results can be used to solve the controller synthesis problem and give a numerical example using the methods defined in [14] and [15].

Having stated the main contributions of this paper, we note that while we show how to enforce the operator inequalities using a slight generalization of existing SOS-based results, the duality results are presented in such a way as to encourage the reader to use other methods of enforcing these inequalities, methods including those contained in [5], or [16]. Indeed, we emphasize that Theorems 1 and 5 are formulated independent of whichever numerical method is used for enforcing the inequalities. In this way, our goal is to simply establish a new class of Lyapunov stability conditions that are well suited to the problem of controller synthesis, leaving the method of enforcement of these conditions to the reader.

Finally, we note that there have been a number of results on dual and adjoint systems [17]. Unfortunately, however, these dual systems are not delay-type systems and there is no clear relationship between the stability of these adjoint and dual systems and the stability of the original delayed system.

This paper is organized as follows. In Sections II and III, we develop a mathematical framework for expressing Lyapunov-based stability conditions as operator inequalities. In Section IV, we show that given additional constraints on the Lyapunov operator, satisfaction of the dual Lyapunov inequality $\langle x, \mathcal{A}\mathcal{P}x\rangle + \langle \mathcal{A}\mathcal{P}x, x\rangle \leq -\epsilon\|x\|^2$ proves the stability of $\dot{x}(t) = \mathcal{A}x(t)$. In Sections VI and V, we define a restricted class of Lyapunov functionals and operators which are valid for the dual stability condition in both the single-delay and multiple-delay cases, applying these classes of operators in Sections VI-B and V-B to obtain dual stability conditions. These dual stability conditions are formulated as positivity and negativity of Lyapunov functionals. In Section VII, we show how SOS-based methods can be used to parameterize positive Lyapunov functionals and thereby enforce the inequality conditions in Sections VI-B and V-B, results which are summarized in Corollary 10. Finally, in Section VIII, we summarize our results with a set of LMI conditions for dual stability in both the single and multiple-delay cases. Section IX describes our MATLAB toolbox, available online, which facilitates construction and solution of the LMIs. Section X applies the results to a variety of stability problems and verifies that the dual stability test is not conservative. Finally, Section XI discusses the problem of full-state feedback controller synthesis and gives a numerical illustration in the case of a single delay.

A. Technical Summary of Results

Before proceeding, we give a brief summary of the main results of Section VI-B using as little mathematical formalism as possible in order to illustrate how these results differ from the classical Lyapunov–Krasovskii stability conditions. These results are stated for systems with a single delay in order to avoid much of the notation and mathematical progression needed for the multiple-delay case. That is, we consider the system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau).$$

1) Classical Lyapunov–Krasovskii Stability Conditions: The standard necessary and sufficient conditions for stability in the single-delay case are the existence of a

$$V(\phi) = \int_{-\tau}^{0} \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix}^T \begin{bmatrix} M_{11} & \tau M_{12}(s) \\ \tau M_{21}(s) & \tau M_{22}(s) \end{bmatrix} \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix} ds + \tau \int_{-\tau}^{0}\int_{-\tau}^{0} \phi(s)^T N(s,\theta)\phi(\theta) d\theta ds$$

such that $V(\phi) \geq \|\phi(0)\|^2$ and

$$\dot{V}(\phi) =$$

$$\int_{-\tau}^{0} \begin{bmatrix} \phi(0) \\ \phi(-\tau) \\ \phi(s) \end{bmatrix}^T \begin{bmatrix} D_{11} + D_{11}^T & D_{12} & \tau D_{13}(s) \\ D_{12}^T & -M_{22}(-\tau) & \tau D_{23}(s) \\ \tau D_{13}(s)^T & \tau D_{23}(s)^T & -\tau \dot{M}_{22}(s) \end{bmatrix} \begin{bmatrix} \phi(0) \\ \phi(-\tau) \\ \phi(s) \end{bmatrix} ds$$

$$-\tau \int_{-\tau}^{0}\int_{-\tau}^{0} \phi(s)^T \left(\frac{d}{ds} N(s,\theta) + \frac{d}{d\theta} N(s,\theta) \right) \phi(\theta) d\theta ds$$

$$\leq -\epsilon \|\phi\|^2$$

$$D_{11} = M_{11}A_0 + M_{12}(0) + \frac{1}{2} M_{22}(0),$$

$$D_{12} = M_{11}A_1 - M_{12}(-\tau),$$

$$D_{23} = A_1^T M_{12}(s) - N(-\tau, s),$$

$$D_{13} = x A_0^T M_{12}(s) - \dot{M}_{12}(s) + N(0, s).$$

2) New Dual Lyapunov–Krasovskii Stability Conditions: As per Corollary 7, the single-delay system is stable if there exists a

$$V(\phi) = \int_{-\tau}^{0} \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix}^T \begin{bmatrix} \tau(R(0,0)+S(0)) & \tau R(0,s) \\ \tau R(s,0) & \tau S(s) \end{bmatrix} \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix} ds$$

$$+ \int_{-\tau}^{0}\int_{-\tau}^{0} \phi(s)^T R(s,\theta) \phi(\theta) d\theta ds \geq \left\| \begin{bmatrix} \phi(0) \\ \phi \end{bmatrix} \right\|^2$$

and

$$V_D(\phi)$$

$$= \int_{-\tau}^{0} \begin{bmatrix} \phi(0) \\ \phi(-\tau) \\ \phi(s) \end{bmatrix}^T \begin{bmatrix} S_{11} + S_{11}^T & S_{12} & \tau S_{13}(s) \\ S_{12}^T & S_{22} & 0_n \\ \tau S_{13}(s)^T & 0_n & \tau \dot{S}(s) \end{bmatrix} \begin{bmatrix} \phi(0) \\ \phi(-\tau) \\ \phi(s) \end{bmatrix} ds$$

$$+ \int_{-\tau}^{0}\int_{-\tau}^{0} \phi(s)^T \left(\frac{d}{ds} R(s,\theta) + \frac{d}{d\theta} R(s,\theta) \right) \phi(\theta) d\theta ds$$

$$\leq -\epsilon \left\| \begin{bmatrix} \phi(0) \\ \phi \end{bmatrix} \right\|$$

where

$$S_{11} := \tau A_0 (R(0,0) + S(0)) + \tau A_1 R(-\tau, 0) + \frac{1}{2} S(0)$$

$$S_{12} := \tau A_1 S(-\tau), \qquad S_{22} := -S(-\tau)$$

$$S_{13}(s) := A_0 R(0,s) + A_1 R(-\tau, s) + \dot{R}(s,0)^T.$$

Although this section only considers the single-delay case, one can see the two primary differences between the primal and dual stability conditions. First, as was the case for delay-free systems, the A_0, A_1 system matrices appear on the left as opposed to the right-hand side of the Lyapunov variables. This allows for controller synthesis via variable substitution as we will demonstrate in Section XI. The second difference is that in the dual stability conditions, the (2, 3) and (3, 2) blocks of the derivative condition are zero. This unexpected structure extends to the multiple-delay case, wherein *ALL* (i, j) blocks are zero for $i, j \neq 1, i \neq j$.

B. Notation

Shorthand notation used throughout this paper includes the Hilbert spaces $L_2^m[X] := L_2(X; \mathbb{R}^m)$ of square integrable functions from X to $\mathbb{R}^m$ and $W_2^m[X] := W^{1,2}(X; \mathbb{R}^m) = H^1(X; \mathbb{R}^m) = \{x : x, \dot{x} \in L_2^m[X]\}$. We use L_2^m and W_2^m when domains are clear from context. We also use the extensions $L_2^{n \times m}[X] := L_2(X; \mathbb{R}^{n \times m})$ and $W_2^{n \times m}[X] := W^{1,2}(X; \mathbb{R}^{n \times m})$ for matrix-valued functions. $\mathcal{C}[X] \supset W_2[X]$ denotes the continuous functions on X. $S^n \subset \mathbb{R}^{n \times n}$ denotes the symmetric matrices. We say an operator $\mathcal{P} : Z \to Z$ is positive on a subset X of Hilbert space Z if $\langle x, \mathcal{P}x \rangle_Z \geq 0$ for all $x \in X$. $\mathcal{P}$ is coercive on X if $\langle x, \mathcal{P}x \rangle_Z \geq \epsilon \|x\|_Z^2$ for some $\epsilon > 0$ and for all $x \in X$. Given an operator $\mathcal{P} : Z \to Z$ and a set $X \subset Z$, we use the shorthand $\mathcal{P}(X)$ to denote the image of $\mathcal{P}$ on subset X. $I_n \in \mathbb{S}^n$ denotes the identity matrix. $0_{n \times m} \in \mathbb{R}^{n \times m}$ is the matrix of zeros with shorthand $0_n := 0_{n \times n}$. We will occasionally denote the intervals $T_i^j := [-\tau_i, -\tau_j]$ and $T_i^0 := [-\tau_i, 0]$. For a natural number $K \in \mathbb{N}$, we adopt the index shorthand notation, which denotes $[K] = \{1, \ldots, K\}$.

II. Standard Results on Lyapunov Stability of Linear Time-Delay Systems

In this paper, we consider the stability of linear discrete-delay systems of the form

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{K} A_i x(t - \tau_i) \quad \text{for all} \quad t \geq 0$$

$$x(t) = \phi(t) \qquad\qquad \text{for all} \quad t \in [-\tau_K, 0], \quad (1)$$

where $A_i \in \mathbb{R}^{n \times n}$, $\phi \in \mathcal{C}[-\tau_K, 0]$, $K \in \mathbb{N}$ and for convenience $\tau_1 < \tau_2 < \cdots < \tau_K$. We associate with any solution x and any time $t \geq 0$, the "state" of System (1), $x_t \in \mathcal{C}[-\tau_K, 0]$, where $x_t(s) = x(t+s)$. For linear discrete-delay systems of Form (1), the system has a unique solution for any $\phi \in \mathcal{C}[-\tau_K, 0]$ and global, local, asymptotic, and exponential stability are all equivalent.

Stability of (1) may be certified through the use of Lyapunov–Krasovskii functionals—an extension of Lyapunov theory to systems with infinite-dimensional state space. In particular, it is known [2] that System (1) is stable if and only if there exist functions M and N, continuous in their respective arguments everywhere except possibly at points $H := \{-\tau_1, \ldots, -\tau_{K-1}\}$, such that the quadratic Lyapunov–Krasovskii functional $V : \mathcal{C}[-\tau_k, 0] \to \mathbb{R}$

$$V(\phi) = \int_{-\tau_K}^{0} \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix}^T M(s) \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix} ds$$

$$+ \int_{-\tau_K}^{0}\int_{-\tau_K}^{0} \phi(s)^T N(s,\theta) \phi(\theta)\, ds\, d\theta \qquad (2)$$

satisfies $V(\phi) \geq \epsilon \|\phi(0)\|^2$ and the Lie (upper Dini) derivative of the functional is negative along any solution x of (1). That is

$$\dot{V}(x_t) = \lim_{h \to 0} \frac{V(x_{t+h}) - V(x_t)}{h} \leq -\epsilon \|x_t(0)\|^2$$

for all $t \geq 0$ and some $\epsilon > 0$.

For the dual stability conditions we propose in this paper, discontinuities in the unknown functions M and N pose challenges, which make this form of Lyapunov–Krasovskii functional poorly suited to controller synthesis. For this reason, we use an alternative formulation of the necessary Lyapunov–Krasovskii functional. Specifically, it has been shown in [19], Theorem 3, that exponential stability is also equivalent to the existence of a Lyapunov–Krasovskii functional of the form

$$\begin{aligned}V(\phi) &= \tau_K \phi(0)^T P\phi(0) + \tau_K \sum_{i=1}^{K}\int_{-\tau_i}^{0} \phi(0)^T Q_i(s)\phi(s)ds \\ &\quad + \tau_K \sum_{i=1}^{K}\int_{-\tau_i}^{0}\phi(s)^T Q_i(s)^T\phi(0)ds \\ &\quad + \tau_K \sum_{i=1}^{K}\int_{-\tau_i}^{0}\phi(s)^T S_i(s)\phi(s) \\ &\quad + \sum_{i,j=1}^{K}\int_{-\tau_i}^{0}\int_{-\tau_j}^{0} \phi(s)^T R_{ij}(s,\theta)\phi(\theta)d\theta \ge \epsilon\left\|\phi(0)\right\|^2 \end{aligned} \tag{3}$$

where $\dot V(x_t) \le -\epsilon\left\|x_t(0)\right\|^2$ for some $\epsilon > 0$ and the functions Q_i, S_i, and R_{ij} may be assumed continuous on their respective domains of definition.

III. Reformulating the Lyapunov Stability Conditions Using Positive Operators

In this section, we introduce the mathematical formalism, which will be used to express both the primal and dual stability conditions. We begin by reviewing the well-established semigroup framework—a generalization of the concept of differential equations. Sometimes known as the "flow map," a "strongly continuous semigroup" is an operator $S(t): Z \to Z$ defined by the Hilbert space Z, which represents the evolution of the state of the system so that for any solution x, $x_{t+s} = S(s)x_t$. Associated with a semigroup on Z is an operator $\mathcal{A}$, called the "infinitesimal generator," which satisfies $\frac{d}{dt}S(t)\phi = \mathcal{A}S(t)\phi$ for any $\phi \in X$. The space $X \subset Z$ is often referred to as the domain of the generator $\mathcal{A}$, and is the space on which the generator is defined and need not be a closed subspace of Z. In this paper we will refer to X as the "state space."

For System (1), we define $Z_{m,n,K} := \{\mathbb{R}^m \times L_2^n[-\tau_1, 0] \times \cdots \times L_2^n[-\tau_K, 0]\}$ and for $\{x, \phi_1, \ldots, \phi_K\} \in Z_{m,n,K}$, we define the following shorthand notation:

$$\begin{bmatrix} x \\ \phi_i \end{bmatrix} := \{x, \phi_1, \ldots, \phi_K\}$$

which allows us to simplify expression of the inner product on $Z_{m,n,K}$, which we define to be

$$\left\langle \begin{bmatrix} y \\ \psi_i \end{bmatrix}, \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{m,n,K}} = \tau_K y^T x + \sum_{i=1}^{K}\int_{-\tau_i}^{0} \psi_i(s)^T\phi_i(s)ds.$$

When $m = n$, we simplify the notation using $Z_{n,K} := Z_{n,n,K}$. We may now conveniently write the state space for System (1) as follows:

$$X := \left\{ \begin{bmatrix} x \\ \phi_i \end{bmatrix} \in Z_{n,K} \ : \ \begin{matrix} \phi_i \in W_2^n[-\tau_i, 0] \text{ and} \\ \phi_i(0) = x \text{ for all } i \in [K] \end{matrix} \right\}.$$

Note that X is a subspace of $Z_{n,K}$, inherits the norm of $Z_{n,K}$, but is not closed in $Z_{n,K}$. We furthermore extend this notation to say

$$\begin{bmatrix} x \\ \phi_i \end{bmatrix}(s) = \begin{bmatrix} y \\ f(s,i) \end{bmatrix}$$

if $x = y$ and $\phi_i(s) = f(s,i)$ for $s \in [-\tau_i, 0]$ and $i \in [K]$. This also allows us to compactly represent the infinitesimal generator $\mathcal{A}$ of (1) as follows:

$$\mathcal{A}\begin{bmatrix} x \\ \phi_i \end{bmatrix}(s) := \begin{bmatrix} A_0 x + \sum_{i=1}^{K} A_i\phi_i(-\tau_i) \\ \dot\phi_i(s) \end{bmatrix}.$$

Using these definitions of $\mathcal{A}$, Z, and X, for matrix P and functions Q_i, S_i, and R_{ij}, we define an operator $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ of the "complete-quadratic" type as follows:

$$\left(\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}\begin{bmatrix} x \\ \phi_i \end{bmatrix}\right)(s) := \begin{bmatrix} Px + \sum_{i=1}^{K}\int_{-\tau_i}^{0} Q_i(s)\phi_i(s)ds \\ \tau_K Q_i(s)^T x + \tau_K S_i(s)\phi_i(s) + \sum_{j=1}^{K}\int_{-\tau_j}^{0} R_{ij}(s,\theta)\phi_j(\theta)\,d\theta \end{bmatrix}.$$

This notation will be used throughout this paper and allows us to associate P, Q_i, S_i, and R_{ij} with the corresponding complete-quadratic functional in (3) as follows:

$$V(\phi) = \left\langle \begin{bmatrix} \phi(0) \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}\begin{bmatrix} \phi(0) \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n,K}}.$$

That is, the Lyapunov functional is defined by the operator $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$, which is a variation of a classical combined multiplier and integral operator whose multipliers and kernel functions are defined by P, Q_i, S_i, R_{ij}.

The upper Dini derivative of the complete-quadratic functional can similarly be represented using complete quadratic operators as follows:

$$\begin{aligned}\dot V(\phi) &= \left\langle \begin{bmatrix} \phi(0) \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}\mathcal{A}\begin{bmatrix} \phi(0) \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n,K}} \\ &\quad + \left\langle \mathcal{A}\begin{bmatrix} \phi(0) \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}\begin{bmatrix} \phi(0) \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n,K}} \\ &= \left\langle \begin{bmatrix} \phi(0) \\ \vdots \\ \phi(-\tau_K) \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{D_1,V_i,\dot S_i,G_{ij}\}}\begin{bmatrix} \phi(0) \\ \vdots \\ \phi(-\tau_K) \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n(K+1),n,K}}\end{aligned}$$

where [20]

$$D_1 = \begin{bmatrix} \Delta_0 & \Delta_1 & \cdots & \Delta_K \\ \Delta_1^T & S_1(-\tau_1) & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ \Delta_K^T & 0 & 0 & S_K(-\tau_K) \end{bmatrix},$$

$$\Delta_0 = PA_0 + A_0^T P + \sum_{k=1}^K Q_k(0) + Q_k(0)^T + S_k(0),$$

$$\Delta_j = PA_j - Q_j(-\tau_j),$$

$$V_i(s) = \begin{bmatrix} \Pi_{0,i}(s)^T & \dots & \Pi_{K,i}(s)^T \end{bmatrix}^T,$$

$$\Pi_{0j}(s) = A_0^T Q_j(s) + \frac{1}{\tau_K}\sum_{k=1}^K R_{jk}^T(s,0) - \dot{Q}_j(s),$$

$$\Pi_{ij}(s) = A_i^T Q_j(s) - \frac{1}{\tau_K} R_{ji}^T(s,-\tau_i),$$

$$G_{ij}(s,\theta) = -\frac{\partial}{\partial s} R_{ij}(s,\theta) - \frac{\partial}{\partial \theta} R_{ij}(s,\theta).$$

In this section, we have reformulated $\mathcal{A}^*\mathcal{P} + \mathcal{P}\mathcal{A} < 0$ as negativity of a multiplier/integral operator on a lifted space. The classical Lyapunov–Krasovskii stability condition, then, states that System (1) is stable if there exists an $\epsilon > 0$, matrix P, and functions Q_i, S_i, and R_{ij} such that $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} \geq \epsilon \hat{I}_1$ and $\mathcal{P}_{\{D_1,V_i,\dot{S}_i,G_{ij}\}} \leq -\epsilon \hat{I}_2$ for suitably defined $\hat{I}_1 = \mathrm{diag}(I_n, 0, \ldots)$ and $\hat{I}_2 = \mathrm{diag}(I_n, 0, \ldots)$.

IV. A Dual Stability Condition for Infinite-Dimensional Systems

Using the notation we have introduced in the preceding section, we compactly represent the dual stability condition that forms the main theoretical contribution of this paper. Note that the results of this section apply to infinite-dimensional systems in general and are not specific to systems with delay.

Theorem 1: Suppose that $\mathcal{A}$ generates a strongly continuous semigroup on Hilbert space Z with domain X. Furthermore, suppose there exists an $\epsilon > 0$ and a bounded, coercive linear operator $\mathcal{P} : X \to X$ with $\mathcal{P}(X) = X$ and which is self-adjoint with respect to the Z inner product and satisfies

$$\langle \mathcal{A}\mathcal{P}z, z\rangle_Z + \langle z, \mathcal{A}\mathcal{P}z\rangle_Z \leq -\epsilon \|z\|_Z^2$$

for all $z \in X$. Then a dynamical system which satisfies $\dot{x}(t) = \mathcal{A}x(t)$ generates an exponentially stable semigroup.

Proof: Because $\mathcal{P}$ is coercive and bounded there exist $\gamma, \delta > 0$ such that $\langle x, \mathcal{P}x\rangle_Z \geq \gamma \|x\|_Z^2$ and $\|\mathcal{P}x\| \leq \delta \|x\|_Z$. By the Lax–Milgram theorem [21], $\mathcal{P}^{-1}$ exists and is bounded and $\mathcal{P}(X) = X$ implies $\mathcal{P}^{-1} : X \to X$. The inverse is self-adjoint since $\mathcal{P}$ is self-adjoint and hence $\left\langle \mathcal{P}^{-1}x, y\right\rangle_Z = \left\langle \mathcal{P}^{-1}x, \mathcal{P}\mathcal{P}^{-1}y\right\rangle_Z = \left\langle x, \mathcal{P}^{-1}y\right\rangle_Z$. Since $\sup_z \frac{\|\mathcal{P}z\|}{\|z\|} = \delta < \infty$, $\inf_y \frac{\|\mathcal{P}^{-1}y\|}{\|y\|} = \inf_x \frac{\|x\|}{\|\mathcal{P}x\|} = \frac{1}{\delta} > 0$ and hence $\left\langle y, \mathcal{P}^{-1}y\right\rangle_Z = \left\langle \mathcal{P}\mathcal{P}^{-1}y, \mathcal{P}^{-1}y\right\rangle_Z \geq \gamma \left\|\mathcal{P}^{-1}y\right\|_Z^2 \geq \frac{\gamma}{\delta^2}\|y\|_Z^2$. Hence, $\mathcal{P}^{-1}$ is coercive.

Define the Lyapunov functional $V(y) = \left\langle y, \mathcal{P}^{-1}y\right\rangle_Z \geq \frac{\gamma}{\delta^2}\|y\|_Z^2$, where positivity holds for any $y \in X$. If $y(t)$ satisfies $\dot{y}(t) = \mathcal{A}y(t)$, then V has time derivative

$$\begin{aligned} \frac{d}{dt}V(y(t)) &= \left\langle \dot{y}(t), \mathcal{P}^{-1}y(t)\right\rangle_Z + \left\langle y(t), \mathcal{P}^{-1}\dot{y}(t)\right\rangle_Z \\ &= \left\langle \mathcal{A}y(t), \mathcal{P}^{-1}y(t)\right\rangle_Z + \left\langle \mathcal{P}^{-1}y(t), \mathcal{A}y(t)\right\rangle_Z. \end{aligned}$$

Now, define $z(t) = \mathcal{P}^{-1}y(t) \in X$ for all $t \geq 0$. Then, $y(t) = \mathcal{P}z(t)$ and since $\mathcal{P}$ is bounded and $\mathcal{P}^{-1}$ is coercive

$$\begin{aligned} \dot{V}(y(t)) &= \left\langle \mathcal{A}y(t), \mathcal{P}^{-1}y(t)\right\rangle_Z + \left\langle \mathcal{P}^{-1}y(t), \mathcal{A}y(t)\right\rangle_Z \\ &= \langle \mathcal{A}\mathcal{P}z(t), z(t)\rangle_Z + \langle z(t), \mathcal{A}\mathcal{P}z(t)\rangle_Z \\ &\leq -\epsilon \|z(t)\|_Z^2 \leq -\frac{\epsilon}{\delta}\langle z(t), \mathcal{P}z(t)\rangle_Z \\ &= -\frac{\epsilon}{\delta}\left\langle y(t), \mathcal{P}^{-1}y(t)\right\rangle_Z \leq -\frac{\epsilon\gamma}{\delta^3}\|y(t)\|_Z^2. \end{aligned}$$

Negativity of the derivative of the Lyapunov function implies exponential stability in the square norm of the state by, e.g., [13] or by the invariance principle. ∎

The constraint $\mathcal{P}(X) = X$ ensures $\mathcal{P}^{-1} : X \to X$ and is satisfied if X is a closed subspace of Z or if X is itself a Hilbert space contained in Z and $\mathcal{P}$ is coercive on the space X with respect to the inner product in which X is closed. For the case of time-delay systems, X is not a closed subspace and we do not wish to constrain $\mathcal{P}$ to be coercive on X, since this space requires the Sobolev inner product in order to be closed. For these reasons, in Lemma 4, we will directly show that for our class of operators (to be defined) $\mathcal{P}(X) = X$.

In the following sections, we discuss how to parameterize operators which satisfy the conditions of Theorem 1, first in the case of multiple delays, and then for the special case of a single delay. We start with the constraints $\mathcal{P} = \mathcal{P}^*$ and $\mathcal{P} : X \to X$. Note that without additional restrictions on P, Q_i, S_i, R_{ij}, the operator $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ satisfies neither constraint.

Before moving to the next section, a natural question is whether the dual stability condition is significantly conservative. That is, does the stability of the system imply that the conditions of Theorem 1 are feasible. We refer to [14, Th. 5.1.3].

Theorem 2: Suppose that $\mathcal{A}$ is the infinitesimal generator of the C_0-semigroup $S(t)$ on the Hilbert space Z with domain $D(\mathcal{A})$. Then, $S(t)$ is exponentially stable if and only if there exists a positive, self-adjoint operator $\mathcal{P} \in \mathcal{L}(Z)$ such that

$$\langle \mathcal{P}\mathcal{A}z, z\rangle_Z + \langle z, \mathcal{P}\mathcal{A}z\rangle_Z = -\langle z, z\rangle_Z \quad \text{for all} \quad z \in D(\mathcal{A}).$$

Absent from the conditions of Theorem 2 is the restriction $\mathcal{P} : D(\mathcal{A}) \to D(\mathcal{A})$ and indeed the uniquely defined operator $\mathcal{P}$ from the proof of the theorem instead maps $D(\mathcal{A}) \to D(\mathcal{A}^*)$, with $D(\mathcal{A}^*)$ the domain defined by $\mathcal{A}^*$ and which has a structure significantly different than that of $D(\mathcal{A})$. Also absent from the conditions is the coercivity of $\mathcal{P}$. Several results show (e.g., [22, Th. 5.5]) that stability implies the existence of a coercive Lyapunov function (using a slightly weaker definition of coercivity). Finally, the image restriction $\mathcal{P}(X) = X$ is not satisfied by the operator in the proof of Theorem 2. However, if $\mathcal{P} : D(\mathcal{A}) \to D(\mathcal{A})$, in the following section we give conditions that guarantee $\mathcal{P}(X) = X$. In summary, however, we conclude that no definitive statement can be made regarding the necessity of Theorem 1.

V. Dual Conditions for Multiple-Delay Systems

In this section, we translate the results of Section IV into positivity and negativity of Lyapunov–Krasovskii-like functionals for systems with multiple delays. First, we give a class of operators $\mathcal{P}$, which satisfy the conditions of Theorem 1. Specifically, we give a parametrization of operators that are self-adjoint with respect to the Hilbert space $Z_{n,K}$, map $X \to X$ and satisfy $\mathcal{P}(X) = X$. Next, we show how the conditions of Theorem 1 can be applied to this class of operators to obtain stability conditions similar to the primal Lyapunov–Krasovskii conditions presented in Section II. Note that in Section VI, we will apply these results specifically to systems with a single delay and the exposition in that section is significantly reduced.

A. A Parametrization of Operators, P, Satisfying the Conditions of Theorem 1 on $Z_{n,K}$

In this section, we parameterize a class of operators which are self-adjoint and map $X \to X$, where

$$X := \left\{ \begin{bmatrix} x \\ \phi_i \end{bmatrix} \in Z_{n,K} \; : \; \begin{matrix} \phi_i \in W_2^n[-\tau_i, 0] \text{ and} \\ \phi_i(0) = x \text{ for all } i \in [K] \end{matrix} \right\}.$$

The following lemma gives constraints on the matrix P and functions Q_i, S_i, and R_{ij} for which $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ is self-adjoint and maps $X \to X$.

Lemma 3: Suppose that $S_i \in W_2^{n\times n}[-\tau_i, 0]$, $R_{ij} \in W_2^{n\times n}[[-\tau_i, 0] \times [-\tau_j, 0]]$ and $S_i(s) = S_i(s)^T$, $R_{ij}(s,\theta) = R_{ji}(\theta, s)^T$, $P = \tau_K Q_i(0)^T + \tau_K S_i(0)$, and $Q_j(s) = R_{ij}(0, s)$ for all $i, j \in [K]$. Then, $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ is a bounded linear operator, maps $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} : X \to X$, and is self-adjoint with respect to the inner product defined on $Z_{n,K}$.

Proof: To simplify the presentation, let $\mathcal{P} := \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$. We first establish that $\mathcal{P} : X \to X$. If

$$\begin{bmatrix} x \\ \phi_i \end{bmatrix} \in X,$$

then $\phi_i \in \mathcal{C}[-\tau_i, 0]$ and $\phi_i(0) = x$. Now, if

$$\begin{bmatrix} y \\ \psi_i(s) \end{bmatrix} = \left(\mathcal{P} \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right)(s),$$

then since $P = \tau_K Q_i(0)^T + \tau_K S_i(0)$ and $Q_j(s) = R_{ij}(0, s)$, we have that

$$\begin{aligned} \psi_i(0) = &\left(\tau_K Q_i(0)^T + \tau_K S_i(0)\right) x \\ &+ \sum_{j=1}^{K} \int_{-\tau_j}^{0} R_{ij}(0,\theta)\phi_j(\theta) d\theta \\ = &\, Px + \sum_{j=1}^{K} \int_{-\tau_j}^{0} Q_j(s)\phi_j(s) ds = y. \end{aligned}$$

Since $S_i \in W_2^{n\times n}[-\tau_i, 0]$ and $R_{ij} \in W_2^{n\times n}[[-\tau_i, 0] \times [-\tau_j, 0]]$, clearly $\psi_i \in W_2^n[-\tau_i, 0]$, and hence we have

$$\begin{bmatrix} y \\ \psi_i \end{bmatrix} \in X.$$

This proves that $\mathcal{P} : X \to X$. Furthermore, boundedness of the functions Q_i, S_i, and R_{ij} implies boundedness of the linear operator $\mathcal{P}$.

Now, to prove that $\mathcal{P}$ is self-adjoint with respect to the inner product $\langle \cdot, \cdot \rangle_{Z_{n,K}}$, we show $\langle y, \mathcal{P}x \rangle_{Z_{n,K}} = \langle \mathcal{P}y, x \rangle_{Z_{n,K}}$ for any $x, y \in Z_{n,K}$. Using the properties $S_i(s) = S_i(s)^T$ and $R_{ij}(s,\theta) = R_{ji}(\theta, s)^T$, we have the following:

$$\begin{aligned} &\left\langle \begin{bmatrix} y \\ \psi_i \end{bmatrix}, \mathcal{P} \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n,K}} \\ &= \tau_K y^T \left(Px + \sum_{i=1}^{K} \int_{-\tau_i}^{0} Q_i(\theta)\phi_i(\theta) d\theta \right) \\ &\quad + \sum_{i=1}^{K} \int_{-\tau_i}^{0} \psi_i(s)^T \Bigg(\tau_K Q_i(s)^T x + \tau_K S_i(s)\phi_i(s) \\ &\qquad\qquad + \sum_{j=1}^{K} \int_{-\tau_j}^{0} R_{ij}(s,\theta)\phi_j(\theta) d\theta \Bigg) \\ &= \tau_K \left(Py + \sum_{j=1}^{K} \int_{-\tau_j}^{0} Q_i(s)\psi_j(s) ds \right)^T x \\ &\quad + \sum_{i=1}^{K} \int_{-\tau_i}^{0} \Bigg(\tau_K Q_i(s)^T y + \tau_K S_i(s)^T \psi_i(s) \\ &\qquad\qquad + \sum_{j=1}^{K} \int_{-\tau_j}^{0} R_{ji}(\theta, s)^T \psi_j(\theta) d\theta \Bigg)^T \phi_i(s)\, ds \\ &= \left\langle \mathcal{P} \begin{bmatrix} y \\ \psi_i \end{bmatrix}, \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n,K}}. \end{aligned}$$

■

Finally, we show that for this class of operators, if $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ is coercive with respect to the L_2-norm, then $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}(X) = X$.

Lemma 4: Suppose that there exist P, Q_i, S_i, and R_{ij} which satisfy the conditions of Lemma 3. If $\langle x, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} x \rangle_{Z_{n,K}} \geq \epsilon \|x\|^2_{Z_{n,K}}$ for all $x \in X$ and some $\epsilon > 0$, then $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}(X) = X$.

Proof: By Lemma 3, $\mathcal{P}$ is self-adjoint and maps $X \to X$. Since $\mathcal{P}$ is coercive, bounded, and self-adjoint, $\mathcal{P}^{-1}$ is coercive, bounded, and self-adjoint. To show $\mathcal{P}(X) = X$, we need only show that $\mathbf{y} = \mathcal{P}\mathbf{x} \in X$ implies that $\mathbf{x} \in X$. First, we show that if

$$\mathbf{y} = \begin{bmatrix} y \\ \psi_i(\theta) \end{bmatrix} \in X,$$

then

$$\mathbf{x} = \begin{bmatrix} x \\ \phi_i(\theta) \end{bmatrix} = \mathcal{P}^{-1}\mathbf{y}$$

satisfies $x = \phi_i(0)$. We proceed by contradiction. Suppose $x - \phi_i(0) \neq 0$ for some i. Then, we have

$$y = P(\phi_i(0) + x - \phi_i(0)) + \sum_{i=1}^{K} \int_{-\tau_i}^{0} Q_i(s)\phi_i(s) ds.$$

Now, since $\mathbf{y} \in X$, $y = \psi_i(0)$, and hence,

$$y = P\phi_i(0) + \sum_{j=1}^{K} \int_{-\tau_j}^{0} R_{ij}(0,\theta)\phi_j(\theta)\, d\theta,$$

which implies $P(x - \phi_i(0)) = 0$. Now, $\langle \mathbf{x}, \mathcal{P}\mathbf{x}\rangle_{Z_{n,K}} \geq \epsilon \|\mathbf{x}\|^2_{Z_{n,K}}$ implies $P \geq \epsilon I$. Hence, $x - \phi(0) \neq 0$ implies $P(x - \phi(0)) \neq 0$, which is a contradiction. We conclude that $x = \phi_i(0)$. Next, we establish $\phi_i \in W_2^n$ for any i by showing $\left\|\dot{\phi}_i\right\|_{L_2} < \infty$. For this, we differentiate ψ_i to obtain

$$\dot{\psi}_i(s) = \tau_K \dot{Q}_i(s)^T x + \tau_K \dot{S}_i(s)\phi_i(s) + \tau_K S_i(s)\dot{\phi}_i(s) + \sum_{j=1}^{K} \int_{-\tau_j}^{0} \partial_s R_{ij}(s,\theta)\phi_j(\theta)\, d\theta,$$

which we reverse to obtain

$$\tau_K S_i(s)\dot{\phi}_i(s) = \dot{\psi}_i(s) - \tau_K \dot{Q}_i(s)^T x - \tau_K \dot{S}_i(s)\phi_i(s) - \sum_{j=1}^{K} \int_{-\tau_j}^{0} \partial_s R_{ij}(s,\theta)\phi_j(\theta)\, d\theta,$$

which is L_2 bounded since $\dot{\psi}_i, \phi_i, \dot{Q}_i \in L_2^n$, and $\dot{S}_i$ and $\partial_s R_{i,j}$ are continuous and thus bounded on $[-\tau_i, 0]$. Now, for $x = 0$ and $\phi_j = 0$ for $j \neq i$, the constraint $\langle \mathbf{x}, \mathcal{P}\mathbf{x}\rangle_{Z_{n,K}} \geq \epsilon \|\mathbf{x}\|^2_{Z_{n,K}}$, implies that the operator on this subspace,

$$\tau_K S_i(s)\phi_i(s) + \int_{-\tau_i}^{0} R_{ii}(s,\theta)\phi_i(\theta)\, d\theta$$

is also coercive. Thus, since integral operators cannot be coercive for L_2-bounded kernels R_{ii}, we have that $S_i(s) \geq \eta I$ for some $\eta > 0$. Therefore, for each i, we conclude $\left\|\dot{\phi}_i\right\|_{L_2} \leq \frac{1}{\eta}\left\|S_i(s)\dot{\phi}_i(s)\right\|_{L_2} < \infty$. Hence, $\mathbf{x} \in X$. We conclude that $\mathcal{P}(X) = X$. ∎

B. Duality Conditions for Multiple Delays

For the multiple-delay case, we apply the operator $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$, with P, Q_i, S_i, and R_{ij} satisfying the conditions of Lemma 4 to the dual stability condition in Theorem 1 and eliminate differential operators from the result. This section provides additional justification for the unique choice of state-space X and Hilbert space $Z_{m,n,K}$ used in this paper. Specifically, the elimination of differential operators and reformulation as negativity of a multiplier/integral operator on $Z_{n(K+1),n,K}$ would not be possible using the more classical state and inner product spaces, which allow for discontinuities in the state.

Theorem 5: Suppose that there exist P, Q_i, S_i, and R_{ij} satisfy the conditions of Lemma 3. If $\langle x, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}x\rangle_{Z_{n,K}} \geq \epsilon \|x\|^2$ for all $x \in Z_{n,K}$ and

$$\left\langle \begin{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{D_1,V_i,\dot{S}_i,G_{ij}\}} \begin{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n(K+1),n,K}} \leq -\epsilon \left\| \begin{bmatrix} y_1 \\ \phi_i \end{bmatrix} \right\|^2_{Z_{n,K}}$$

for all $y_1 \in \mathbb{R}^n$ and

$$\begin{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \phi_i \end{bmatrix} \in Z_{n(K+1),n,K},$$

where

$$D_1 := \begin{bmatrix} C_0 + C_0^T & C_1 & \cdots & C_k \\ C_1^T & -S_1(-\tau_1) & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ C_{k^T} & 0 & 0 & -S_k(-\tau_K) \end{bmatrix}$$

$$C_0 := A_0 P + \sum_{i=1}^{K} \left(\tau_K A_i Q_i(-\tau_i)^T + \frac{1}{2} S_i(0) \right)$$

$$C_i := \tau_K A_i S_i(-\tau_i), \quad i \in [K]$$

$$V_i(s) := \begin{bmatrix} B_i(s)^T & 0 & \cdots & 0 \end{bmatrix}^T, \quad i \in [K]$$

$$B_i(s) := A_0 Q_i(s) + \dot{Q}_i(s) + \sum_{j=1}^{K} A_j R_{ji}(-\tau_j, s), \ \ i \in [K]$$

$$G_{ij}(s,\theta) := \frac{\partial}{\partial s} R_{ij}(s,\theta) + \frac{\partial}{\partial \theta} R_{ji}(s,\theta)^T, \quad i,j \in [K]$$

then the system defined by (1) is exponentially stable.

Proof: Define the operators $\mathcal{A}$ and $\mathcal{P} = \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ as aforementioned. By Lemma 3, $\mathcal{P}$ is self-adjoint and maps $X \to X$. Since $\mathcal{P}$ is coercive by assumption, this implies by Theorem 1 and Lemma 4 that the system is exponentially stable if

$$\left\langle \mathcal{A}\mathcal{P} \begin{bmatrix} x \\ \phi_i \end{bmatrix}, \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n,K}} + \left\langle \begin{bmatrix} x \\ \phi_i \end{bmatrix}, \mathcal{A}\mathcal{P} \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n,K}} \leq -\epsilon \left\| \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\|^2_{Z_{n,K}}$$

for all

$$\begin{bmatrix} x \\ \phi_i \end{bmatrix} \in X.$$

We begin by constructing

$$\begin{bmatrix} y \\ \psi_i(s) \end{bmatrix} := \mathcal{A}\mathcal{P} \begin{bmatrix} x \\ \phi_i \end{bmatrix},$$

where

$$y = A_0 P x + \sum_{i=1}^{K} \int_{-\tau_i}^{0} A_0 Q_i(s)\phi_i(s) ds + \sum_{i=1}^{K} A_i \left(\tau_K Q_i(-\tau_i)^T x + \tau_K S_i(-\tau_i)\phi_i(-\tau_i) + \sum_{j=1}^{K} \int_{-\tau_j}^{0} R_{ij}(-\tau_i,\theta)\phi_j(\theta) d\theta \right)$$

$$\psi_i(s) = \tau_K \dot{Q}_i(s)^T x + \tau_K \dot{S}_i(s)\phi_i(s) + \tau_K S_i(s)\dot{\phi}_i(s) + \sum_{j=1}^{K} \int_{-\tau_j}^{0} \frac{d}{ds} R_{ij}(s,\theta)\phi_j(\theta) d\theta.$$

Now, divide the expression into terms as follows:

$$\left\langle \begin{bmatrix} x \\ \phi_i \end{bmatrix}, \mathcal{A}\mathcal{P}\begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n,K}} := \tau_K x^T y + \sum_{i=1}^{K}\int_{-\tau_i}^{0} \phi_i(s)^T \psi_i(s) ds.$$

Examining the first term and using $x = \phi_i(0)$, we have

$$\begin{aligned} x^T y = x^T A_0 P x &+ \sum_{i=1}^{K}\int_{-\tau_i}^{0} x^T A_0 Q_i(s)\phi_i(s) ds \\ &+ \sum_{i=1}^{K} \tau_K x^T A_i Q_i(-\tau_i)^T x \\ &+ \sum_{i=1}^{K} \tau_K x^T A_i S_i(-\tau_i)\phi_i(-\tau_i) \\ &+ \sum_{i=1}^{K}\int_{-\tau_i}^{0}\sum_{j=1}^{K} x^T A_j R_{ji}(-\tau_j,\theta)\phi_i(\theta) d\theta. \end{aligned}$$

Next, we examine the second term and use integration by parts to eliminate $\dot{\phi}$:

$$\begin{aligned} \sum_{i=1}^{K}\int_{-\tau_i}^{0}\phi_i(s)^T\psi_i(s)ds &= \sum_{i=1}^{K}\tau_K\int_{-\tau_i}^{0}\phi_i(s)^T\dot{Q}_i(s)^T x\, ds \\ &+ \sum_{i=1}^{K}\tau_K\int_{-\tau_i}^{0}\phi_i(s)^T\dot{S}_i(s)\phi_i(s)ds \\ &+ \sum_{i=1}^{K}\tau_K\int_{-\tau_i}^{0}\phi_i(s)^T S_i(s)\dot{\phi}_i(s)ds \\ &+ \sum_{i,j}\int_{-\tau_i}^{0}\int_{-\tau_j}^{0}\phi_i(s)^T\frac{\partial}{\partial s}R_{ij}(s,\theta)\phi_j(\theta)\, ds\, d\theta \\ = \sum_{i=1}^{K}\tau_K\int_{-\tau_i}^{0}\phi_i(s)^T\dot{Q}_i(s)^T x\, ds & \\ &+ \frac{\tau_K}{2}\sum_{i=1}^{K}\int_{-\tau_i}^{0}\phi_i(s)^T\dot{S}_i(s)\phi_i(s)ds \\ &+ \frac{\tau_K}{2}x^T\sum_{i=1}^{K}S_i(0)x \\ &- \frac{\tau_K}{2}\sum_{i=1}^{K}\phi_i(-\tau_i)^T S_i(-\tau_i)\phi_i(-\tau_i) \\ &+ \sum_{i,j}\int_{-\tau_i}^{0}\int_{-\tau_j}^{0}\phi_i(s)^T\frac{\partial}{\partial s}R_{ij}(s,\theta)\phi_j(\theta)\, ds\, d\theta. \end{aligned}$$

Combining both terms, we obtain

$$\begin{aligned} \left\langle \begin{bmatrix} x \\ \phi_i \end{bmatrix}, \mathcal{A}\mathcal{P}\begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n,K}} &= \tau_K x^T y + \sum_{i=1}^{K}\int_{-\tau_i}^{0}\phi_i(s)^T\psi_i(s)ds \\ &= x^T\left(\tau_K A_0 P + \sum_{i=1}^{K}\tau_K^2 A_i Q_i(-\tau_i)^T + \frac{\tau_K}{2}\sum_{i=1}^{K}S_i(0)\right)x \\ &+ \tau_K^2\sum_{i=1}^{K}x^T A_i S_i(-\tau_i)\phi_i(-\tau_i) \\ &- \frac{\tau_K}{2}\sum_{i=1}^{K}\phi_i(-\tau_i)^T S_i(-\tau_i)\phi_i(-\tau_i) \\ &+ \tau_K\sum_{i=1}^{K}\int_{-\tau_i}^{0}x^T\Bigg(A_0 Q_i(s) + \dot{Q}_i(s) \\ &\qquad + \sum_{j=1}^{K}A_j R_{ji}(-\tau_j,s)\Bigg)\phi_i(s)ds \\ &+ \frac{\tau_K}{2}\sum_{i=1}^{K}\int_{-\tau_i}^{0}\phi_i(s)^T\dot{S}_i(s)\phi_i(s)ds \\ &+ \sum_{i,j}\int_{-\tau_i}^{0}\int_{-\tau_j}^{0}\phi_i(s)^T\frac{\partial}{\partial s}R_{ij}(s,\theta)\phi_j(\theta)\, ds\, d\theta. \end{aligned}$$

Combining the expression with its adjoint, we recover

$$\begin{aligned} &\left\langle \mathcal{A}\mathcal{P}\begin{bmatrix} x \\ \phi_i \end{bmatrix}, \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n,K}} + \left\langle \begin{bmatrix} x \\ \phi_i \end{bmatrix}, \mathcal{A}\mathcal{P}\begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n,K}} \\ &= \left\langle \begin{bmatrix} \begin{bmatrix} x \\ \phi_1(-\tau_1) \\ \vdots \\ \phi_k(-\tau_K) \end{bmatrix} \\ \phi_i \end{bmatrix}, \mathcal{D}\begin{bmatrix} \begin{bmatrix} x \\ \phi_1(-\tau_1) \\ \vdots \\ \phi_k(-\tau_K) \end{bmatrix} \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n(K+1),n,K}} \\ &\le -\epsilon\left\| \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\|_{Z_{n,K}}^2, \end{aligned}$$

where $\mathcal{D} := \mathcal{P}_{\{D_1,V_i,\dot{S}_i,G_{ij}\}}$. We conclude that all conditions of Theorem 1 are satisfied and hence System (1) is stable. ■

Theorem 5 provides stability conditions expressed as the positivity of $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ and negativity of the multiplier/integral operator $\mathcal{D} = \mathcal{P}_{\{D_1,V_i,\dot{S}_i,G_{ij}\}}$. Note that positivity is defined with respect to the inner product $Z_{m,n,K}$. In Section VII, we will show how to reformulate positivity on $Z_{m,n,K}$ as an equivalent positivity condition on the space $Z_{m,nK,1}$. Positive operators on $Z_{m,nK,1}$ are then parameterized using LMIs, as also described in Section VII. Before moving to the next section, we note that the derivative operator $\mathcal{D} = \mathcal{P}_{\{D_1,V_i,\dot{S}_i,G_{ij}\}}$ is sparse in the sense that no terms of the form $\phi(-\tau_i)^T\phi_j(-\tau_j)$ for $i \neq j$ or $\phi_i(-\tau_i)^T\phi_i(s)$ for any i appear in $\left\langle \phi, \mathcal{P}_{\{D_1,V_i,\dot{S}_i,G_{ij}\}}\phi \right\rangle$. This is extraordinary, as all such terms do appear in the similar formulation of the primal stability conditions (i.e., $\left\langle \phi, \mathcal{P}_{\{D_1,V_i,\dot{S}_i,G_{ij}\}}\phi \right\rangle$ from Section III). To emphasize this difference, we fully expand both versions of the form $\left\langle \phi, \mathcal{P}_{\{D_1,V_i,\dot{S}_i,G_{ij}\}}\phi \right\rangle$ to obtain the following.

1) Dual Lyapunov–Krasovskii Form: Theorem 5 implies that system (1) is stable if there exists a

$$\begin{aligned}V(\phi) &= \tau_K \phi(0)^T P\phi(0) + \tau_K \sum_{i=1}^{K} \int_{-\tau_i}^{0} \phi(0)^T Q_i(s)\phi(s)ds \\ &\quad + \tau_K \sum_{i=1}^{K} \int_{-\tau_i}^{0} \phi(s)^T Q_i(s)^T \phi(0)ds \\ &\quad + \tau_K \sum_{i=1}^{K} \int_{-\tau_i}^{0} \phi(s)^T S_i(s)\phi(s)ds \\ &\quad + \sum_{i,j=1}^{K} \int_{-\tau_i}^{0} \int_{-\tau_j}^{0} \phi(s)^T R_{ij}(s,\theta)\phi(\theta)d\theta,\end{aligned}$$

such that

$$V(\phi) \geq \epsilon \left\| \begin{bmatrix} \phi(0) \\ \phi_i \end{bmatrix} \right\|_{Z_{n,K}}^2$$

and

$$\begin{aligned}V_D(\phi) &= \tau_K \phi(0)^T (C_0 + C_0^T)\phi(0) + 2\tau_K \sum_{i=1}^{K} \phi(0)^T C_i \phi_i(-\tau_i) \\ &\quad - \tau_K \sum_{i=1}^{K} \phi_i(-\tau_i)^T S_i(-\tau_i)\phi_i(-\tau_i) \\ &\quad + 2\tau_K \sum_{i=1}^{K} \int_{-\tau_i}^{0} \phi(0)^T B_i(s)\phi_i(s)ds \\ &\quad + \tau_K \sum_{i=1}^{K} \int_{-\tau_i}^{0} \phi_i(s)^T \dot{S}_i(s)\phi_i(s)ds \\ &\quad + \sum_{i,j=1}^{K} \int_{-\tau_i}^{0} \int_{-\tau_j}^{0} \phi_i(s)^T G_{ij}(s,\theta)\phi_i(\theta)\, ds\, d\theta \\ &\quad \leq -\epsilon \left\| \begin{bmatrix} \phi(0) \\ \phi_i \end{bmatrix} \right\|_{Z_{n,K}}^2 .\end{aligned}$$

2) Primal Lyapunov–Krasovskii Form: Now, compare with the associated primal classical Lyapunov–Krasovskii derivative condition [20] from Section III, which states that system (1) is stable if there exists a

$$\begin{aligned}V(\phi) &= \phi(0)^T P\phi(0) + \sum_{i=1}^{K} \int_{-\tau_i}^{0} \phi(0)^T Q_i(s)\phi(s)ds \\ &\quad + \sum_{i=1}^{K} \int_{-\tau_i}^{0} \phi(s)^T Q_i(s)^T \phi(0)ds \\ &\quad + \sum_{i=1}^{K} \int_{-\tau_i}^{0} \phi(s)^T S_i(s)\phi(s)ds \\ &\quad + \sum_{i,j=1}^{K} \int_{-\tau_i}^{0} \int_{-\tau_j}^{0} \phi(s)^T R_{ij}(s,\theta)\phi(\theta)d\theta\end{aligned}$$

such that $V(\phi) \geq \epsilon \|\phi(0)\|^2$ and

$$\begin{aligned}\dot{V}(\phi) &= \phi(0)^T \Delta_0 \phi(0) + \sum_{i=1}^{K} \phi_i(-\tau_i)^T S_i(-\tau_i)\phi_i(-\tau_i) \\ &\quad + 2\sum_{i=1}^{K} \phi(0)^T \Delta_i \phi_i(-\tau_i) \\ &\quad + 2\sum_{i=1}^{K} \int_{-\tau_i}^{0} \phi(0)^T \Pi_{0i}(s)\phi_i(s)ds \\ &\quad + \sum_{i=1}^{K} \int_{-\tau_i}^{0} \phi_i(s)^T \dot{S}_i(s)\phi_i(s)ds \\ &\quad + 2\sum_{i,j=1}^{K} \int_{-\tau_i}^{0} \phi_i(-\tau_i)^T \Pi_{ij}(s)\phi_j(s)ds \\ &\quad - \sum_{i,j=1}^{K} \int_{-\tau_i}^{0} \int_{-\tau_j}^{0} \phi_i(s)^T G_{ij}(s,\theta)\phi_i(\theta)\, ds\, d\theta \\ &\quad \leq -\epsilon \|\phi(0)\|^2 .\end{aligned}$$

From this comparison, we see that the structure of the dual stability condition is very similar to the structure of the primal except for the fifth line of the derivative, which is absent from the dual. Roughly speaking, it is as if all the Π_{ij} terms in the primal form have been combined in Π_{0i}. This sparsity pattern yields a multiplier of the form

$$\begin{bmatrix} \cdot & \cdots \\ \vdots & \ddots \end{bmatrix}$$

consisting of a single row, single column, and diagonal. For an example of how to exploit such sparsity, the positivity of such a multiplier would be equivalent to positivity of the diagonal and positivity of the scalar $[\cdot] - \cdots \begin{bmatrix} \ddots \end{bmatrix}^{-1} \vdots$.

VI. Duality Conditions for Single-Delay Systems

In this section, we simplify the results of Section VIII-A for systems with a single delay. We find that in the case of single delay the parametrization of the operator $\mathcal{P}$ is direct (it does not rely on equality constraints to enforce the mapping conditions of Theorem 1), which allows us to arrive at the explicit forms described in Section I-A.

A. Parametrization of Operators, $\mathcal{P}$, Satisfying the Conditions of Theorem 1 on $Z_{n,1}$

First, we consider a class of operators that are self-adjoint with respect to Z and map $X \to X$. This is simplified in the case of a single-delay case partially due to the fact that $Z = Z_{n,1} = \mathbb{R}^n \times L_2^n$ equipped with the L_2^{2n} inner product and subspace $X := \{\{x, \phi\} \in \mathbb{R}^n \times W_2^n[-\tau, 0] : \phi(0) = x\}$. Specifically, given functions $S, R \in W_2^{n\times n}[-\tau, 0]$, in this section, we will define $\mathcal{P}$ as follows:

$$\left(\mathcal{P}\begin{bmatrix} x \\ \phi \end{bmatrix}\right)(s) := \begin{bmatrix} \tau(R(0,0)+S(0))x + \int_{-\tau}^{0} R(0,s)\phi(s)ds \\ \tau R(s,0)\phi(0) + \tau S(s)\phi(s) + \int_{-\tau}^{0} R(s,\theta)\phi(\theta)d\theta \end{bmatrix}. \tag{4}$$

Clearly, we have that $\mathcal{P}$ is a bounded linear operator and since S, R are continuous, it is trivial to show that $\mathcal{P} : X \to X$. Furthermore, $\mathcal{P}$ is self-adjoint with respect to the L_2^{2n} inner product, as indicated in the following lemma.

Lemma 6: Suppose $S \in W_2^{n\times n}[-\tau, 0]$, $R \in W_2^{n\times n}[[-\tau, 0] \times [-\tau, 0]]$, $R(s,\theta) = R(\theta, s)^T$, and $S(s) \in \mathbb{S}^n$. Then, the operator $\mathcal{P}$, as defined in (5), is self-adjoint with respect to the L_2^{2n} inner product. Furthermore, if there exists $\epsilon > 0$ such that $\langle x, \mathcal{P}x\rangle_{L_2^{2n}} \geq \epsilon \|x\|^2$ for all $x \in X$, then $\mathcal{P}(X) = X$.

Proof: The proof is a direct application of Lemma 3. First, we note that $\mathcal{P} = \mathcal{P}_{\{P,Q,S,R\}}$ where $P = \tau(R(0,0) + S(0))$ and $Q(s) = R(0, s)$. Noting that $P = \tau(R(0,0) + S(0)) = \tau Q(0)^T + \tau S(0)$, we see that $\mathcal{P}_{\{P,Q,S,R\}}$ satisfies the conditions of Lemma 3. ∎

Note that the constraints $\mathcal{P} : X \to X$ and $\mathcal{P} = \mathcal{P}^*$ significantly reduce the number of free variables. In the single-delay case, we make this explicit by replacing P and Q with $P = \tau(R(0,0) + S(0))$ and $Q(s) = R(0, s)$.

Having introduced a parametrization of $\mathcal{P}$ and established properties of this operator, we now apply this structured operator to Theorem 1 to obtain Lyapunov-like conditions on S and R for which stability holds.

B. Dual Stability Conditions: Single Delay

In this section, we specialize the results of Theorem 5 to single-delay systems. First, recall that the dynamics of the single-delay system are represented by the infinitesimal generator $\mathcal{A}$ defined as follows:

$$\left(\mathcal{A}\begin{bmatrix} x \\ \phi \end{bmatrix}\right)(s) = \begin{bmatrix} A_0 x + A_1\phi(-\tau) \\ \frac{d}{ds}\phi(s) \end{bmatrix}.$$

Then, we have the following.

Corollary 7: Suppose S and R satisfy the conditions of and Lemma 6 and there exists $\epsilon > 0$ such that

$$\langle x, \mathcal{P}_{\{P,Q,S,R\}}x\rangle_{L_2^{2n}} \geq \epsilon \|x\|^2_{L_2^{2n}}$$

for all $x \in \mathbb{R}^n \times L_2^n[-\tau, 0]$ where $P = \tau(R(0,0) + S(0))$ and $Q(s) = R(0, s)$. Furthermore, suppose

$$\left\langle \begin{bmatrix} x \\ y \\ \phi \end{bmatrix}, \mathcal{D}\begin{bmatrix} x \\ y \\ \phi \end{bmatrix}\right\rangle_{L_2^{3n}} \leq -\epsilon \left\| \begin{bmatrix} x \\ \phi \end{bmatrix}\right\|^2_{L_2^{2n}}$$

for all

$$\begin{bmatrix} x \\ y \\ \phi \end{bmatrix} \in \mathbb{R}^n \times \mathbb{R}^n \times L_2^n[-\tau, 0]$$

where $\mathcal{D} = \mathcal{P}_{\{D_1, V, \dot{S}, G\}}$ and

$$D_1 := \begin{bmatrix} C_0 + C_0^T & C_1 \\ C_1^T & -S(-\tau) \end{bmatrix}, \quad V(s) = \begin{bmatrix} B(s) \\ 0 \end{bmatrix}$$

$$C_0 := \tau A_0(R(0,0) + S(0)) + \tau A_1 R(-\tau, 0) + \frac{1}{2}S(0)$$

$$C_1 := \tau A_1 S(-\tau)$$

$$B(s) := A_0 R(0, s) + A_1 R(-\tau, s) + \dot{R}(s, 0)^T$$

$$G(s,\theta) := \frac{d}{ds}R(s,\theta) + \frac{d}{d\theta}R(s,\theta).$$

Then, the system defined by (1) in the case $K = 1$ with $\tau_1 = \tau$ is exponentially stable.

Proof: The proof is a direct application of Lemma 6 and Theorem 5. ∎

Note that expanding the term

$$\left\langle \begin{bmatrix} \phi(0) \\ \phi(-\tau) \\ \phi \end{bmatrix}, \mathcal{D}\begin{bmatrix} \phi(0) \\ \phi(-\tau) \\ \phi \end{bmatrix}\right\rangle_{L_2^{3n}}$$

from Corollary 7 yields the new dual stability conditions previously described in Section I-A.

VII. Using LMIs to Solve Linear Operator Inequalities (LOIs) on $Z_{m,n,K}$

In previous sections, we have formulated dual stability conditions, with decision variables parameterized by the matrix P and functions Q_i, S_i, and R_{ij}. The dual stability conditions were reformulated as the positivity of

$$\langle x, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}x\rangle_{Z_{n,K}} \geq \epsilon \|x\|^2_{Z_{n,K}}$$

for all $x \in Z_{n,K}$ and the negativity of

$$\left\langle \begin{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{D_1,V_i,\dot{S}_i,G_{ij}\}}\begin{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \phi_i \end{bmatrix}\right\rangle_{Z_{n(K+1),n,K}} \leq -\epsilon \left\| \begin{bmatrix} y_1 \\ \phi_i \end{bmatrix}\right\|^2_{Z_{n,K}}$$

for $y_1 \in \mathbb{R}^n$ and

$$\begin{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \phi_i \end{bmatrix} \in Z_{n(K+1,n,K)}$$

where $D_1, V_i, \dot{S}_i$, and G_{ij} are as defined in Theorem 5. Operator feasibility conditions of this form are termed linear operator inequalities and, in this section, we will show how LMIs can be used to solve LOIs under the presumption that the functions Q_i, S_i, and R_{ij} are polynomial (which implies $D_1, V_i, \dot{S}_i$, and G_{ij} are polynomial). Specifically, the variables in this case become the coefficients of the polynomials Q_i, S_i, and R_{ij} and the goal of the section is to find LMI constraints on P and these polynomial coefficients, which ensure that

$$\langle x, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}x\rangle_{Z_{m,n,K}} \geq 0.$$

Our approach to solving LOIs on $Z_{m,n,K}$ is to construct an equivalent feasibility condition using operators on $Z_{m,nK,1} = \mathbb{R}^m \times L_2^{nK}[-\tau_K, 0]$. This is accomplished in two steps. First, in Section VII-A, we construct polynomials $\hat{Q}$, $\hat{S}$, and $\hat{R}$

such that $\mathcal{P}_{\{P,\hat{Q},\hat{S},\hat{R}\}}$ is coercive on $Z_{m,nK,1}$ if and only if $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ is coercive on $Z_{m,n,K}$. Second, in Section VII-B, we impose LMI constraints on P and the coefficients of these polynomials $\hat{Q}$, $\hat{S}$, and $\hat{R}$, constraints which are denoted $\{P,\hat{Q},\hat{S},\hat{R}\} \in \Xi_{d,m,nK}$ and which ensure that $\mathcal{P}_{\{P,\hat{Q},\hat{S},\hat{R}\}}$ is coercive on $Z_{m,nK,1}$.

Both steps are combined into a single summarizing statement in Corollary 10.

A. Equivalence Between $Z_{m,n,K}$ and $Z_{m,nK,1}$

In this section, we address the positivity of $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ on $Z_{m,n,K}$ by constructing a linear map from the matrix P and coefficients of Q_i, S_i, and R_{ij} to the coefficients of new polynomial variables $\hat{Q}$, $\hat{S}$, and $\hat{R}$, where the coercivity of $\mathcal{P}_{\{P,\hat{Q},\hat{S},\hat{R}\}}$ on $Z_{m,nK,1}$ is equivalent to the coercivity of $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ on $Z_{m,n,K}$.

Given matrix P and polynomials Q_i, S_i, and R_{ij}, define the linear map $\mathcal{L}_1$ by

$$\{\hat{P},\hat{Q},\hat{S},\hat{R}\} := \mathcal{L}_1(P,Q_i,S_i,R_{ij}) \tag{5}$$

if $a_i = \frac{\tau_i}{\tau_K}$, $\hat{P} = P$ and

$$\hat{Q}(s) := \left[\sqrt{a_1}Q_1(a_1 s) \quad \cdots \quad \sqrt{a_K}Q_K(a_K s)\right]$$

$$\hat{S}(s) := \begin{bmatrix} S_1(a_1 s) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & S_K(a_K s) \end{bmatrix}$$

$$\hat{R}(s,\theta) :=$$
$$\begin{bmatrix} \sqrt{a_1 a_1}R_{11}(sa_1,\theta a_1) & \cdots & \sqrt{a_1 a_K}R_{1K}(sa_1,\theta a_K) \\ \vdots & \cdots & \vdots \\ \sqrt{a_K a_1}R_{K1}(sa_K,\theta a_1) & \cdots & \sqrt{a_K a_K}R_{KK}(sa_K,\theta a_K) \end{bmatrix}.$$

Then, we have the following result.

Lemma 8: Let $\{\hat{P},\hat{Q},\hat{S},\hat{R}\} := \mathcal{L}_1(P,Q_i,S_i,R_{ij})$. Then

$$\left\langle \begin{bmatrix} x \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{m,n,K}} \geq \alpha \left\| \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\|_{Z_{m,n,K}}$$

for all $\begin{bmatrix} x \\ \phi_i \end{bmatrix} \in Z_{m,n,K}$ if and only if

$$\left\langle \begin{bmatrix} x \\ \hat{\phi} \end{bmatrix}, \mathcal{P}_{\{\hat{P},\hat{Q},\hat{R},\hat{S}\}} \begin{bmatrix} x \\ \hat{\phi} \end{bmatrix} \right\rangle_{Z_{m,nK,1}} \geq \alpha \left\| \begin{bmatrix} \hat{x} \\ \hat{\phi} \end{bmatrix} \right\|_{Z_{m,nK,1}}$$

for all $\begin{bmatrix} \hat{x} \\ \hat{\phi} \end{bmatrix} \in Z_{m,nK,1}$.

Proof: The proof is straightforward. For necessity, let

$$\hat{\phi} = \begin{bmatrix} \sqrt{a_i}\phi_1(sa_1) \\ \vdots \\ \sqrt{a_K}\phi_K(sa_K) \end{bmatrix}.$$

Then,

$$\begin{bmatrix} x \\ \hat{\phi} \end{bmatrix} \in Z_{m,nK,1}$$

and define the change of variables $s_i' = \frac{\tau_K}{\tau_i} s_i = \frac{1}{a_i} s_i$. Then, $s_i = \frac{\tau_i}{\tau_K} s_i' = a_i s_i'$ and $ds_i = a_i ds_i'$ and

$$\begin{aligned}
\left\| \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\|_{Z_{m,n,K}} &= \tau_K x^T x + \sum_{i=1}^K \int_{-\tau_i}^0 \|\phi_i(s_i)\|^2 ds_i \\
&= \tau_K x^T x + \sum_{i=1}^K \int_{-\tau_K}^0 \|\sqrt{a_i}\phi_i(s_i' a_i)\|^2 ds_i' \\
&= \tau_K x^T x + \int_{-\tau_K}^0 \left\|\hat{\phi}(s)\right\|^2 ds = \left\| \begin{bmatrix} x \\ \hat{\phi} \end{bmatrix} \right\|_{Z_{m,nK,1}}.
\end{aligned}$$

Now, using a similar change of integration variables, we have the following:

$$\begin{aligned}
&\left\langle \begin{bmatrix} x \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{m,n,K}} \\
&= \tau_K x^T P x + 2\tau_K \int_{-\tau_K}^0 \sum_{i=1}^K x^T \sqrt{a_i} Q_i(sa_i)\hat{\phi}_i(s) ds \\
&\quad + \tau_K \int_{-\tau_K}^0 \sum_{i=1}^K \hat{\phi}_i(s)^T S_i(sa_i)\hat{\phi}_i(s) ds \\
&\quad + \int_{-\tau_K}^0 \int_{-\tau_K}^0 \sum_{i,j=1}^K \hat{\phi}_i(s)^T \sqrt{a_i a_j} R_{ij}(sa_i,\theta a_j)\hat{\phi}_j(\theta) d\theta ds \\
&= \int_{-\tau_K}^0 \begin{bmatrix} x \\ \hat{\phi}(s) \end{bmatrix}^T \begin{bmatrix} P & \tau_K \hat{Q}(s) \\ \tau_K \hat{Q}(s)^T & \tau_K \hat{S}(s) \end{bmatrix} \begin{bmatrix} x \\ \hat{\phi}(s) \end{bmatrix} ds \\
&\quad + \int_{-\tau_K}^0 \int_{-\tau_K}^0 \hat{\phi}(s)^T \hat{R}(s,\theta) \hat{\phi}(\theta) d\theta ds \\
&= \left\langle \begin{bmatrix} x \\ \hat{\phi} \end{bmatrix}, \mathcal{P}_{\{P,\hat{Q},\hat{R},\hat{S}\}} \begin{bmatrix} x \\ \hat{\phi} \end{bmatrix} \right\rangle_{Z_{m,nK,1}} \\
&\geq \alpha \left\| \begin{bmatrix} x \\ \hat{\phi} \end{bmatrix} \right\|_{Z_{m,nK,1}} = \alpha \left\| \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\|_{Z_{m,n,K}}.
\end{aligned}$$

For the sufficiency, we reverse the steps using

$$\phi_i(s) = \frac{1}{\sqrt{a_i}} \hat{\phi}_i\left(\frac{s}{a_i}\right). \qquad \blacksquare$$

Note that if Q_i, S_i, and R_{ij} are polynomials whose coefficients are variables in the optimization problem, then the constraint $\{\hat{P},\hat{Q},\hat{S},\hat{R}\} = \mathcal{L}_1(P,Q_i,S_i,R_{ij})$ defines a linear equality constraint between the coefficients of Q_i, S_i, and R_{ij} and the coefficients of the polynomials that define $\hat{Q}$, $\hat{S}$, and $\hat{R}$. In the following section, we will discuss how to enforce the positivity of operators on $Z_{m,nK,1}$.

B. LMI Conditions for the Positivity of Multiplier and Integral Operators on $Z_{m,nK,1}$

In this section, we define LMI-based conditions for the positivity of operators $\mathcal{P}_{\{P,Q,R,S\}}$ on $Z_{m,nK,1}$ where Q, S, and R are continuous on $[-\tau_K, 0]$.

Our approach to positivity is based on the observation that a positive operator will always have a square root. If we assume that this square root is also of the form $\mathcal{P}_{\{P,Q,R,S\}}$ with functions Q, S, and R polynomial of bounded degree, then the results of this section give necessary and sufficient conditions. Note that although this assumption is restrictive, it is unclear whether it implies conservatism. For example, while not all positive polynomials are sum-of-squares, any positive polynomial can be approximated arbitrarily well in the sup norm on a bounded domain by a polynomial with a polynomial "root." Specifically, the following theorem assumes a square root of the form

$$\left(\mathcal{P}^{\frac{1}{2}}\begin{bmatrix} x \\ \phi \end{bmatrix}\right)(s) := N_1\sqrt{g(s)}x + N_2\sqrt{g(s)}Y_1(s)\phi(s) + \int_{-\tau_K}^{0} N_3\sqrt{g(s)}Y_2(s,\theta)\phi(\theta)d\theta$$

where the matrices N_i are unknown, the matrix-valued functions, Y_i are chosen apriori, and g is either $g(s) = 1$ or $g(s) = -s(s+\tau_K)$ (meaning $g(s)$ is nonnegative on the interval $[-\tau_K, 0]$).

Theorem 9: For any functions $Y_1 : [-\tau_K, 0] \to \mathbb{R}^{m_1\times n}$ and $Y_2 : [-\tau_K, 0]\times[-\tau_K, 0] \to \mathbb{R}^{m_2\times n}$, square integrable on $[-\tau_K, 0]$ with $g(s) \ge 0$ for $s \in [-\tau_K, 0]$, suppose that

$$P = M_{11}\cdot\frac{1}{\tau_K}\int_{-\tau_K}^{0} g(s)ds$$

$$Q(s) = \frac{1}{\tau_K}\left(g(s)M_{12}Y_1(s) + \int_{-\tau_K}^{0} g(\eta)M_{13}Y_2(\eta,s)d\eta\right)$$

$$S(s) = \frac{1}{\tau_K}g(s)Y_1(s)^T M_{22}Y_1(s)$$

$$R(s,\theta) = g(s)Y_1(s)^T M_{23}Y_2(s,\theta) + g(\theta)Y_2(\theta,s)^T M_{32}Y_1(\theta) + \int_{-\tau_K}^{0} g(\eta)Y_2(\eta,s)^T M_{33}Y_2(\eta,\theta)d\eta$$

where $M_{11}\in\mathbb{R}^{m\times m}$, $M_{22}\in\mathbb{R}^{m_1\times m_1}$, $M_{33}\in\mathbb{R}^{m_2\times m_2}$, and

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \ge 0.$$

Then, $\langle x, \mathcal{P}_{\{P,Q,R,S\}}x\rangle_{Z_{m,n,1}} \ge 0$ for all $x \in Z_{m,n,1}$.

Proof: Since $M \ge 0$, there exists a matrix $N = [\,N_1 \quad N_2 \quad N_3\,]$ such that $M = N^T N$ where $N_1 \in \mathbb{R}^{m+m_1+m_2\times m}$, $N_2\in\mathbb{R}^{m+m_1+m_2\times m_2}$, and $N_3\in\mathbb{R}^{m+m_1+m_2\times m_2}$. Using the definition of $\mathcal{P}^{\frac{1}{2}}$ introduced previously, it is straightforward to show that

$$\langle x, \mathcal{P}_{\{P,Q,R,S\}}x\rangle_{Z_{m,n,1}} = \left\langle \mathcal{P}^{\frac{1}{2}}x, \mathcal{P}^{\frac{1}{2}}x\right\rangle_{L_2^{m+m_1+m_2}} \ge 0. \quad\blacksquare$$

Theorem 9 gives a linear parametrization of a cone of positive operators using positive semidefinite matrices. Inclusion of g is inspired by the Positivstellensatz approach to local positivity of polynomials, as can be found in, e.g., [23]–[25]. For example, under mild conditions, Putinar's P-Satz states that a polynomial $p(x)$ is positive for all $x \in \{x : g(x) \ge 0\}$ if and only if it can be represented as $p(x) = s_1(x) + g(x)s_2(x)$ for some sum-of-squares polynomials s_1, s_2. In this way, Theorem 9 can be seen as an operator-valued version of this classical result. Note, however, in our case g is a function of the variable of integration and not the state and so the analogy is somewhat specious. Furthermore, for this paper, we restrict ourselves to *linear* maps of the state space. A partial discussion of parametrization of positive nonlinear operators for the stability of nonlinear time-delay systems can be found in [26] and [27].

Note that there are few constraints on the matrix-valued functions Y_1 and Y_2, functions whose elements are a basis for the multiplier and kernel functions found in $\mathcal{P}^{\frac{1}{2}}$. In this paper, these are chosen as $Y_1(s) = Z_d(s)\otimes I_n$ and $Y_2(s,\theta) = Z_d(s,\theta)\otimes I_n$, where Z_d is the vector of monomials of degree d or less in variables s and s,θ, respectively. Likewise, as mentioned, g is chosen as both $g(s) = 1$ and $g(s) = -s(s+\tau_K)$, with the resulting P, Q, R, S being the sum of the results of applying Theorem 9 to each case. To simplify notation, throughout the remainder of this paper, we will use the notation $\{P, Q, S, R\} \in \Xi_{d,m,n}$ to denote the LMI constraints on the coefficients of the polynomials P, Q, R, S implied by the conditions of Theorem 9 using both $g_i(s) = 1$ and $g_i = -s(s+\tau_K)$ as follows:

$$\Xi_{d,m,n} := \left\{\{P,Q,R,S\} : \begin{array}{c} \{P,Q,S,R\}=\{P_1,Q_1,S_1,R_1\}+\{P_2,Q_2,S_2,R_2\}, \\ \text{where } \{P_1,Q_1,S_1,R_1\} \text{ and } \{P_2,Q_2,S_2,R_2\} \text{ satisfy} \\ \text{Theorem 9 with } g{=}1 \text{ and } g{=}-s(s+\tau_K), \text{ respectively.}\end{array}\right\}$$

C. Summary of Conditions for Positivity on $Z_{m,n,K}$

The following corollary summarizes the main result of this section.

Corollary 10: Suppose there exist $d\in\mathbb{N}$, constant $\epsilon > 0$, matrix $P\in\mathbb{R}^{m\times m}$, polynomials Q_i, S_i, and R_{ij} for $i,j\in[K]$ such that

$$\mathcal{L}_1(P, Q_i, S_i, R_{ij}) \in \Xi_{d,m,nK}.$$

Then, $\langle x, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}x\rangle_{Z_{m,n,K}} \ge 0$ for all $x\in Z_{m,n,K}$.

Proof: Define $\{\hat{P},\hat{Q},\hat{S},\hat{R}\} = \mathcal{L}_1(P,Q_i,S_i,R_{ij})$. $\{\hat{P},\hat{Q},\hat{S},\hat{R}\}\in\Xi_{d,m,nK}$, by Theorem 9, $\left\langle x, \mathcal{P}_{\{\hat{P},\hat{Q},\hat{S},\hat{R}\}}x\right\rangle_{Z_{m,nK,1}} \ge 0$ for all $x\in Z_{m,nK,1}$. Next, since $\{\hat{P},\hat{Q},\hat{S},\hat{R}\} = \mathcal{L}_1(P,Q_i,S_i,R_{ij})$, by Lemma 8, $\langle x, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}x\rangle_{Z_{m,n,K}} \ge 0$ for all $x\in Z_{m,n,K}$. $\blacksquare$

To simplify presentation, the main results of the following section will reference Corollary 10 instead of the individual lemma and theorem statements, which it combines.

VIII. LMI Formulation of the Dual Stability Test

In this section, we apply the positivity conditions developed in Section VII to the operators parameterized in Section V-B, yielding a computational method for verification of the dual stability conditions of Theorem 5 and Corollary 7.

A. LMI Test for Dual Stability With Multiple Delays

We first consider the case of systems with multiple delays. The variables in the LMI are the matrix P and the coefficients of the polynomial functions Q_i S_i, and R_{ij}. The polynomial constraints $\in \Xi_{d,n,nK}$ and $\in \Xi_{d,n(K+1),nK}$ represent LMI constraints on the coefficients of the polynomials as per Theorem 9.

Theorem 11: Suppose there exist $d \in \mathbb{N}$, constant $\epsilon > 0$, matrix $P \in \mathbb{R}^{n\times n}$, polynomials $S_i, Q_i \in W_2^{n\times n}[T_i^0]$ and $R_{ij} \in W_2^{n\times n}\left[T_i^0 \times T_j^0\right]$ for $i,j \in [K]$ such that

$$\mathcal{L}_1(P-\epsilon I_n, Q_i, S_i - \epsilon I_n, R_{ij}) \in \Xi_{d,n,nK}$$

$$\mathcal{L}_1(D_1+\epsilon\hat{I}, V_i, \dot{S}_i+\epsilon I_n, G_{ij}) \in \Xi_{d,n(K+1),nK}$$

where $\hat{I} = \operatorname{diag}(I_n, 0_{nK})$, $\mathcal{L}_1$ is as defined in (6), and where P_1, V_1, G_{ij} are as defined in Theorem 5.

Furthermore, suppose

$$P = \tau_K Q_i(0)^T + \tau_K S_i(0) \quad \text{for } i \in [K]$$

$$S_i(s) = S_i(s)^T, \qquad R_{ij}(s,\theta) = R_{ji}(\theta,s)^T \quad \text{for } i,j \in [K]$$

$$Q_j(s) = R_{ij}(0,s) \quad \text{for } i,j \in [K].$$

Then, the system defined by (1) is exponentially stable.

Proof: Clearly, $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ satisfies the conditions of Lemma 3. By Corollary 10, we have

$$\begin{aligned}&\left\langle x, \mathcal{P}_{\{P-\epsilon I_n, Q_i, S_i-\epsilon I_n, R_{ij}\}} x\right\rangle_{Z_{n,K}}\\ &\quad= \left\langle x, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} x\right\rangle_{Z_{n,K}} - \epsilon\|x\|^2_{Z_{n,K}} \geq 0\end{aligned}$$

for all $x \in Z_{n,K}$. Similarly, we have

$$\begin{aligned}&\left\langle \begin{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix}\\ \phi_i\end{bmatrix}, \mathcal{P}_{\{D_1+\epsilon\hat{I}, V_i, \dot{S}_i+\epsilon I_n, G_{ij}\}} \begin{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix}\\ \phi_i\end{bmatrix}\right\rangle_{Z_{n(K+1),n,K}}\\ &= \left\langle \begin{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix}\\ \phi_i\end{bmatrix}, \mathcal{P}_{\{D_1, V_i, \dot{S}_i, G_{ij}\}} \begin{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix}\\ \phi_i\end{bmatrix}\right\rangle_{Z_{n(K+1),n,K}}\\ &\quad + \epsilon \left\| \begin{bmatrix} y_1\\ \phi_i\end{bmatrix}\right\|^2_{Z_{n,K}} \leq 0.\end{aligned}$$

Hence, Theorem 5 establishes the exponential stability of (1). ■

B. LMI for Dual Stability of Single-Delay Systems

We now state an LMI representation of the dual stability condition for a single delay ($\tau_1 = \tau_K = \tau$). This is a simplified version of Theorem 11, where we have eliminated the variables P and Q.

Theorem 12: Suppose there exist $d \in \mathbb{N}$, constant $\epsilon > 0$, polynomials $S \in W_2^{n\times n}[-\tau, 0]$ and $R \in W_2^{n\times n}[[-\tau,0]\times[-\tau,0]]$, with $R(s,\theta) = R(\theta,s)^T$ and $S(s) \in \mathbb{S}^n$ such that

$$\begin{aligned}\{\tau(R(0,0)+S(0)) - \epsilon I_n, R(0,\cdot), S-\epsilon I_n, R\} &\in \Xi_{d,2n,1}\\ -\left\{D_1+\epsilon I_n, V, \dot{S}+\epsilon I_n, G\right\} &\in \Xi_{d,2n,n}\end{aligned}$$

where D_1, V, and G are as defined in Corollary 7.

Then, the system defined by (1) in the case $K = 1$ with $\tau_1 = \tau$ is exponentially stable.

Proof: The proof follows from Theorem 11 by defining $P = \tau(R(0,0) + S(0))$ and $Q(s) = R(0,s)$ and noting that when $K = 1$

$$\{P,Q,S,R\} = \mathcal{L}_1(P,Q,S,R). \qquad ■$$

IX. MATLAB Toolbox Implementation

To assist with the application of these results, we have created a library of functions for verifying the stability conditions described in this paper. These libraries make use of modified versions of the SOSTOOLS [28] and MULTIPOLY toolboxes coupled with either SeDuMi [29] or Mosek. A complete package can be downloaded from [30] or [31] and all scripts and functions are well documented and commented. Key examples of functions included are as follows.

1) `sosjointpos_mat_ker_ndelay_PQRS_vZ.m`
   a) Declares a $[P, Q_i, R_{ij}, S_i]$ that defines an operator, which is positive on $Z_{m,n,K}$.
2) `sosmateq.m`
   a) Declare a matrix-valued equality constraint.
3) `solver_ndelay_dual_joint_nd_RL2.m`
   a) A script that combines the functions listed previously to test the stability of a user-defined problem.

These functions are implemented within the pvar framework of SOSTOOLS and are available on Code Ocean.

Pseudocode: The following is a pseudocode implementation of the conditions of Theorem 11.

(a) `[P,Q,R,S]=sosjointpos_mat_ker_ndelay_PQRS`
(b) `[D,E,G,H]=F(P,Q,R,S)`
(c) `[L,M,N,O]=sosjointpos_mat_ker_ndelay_PQRS`
(d) `sosmateq(D+L); sosmateq(E+M)`
(e) `sosmateq(G+N); sosmateq(H+O)`

Here, we use the function `F` to represent the derivative construction defined in Theorem 11. This is not an actual function in the toolbox. The derivative construction can be found in `solver_ndelay_dual_joint_nd_RL2`, however.

X. Numerical Validation

In the preceding sections, we proposed a sufficient condition for stability. However, as discussed, this condition is not necessary and there are several potential sources of conservatism, including the constraint $\mathcal{P}(X) = X$ and the assumption of an SOS representation of the positive operator. In this section, we apply the dual stability condition to a battery of numerical examples in order to determine whether this potential conservatism is significant.

In each case, a table is given that lists the maximum provably stable value of a specified parameter for each degree d. This maximum value is found using bisection on the parameter. In each case d is increased until the maximum parameter value converges to several decimal places. The true maximum is also provided as either the "limit" or "analytic" value, depending on whether this limiting value is known analytically or is a best estimate based on simulation. The computation time is also listed in CPU seconds on an Intel i7-5960X 3.0-GHz processor. This time corresponds to the interior-point (IPM) iteration in SeDuMi and does not account for preprocessing, postprocessing, or for the time spent on polynomial manipulations formulating the SDP using SOSTOOLS. Such polynomial manipulations can significantly exceed SDP computation time for small problems.

b) Example A: First, we consider a simple example that is known to be stable for $\tau \le \frac{\pi}{2}$:

$$\dot{x}(t) = -x(t-\tau)$$

d	1	2	3	analytic
$\tau_{\max}$	1.558	1.5707	1.5707	1.5707
CPU sec	0.309	0.516	0.776	

c) Example B: Next, we consider a well-studied two-dimensional (2-D), single-delay system:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & .1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t-\tau)$$

d	1	2	3	limit
$\tau_{\max}$	1.693	1.7176	1.71785	1.71785
$\tau_{\min}$	0.10018	0.100174	0.100174	0.100174
CPU sec	0.478	0.879	2.48	

d) Example C: We consider a scalar, two-delay system:

$$\dot{x}(t) = ax(t) + bx(t-1) + cx(t-2).$$

In this case, we fix $a = -2$ and $c = -1$ and search for the maximum b, which is 3 [32]–[34]:

d	1	2	3	analytic
$b_{\max}$	0.829	2.999	2.999	3
CPU sec	0.603	1.50	3.89	

e) Example D: We consider a 2-D, two-delay system where $\tau_1 = \tau_2/2$ and search for the maximum stable τ_2:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & .1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} x(t-\tau/2)$$
$$+ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t-\tau)$$

d	1	2	3	limit
$\tau_{\max}$	1.354	1.3722	1.3722	1.3722
CPU sec	1.75	7.51	27.2	

f) Example E: Next, we consider a 4-D, one-delay static output feedback system which, in [35], was found to be challenging for SOS-based methods. This example considers the static feedback system

$$\dot{x}(t) = (A - BKC)x(t) + BKCx(t-\tau)$$

where

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10 & 10 & 0 & 0 \\ 5 & -15 & 0 & -.25 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T.$$

TABLE I
COMPUTATION TIME (IN CPU SEC) INDEXED BY THE NUMBER OF STATES (n) AND THE NUMBER OF DELAYS (K)

$K \downarrow n \rightarrow$	1	2	3	5	10
1	.366	.094	.158	.686	12.8
2	.112	.295	1.260	10.83	61.05
3	.177	1.311	6.86	96.85	5223
5	.895	13.05	124.7	2014	80950
10	13.09	59.5	5077	80231	NA

In this case, we take $K = 1$. It has been reported that it requires polynomials of degree 10 even in the primal case to prove stability of $h = 3$. However, using the dual stability condition, we find a stability proof for degree $d = 4$, perhaps due to the use of the new parametrization of positive operators. The computation times for increasing degrees are listed in the following table:

d	1	2	3
CPU sec	2.23	7.45	21.6
Stability?	no	yes	yes

g) Example F: In this example, we consider a generalized n-D system with K delays and examine the computational scalability of the stability test. Our system has the form

$$\dot{x}(t) = -\sum_{i=1}^{K} \frac{x(t-i/K)}{K}.$$

For this example, we only search for polynomials of degree 2 and leave off the second kernel function. All results indexed in Table I list IPM computation time in seconds and all establish the stability of the system. The table is jointly indexed by the number of states and the number of delays.

These numerical examples indicate little, if any conservatism in the LMI implementation of the dual stability conditions, and moreover, the method is accurate for relatively low degree. Example E shows that computational complexity is a function of nK and that the results scale well to high-dimensional systems and large numbers of delay. Specifically, current desktop computers with 128-GB RAM can solve problems where $\cong nK \le 50$. This scaling can be improved if the delay channel is low dimensional through the use of the differential-difference framework [19]. In the following section, we introduce a controller synthesis condition. Note that adding the controller to the optimization problem does *not* significantly change the computational complexity of the problem.

XI. LMI CONTROLLABILITY TEST

Establishment of dual stability conditions is the first step in developing full-state feedback controller synthesis conditions. Obtaining the stabilizing controller requires two more steps. Specifically, consider the system $\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(t)$, where $u(t) \in \mathbb{R}^m$. First, we define the controllability test.

Theorem 13: Suppose there exist $d \in \mathbb{N}$, constant $\epsilon > 0$, matrix $P \in \mathbb{R}^{n\times n}$, polynomials $S_i, Q_i \in W_2^{n\times n}[T_i^0]$, $R_{ij} \in W_2^{n\times n}\left[T_i^0 \times T_j^0\right]$ for $i, j \in [K]$, matrices $W_i \in \mathbb{R}^{m\times n}$, and polynomials $Y_i \in W_2^{m\times n}$ for $i \in [K]$ such that

$$\mathcal{L}_1(P - \epsilon I_n, Q_i, S_i - \epsilon I_n, R_{ij}) \in \Xi_{d,n,nK}$$
$$-\mathcal{L}_1(D_1 + W + \epsilon \hat{I}, V_i + BY_i, \dot{S}_i + \epsilon I_n, G_{ij}) \in \Xi_{d,n(K+1),nK}$$

where $\hat{I}$, D_1, V_i, and G_{ij} are as defined in Theorem 5, $\mathcal{L}_1$ is as defined in (6), and

$$W = \begin{bmatrix} BW_0 + W_0^T B^T & BW_1 & \dots & BW_K \\ W_1^T B^T & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ W_K B^T & 0 & \dots & 0 \end{bmatrix}.$$

Furthermore, suppose P, Q_i, S_i, and R_{ij} satisfy the conditions of Lemma 3. Then, the system $\dot{x}(t) = A_0 x(t) = \sum_i A_i x(t-\tau_i) + Bu(t)$ is exponentially stabilizable and $u(t) = \mathcal{Z}\mathcal{P}^{-1}x(t)$ is an exponentially stabilizing controller where

$$\left(\mathcal{Z}\begin{bmatrix} x \\ \phi_i \end{bmatrix}\right)(s) := W_0 x + \sum_{i=1}^{K} W_i \phi_i(-\tau_i) + \sum_{i=1}^{K}\int_{-\tau_i}^{0} Y_i(s)\phi_i(s)ds.$$

Proof: If $u(t) = \mathcal{Z}\mathcal{P}^{-1}x(t)$, then $\dot{x}(t) = \left(\mathcal{A} + \mathcal{B}\mathcal{Z}\mathcal{P}^{-1}\right)x(t)$ where

$$(\mathcal{B}u)(s) = \begin{bmatrix} Bu(t) \\ 0 \end{bmatrix}.$$

Hence, as in Theorem 5, the closed-loop system is stable if

$$\left\langle \left(\mathcal{A} + \mathcal{B}\mathcal{Z}\mathcal{P}^{-1}\right)\mathcal{P}\begin{bmatrix} x \\ \phi_i \end{bmatrix}, \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n,K}}$$
$$+ \left\langle \begin{bmatrix} x \\ \phi_i \end{bmatrix}, \left(\mathcal{A} + \mathcal{B}\mathcal{Z}\mathcal{P}^{-1}\right)\mathcal{P}\begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n,K}}$$
$$= \left\langle \begin{bmatrix} \begin{bmatrix} x \\ \phi_1(-\tau_1) \\ \vdots \\ \phi_k(-\tau_K) \end{bmatrix} \\ \phi_i \end{bmatrix}, \mathcal{D} + \mathcal{D}_Z \begin{bmatrix} \begin{bmatrix} x \\ \phi_1(-\tau_1) \\ \vdots \\ \phi_k(-\tau_K) \end{bmatrix} \\ \phi_i \end{bmatrix} \right\rangle_{Z_{n(K+1),n,K}}$$
$$\le -\epsilon \left\| \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\|_{Z_{n,K}}^2 \quad \forall \begin{bmatrix} x \\ \phi_i \end{bmatrix} \in X$$

where

$$\mathcal{D}_Z := \mathcal{P}_{\{W, BY_i, 0, 0\}} \quad \text{and} \quad \mathcal{D} := \mathcal{P}_{\{D_1, V_i, \dot{S}_i, G_{ij}\}}.$$

Now, from Corollary 10, we have

$$\mathcal{P}_{\{D_1 + W + \epsilon\hat{I}, V_i + BY_i, \dot{S}_i + \epsilon I_n, G_{ij}\}} \le 0$$

and hence

$$\left\langle \begin{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{D_1 + W + \epsilon\hat{I}, V_i + BY_i, \dot{S}_i + \epsilon I_n, G_{ij}\}} \begin{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \phi_i \end{bmatrix} \right\rangle$$
$$= \left\langle \begin{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \phi_i \end{bmatrix}, (\mathcal{D} + \mathcal{D}_Z) \begin{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \phi_i \end{bmatrix} \right\rangle + \epsilon \left\| \begin{bmatrix} y_1 \\ \phi_i \end{bmatrix} \right\|_{Z_{n,K}}^2$$
$$\le 0.$$

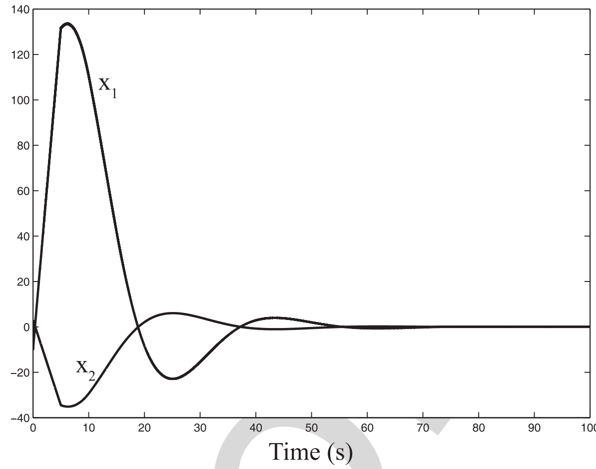

Fig. 1. MATLAB DDE23 simulation of System (6) and Controller (8) and delay $\tau = 5$ s.

Therefore, by Theorem 5, the closed-loop system is exponentially stable. ■

The second step in controller synthesis is the construction of the stabilizing controller $u(t) = \mathcal{Z}\mathcal{P}^{-1}_{\{P,Q_i,S_i,R_{ij}\}}$, which requires inversion of the operator $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$—a topic which is addressed in the sequel to this paper [36]. We illustrate these results in the single-delay case using the well-studied system

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -2 & -.5 \\ 0 & -1 \end{bmatrix} x(t-\tau) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t). \tag{6}$$

For $\tau = 5$ using simple polynomials of degree 2, we obtained the following exponentially stabilizing controller:

$$u(t) = \begin{bmatrix} -3601 \\ -944 \end{bmatrix}^T x(t) + \begin{bmatrix} -.00891 \\ .872 \end{bmatrix}^T x(t-\tau)$$
$$+ \int_{-5}^{0} \begin{bmatrix} 52.1 + 6.98s + .00839s^2 - .0710s^3 \\ 12.7 + 1.50s - .0407s^2 - .0190s^3 \end{bmatrix}^T x(t+s)ds. \tag{7}$$

Simulations for fixed initial conditions were performed and can be seen in Fig. 1.

XII. Conclusion

We have proposed a new form of dual Lyapunov stability condition that allows the convexification of the controller synthesis problem for delayed and other infinite-dimensional systems. This duality principle requires a Lyapunov operator that is positive, invertible, and self-adjoint and preserves the structure of the state space. We have proposed such a class of operators and used them to create stability conditions that can be expressed as positivity and negativity of quadratic Lyapunov functions. These dual stability conditions have a tridiagonal structure, which is distinct from standard Lyapunov–Krasovskii forms and may be exploited to increase performance when studying systems with large numbers of delays. The dual stability condition is presented in a format that can be adapted to many existing computational methods for Lyapunov stability analysis. We have applied the sum-of-squares approach to enforce the positivity of the quadratic forms and tested the stability condition in both the single- and multiple-delay cases. Numerical testing on several examples indicates the method is not likely to be conservative. The contribution of this paper is not in the efficiency of the stability test, however, as these are likely less efficient when compared to, e.g., previous SOS results due to the structural constraints imposed upon the operator. Rather, the contribution is in the convexification of the synthesis problem, which opens the door for dynamic output-feedback H_∞ synthesis for infinite-dimensional systems. This potential is demonstrated in the numerical example of controller synthesis for a single-delay system.

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